Discrete Comput Geom 21:437–447 (1999)

Discrete & Computational
Geometry

Vertex Degrees of Steiner Minimal Trees in ℓ_p^d and Other Smooth Minkowski Spaces

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Abstract. We find upper bounds for the degrees of vertices and Steiner points in Steiner Minimal Trees (SMTs) in the d-dimensional Banach spaces ℓ_p^d independent of d. This is in contrast to Minimal Spanning Trees, where the maximum degree of vertices grows exponentially in d [19]. Our upper bounds follow from characterizations of singularities of SMTs due to Lawlor and Morgan [14], which we extend, and certain ℓ_p-inequalities. We derive a general upper bound of $d+1$ for the degree of vertices of an SMT in an arbitrary smooth d-dimensional Banach space (i.e. Minkowski space); the same upper bound for Steiner points having been found by Lawlor and Morgan. We obtain a second upper bound for the degrees of vertices in terms of 1-summing norms.

1. Introduction

Given a metric space (X, ρ) and a set $S \subseteq X$, a *Minimal Spanning Tree* (*MST*) of S is a tree T with vertex set $V(T) = S$ and edge set $E(T)$ such that

$$\sum_{\{x,y\}\in E(T)} \rho(x, y)$$

is minimal among all trees on S.

A *Steiner Minimal Tree* (*SMT*) of S is a tree T with vertex set $V(T)$ satisfying $S \subseteq V(T) \subseteq X$ such that

$$\sum_{\{x,y\}\in E(T)} \rho(x, y)$$

is minimal among all trees on S with vertex sets satisfying $S \subseteq V(T) \subseteq X$. The elements of S are *vertices*, and the elements of $V(T)\backslash S$ are *Steiner points* of the SMT.

Estimates for the largest degrees of MSTs and SMTs have consequences for the complexities of algorithms that find such trees. For example, it is known that an MST on n points can be calculated in polynomial time [2], while calculating the SMT in the euclidean or rectilinear planes is NP-hard [7], [8]. Upper bounds for the degrees of vertices and Steiner points are used to reduce the search space of known exponential time algorithms.

Distance functions other than euclidean or rectilinear are sometimes used. The ℓ_p metrics have been found useful; see [15]. We consider general Minkowski spaces, i.e. finite-dimensional Banach spaces, and then specialize to ℓ_p^d, d-dimensional real linear space with norm

$$\|(x_1, \ldots, x_d)\|_p = \left(\sum_{i=1}^{d} |x_i|^p \right)^{1/p}.$$

It is known that in a Minkowski space, the largest degree of an MST is equal to the so-called Hadwiger number $H(B)$ of the unit ball B of the space [3]. For each $1 \leq p \leq \infty$ there is an exponential lower bound for the Hadwiger number of ℓ_p^d, $H(B_p^d) > (1+\varepsilon_p)^d$ [19].

In contrast to this, we show in Section 4 that the degrees of both vertices and Steiner points of an SMT in ℓ_p^d ($1 < p < \infty$) are bounded above by functions of p alone, independent of d. For $p > 2$ we derive a general upper bound of 7, with various sharper values for specific p. For $1 < p < 2$, however, we find an upper bound exponential in $p^* := p/(p-1)$, and a lower bound linear in p^*, as p tends to 1. Thus with respect to the SMT problem, ℓ_p^d behaves very similarly to euclidean space, where both vertices and Steiner points have degree at most 3.

For general d-dimensional smooth Minkowski spaces, it is known that the degree of a Steiner point is at most $d+1$ [14]. In Section 3 we show that this upper bound also holds for the degree of a vertex in an SMT. The proof has two ingredients. Firstly, in Section 2 we derive a characterization of the local structure of a vertex in an SMT (Theorem 2) similar to the characterization of Steiner points due to Lawlor and Morgan [14]. We also rederive their characterization, paying attention to some combinatorial subtleties (Theorem 1). Both derivations are completely elementary. The second ingredient is Theorem 4, which generalizes a result of [6] and [14], thus answering a question in [21].

In Theorem 5 we also obtain an upper bound for the degrees of vertices and Steiner points in terms of the 1-summing norm of the dual of the space.

2. Derivation of the Singularity Characterizations

Theorem 1 below, due to [14], provides a characterization of the structure of the neighbourhood of a Steiner point in an SMT in a smooth Minkowski space. We give a similar characterization of the structure of the neighbourhood of a vertex in an SMT in Theorem 2. Both characterizations are in terms of unit vectors in the dual of the Minkowski space.

We now recall some facts about dual spaces. Note that the discussion below pertains to finite-dimensional Banach spaces, i.e. Minkowski spaces; see [23].

For any d-dimensional real vector space X, the *dual* of X, denoted by X^*, is the vector space of linear functionals $x^*: X \to \mathbb{R}$. This dual is also a d-dimensional vector space. We denote application of $x^* \in X^*$ to $x \in X$ by $\langle x^*, x\rangle$. If X is furthermore a Minkowski space with norm $\|\cdot\|$, then $\|x^*\|^* = \sup_{\|x\|\leq 1}\langle x^*, x\rangle$ defines a norm on X^*.

We say that a Minkowski space is *smooth* if

$$\lim_{t\to 0}\frac{\|x+th\|-\|x\|}{t} =: f_x(h)$$

exists for all $x, h \in X$ with $x \neq 0$. It follows easily that $f_x \in X^*$, $\|f_x\|^* = 1$ and $\langle f_x, x\rangle = \|x\|$. A linear functional $x^* \in X^*$ is a *norming functional* of x if x^* satisfies $\langle x^*, x\rangle = \|x\|$ and $\|x^*\|^* = 1$. Each non-zero vector in a Minkowski space has a norming functional (the Hahn–Banach theorem). A Minkowski space is smooth iff each non-zero vector has a unique norming functional.

A Minkowski space X is *strictly convex* if $\|x\| = \|y\| = 1$ and $x \neq y$ imply that $\|\frac{1}{2}(x+y)\| < 1$, equivalently, that the boundary of the unit ball of X does not contain any straight line segment. A Minkowski space X is smooth (resp. strictly convex) iff X^* is strictly convex (resp. smooth).

The balancing and collapsing conditions in Theorems 1 and 2 thus occur in a strictly convex space. We say that a finite set of unit vectors $x_1, \dots, x_m \in X$ satisfies the *balancing condition* if

$$\sum_{i=1}^{m} x_i = 0, \tag{1}$$

and satisfies the *collapsing condition* if

$$\left\|\sum_{i\in J} x_i\right\| \leq 1 \qquad \text{for each} \quad J \subseteq \{1,\dots,m\}. \tag{2}$$

Note that the above balancing condition is the characterization of the so-called Fermat point of a set of points in a smooth Minkowski space in the non-absorbing case (i.e. where the Fermat point differs from the given points) in terms of norming functionals [1].

Theorem 1 [14]. *Let $a_1, \dots, a_m$ be distinct non-zero points in a smooth Minkowski space X. For each $i = 1, \dots, m$, let a_i^* be the norming functional of a_i. Then the tree connecting each a_i to 0 is an SMT of $S = \{a_1, \dots, a_m\}$ iff $\{a_1^*, \dots, a_m^*\}$ satisfies the balancing and collapsing conditions in X^*.*

Proof. ($\Rightarrow$) Since we have an SMT, for any $x \in X$,

$$\sum_{i=1}^{m}\|a_i - x\| \geq \sum_{i=1}^{m}\|a_i\|,$$

i.e. for any unit vector $e \in X$ the function

$$\varphi_e(t) := \sum_{i=1}^{m}(\|a_i + te\| - \|a_i\|) \geq 0$$

attains a minimum at $t = 0$. For sufficiently small t, $a_i + te \neq 0$, and $\varphi_e(t)$ is differentiable at 0, with $\varphi_e'(0) = 0$. However,

$$\varphi_e'(0) = \lim_{t\to 0} \sum_{i=1}^{m} \frac{\|a_i + te\| - \|a_i\|}{t} = \sum_{i=1}^{m} \langle a_i^*, e\rangle.$$

Therefore, $\sum_{i=1}^{m} a_i^* = 0$.

Secondly, given $J \subseteq \{1, \dots, m\}$, define a tree T_J as follows: Connect $\{a_i : i \in J\}$ to an arbitrary point x, connect $\{a_i : i \notin J\}$ to 0, and connect x to 0. Then the total length of T_J is not smaller than $\sum_{i=1}^{m} \|a_i\|$:

$$\sum_{i\in J} \|a_i - x\| + \sum_{i\notin J} \|a_i\| + \|x\| \geq \sum_{i=1}^{m} \|a_i\|,$$

i.e. for any unit vector e the function

$$\psi_e(t) := \sum_{i\in J} (\|a_i - te\| - \|a_i\|) + |t| \geq 0$$

attains a minimum at $t = 0$. However, ψ_e is not differentiable at 0. Circumventing this difficulty, we calculate

$$\begin{aligned} 0 \ &\leq \ \lim_{t\to 0^+} \frac{\psi_e(t)}{t} = \lim_{t\to 0^+} \sum_{i\in J} \frac{\|a_i - te\| - \|a_i\|}{t} + 1 \\ &= \sum_{i\in J} \langle a_i^*, -e\rangle + 1 \end{aligned}$$

and $\left\langle \sum_{i\in J} a_i^* \right\rangle e \leq 1$ for all unit e. Thus $\| \sum_{i\in J} a_i^* \|^* \leq 1$.

($\Leftarrow$) Let $a_1^*, \dots, a_m^* \in X^*$ satisfy (1) and (2), and let T be any SMT of $\{a_1, \dots, a_m\}$. We have to show that

$$\sum_{\{x,y\}\in E(T)} \|x - y\| \geq \sum_{i=1}^{m} \|a_i\|.$$

For $i \geq 2$, let P_i be any non-overlapping path in T from a_1 to a_i, i.e. $P_i = x_1^{(i)} x_2^{(i)} \cdots x_{k_i}^{(i)}$ with $x_1^{(i)} = a_1$, $x_{k_i}^{(i)} = a_i$ and $\{x_j^{(i)}, x_{j+1}^{(i)}\}$ distinct edges in $E(T)$ for $j = 1, \dots, k_i - 1$. Note that each edge of T is used in some P_i, since the union of the paths is a connected subgraph of T. For each edge $e \in E(T)$ we assign a direction depending on the way e is traversed in some P_i containing e. This direction is unambigious, since if two paths would give conflicting directions, their union would contain a cycle. We denote a directed edge from x to y by $(x, y) = \vec{e}$ and the set of directed edges by $\vec{E}(T)$. For each $\vec{e} \in \vec{E}(T)$, let $S_{\vec{e}} := \{i \geq 2 : \vec{e} \in P_i\}$. Then

$$\begin{aligned} \sum_{i=1}^{m} \|a_i\| \ &= \ \sum_{i=1}^{m} \langle a_i^*, a_i\rangle \\ &= \sum_{i=2}^{m} \langle a_i^*, a_i - a_1\rangle \qquad \text{(by the balancing condition)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=2}^{m} \sum_{j=2}^{k_i-1} \langle a_i^*, x_{j+1}^{(i)} - x_j^{(i)} \rangle \\
&= \sum_{\vec{e}=(x,y)\in\vec{E}(T)} \sum_{i\in S_{\vec{e}}} \langle a_i^*, y - x \rangle \\
&\le \sum_{\vec{e}=(x,y)\in\vec{E}(T)} \left\| \sum_{i\in S_{\vec{e}}} a_i^* \right\|^* \|x - y\| \\
&\le \sum_{(x,y)\in\vec{E}(T)} \|x - y\| \qquad \text{(by the collapsing condition).} \qquad \square
\end{aligned}
$$

As mentioned in [14], the balancing and collapsing conditions are still sufficient for the tree in the above theorem to be an SMT in non-smooth spaces, if (1) and (2) holds for *some* norming functional a_i^* for each a_i. A similar remark holds for the next theorem.

Theorem 2. *Given points $a_1, \dots, a_m \neq 0$ in a smooth Minkowski space X, let a_i^* be the norming functional of a_i. Then the tree connecting each a_i to 0 is an SMT of $S = \{0, a_1, \dots, a_m\}$ iff $\{a_1^*, \dots, a_m^*\}$ satisfies the collapsing condition in X^*.*

Proof. Similar to the proof of the previous theorem. Note that there is no balancing condition, since we cannot perturb 0, as 0 is in this case a vertex of the SMT. □

3. Upper Bounds for Smooth Minkowski Spaces

For a Minkowski space X, let $v(X)$ be the largest degree of a vertex of an SMT in X, and let $s(X)$ be the largest degree of a Steiner point in an SMT.

In [14] it is shown that $s(X) \le d+1$ if X is smooth and d-dimensional. This inequality is sharp in the sense that there are spaces and SMTs where the degree of $d+1$ is attained. We give a similar bound for $v(X)$:

Theorem 3. *For a smooth Minkowski space X of dimension $d \ge 2$,*

$$3 \le s(X) \le v(X) \le d + 1.$$

The outer inequalities are sharp in general.

Proof. Theorems 1 and 2 immediately imply $s(X) \le v(X)$.

In any two-dimensional subspace of the dual X^* we can find two unit vectors x^*, y^* such that $\|x^* - y^*\|^* = 1$. Then the set $\{x^*, -y^*, y^* - x^*\}$ satisfies (1) and (2).

The euclidean spaces $X = \ell_2^d$ are examples where $s(X) = v(X) = 3$.

The rest of the theorem now follows from Theorem 2 above and Theorem 4 below. An example where $v(X) = d + 1$ may be constructed in the same way as for $s(X)$, as is done in Lemma 4.3 of [14]. □

The following theorem, suggested in [21], sharpens results from [6] and [14] by eliminating the balancing condition from the hypotheses.

Theorem 4. *Let X be a strictly convex d-dimensional Minkowski space. If $x_1, \dots, x_m \in X$ are unit vectors satisfying the collapsing condition, then $m \le d+1$. Furthermore, if the balancing condition is not satisfied, i.e. $\sum_{i=1}^m x_i \ne 0$, then $m \le d$.*

Proof. Let $x_i^* \in X^*$ be norming functionals of x_i. Firstly, for $i \ne j$ we have

$$1 + \langle x_i^*, x_j \rangle = \langle x_i^*, x_i + x_j \rangle \le \|x_i + x_j\| \le 1$$

by the collapsing condition, and thus

$$\langle x_i^*, x_j \rangle \le 0 \qquad \text{for} \quad i \ne j.$$

Secondly,

$$0 \le \left\langle x_i^*, -\sum_{j\ne i} x_j \right\rangle \le \left\| \sum_{j\ne i} x_j \right\| \le 1.$$

If $\langle x_i^*, -\sum_{j\ne i} x_j \rangle = 1$, then x_i^* is also a norming functional of $-\sum_{j\ne i} x_j$, which is now a unit vector. Then, since X is strictly convex, it easily follows that $x_i = -\sum_{j\ne i} x_j$.

Thus, if $\sum_{i=1}^m x_i \ne 0$, then

$$0 \le \left\langle x_i^*, -\sum_{j\ne i} x_j \right\rangle < 1,$$

and the diagonal of the matrix $A = [\langle x_i^* \rangle x_j]_{i,j=1}^m$ majorizes the rows. Thus A is invertible. Since A has rank at most d, we obtain $m \le d$.

If, however, $\sum_{i=1}^m x_i = 0$, the above argument applied to $x_1, \dots, x_{m-1}$ gives $m - 1 \le d$. □

Note that in the above proof, we do not use the full force of the collapsing condition.

For the next bound, we recall a notion from the local theory of Banach spaces. The *absolutely summing constant* or the 1-*summing norm* (of the identity operator on) a Minkowski space X is defined to be

$$\pi_1(X) := \inf \left\{ c > 0 : \forall x_1, \dots, x_m \in X : \sum_{i=1}^m \|x_i\| \le c \max_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^m \varepsilon_i x_i \right\| \right\}.$$

This notion has been studied extensively; see, e.g. [16], [5], [20], [12], [9], and [13]. Note that the quantity $(2\pi_1(X))^{-1}$ has also been called the *Macphail constant* in the literature.

Theorem 5. *For a smooth Minkowski space X,*

$$s(X) \le v(X) \le 2\pi_1(X^*).$$

Proof. Let $x_1^*, \dots, x_m^* \in X^*$ be unit vectors satisfying the collapsing condition, with $m = v(X)$. Then, for any sequence of signs $\varepsilon_i = \pm 1, i = 1, \dots, m$, we have $\| \sum_i \varepsilon_i x_i^* \|^* \leq 2$, hence

$$m = \sum_{i=1}^m \|x_i^*\|^* \geq \frac{m}{2} \max_{\varepsilon_i=\pm 1} \left\| \sum_{i=1}^m \varepsilon_i x_i^* \right\|^*,$$

implying that $m/2 \leq \pi_1(X^*)$. □

It is known that $\sqrt{d} \leq \pi_1(X) \leq d$ for any d-dimensional X [12]. We thus obtain an upper bound worse than that of Theorem 3, although it is of the same order. It is, however, possible in principle to obtain bounds better than that of Theorem 3 for specific spaces. However, we cannot do better than $2\sqrt{d}$.

4. Upper Bounds for ℓ_p^d

Restricting ourselves to the smooth case $1 < p < \infty$, we recall that the dual of ℓ_p^d is $(\ell_p^d)^* = \ell_{p^*}^d$, where $1/p + 1/p^* = 1$. We use the Khinchin inequalities with the best constants, due to [22], [10], and [11].

Khinchin's Inequalities. *For any* $1 \leq q < \infty$ *there exist constants* $A_q, B_q > 0$ *such that for any* $a_1, \dots, a_n \in \mathbb{R}$ *we have*

$$A_q \left(\sum_{i=1}^n a_n^2 \right)^{1/2} \leq \left(2^{-n} \sum_{\varepsilon_i=\pm 1} \left| \sum_{i=1}^n \varepsilon_i a_i \right|^q \right)^{1/q} \leq B_q \left(\sum_{i=1}^n a_n^2 \right)^{1/2}.$$

For $q \geq 2$ *we have* $A_q = 1$, $B_q = \sqrt{2}(\Gamma((q+1)/2)/\sqrt{\pi})^{1/q}$, *and for* $1 \leq q \leq 2$ *we have* $B_q = 1$,

$$A_q = \begin{cases} 2^{1/2-1/q} & \text{if} \quad q < q_0, \\ \sqrt{2} \left(\dfrac{\Gamma((q+1)/2)}{\sqrt{\pi}} \right)^{1/q} & \text{if} \quad q \geq q_0, \end{cases}$$

where $q_0 \approx 1.8474$ *is defined by* $\Gamma((q_0+1)/2) = \sqrt{\pi}/2$, $1 < q_0 < 2$.

The following lemma is analogous to Hilfsatz 4 of [4]. We omit the proof, which easily follows from calculus.

Lemma 6. *Let* $x, y \in \mathbb{R}$ *and* $1 \leq q \leq 2$. *Then*

$$|x + y|^q \geq 2^{q-2}(|x|^{q/2} \operatorname{sgn} x + |y|^{q/2} \operatorname{sgn} y)^2.$$

The earliest reference we could find to the following lemma is [17].

Lemma 7. *Let* $x_1, \dots, x_m \in \ell_2^d$ *satisfy* $\|x_i\|_2 = 1$ *and* $\langle x_i, x_j \rangle < -1/n$ *for* $i \neq j$, *where* n *is a positive integer. Then* $m \leq n$.

Proof.

$$0 \le \left\| \sum_{i=1}^{m} x_i \right\|_2^2 = \sum_{i=1}^{m} \|x_i\|_2^2 + 2\sum_{i<j} \langle x_i, x_j \rangle$$
$$< m - \frac{m(m-1)}{n}. \qquad \square$$

The next two theorems show that the largest degree of a vertex $v(\ell_p^d)$ and the largest degree of a Steiner point $s(\ell_p^d)$ in an SMT in ℓ_p^d are both relatively small and independent of d. In particular, for $p \ge 2$ we have a general upper bound of 7. For $2 \le p \lesssim 3.40942$ we furthermore obtain the exact values of $v(\ell_p^d)$ and $s(\ell_p^d)$. For $p < 2$ we only obtain a lower bound linear in p^* and an upper bound exponential in p^*. It is not clear what the correct order of growth should be case.

Theorem 8. *Let* $2 \le p < \infty$ *and* $d \ge 3$. *Then*

$$s(\ell_p^d) = v(\ell_p^d) = 3 \quad \textit{for} \quad 2 \le p < \frac{\log 3}{\log 3 - \log 2} \approx 2.70951, \tag{3}$$

$$s(\ell_p^d) = v(\ell_p^d) = 4 \quad \textit{for} \quad \frac{\log 3}{\log 3 - \log 2} \le p < \frac{\log 8 - \log 3}{\log 4 - \log 3} \approx 3.40942, \tag{4}$$

$$4 \le s(\ell_p^d) \le v(\ell_p^d) \le 5 \quad \textit{for} \quad \frac{\log 3}{\log 3 - \log 2} \le p < \frac{\log 4}{\log 4 - \log 3} \approx 4.81884, \tag{5}$$

$$4 \le s(\ell_p^d) \le v(\ell_p^d) \le 6 \quad \textit{for} \quad \frac{\log 3}{\log 3 - \log 2} \le p < \frac{\log 4}{\log 8 - \log 7} \approx 10.3818, \tag{6}$$

$$4 \le s(\ell_p^d) \le v(\ell_p^d) \le 7 \quad \textit{for all} \quad p \ge \frac{\log 3}{\log 3 - \log 2}. \tag{7}$$

Proof. Let $q := p^* = p/(p-1)$. The lower bound of 3 for $s(X)$ and $v(X)$ comes from Theorem 3. For $p \ge (\log 3)/(\log 3 - \log 2)$, i.e. for $q \le (\log 3)/(\log 2)$, we obtain four unit vectors in ℓ_q^d satisfying the balancing and collapsing conditions as follows:

$$x_1 := 3^{-1/q}(1,1,1), \qquad x_2 := 3^{-1/q}(1,-1,-1),$$
$$x_3 := 3^{-1/q}(-1,1,-1), \qquad x_4 := 3^{-1/q}(-1,-1,1).$$

For the upper bounds, let $x_1, \dots, x_m \in \ell_q^d$ be unit vectors satisfying the collapsing condition.

We first use a “twisting” technique used in the Geometry of Numbers; see [18]. Denote the coordinates of x_i as $x_i = (x_{i,1}, x_{i,2}, \dots, x_{i,d})$. Define $\tilde{x}_i = (\tilde{x}_{i,1}, \tilde{x}_{i,2}, \dots, \tilde{x}_{i,d})$ by $\tilde{x}_{i,n} := |x_{i,n}|^{q/2} \operatorname{sgn} x_{i,n}$. Note that $\|\tilde{x}_i\|_2 = 1$, i.e. we have twisted x_i to become a euclidean unit vector. By Lemma 6 we obtain for $i \ne j$ that

$$1 \ge \|x_i + x_j\|_q^q \ge 2^{q-2}\|\tilde{x}_i + \tilde{x}_j\|_2^2 = 2^{q-2}(2 + 2\langle \tilde{x}_i, \tilde{x}_j \rangle),$$

where $\langle \cdot, \cdot \rangle$ denotes the standard euclidean inner product. Thus $\langle \tilde{x}_i, \tilde{x}_j \rangle \le 2^{1-q} - 1 < 0$. If $p < (\log 3)/(\log 3 - \log 2)$, i.e. $q > (\log 3)/(\log 2)$, then $2^{1-q} - 1 < -\frac{1}{3}$. By Lemma 7 we obtain $m \le 3$, and (3) follows. Similarly, if $p < (\log 8 - \log 3)/(\log 4 - \log 3)$, then $2^{1-q} - 1 < -\frac{1}{4}$, hence $m \le 4$, and (4) follows.

For the remaining estimates we apply Khinchin's inequalities. We may assume in the light of (3) and (4) that $p \ge (\log 8 - \log 3)/(\log 4 - \log 3)$, i.e. $q \le (\log 8 - \log 3)/(\log 2) < q_0$. Thus $A_q = 2^{1/2-1/q}$. By (2) we have for any sequence of signs $\varepsilon_i = \pm 1$, $i = 1, \dots, m$, that $\|\sum_{i=1}^m \varepsilon_i x_i\|_q \le 2$. Therefore,

$$\begin{aligned}
2^q &\ge \sum_{n=1}^d 2^{-m} \sum_{\varepsilon_i=\pm 1} \left| \sum_{i=1}^m \varepsilon_i x_{i,n} \right|^q \\
&\ge \sum_{n=1}^d A_q^q \left(\sum_{i=1}^m x_{i,n}^2 \right)^{q/2} \qquad \text{(Khinchin's inequality)} \\
&= A_q^q \sum_{n=1}^d \|(|x_{i,n}|^q)_i\|_{2/q} \qquad \text{(where } (|x_{i,n}|^q)_{i=1}^m \in \ell_{2/q}^m\text{)} \\
&\ge A_q^q \left\| \left(\sum_{n=1}^d |x_{i,n}|^q \right)_i \right\|_{2/q} \qquad \text{(triangle inequality in } \ell_{2/q}^m\text{)} \\
&= A_q^q \left(\sum_{i=1}^m \|x_i\|_q^2 \right)^{q/2} \\
&= A_q^q m^{q/2},
\end{aligned}$$

and $m \le 4/A_q^2 = 2^{3-2/p} < 8$. Estimates (5), (6) and (7) now follow. □

Theorem 9. *Let* $1 < p < 2$ *and* $d \ge 3$. *Then*

$$\min(d, f(p^*)) \le s(\ell_p^d), v(\ell_p^d) \le \min(d+1, 2^{p^*}), \tag{8}$$

where, for $q > 2$,

$$f(q) := \max\{d : 2(d-2)^q + (d-2)2^q \le (d-1)^q + d - 1\}.$$

In particular,

$$\begin{aligned}
f(q) &\ge 3 && \textit{for} \quad q > 2, \\
f(q) &\ge 4 && \textit{for} \quad q \ge 3.21067, \\
f(q) &\ge 5 && \textit{for} \quad q \ge 3.40093, \\
f(q) &\ge \left\lceil \frac{q}{\log 2} \right\rceil && \textit{for} \quad q \ge 3.69247.
\end{aligned}$$

Proof. Let $q := p^* = p/(p-1)$.

The upper bound follows from Theorem 3 and an application of Khinchin's inequalities:

$$2^q \ge \sum_{n=1}^d 2^{-m} \sum_{\varepsilon_i=\pm 1} \left| \sum_{i=1}^m \varepsilon_i x_{i,n} \right|^q$$

$$\begin{aligned}
&\geq \sum_{n=1}^{d} \left(\sum_{i=1}^{m} x_{i,n}^2 \right)^{q/2} \qquad \text{(Khinchin's inequality)} \\
&= \sum_{n=1}^{d} \|x_i\|_2^q \\
&\geq \sum_{n=1}^{d} \|x_i\|_q^q = m \qquad \text{(monotonicity of } q\text{-norms).}
\end{aligned}$$

For the lower bound we may assume that $d \geq 4$. Let x_i be the vector in ℓ_q^d with $d-1$ in its ith coordinate, and -1 in the remaining coordinates, for $i = 1, \dots, d$. Let $\hat{x}_i := \|x_i\|_q^{-1} x_i$. Then $\{\hat{x}_i : i = 1, \dots, d\}$ satisfies the balancing condition (1). This set will also satisfy the collapsing condition iff, for all $2 \leq k \leq d/2$,

$$g(k, d, q) := k(d-k)^q + (d-k)k^q \leq (d-1)^q + d - 1 = g(1, d, q).$$

By differentiating with respect to q and using $2 \leq k \leq d/2$, it is easily seen that if $g(k, d, q) \leq g(1, d, q)$ holds for some $q = q'$, then it will hold for all $q \geq q'$. The following numerical facts are easily verified:

$$\begin{aligned}
g(k, d, q) &\leq g(2, d, q) \qquad \text{for} \quad 4 \leq d \leq 7, \quad 2 \leq k \leq \frac{d}{2} \quad \text{and} \quad p \geq 3.2, \\
g(2, 4, q) &\leq g(1, 4, q) \qquad \text{for} \quad q \geq 3.21066\dots, \\
g(2, 5, q) &\leq g(1, 5, q) \qquad \text{for} \quad q \geq 3.40092\dots, \\
g(2, 6, q) &\leq g(1, 6, q) \qquad \text{for} \quad q \geq 3.69246\dots,
\end{aligned}$$

and

$$g(2, 7, q) \leq g(1, 7, q) \qquad \text{for} \quad q \geq 4.09345\dots.$$

It is now sufficient to show for $d \geq 8$ and $q = (d-1)\log 2$ that $g(k, d, q) \leq g(2, d, q) \leq g(1, d, q)$ for all $2 \leq k \leq d/2$. Firstly, note that in this case $g(2, d, q) \leq g(1, d, q)$ is equivalent to

$$2^{1+(d-1)\log(d-2)} + (d-2)2^{(d-1)\log 2} \leq 2^{(d-1)\log(d-1)} + d - 1,$$

which is easily verified for $d \geq 8$.

Secondly, to show that $g(k, d, q) \leq g(2, d, q)$ it is sufficient to show that

$$f(x) := x(1-x)^q + (1-x)x^q, \qquad \frac{2}{d} \leq x \leq \tfrac{1}{2},$$

attains its maximum at $x = 2/d$. To see this, it is in turn sufficient to show that $f'(x) \leq 0$ for $2/d \leq x \leq \frac{1}{2}$. By setting $y = (1-x)/x$ we find that it is sufficient to show that, for $1 \leq y \leq d/2 - 1$,

$$x^{-q} f'(x) = y^q - qy^{q-1} - 1 + qy =: h(y) \leq 0.$$

By calculating the first and second derivatives of $h(y)$ and recalling that $q > 3$, it is seen that $h(y)$ does not attain its maximum if $1 < y < d/2 - 1$. Since $h(1) = 0$, we only have to show that $h(d/2 - 1) \leq 0$, which easily follows from $q \geq 4$ and $d \geq 8$. $\square$

Acknowledgment

This paper is part of the author's Ph.D. thesis being written under the supervision of Prof. W. L. Fouché at the University of Pretoria.

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Received April 22, 1997, *and in revised form October* 1, 1997.