Intersection Properties of Families of Convex (n, d) -Bodies*

T. Kaiser¹ and Y. Rabinovich²

¹ Department of Applied Mathematics, Charles University,
Malostranské nám. 25, 118 00 Praha, Czech Republic
kaiser@kam.ms.mff.cuni.cz

² Department of Mathematics and Computer Science,
Ben-Gurion University of the Negev,
Beer-Sheva 84105, Israel
uri@cs.bgu.ac.il

Abstract. We study a multicomponent generalization of Helly's theorem.

An (n, d) -body K is an ordered n -tuple of d -dimensional sets, $K = \langle K_1, \dots, K_n \rangle$. A family $\mathcal{F}$ of (n, d) -bodies is *weakly intersecting* if there exists an n -point $\mathbf{p} = \langle p_1, \dots, p_n \rangle$ such that for every $K \in \mathcal{F}$ there exists an index $1 \leq i \leq n$ for which $p_i \in K_i$. A family $\mathcal{F}$ of (n, d) -bodies is *strongly intersecting* if there exists an index i such that $\bigcap_{K \in \mathcal{F}} K_i \neq \emptyset$. The main question addressed in this paper is:

What is the smallest number $H(n, d)$, such that for every finite family of convex (n, d) -bodies, if every $H(n, d)$ of them are strongly intersecting, then the entire family is weakly intersecting?

We establish some basic facts about $H(n, d)$, and also prove an upper bound $H(n, d) \leq ([\log_2(n+1)] + 1)^d$. In addition, we introduce and discuss two interesting related questions of a combinatorial-topological nature.

1. Introduction

Helly's theorem is a fundamental result in convex analysis (see, e.g., [DGK], or Eckhoff's survey in [GW], for a detailed exposition). Call a family of bodies *intersecting* if its members have a nonempty intersection. Helly's theorem claims that a finite family $\mathcal{F}$ of convex bodies in $\mathbb{R}^d$ is intersecting if and only if every subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size $d + 1$ is intersecting. Over the years many different generalizations of this theorem have

---

* This paper was completed while the second author was visiting Cornell University, Ithaca, New York, USA.

appeared; the one closest to our framework deals with intersection properties of new objects, each being a union of n or less disjoint convex sets in $\mathbb{R}^d$. Since the families of such objects do not in general have the Helly property (i.e., there exists no fixed number k such that the family is intersecting if and only if all its size- k subfamilies are), an additional constraint was introduced, namely, that the intersection of any subset of the members of the family is also a union of n or less pairwise disjoint convex bodies. Under these conditions it was proven that such families have Helly number $n(d+1)$ (see [Mo]).

However, this additional constraint seems to be somewhat unnatural for $\mathbb{R}^d$. Slightly modifying the definition of the basic object, which we call the (n, d) -body, we cause the constraint to become automatically satisfied:

Definition 1.1. (n, d) -body K is an ordered n -tuple of d -dimensional sets $K = \langle K_1, \dots, K_n \rangle$. Often K_i is called the i th component of K . Operations (i.e., unions, intersections, etc.) on (n, d) -bodies are defined componentwise. Thus, for example, $\langle K_1, \dots, K_n \rangle \cap \langle Q_1, \dots, Q_n \rangle = \langle K_1 \cap Q_1, \dots, K_n \cap Q_n \rangle$. An (n, d) -body K is called convex (open, closed, empty, etc.) if all components of K are such. It is convenient to depict (n, d) -bodies as subsets of a space with n connected components, each being a copy of $\mathbb{R}^d$. The i th component of K is then the part of K in $\mathbb{R}_i^d$, the i th component of this space.

What is the Helly number of families of convex (n, d) -bodies? The answer is the same as before, $n(d+1)$. However, unlike before, the proof of this fact is rather obvious, and poses no special difficulties. In this paper we study a different, also natural, generalization of the classical Helly theorem to (n, d) -bodies, which leads to deeper and more interesting questions. We first generalize the notion of an intersecting family.

Definition 1.2. A family $\mathcal{F}$ of (n, d) -bodies is called *weakly intersecting* if there exists an n -point $\mathbf{p} = \langle p_1, \dots, p_n \rangle$ where every p_i is a point in $\mathbb{R}_i^d$, such that, for every $K \in \mathcal{F}$, $K \cap \mathbf{p} \neq \emptyset$.

Weak intersection is opposed to the *strong* one, which has the usual meaning. In what follows, the word “strong” will sometimes be omitted. The notion of weak intersection differs from that of the strong one in many ways. For instance, families of (n, d) -bodies do not in general have the Helly property with respect to weak intersection. The main question studied in this paper is:

What is the smallest number $H(n, d)$ such that for every finite family $\mathcal{F}$ of convex (n, d) -bodies, if any $H(n, d)$ members of the family are strongly intersecting, then $\mathcal{F}$ is weakly intersecting?

Notice that by the above discussion, $H(n, d)$ exists and is $\leq n(d+1)$. As we shall see later, this bound can be improved to $H(n, d) \leq nd$ for $n > 1$.

A special interesting case of (n, d) -bodies are $(n, 1)$ -bodies, often called n -intervals. These objects were first introduced in [GL], and subsequently studied, e.g., in [R] and [T]. In [T] an elegant result about the intersection properties of 2-intervals is shown. In particular, it claims that if a finite family $\mathcal{F}$ of 2-intervals has the property that no k members of $\mathcal{F}$ are pairwise disjoint, then there exist $2k$ points, k in each component, such

that every $I \in \mathcal{F}$ contains at least one of these points. This result has subsequently been extended in [K] to the case of n -intervals for general n ; the number of points necessary to hit every member of $\mathcal{F}$ is then at most $O(n^2k)$. Both of these proofs are topological. Recently, Alon gave in [A] an elementary proof of this fact.

Our definition was prompted in part by the surprising fact that $H(2, 1) = H(1, 1)$, i.e., is 2. This may lead one to conjecture that $H(n, 1)$ is always 2. The conjecture is however false, and it turns out that $H(3, 1) = 3$. The existence of a universal constant c_d such that $H(n, d) \leq c_d$ for all n , remains open for all $d \geq 1$. In this paper we make a step toward the solution of this problem, and show that $H(n, d)$ is at most polylogarithmic in n . More precisely, $H(n, d) \leq (\lfloor \log_2(n+1) \rfloor + 1)^d$.

In addition to discussing (n, d) -bodies, we also introduce and partially answer two interesting related combinatorial-topological questions concerning the structure of the nerve of certain open covers of polytopes.

2. Basic Properties of $H(n, d)$

Observe first that $H(n, d)$ is monotone nondecreasing in both n and d , since both $(n-1, d)$ -bodies and $(n, d-1)$ -bodies are special cases of (n, d) -bodies.

Notice also that $H(1, d) = d + 1$; this is precisely the statement of Helly's theorem.

Theorem 2.1. *$H(n, d)$ is subadditive in n , i.e.,*

$$H(n_1 + n_2, d) \leq H(n_1, d) + H(n_2, d).$$

For $n_2 = 1$, a stronger inequality holds:

$$H(n + 1, d) \leq H(n, d) + d.$$

Proof. We prove here only the first part of the theorem, delaying the proof of the second till the next section.

Let $\mathcal{F}$ be a family of $(n_1 + n_2, d)$ -bodies with the property that every subfamily of $\mathcal{F}$ of size $H(n_1, d) + H(n_2, d)$ is (strongly) intersecting. The goal is to show that the entire $\mathcal{F}$ is weakly intersecting. Distinguish between the following two cases: either every subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size $H(n_1, d)$ is intersecting with respect to the first n_1 components, or there exists $\mathcal{G} \subseteq \mathcal{F}$, $|\mathcal{G}| \leq H(n_1, d)$, such that the restriction of $\bigcap_{K \in \mathcal{G}} K$ to the first n_1 components is empty.

In the former case, by the definition of $H(n_1, d)$, the restriction of $\mathcal{F}$ to the first n_1 components is weakly intersecting, and thus $\mathcal{F}$ is weakly intersecting. In the latter case, we claim that every subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size $H(n_2, d)$ is intersecting with respect to the last n_2 components. Indeed, consider the subfamily $\mathcal{F}' \cup \mathcal{G}$. Its size is at most $H(n_1, d) + H(n_2, d)$, and by our assumptions it is intersecting. However, since $\bigcap_{K \in \mathcal{G}} K$ is empty with respect to the first n_1 components, the subfamily $\mathcal{F}' \cup \mathcal{G}$ must have a nonempty intersection with respect to the last n_2 components. The same, of course, applies to $\mathcal{F}'$. By the definition of $H(n_2, d)$, this implies that the entire $\mathcal{F}$ is weakly intersecting with respect to the last n_2 components. We conclude that $\mathcal{F}$ is weakly intersecting in this case as well. $\square$

As an immediate corollary of the second part of Theorem 2.1 (whose proof is given in the next section), and the fact that $H(1, d) = d + 1$, one gets an improvement over the trivial bound $H(n, d) \leq n(d + 1)$ mentioned in the Introduction:

Corollary 2.2. $H(n, d) \leq nd + 1$.

In what way does $H(n, d)$ depend on d ? At the moment we can show only the following:

Theorem 2.3. $H(n, d)$ is submultiplicative in d , i.e.,

$$H(n, d_1 + d_2) \leq H(n, d_1) \cdot H(n, d_2).$$

Proof. Let $\mathcal{F}$ be a family of $(n, d_1 + d_2)$ -bodies with the property that every subfamily of $\mathcal{F}$ of size $H(n, d_1) \cdot H(n, d_2)$ is intersecting. In each component i choose an arbitrary affine subspace A_i of dimension d_1 . For any $(n, d_1 + d_2)$ -body $K = \langle K_1, K_2, \dots, K_n \rangle \in \mathcal{F}$, let K_A be a componentwise orthogonal projection of K on $A = \langle A_1, \dots, A_n \rangle$. Moreover, for each subfamily $\mathcal{G} \subseteq \mathcal{F}$ of size $H(n, d_2)$, define

$$(\mathcal{G})_A = \left\{ \text{the orthogonal componentwise projection of } \bigcap_{K \in \mathcal{G}} K \text{ on } A \right\}.$$

Clearly, the orthogonal projection preserves convexity, and while it may create new intersections, it preserves the old ones. Now, since every subfamily of $\mathcal{F}$ of size $H(n, d_1) \cdot H(n, d_2)$ is intersecting, we conclude that any set of size $H(n, d_1)$ of $(\mathcal{G})_A$'s must be intersecting as well. However, each $(\mathcal{G})_A$ is a convex (n, d_1) -body, and therefore, by the definition of $H(n, d_1)$, there exists an n -tuple of points $\mathbf{p} = \langle p_1, \dots, p_n \rangle \in \langle A_1, \dots, A_n \rangle$ such that, for every $\mathcal{G}$, the intersection $(\mathcal{G})_A \cap \mathbf{p}$ is not empty.

In the second stage of the proof, define $A^\perp = \langle A_1^\perp, \dots, A_n^\perp \rangle$, where $A_i^\perp$ is an affine space of dimension d_2 in the i th component, orthogonal to A_i and passing through p_i . Let $\mathcal{F}(A^\perp) = \{K(A^\perp) \mid K \in \mathcal{F}\}$, where

$$K(A^\perp) = A^\perp \cap K = \langle A_1^\perp \cap K_1, \dots, A_n^\perp \cap K_n \rangle.$$

Obviously, each $K(A^\perp)$ is a convex (n, d_2) -body. We claim that any subfamily $\mathcal{F}'(A^\perp) \subseteq \mathcal{F}(A^\perp)$ of size $H(n, d_2)$ is intersecting. This is tantamount to saying that $A^\perp \cap (\mathcal{F}')_A$ is not empty, which is implied by our construction. Therefore, by the definition of $H(n, d_2)$, $\mathcal{F}(A^\perp)$ (i.e., the restriction of $\mathcal{F}$ to $A^\perp$) is weakly intersecting. Keeping in mind that $K(A^\perp) \subseteq K$, the proof of the theorem is now complete. $\square$

3. A Reformulation

In this section we reformulate our main question in different terms, and use the new formulation to prove additional properties of (n, d) -bodies.

Call an (n, d) -body $H = \langle H_1, \dots, H_n \rangle$ an n -halfspace if every $H_i \in \mathbb{R}_i^d$ is a convex (i.e., open or closed) halfspace. The *type* of H is a vector $t(H) = (u_1, \dots, u_n)$ where each

u_i is a unit-length d -dimensional vector such that either $H_i = \{x \in \mathbb{R}_i^d, \langle u_i, x \rangle > a_i\}$ or $H_i = \{x \in \mathbb{R}_i^d, \langle u_i, x \rangle \geq a_i\}$ for some real a_i .

A family $\mathcal{F}$ of n -halfspaces *strongly covers* $\langle \mathbb{R}_1^d, \dots, \mathbb{R}_n^d \rangle$ if for every n -point $\mathbf{p}$ there exists $H \in \mathcal{F}$ such that $\mathbf{p} \in H$. Respectively, $\mathcal{F}$ *weakly covers* the space if, for every component i , the restriction $\mathcal{F}_i$ of $\mathcal{F}$ to this component strongly covers it.

Proposition 3.1. *Let $h(n, d)$ be the smallest natural number such that for any finite family $\mathcal{F}$ of n -halfspaces which strongly covers $\langle \mathbb{R}_1^d, \dots, \mathbb{R}_n^d \rangle$, there exists a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size at most $h(n, d)$ which weakly covers it. Then*

$$H(n, d) = h(n, d).$$

Proof. First, observe that, without loss of generality, the convex (n, d) -bodies appearing in the definition of $H(n, d)$ can be assumed to be n -halfspaces. This can be shown by the following standard argument. Approximate each $K = \langle K_1, \dots, K_n \rangle$ in $\mathcal{F}$ by an n -polytope $P = \langle P_1, \dots, P_n \rangle$ such that the new family has the same intersection pattern as $\mathcal{F}$. Then replace each P by the set of all n -halfspaces $H = \langle H_1^{i_1}, \dots, H_n^{i_n} \rangle$ where, for each k , $P_k \subseteq H_k^{i_k}$ is the halfspace defined by the i_k th facet of P_k . It is readily checked that the resulting family is strongly/weakly intersecting if and only if the original $\mathcal{F}$ is.

Let $\mathcal{F}$ be a family of n -halfspaces. Define $\mathcal{F}^c$ as the family of n -halfspaces obtained by replacing each $H = \langle H_1, \dots, H_n \rangle$ by $H^c = \langle H_1^c, \dots, H_n^c \rangle$, where H_i^c is a complement of H_i . It is easy to verify that:

- $\mathcal{F}$ is *not* intersecting with respect to the i th component, if and only if $\mathcal{F}^c$ is weakly covering this component;
- $\mathcal{F}$ is *not* strongly intersecting, if and only if $\mathcal{F}^c$ is weakly covering the space (i.e., $\langle \mathbb{R}_1^d, \dots, \mathbb{R}_n^d \rangle$);
- $\mathcal{F}$ is *not* weakly intersecting if and only if $\mathcal{F}^c$ is strongly covering the space.

Using those observations as a tool for translating claims about the intersection properties of n -halfspaces to claims about the cover properties of complementary n -halfspaces, the definition of $h(n, d)$ becomes:

“ $h(n, d)$ is the smallest natural number such that every finite family $\mathcal{F}$ of n -halfspaces which is not weakly intersecting, contains a subfamily of size $h(n, d)$ which is not strongly intersecting.”

However, this is just the contrapositive of the definition of $H(n, d)$. □

An important remark is that since the family $\mathcal{F}$ in the definition of $H(n, d)$ is finite, without loss of generality one may assume the involved (n, d) -bodies to be all closed or all open. Consequently, the n -halfspaces in the definition of $h(n, d)$ can be assumed to be all open or all closed. In what follows we pick the more convenient possibility.

An interesting and useful result concerning covering families of n -halfspaces is:

Lemma 3.2. *Let $\mathcal{F}$ be a strongly covering family of open (equivalently, closed) n -halfspaces. Then $\mathcal{F}$ contains a subfamily $\mathcal{F}'$, such that $\mathcal{F}'$ is weakly covering, and is intersecting with respect to every component.*

Proof. We argue by induction, and start with the base case $n = 1$. The following fact is a weakened version of a theorem by Wegner (see [W]):

Fact. *In any finite family of closed convex bodies $\mathcal{B} = \{B^i\}$ in $\mathbb{R}^d$, there exists an intersecting subfamily $\mathcal{B}' \subseteq \mathcal{B}$, and a point p , with $p \in \partial B'$ for every $B' \in \mathcal{B}'$, such that, for every $B \in \mathcal{B} - \mathcal{B}'$,*

$$B \cap \left(\bigcap_{i=1}^r B^i \right) = \emptyset \iff p \notin B.$$

Let $\mathcal{F}$ be a covering family of open halfspaces. Applying Wegner's theorem to the family $\mathcal{F}^c$, we conclude that there exist $H^1, \dots, H^r \in \mathcal{F}$, and a point p in the closure of $\bigcap_{i=1}^r H^i$, such that, for any $H \in \mathcal{F} - \{H^i\}_{i=1}^r$,

$$H \cup \{H^1 \cup \dots \cup H^r\} = \mathbb{R}^d \iff p \in H.$$

Observe that H with $p \in H$ must exist: otherwise, $\mathcal{F}$ does not cover p . However, then H must have a nonempty intersection with $\bigcap_{i=1}^r H^i$. Adding H to $\{H^i\}_{i=1}^r$ we obtain the desired subfamily. The base case is established.

Assume now the lemma is true for $n - 1$; we want to show it holds for n as well. Consider the n th component. The restriction of $\mathcal{F}$ to this component is strongly covering, and therefore there exists a point $p_n \in \mathbb{R}_n^d$ such that the subfamily

$$\mathcal{F}' = \{H \in \mathcal{F} \mid p_n \in H_n\}$$

covers this component. Since $\mathcal{F}$ is strongly covering, the restriction of $\mathcal{F}'$ to the first $n - 1$ components is also strongly covering. Applying the induction hypothesis, we conclude that the lemma holds for n as well. $\square$

Let $\mathcal{F}$ be an intersecting family of halfspaces in $\mathbb{R}^d$. Define the polytope $P(\mathcal{F}) \subseteq \mathbb{R}^d$ as the convex hull of the set of types $t(\mathcal{F})$ of the elements in $\mathcal{F}$. Since the types are unit-length vectors, the vertices of P correspond to the distinct types in $t(\mathcal{F})$. Observe that $\mathcal{F}$ is covering if and only if $P(\mathcal{F})$ contains 0.

To demonstrate the usefulness of the new formulation and of Lemma 3.2, let us prove the second part of Theorem 2.1:

Proof of $H(n + 1, d) \leq H(n, d) + d$. Let $\mathcal{F}$ be a strongly covering family of open d -dimensional $(n + 1)$ -halfspaces. Consider its last component. $\mathcal{F}$ is covering it, and by Lemma 3.2 there exists a point $p \in \mathbb{R}_{n+1}^d$ such that the subfamily $\mathcal{F}' \subseteq \mathcal{F}$ defined by $\mathcal{F}' = \{H \in \mathcal{F} \mid p \in H_{n+1}\}$ also covers it. Since $\mathcal{F}$ is strongly covering, $\mathcal{F}'$ is strongly covering with respect to the first n components. By the definition of $H(n, d)$ there exists $\mathcal{F}'' \subseteq \mathcal{F}'$, $|\mathcal{F}''| \leq H(n, d)$, which weakly covers those components. We want to show that by adding at most d additional members to $\mathcal{F}''$, one can weakly cover the $(n + 1)$ th component as well.

Consider the polytope $P = \text{conv}\{t(H_{n+1}) \mid H \in \mathcal{F}'\}$. Let H be a member of $\mathcal{F}''$, and let v be a vertex of P which corresponds to H . The projection of $0 \in P$ from v to the boundary of P belongs to a face of P of dimension less than d . Therefore,

by Carathéodory's theorem, there exist vertices $v_1, \dots, v_d$ in this face, such that $0 \in \text{conv}(v, v_1, \dots, v_d)$. Adding to $\mathcal{F}'$ the n -halfspaces in $\mathcal{F}'$ corresponding to $v_1, \dots, v_d$, we obtain the desired weakly covering family. $\square$

Next, we would like to show that $H(n, d) \leq nd$ for $n > 1$. In order to do so, we need the following geometrical lemma:

Lemma 3.3. *Let $P \subseteq \mathbb{R}^d$ be a convex polytope whose vertices are $V = \{v_1, \dots, v_r\}$, and let $p \in P$. Call a vertex v_i essential with respect to p , if $p \notin \text{conv}(V - \{v_i\})$. Then the essential vertices of P with respect to p form a simplicial face of P (possibly empty).*

Proof. Given a polytope Q and $q \in Q$, let $\text{Ess}(Q, q)$ denote the set of vertices of Q essential with respect to q . Consider the following procedure:

Let $B^0 = P$, and let $p^0 = p$. If all the vertices of B^0 are essential with respect to p^0 , stop. Otherwise, there exists a nonessential vertex v in B^0 . Let p^1 be the projection of p^0 from v to the boundary of B^0 . Let B^1 be the (minimal) face containing p^1 . Proceeding in a similar manner on B^1 and so on, we obtain a sequence of pairs $(B^0, p^0), (B^1, p^1), \dots, (B^r, p^r)$, where all B^i 's are a decreasing sequence of faces of P , and $p^i \in B^i$. By the termination condition, all the vertices of B^r are essential with respect to p^r . Clearly, $\text{Ess}(B^i, p^i) \subseteq \text{Ess}(B^{i+1}, p^{i+1})$. Therefore, $\text{Ess}(P, p) \subseteq \text{Ess}(B^r, p^r)$.

Consider (B^r, p^r) ; let the dimension of this face be k . By Carathéodory's theorem there exist $k + 1$ vertices of B^r which span the point p^r . Since all the vertices of B^r are essential with respect to p^r , B^r has at most (in fact, exactly) $k + 1$ vertices, and must be simplicial.

We conclude that $\text{Ess}(P, p)$ is a subset of vertices of a simplicial face of P , and therefore forms a simplicial face of P as well. $\square$

Theorem 3.4. *For $n \geq 2$,*

$$H(n, d) \leq nd.$$

Proof. Let $\mathcal{F}$ be a strongly covering family of closed d -dimensional n -halfspaces. By Lemma 3.2, there exists a weakly covering subfamily $\mathcal{F}' \subseteq \mathcal{F}$, which is intersecting with respect to every component. The goal is to extract from $\mathcal{F}'$ a subfamily $\mathcal{F}''$ of size nd or less, which still has the same weakly covering property.

Let P_i be a polytope corresponding to the i th component of $\mathcal{F}'$, with vertices marked by the corresponding members of $\mathcal{F}'$ (one vertex may have many marks). From the assumptions on $\mathcal{F}'$, for all $i = 1, \dots, n$, P_i contains the origin of the coordinates. Call a member of $\mathcal{F}'$ i -essential if it is the only mark on an essential vertex of P_i with respect to 0. Observe that $H \in \mathcal{F}'$ which is not essential for any i , can be deleted without affecting the properties of $\mathcal{F}'$ (i.e., weakly covering and intersecting on every component). We claim that if $\mathcal{F}' > nd$, then there necessarily exists such an H . Clearly, this would imply the theorem.

Let $H^1, H^2, \dots, H^k$ be the i -essential elements of $\mathcal{F}'$. In view of Lemma 3.3, $k \leq d + 1$. Observe that the equality $k = d + 1$ is possible only when P_i is a d -dimensional

simplex, with every vertex marked by a unique element of $\mathcal{F}'$. In this case $|\mathcal{F}'| = d + 1 < nd$, and we are done.

Otherwise, there are at most d essential elements for every i . In this case, the number of essential members of $\mathcal{F}'$ is at most nd , as claimed. $\square$

4. On n -Intervals

This section is dedicated to the study of $H(n, 1)$.

A convex $(n, 1)$ -body is called simply an n -interval. If an $(n, 1)$ -body is in fact an n -halfspace, we call it simply an n -ray. Recall that the *type* of an n -ray R is a vector $t(R) = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$, there $\varepsilon_i = +1$ if R 's i th component ray contains $+\infty$, and -1 otherwise.

Observation 4.1. Let $\mathcal{F}$ be a family of n -rays which is intersecting with respect to every component, and let $t(\mathcal{F}) \subseteq \{-1, 1\}^n$ be the family of corresponding types. Let $S \subseteq \{1, \dots, n\}$ be the set of coordinates on which the members of $t(\mathcal{F})$ differ (i.e., are not all identically the same). Then $\mathcal{F}$ weakly covers the coordinates of S .

The goal is to show that for any strongly covering family of n -rays there exists a small subfamily which is intersecting with respect to every component, and whose types differ on *all* coordinates. The main tool is the following lemma, to be proven in the next section:

Lemma 4.2. *Let $\mathcal{F}$ be a strongly covering finite family of open (equivalently, closed) n -rays. Then there exists a subfamily $\mathcal{H} = \{H_1, \dots, H_k\} \subseteq \mathcal{F}$, and some nonnegative weights α_i such that:*

- $\mathcal{H}$ is intersecting with respect to every coordinate;
- $\sum \alpha_i t(H_i) = 0$, while $\sum \alpha_i = 1$.

In addition, we also need the following result:

Lemma 4.3. *Let $S \subseteq \{-1, 1\}^n$ be a set of vectors such that 0 is a convex combination of members of S , i.e., $\sum_{v \in S} \alpha_v v = 0$ for some real nonnegative weights α_v with $\sum \alpha_v = 1$. Then, for every $i = 1, 2, \dots, \lfloor \log_2(n+1) \rfloor + 1$, there exists a subset S_i of S such that*

- $|S_i| \leq i$;
- *members of S_i differ at at least $(n+1)(1 - 2^{1-i})$ coordinates.*

In particular, for $i = \lfloor \log_2(n+1) \rfloor + 1$, the corresponding set S_i differs on all n coordinates.

Proof. The statement is trivially true for $i = 1$. Assume by induction it holds for certain i , i.e., there exists a subset S_i with the required properties. The goal is to construct a new subset S_{i+1} with such properties.

Let $T \subset S$ be an arbitrary subset of S , and assume its members agree on a set of coordinates A . Then there exists a vector $v \in S$ which disagrees with members of T on

more than half of the coordinates in A . Indeed, let X be a random variable in $\mathbb{R}^n$ obtained by picking a vector $v \in S - T$ with probability $\alpha_v / \sum_{u \in S-T} \alpha_u$. Let X_j denote the j th coordinate of X . By assumptions of the lemma, for every $j \in A$,

$$\Pr[X_j \text{ differs from the common value of the } j\text{th coordinate in } T] > \frac{1}{2}.$$

The expected number of coordinates on which X differs from A is

$$E \left[\bigcup_{j \in A} \{X_j \neq A_j\} \right] = \sum_{j \in A} E[\{X_j \neq A_j\}] > \frac{|A|}{2},$$

and the required v must exist.

Back to S_i : by the inductive assumption, it has size i or less, and its members differ on at least $(n+1)(1-2^{1-i})$ coordinates. Let D be the set of these coordinates. By the above argument, there exists a vector $v \in S - S_i$ which differs from more than a half of the remaining coordinates. Define $S_{i+1} = S_i \cup \{v\}$. Members of S_{i+1} differ at least

$$\begin{aligned} |D| + \frac{1}{2} \{n - |D| + 1\} &= \frac{1}{2} \{n + 1 + |D|\} \\ &\geq \frac{1}{2} \left\{ n + 1 + (n + 1) \left(1 - \frac{1}{2^{i-1}} \right) \right\} = (n + 1) \left(1 - \frac{1}{2^i} \right) \end{aligned}$$

of the coordinates, as claimed. $\square$

Remark 4.4. The statement of Lemma 4.3 cannot in general be improved, as shown by the following example. Consider $\mathbb{Z}_2^k$, a k -dimensional linear space over $\mathbb{Z}_2$. For every hyperplane $I \subset \mathbb{Z}_2^k$ define a real-valued vector $v_I \in \mathbb{R}^{2^k-1}$, whose coordinates correspond to the nonzero members of $\mathbb{Z}_2^k$, and

$$v_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ -1 & \text{otherwise.} \end{cases}$$

Let also v_{all} be a vector which is identically 1. The family consisting of all these vectors sums up to 0, and it is readily checked that any subset of it of size i , $i = 1, 2, \dots, k+1$, agrees on $2^{k-i+1} - 1$ positions or more. This matches the upper bound of Lemma 4.3.

As an easy consequence of Lemmas 4.2 and 4.3 we obtain the main result of this section:

Theorem 4.5. $H(n, 1) \leq \lfloor \log_2(n+1) \rfloor + 1$.

Proof. By Proposition 3.1, it suffices to bound $h(n, 1)$. Let $\mathcal{F}$ be a strongly covering family of n -rays. Then $\mathcal{F}$ contains a subfamily $\mathcal{H}$ as described in Lemma 4.2. Applying Lemma 4.3 to $\mathcal{H}$, we conclude that there exists a subfamily $\mathcal{G}$ of $\mathcal{H}$ of size $|\mathcal{G}| \leq \lfloor \log_2(n+1) \rfloor + 1$, such that $\mathcal{G}$ is intersecting with respect to every component, and the members of $t(\mathcal{G})$ differ on every coordinate. By Observation 4.1, $\mathcal{G}$ is weakly covering. Thus, $\mathcal{F}$ contains a weakly covering family of size $\lfloor \log_2(n+1) \rfloor + 1$. $\square$

As an unexpected by-product of our approach, we also obtain the following theorem, which seems to be interesting in its own right:

Theorem 4.6. *Let $\mathcal{F}$ be a finite family of n -intervals such that every pair of them intersects in at least $\lfloor (n+1)/2 \rfloor$ of the components. Then $\mathcal{F}$ is weakly intersecting.*

Proof. Replacing intervals by rays, and performing “translation” to the language of coverings, one arrives at the following equivalent statement:

If a family $\mathcal{F}$ of n -rays strongly covers the entire space, then there exist a pair of rays in $\mathcal{F}$ which weakly covers at least $1+n-\lfloor (n+1)/2 \rfloor = \lceil (n+1)/2 \rceil$ of the coordinates.

Arguing along the same lines as in the proof of Theorem 4.5, and choosing this time the value $i = 2$ in Lemma 4.2, the latter statement follows. $\square$

To conclude this section we show that $H(3, 1) = 3$. By Theorem 4.5, $H(3, 1) \leq 3$. To see that $H(3, 1)$ is not 2 (a fact observed previously in [GL], and independently in [R]), consider the following explicit construction:

$$\mathcal{F} = \left\{ \begin{array}{lll} \langle [1, 2]; & [5, 6]; & [5, 8] \rangle & \langle [3, 4]; & [7, 8]; & [5, 8] \rangle \\ \langle [5, 8]; & [1, 2]; & [5, 6] \rangle & \langle [5, 8]; & [3, 4]; & [7, 8] \rangle \\ \langle [5, 6]; & [5, 8]; & [1, 2] \rangle & \langle [7, 8]; & [5, 8]; & [3, 4] \rangle \\ \langle [1, 4]; & [1, 4]; & [1, 4] \rangle & & & \end{array} \right\}.$$

A straightforward check is omitted.

As it happens, this is the best lower bound we can show at the moment for any $H(n, 1)$.

5. A Topological Theorem

It is convenient to recast Lemma 4.2 in different, more accessible, terms. Let $R = \langle R_1, \dots, R_n \rangle \subseteq \langle \mathbb{R}_1, \dots, \mathbb{R}_n \rangle$ be an n -ray. The associated n -corner $C(R) \subseteq \mathbb{R}^d$ is defined as $C(R) = R_1 \times \dots \times R_n$. Let the type $t(C)$ of the n -corner $C(R)$ be the same as the type of the corresponding n -ray R . Re-interpreting Lemma 4.2 in terms of n -corners, we obtain the following equivalent statement:

“Let $\mathcal{C}$ be a finite family of open n -corners which covers the entire $\mathbb{R}^n$. Then there exists an intersecting subfamily $\mathcal{C}' \subseteq \mathcal{C}$, such that $\sum_{C \in \mathcal{C}'} \alpha_C t(C) = 0$ for some positive numbers α_C .”

Let K be a huge hypercube in $\mathbb{R}^n$, such that its interior contains the apexes of all the n -corners in $\mathcal{C}$. Assume for convenience that $K = [-1, 1]^n$. Then the type of a corner $C \in \mathcal{C}$ is just the (unique!) vertex of K which it contains. Thus, the above statement is a corollary to the following more general theorem:

Theorem 5.1. *Let K be a convex polytope in $\mathbb{R}^n$. Let $\mathcal{U}$ be an open cover of K such that, for every $U \in \mathcal{U}$,*

- *U contains a unique vertex v_U of K ;*
- *for a facet B of K , $v_U \notin B \Rightarrow U \cap B = \emptyset$.*

Then, for every point $q \in K$, there exists an intersecting subfamily $\mathcal{U}' \subseteq \mathcal{U}$, such that

$$q \in \text{conv}\{v_U \mid U \in \mathcal{U}'\}.$$

Remark 5.2. This theorem can be viewed as a generalization of the famous Sperner lemma from combinatorial topology (see, e.g., [AH]), which claims that if a similarly defined family $\mathcal{U}$ of open sets covers an n -dimensional simplex, then the entire $\mathcal{U}$ is intersecting.

Proof. For every vertex v of K , define U^v as the (pointwise) union of all members of $\mathcal{U}$ which contain v . It suffices to prove the theorem for the family $\{U^v\}_{v \in V(K)}$.

For a point $p \in K$ and a vertex v , let $p(v) = \text{dist}(p, K - U^v)$, where *dist* is the usual Euclidean distance. Let $w(p) = \sum_{v \in V(K)} p(v)$. Define function $f: K \rightarrow K$ as follows:

$$f(p) = \sum_{v: \text{vertices}} \frac{p(v)}{w(p)} v.$$

Obviously, f is continuous. The crucial property of f is that it maps every face B of K to itself: only the sets U^v with $v \in B$ may have nonempty intersection with this face. We claim that f is onto. Clearly, this would imply the theorem.

The claim can be deduced from the observation that f maps the boundary of K to itself, and its restriction to the boundary is homotopic to identity. Here we present a more direct argument.

By continuity of f , it suffices to show that $\text{Im}(f)$ contains every interior point of K . Let q be such a point. Define a new function $F: K \rightarrow K$ as follows:

$$F(p) = \begin{cases} q + p - f(p) & \text{if } q + p - f(p) \in K, \\ q + \max(\alpha)(p - f(p)) & \text{otherwise,} \end{cases}$$

where $\max(\alpha)$ is the maximum α such that $q + \alpha(p - f(p)) \in K$. It is easy to verify that F is continuous, and therefore, by the Brouwer Fixed Point Theorem, there exists p such that $F(p) = p$. We claim that this implies $f(p) = q$. Distinguish between two cases: p is in the interior, and p is on the boundary. In the former case, $F(p) = q + p - f(p) = p$, and $f(p) = q$ follows immediately. The latter case turns out to be impossible: since both p and $f(p)$ belong to the same facet, the ray from q in the direction $p - f(p)$ is parallel to this facet, and cannot intersect it at all (and, in particular, at p). $\square$

In the special case when K is a hypercube, combining Theorem 5.1 with Lemma 4.3 ($i = 2$), one arrives at the following interesting new result:

Theorem 5.3. *Let K be a hypercube, and let $\mathcal{U}$ be an open cover of K as in the previous theorem. Then there exist two intersecting sets $U_1, U_2 \in \mathcal{U}$ such that the corresponding types $t(U_1)$ and $t(U_2)$ differ in at least $\lceil (n+1)/2 \rceil$ places.*

6. Implications to $d > 1$

As an immediate consequence of Theorems 4.5 and 2.3, we obtain the main result of this paper:

Theorem 6.1.

$$H(n, d) \leq (\lfloor \log_2(n+1) \rfloor + 1)^d.$$

Using a suitable equivalent of Theorem 2.3, it is not hard to extend Theorem 4.6 in a similar fashion (we omit the proof):

Theorem 6.2. *Let $\mathcal{F}$ be a family of convex (n, d) -bodies such that any subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size 2^d is intersecting with respect to at least $\lfloor (n+1)/2 \rfloor$ of the components. Then $\mathcal{F}$ is weakly intersecting.*

Remark 6.3. It is natural to ask whether the straightforward generalization of Lemma 4.2 holds for $d > 1$, and whether it can lead to a result stronger than that of Theorem 3.4. Leaving the first question aside, we will show that the stronger statement cannot give, for instance, an upper bound better than n for $H(n, 2)$. This is far away from the logarithmic bound of Theorem 4.5. In general, we conjecture that it cannot improve on Theorem 3.4 at all.

Following the discussion of Section 3, it suffices to construct a family of vectors $\mathcal{V} = \{V_i = (v_1^i, \dots, v_n^i)\}$ such that

1. every v_j^i is a unit length vector in $\mathbb{R}^2$;
2. $\sum V_i = 0$;
3. for every subfamily $\mathcal{V}' \subseteq \mathcal{V}$ of size less than n , there exists a coordinate j such that 0 is not in the convex hull of $\{v_j^i | V_i \in \mathcal{V}'\}$.

Let u_0, u_1, u_2 be three unit-length vectors in $\mathbb{R}^2$ such that $u_0 = ((1-n)/2)(u_1 + u_2)$. Define $\mathcal{V}$ to be the family of vectors V as above, such that any of the n coordinates of V can be either u_0, u_1 , or u_2 , the only restriction being that u_0 should appear exactly once. Clearly, $\mathcal{V}$ satisfies the first condition. A simple counting shows the second condition is satisfied as well. To see that the third condition is also satisfied, observe that u_0 is essential for every coordinate, and thus every coordinate must contain it; however, any vector in our $\mathcal{V}$ may introduce only one u_0 .

7. Open Problems

This paper makes only the first steps in the study of $H(n, d)$. The most intriguing unresolved questions are (in our opinion) the following:

Problem 7.1. How does $H(n, d)$ depend on n ? In particular, is $\lim_{n \rightarrow \infty} H(n, d) = \infty$?

In view of Theorem 2.3, it suffices to answer this question for $H(n, 1)$. At the moment we do not even know whether $H(n, 1) > 3$ for large n 's. By Theorem 4.5, such n cannot be smaller than 7.

Problem 7.2. How does $H(n, d)$ depends on d ? Can one expect a linear rather than exponential dependence, i.e., a statement of the sort $H(n, d) \leq (d + 1)H(n, 1)$?

The simplest unresolved problem for $d > 1$ is determining the value of $H(2, 2)$. Is it 3 or 4?

Finally, Theorem 5.3 claims that for any open cover as in Theorem 5.1 of the n -dimensional cube, there exist two intersecting U_1, U_2 whose types differ at $\lceil (n + 1)/2 \rceil$ places or more.

Problem 7.3. Is $\lceil (n + 1)/2 \rceil$ the best possible answer to this question?

This problem appears to be interesting in its own right. An improvement over $\lceil (n + 1)/2 \rceil$ would imply a stronger upper bound on $H(n, d)$.

Acknowledgments

A part of this work was carried out while the second author was an M.Sc. student at the Hebrew University, Jerusalem, advised by Gil Kalai. The second author would like to thank Gil for much help and support during his studies. Many thanks to Gábor Tardos, Yuval Peres, Noga Alon, Igal Galperin, and Eran London for helpful discussions and suggestions.

References

- [A] N. Alon, Piercing d -intervals, *Discrete Comput. Geom.* **19** (1998), 333–334.
- [AH] P. Alexandroff and H. Hopf, *Topologie*, vol. 1, Springer-Verlag, Berlin, 1935.
- [DGK] L. Danzer, B. Grünbaum, and V. Klee, Helly theorem and its relatives, in *Convexity*, Proceedings of Symposia in Pure Mathematics 7, AMS, Providence, RI, 1963, pp. 101–180.
- [GL] A. Gyárfás and L. Lehel, Helly-type theorem in trees, in *Combinatorial Theory and Its Applications*, Balatonfüred (Hungary), Colloquia Mathematica Societatis János Bolyai 4, North-Holland, Amsterdam, 1969, pp. 571–584.
- [GW] P. M. Gruber and J. M. Wills, editors, *Handbook of Convex Geometry*, North-Holland, Amsterdam, 1993.
- [K] T. Kaiser, Transversals of d -intervals, *Discrete Comput. Geom.* **18** (1997), 195–203.
- [Mo] H. Morris, Two pigeonhole principles and unions of convexly disjoint sets, Ph.D. thesis, California Institute of Technology, 1973.
- [R] Y. Rabinovich, Intersection and partition properties of families of convex bodies in $\mathbb{R}^d$, M.Sc. thesis, The Hebrew University, Jerusalem, 1989. In Hebrew.
- [T] G. Tardos, Transversals of 2-intervals, a topological approach, *Combinatorica* **15**(1) (1995), 123–134.
- [W] G. Wegner, d -Collapsing and nerves of families of convex sets, *Arch. Math.* **26** (1975), 317–327.

Received February 9, 1996, and in revised form November 6, 1996, December 16, 1996, and January 7, 1998.