Approximate Shortest Paths and Geodesic Diameter on a Convex Polytope in Three Dimensions*

S. Har-Peled

School of Mathematical Sciences, Tel Aviv University,
Tel Aviv 69978, Israel
sariel@math.tau.ac.il

Abstract. Given a convex polytope P with n edges in $\mathbb{R}^3$, we present a relatively simple algorithm that preprocesses P in $O(n)$ time, such that, given any two points $s, t \in \partial P$, and a parameter $0 < \varepsilon \leq 1$, it computes, in $O((\log n)/\varepsilon^{1.5} + 1/\varepsilon^3)$ time, a distance $\Delta_P(s, t)$, such that $d_P(s, t) \leq \Delta_P(s, t) \leq (1 + \varepsilon)d_P(s, t)$, where $d_P(s, t)$ is the length of the shortest path between s and t on ∂P . The algorithm also produces a polygonal path with $O(1/\varepsilon^{1.5})$ segments that avoids the interior of P and has length $\Delta_P(s, t)$.

Our second related result is: Given a convex polytope P with n edges in $\mathbb{R}^3$, and a parameter $0 < \varepsilon \leq 1$, we present an $O(n + 1/\varepsilon^5)$ -time algorithm that computes two points $s, t \in \partial P$ such that $d_P(s, t) \geq (1 - \varepsilon)\mathcal{D}_P$, where $\mathcal{D}_P = \max_{s, t \in \partial P} d_P(s, t)$ is the geodesic diameter of P .

1. Introduction

The *three-dimensional Euclidean shortest-path problem* is defined as follows: Given a set of pairwise-disjoint polyhedral objects in $\mathbb{R}^3$ and two points s and t , compute the shortest path between s and t which avoids the interiors of the given polyhedral “obstacles.” This problem has received considerable attention in computational geometry. It was shown to be NP-hard by Canny and Reif [3], and the fastest available algorithms for this problem run in time that is exponential in the total number of obstacle vertices (which we denote by n) [15], [16]. The intractability of the problem has motivated researchers to develop

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polynomial-time algorithms for computing approximate shortest paths and for computing shortest paths in special cases.

In the *approximate three-dimensional Euclidean shortest-path* problem, we are given an additional parameter $\varepsilon > 0$, and the goal is to compute a path between s and t that avoids the interiors of the obstacles and whose length is at most $(1 + \varepsilon)$ times the length of the shortest obstacle-avoiding path (we call such a path an ε -approximate path). Approximation algorithms for the three-dimensional shortest-path problem were first studied by Papadimitriou [13], who gave an $O(n^4(L + \log(n/\varepsilon))^2/\varepsilon^2)$ -time algorithm for computing an ε -approximate shortest path, where L is the number of bits of precision in the model of computation. A rigorous analysis of Papadimitriou's algorithm was recently given by Choi et al. [5]. A different approach was taken by Clarkson [6], whose algorithm computes an ε -approximate shortest path in close to $O(n^2/\varepsilon^4 \log^{O(1)} n)$ time (the complexity of Clarkson's algorithm depends also on an additional accuracy parameter).

In a companion paper [9], we present an improved solution to an extended variant of the problem: Given a polyhedral environment V , and a point $s \in V$, and a parameter $\varepsilon > 0$, preprocess them into a data-structure that supports fast ε -approximate shortest-path queries from s to any $t \in V$. The preprocessing takes roughly $O((n^4/\varepsilon^6) \log^2(n) \log^2(1/\varepsilon) + (n^2/\varepsilon^8) \log^5(1/\varepsilon))$ time and the size of the data structure is $O(n^2/\varepsilon^{4+\delta})$, for any $\delta > 0$ (the time complexity of the algorithm depends also on an additional accuracy parameter, since it uses Clarkson's algorithm [6]), and a query can be answered in $O(\log(n/\varepsilon))$ time.

The problem of computing a shortest path between two points along the surface of a single convex polytope is an interesting special case of the three-dimensional Euclidean shortest-path problem. Sharir and Schorr [17] gave an $O(n^3 \log n)$ algorithm for this problem, exploiting the property that a shortest path on a polyhedron *unfolds* into a straight line segment. Mitchell et al. [12] improved the running time to $O(n^2 \log n)$; their algorithm works for nonconvex polyhedra (or polyhedral surfaces) as well. Chen and Han [4] gave another algorithm with an improved running time of $O(n^2)$. It is a rather long-standing and intriguing open problem whether the shortest path on a convex polytope can be computed in subquadratic time. This has motivated the problem of finding near-linear algorithms that produce only an approximation of the shortest path. That is, we are given a convex polytope P with n vertices, two points s and t on its surface, and a positive real number ε . Let $\pi_P(s, t)$ denote any shortest path between s and t along the surface of P , and let $d_P(s, t)$ denote its length ($\pi_P(s, t)$ is usually, but not always, unique). We want to compute a path on the surface of P between s and t whose length is at most $(1 + \varepsilon)d_P(s, t)$.

The first result in this direction is by Hershberger and Suri [11]. They present a simple algorithm that runs in $O(n)$ time, and computes a path whose length is at most $2d_P(s, t)$. Using the algorithm of [11], Agarwal et al. [2] present a relatively simple algorithm that computes an ε -approximate shortest path, this is a path on ∂P between the points $s, t \in \partial P$ whose length is at most $(1 + \varepsilon)d_P(s, t)$, for any prescribed $0 < \varepsilon \leq 1$, where the running time of the algorithm is $O(n \log(1/\varepsilon) + 1/\varepsilon^3)$.

In this paper we extend the analysis of [2] to solve two problems involving approximate shortest paths on convex polytopes in $\mathbb{R}^3$. In both problems, our solutions are much faster than the best known algorithms for the corresponding exact problems.

Approximate Shortest-Paths Queries. In Section 3.1 we present an alternative to the algorithm of [11] that receives as input a convex polytope P with n edges and preprocesses P in $O(n)$ time, such that, for any $s, t \in \partial P$, one can compute, in $O(\log n)$ time, a number $\Delta_P(s, t)$, such that $d_P(s, t) \leq \Delta_P(s, t) \leq 8d_P(s, t)$. In Section 3.2 we modify the algorithm of [2], so that it uses the new algorithm instead of the algorithm of [11]. This results in an algorithm with $O(n)$ preprocessing time, such that, for any $s, t \in \partial P$ and $0 < \varepsilon \leq 1$, it computes a length $\Delta_P(s, t)$ of an ε -approximate shortest path between s and t on ∂P , in $O((\log n)/\varepsilon^{1.5} + 1/\varepsilon^3)$ time. In contrast, the fastest known algorithm for computing a data-structure that supports queries of computing the length of the exact shortest path between any pair of points on ∂P is due to [1]; it requires $O(n^6 m^{1+\delta})$ space and preprocessing time, for any $\delta > 0$, and answers a query in $O((\sqrt{n}/m^{1/4}) \log m)$ time, where $1 \leq m \leq n^2$ (the constants of proportionality depends on δ).

Approximate Geodesic Diameter. We present in Section 4 an algorithm that computes, for a given convex polytope P with n edges in $\mathbb{R}^3$, and a parameter $0 < \varepsilon \leq 1$, two points $s, t \in \partial P$ such that $d_P(s, t) \geq (1 - \varepsilon)\mathcal{D}_P$, where $\mathcal{D}_P = \max_{s, t \in \partial P} d_P(s, t)$ is the *geodesic diameter* of P . The running time of the algorithm is $O(n + 1/\varepsilon^5)$. In contrast, the fastest known algorithm for the exact geodesic diameter takes $O(n^8 \log n)$ time (see [1]).

The paper is organized as follows. Section 2 introduces the required terminology and establishes some initial properties, mostly taken or adapted from [2]. In Section 3 we describe the algorithm for answering arbitrary two-point approximate shortest-path queries. In Section 4 we derive an algorithm for approximating the geodesic diameter of a convex polytope in $\mathbb{R}^3$. We conclude in Section 5 by mentioning a few open problems.

2. Preliminaries

We begin with some terminology and some initial observations, most of them taken or adapted from [2]. Let P be a convex polytope with n edges in $\mathbb{R}^3$. For a face f of P , we denote by H_f the plane passing through f . Given a set $A \subseteq \mathbb{R}^3$, and a real number $r \geq 0$, let A_r denote the *Minkowski sum* $A \oplus B_r$, where B_r is a ball of radius r about the origin; that is,

$$A_r = \left\{ x \mid \inf_{y \in A} |x - y| \leq r \right\}.$$

For a convex polytope P and for $x \in \partial P$, let $F_{P,r}(x) = \{y \mid y \in \partial P_r, |x - y| = r\}$ be the set of points of ∂P_r “corresponding” to x . See Fig. 1 for an example of such an inflated polytope, and see [2] for (straightforward) details concerning the structure of P_r and of $F_{P,r}$.

For any plane H that avoids the interior of P , the *positive half-space* H^+ (resp. *negative half-space* H^-) bounded by H is the one containing P (resp. avoiding the interior of P); these half-spaces are assumed to be closed. Such a plane H is a *supporting plane* of P if $\partial P \cap H \neq \emptyset$.

Fig. 1. A cube P and the inflated set P_r .

Given a positive number $r \in \mathbb{R}$, we denote by $T_r(p) = r \cdot p$ the linear scaling of $\mathbb{R}^3$ by the factor r . Given two points $p, q \in \mathbb{R}^3$ such that $q \neq 0$, we denote by $\text{ray}(p, q)$ the ray emanating from p in the direction of q ; that is, $\text{ray}(p, q) = \{p + tq \mid t \geq 0\}$.

Given two curves¹ γ_1, γ_2 in $\mathbb{R}^3$ that share an endpoint, we denote by $\gamma_1 \parallel \gamma_2$ the curve resulting from concatenating γ_1 to γ_2 . For a curve $\gamma \subset \mathbb{R}^3$ and for $a, b \in \gamma$, we denote by $\gamma(a, b)$ the portion of γ between a and b . An *outer path* of a convex body K is a curve γ connecting two points on ∂K and disjoint from the interior of K . The length of a curve γ is denoted by $|\gamma|$.

The following well-known theorem implies that any outer path of P connecting two points $s, t \in \partial P$ must be of length at least $d_P(s, t)$, where, as above, $d_P(s, t)$ denotes the length of the shortest path between s and t on ∂P .

Theorem 2.1 [14]. *Let F be the boundary of a convex body K , and let γ be a curve that does not meet the interior of K and connects points s and t on F . Then the length of γ is at least the shortest distance $d_K(s, t)$ between its endpoints along the surface F ; it is strictly greater than this distance if the curve does not lie entirely on F .*

Lemma 2.2 [2]. *Let P be a convex polytope in $\mathbb{R}^3$, let s, t be any two points on ∂P , let $0 < \varepsilon < 1$ be an arbitrary parameter, and let $r > 0$. For any $s_r \in F_{P,r}(s)$,*

¹ By a curve we mean the image of a 1–1 continuous mapping of the unit interval $[0, 1]$ into $\mathbb{R}^3$.

$t_r \in F_{P,r}(t)$, there exists an outer path of P_r connecting s_r with t_r , whose length is at most $(1 + \varepsilon/2)d_P(s, t) + 2\pi r + 100r/\sqrt{\varepsilon}$.

Lemma 2.2 implies the following:

Lemma 2.3. *Let P, Q be convex polytopes in $\mathbb{R}^3$, let $0 < \varepsilon < 1$ be an arbitrary parameter, and let $r > 0$. If $P \subset Q \subset P_r$, then for any two points $s, t \in \partial P$ we have $d_P(s, t) \leq d_Q(s', t') + 2r \leq (1 + \varepsilon/2)d_P(s, t) + (2\pi + 4)r + 100r/\sqrt{\varepsilon}$, where $s' = \text{ray}(s, \eta_s) \cap \partial Q$, $t' = \text{ray}(t, \eta_t) \cap \partial Q$, and η_s, η_t denote the outward normals of P at s, t , respectively.*

Proof. Let $s_r = \text{ray}(s, \eta_s) \cap F_{P,r}(s)$ and $t_r = \text{ray}(t, \eta_t) \cap F_{P,r}(t)$. Clearly, $|ss_r| = |tt_r| = r$.

Let σ' be a shortest path connecting s_r to t_r on ∂P_r . By Lemma 2.2 and Theorem 2.1, we have $|\sigma'| \leq (1 + \varepsilon/2)d_P(s, t) + 2\pi r + 100r/\sqrt{\varepsilon}$. Let $\sigma = s's_r \parallel \sigma' \parallel t_r t'$. Clearly, $\sigma \cap \text{int } Q = \emptyset$, and $|\sigma| \leq (1 + \varepsilon/2)d_P(s, t) + (2\pi + 2)r + 100r/\sqrt{\varepsilon}$.

By Theorem 2.1,

$$\begin{aligned} d_P(s, t) &\leq |ss'| + d_Q(s', t') + |t't| \\ &\leq |\sigma| + 2r \leq (1 + \varepsilon/2)d_P(s, t) + (2\pi + 4)r + 100r/\sqrt{\varepsilon}. \quad \square \end{aligned}$$

Let $P \subseteq \mathbb{R}^3$ be a convex polytope. A *point-normal pair* on P is an ordered pair (p, η_p) such that (1) $p \in \partial P$, (2) the plane $H(p, \eta_p)$ that passes through p and whose normal is η_p is a supporting plane of P , and (3) η_p is an outward normal to P at p . As above, we denote by $H^+(p, \eta_p)$ the closed half-space containing P that is bounded by $H(p, \eta_p)$. For any $\delta > 0$, we call a collection S of point-normal pairs on P a δ -dense set if for any point-normal pair (q, η_q) on P , there exists a $(p, \eta_p) \in S$ such that $|pq| \leq \delta$, and the angle between η_p and η_q is at most δ . Let $P(S)$ denote the intersection of all the half-spaces corresponding to elements of S , i.e., $P(S) = \bigcap_{(p, \eta_p) \in S} H^+(p, \eta_p)$.

Lemma 2.4 [2]. *Given a convex polytope P in $\mathbb{R}^3$ having n edges and contained in a unit ball, then one can construct, in $O(n + (1/\varepsilon^2) \log(1/\varepsilon))$ time, an ε -dense set S of P , such that $|S| = O(1/\varepsilon^2)$ and $P \subseteq P(S) \subseteq P_{2\varepsilon^2}$.*

3. Approximate Shortest-Path Queries on a Convex Polytope

Let P be a given convex polytope in $\mathbb{R}^3$ with n edges. In this section we present an algorithm that preprocesses P in linear time, into a linear-size data structure, such that given any two query points $s, t \in \partial P$ and a parameter $0 < \varepsilon \leq 1$, one can compute, in $O((\log n)/\varepsilon^{1.5} + 1/\varepsilon^3)$ time, an outer path connecting s to t whose length $\Delta_P(s, t)$ satisfies $d_P(s, t) \leq \Delta_P(s, t) \leq (1 + \varepsilon)d_P(s, t)$. In Section 3.1 we describe an algorithm with linear preprocessing time that can answer, in $O(\log n)$ time, a factor 8 approximate shortest-path query. In Section 3.2 we show how to plug this new algorithm into the algorithm of [2], resulting in the specified algorithm.

3.1. A Constant Factor Approximation Algorithm

Given two points $s, t \in \partial P$, we assume, without loss of generality, that $s = (0, 0, -l)$ and $t = (0, 0, l)$. For $r \geq 0$, let $B(r)$ denote the axis parallel cube of side $2r$ centered at the origin. Let $C(r) = \partial B(r) \setminus \text{int } P$. Let $s(r)$ and $t(r)$ denote the lower and upper intersection points of $C(r)$ with the z -axis, respectively. Note that these points are well defined for $r \geq l$.

Lemma 3.1. *For any $r \geq l$, there exists a path σ between s and t on ∂P such that $\sigma \subset B(r)$, if and only if $s(r)$ and $t(r)$ are connected in $C(r)$.*

Proof. If the origin o lies on the boundary of P , then $st \subseteq \partial P$ and the lemma follows easily. Otherwise, for any $x \in \partial B(r)$, let $f_r(x)$ denote the point $\text{ray}(o, x) \cap \partial P$. It is easy to verify that the function f_r is a continuous bijective deformation of $\partial B(r)$ onto ∂P . In particular, given a path $\sigma \subset B(r)$ connecting s to t on ∂P , then $\sigma^{-1} = f_r^{-1}(\sigma)$ is connected, $\sigma^{-1} \subset C(r)$, and $s' = f_r^{-1}(s), t' = f_r^{-1}(t) \in \sigma^{-1}$, implying that σ^{-1} connects s' to t' in $C(r)$.

Conversely, if $s(r)$ and $t(r)$ are connected in $C(r)$, then there exists a path γ on $C(r)$ connecting $s(r)$ to $t(r)$. Clearly, $f_r(\gamma)$ is connected, connects s to t on ∂P , and $f_r(\gamma) \subset B(r)$. $\square$

Consider the set $V = \{-1, 0, 1\} \times \{-1, 0, 1\} \times \{-1, 1\}$ of 18 points on $B(1)$. Let $s' = (0, 0, -1), t' = (0, 0, 1)$. Define a set E of 28 straight segments, such that $vw \in E$ if and only if $v \neq w \in V$, and either $\|v - w\| = 1$ or $x(v) = x(w) = \pm 1$, and $y(v) = y(w) = \pm 1$. The resulting graph $G = (V, E)$ is depicted in Fig. 2. For $r \geq 0$, let $G(r) = (V, E(r))$, where $vw \in E(r)$ if $vw \in E$ and the segment $T_r(vw)$ does not intersect $\text{int } P$, where $T_r(a) = ra$ is the linear scaling of $\mathbb{R}^3$ by the factor r .

Fig. 2. The graph G defined on the boundary of the cube $B(1)$.

Clearly, $G(r)$ is monotone increasing, that is, $G(r') \subseteq G(r)$, for any $0 \leq r' \leq r$. Moreover, $G(0) = (V, \emptyset)$ and $G(r) = G$, for r sufficiently large.

Lemma 3.2. *For $r \geq l$, the vertices s', t' in $G(r)$ are connected if and only if $s(r)$ and $t(r)$ are connected in $C(r)$.*

Proof. If s' and t' are connected by a path σ in $G(r)$, then the path $T_r(\sigma)$ connects $s(r)$ to $t(r)$, and satisfies $T_r(\sigma) \subset C(r)$.

Conversely, suppose that $s(r)$ and $t(r)$ are connected in $C(r)$. Let $G_r = T_r(G)$ denote the natural embedding of the graph G into the boundary of $B(r)$.

Let σ be a path connecting $s(r)$ to $t(r)$ in $C(r)$ (since $C(r)$ is a polyhedral surface, we may assume that σ intersects each edge of G_r a finite number of times). For each face f of G_r (f is an axis-parallel square of side 1 or 2), such that $\sigma \cap \text{int } f \neq \emptyset$, let p_f and q_f be two intersection points of σ with ∂f , such that $\text{int } \sigma(p_f, q_f) \subseteq \text{int } f$. Partition ∂f into two Jordan arcs γ_1, γ_2 with disjoint relative interiors such that the points p_f and q_f are their common endpoints.

By the convexity of P , it follows that either γ_1 or γ_2 avoids the interior of P , because $\sigma(p_f, q_f) \cap \text{int } P = \emptyset$. Assume, without loss of generality, that γ_1 avoids the interior of P (see Fig. 3). We then replace σ by the curve $\sigma(s(r), p_f) \parallel \gamma_1 \parallel \sigma(q_f, t(r))$. We repeat this replacement process for each connected portion of $\sigma \cap \text{int } f$, until $\sigma \cap \text{int } f = \emptyset$, and apply this procedure for all faces f of G_r . Let σ' be the resulting path.

Clearly, σ' lies completely on the edges of G_r , and it connects $s(r)$ to $t(r)$. Moreover, σ' avoids the interior of P . This implies that $s(r)$ and $t(r)$ are connected in G_r , by a path that avoids the interior of P . Thus s' and t' are connected in $G(r)$. $\square$

It is easy to verify the following:

Lemma 3.3. *The length of the shortest path between s' and t' in any subgraph of G in which s' and t' are connected is at most 18, where all the horizontal edges of G have weight 1 and all the vertical edges have weight 2.*

Fig. 3. Either γ_1 or γ_2 must avoid the interior of P .

Fig. 4. A path realizing the longest possible shortest path (of length 14) in a subgraph $G(r)$.

Actually, the subgraphs $G(\cdot)$ of G have additional conditions imposed on them because of the convexity of P . By checking all possible subgraphs of G that comply with these “convexity” conditions, we get the following improvement, which only affects the constants of proportionality in the resulting algorithm.

Lemma 3.4. *The length of the shortest path between s' and t' in any subgraph $G(r)$ of G in which s' and t' are connected is at most 14, where the edges of G have the same weights as above. Moreover, there exist a convex polytope P , a pair of points $s, t \in \partial P$, and $r > 0$ such that the length of the shortest path between s' and t' in $G(r)$ is 14.*

The upper bound follows from an exhaustive search through all possible subgraphs $G(r)$ that we performed by a computer program. This can also be verified manually, but it is rather tedious to do so. The lower bound is depicted in Fig. 4. The polytope P is the convex hull of all the black dots in that figure (in fact, we have to inflate P a little so that the black dots lie in its interior).

Lemma 3.5. *Let r^* be the minimal value of r for which s' and t' belong to the same connected component of $G(r)$. Then $2r^* \leq d_P(s, t) \leq 16r^*$.*

Proof. Let γ' be the path on $C(r^*)$ corresponding to the shortest path between s' and t' in $G(r^*)$. By Lemma 3.4, the length of γ' is at most $14r^*$. Let $\gamma = ss(r^*) \parallel \gamma' \parallel t(r^*)t$. Clearly, $|\gamma| \leq 16r^*$. Since γ is an outer path of P connecting s to t , Theorem 2.1 implies that $d_P(s, t) \leq 16r^*$.

Lemmas 3.2 and 3.1 imply that there exists a path $\sigma' \subset B(r)$ connecting s to t on ∂P , if and only if s' and t' are connected in $G(r)$.

Let σ be a shortest path between s and t on ∂P . We claim that σ must intersect

$\partial B(r^*)$, for otherwise s' and t' are connected in $G(r^*)$, and this also holds for $r' < r^*$ sufficiently close to it. Hence the vertices s' and t' are connected in $G(r')$, a contradiction to the choice of r^* .

If σ intersects one of the vertical faces of $B(r^*)$ then its length is at least $2r^*$. If σ intersects the top or bottom face of $B(r^*)$, then its length is also at least $(r^* - l) + (r^* + l) = 2r^*$. $\square$

Lemma 3.6. *For an edge $e \in E$, let $r(e)$ denote the minimal value of r for which $e \in E(r)$. Let $e_1, \dots, e_{28}$ be the edges of E ordered in increasing value of $r(\cdot)$. Then $r^* = r(e_i)$, where i is the smallest index for which s' and t' are connected in the graph $G(r(e_i)) = (V, \{e_1, \dots, e_i\})$.*

Proof. Obvious. $\square$

Lemma 3.7. *A convex polytope P with n edges in $\mathbb{R}^3$ can be preprocessed in linear time, into a linear-size data structure, such that given any two points $s, t \in \partial P$, one can calculate, in $O(\log n)$ time, a value $\Delta_P(s, t)$ satisfying $d_P(s, t) \leq \Delta_P(s, t) \leq 8d_P(s, t)$.*

Proof. We construct, in $O(n)$ time, the Dobkin–Kirkpatrick hierarchical decomposition of P [7], [8].

By Lemma 3.6, all we have to do is to compute, for each edge $e = pq \in E$, the value of $r(e)$, which is the minimal value of r for which $e \in G(r)$. Once these values are available, r^* can be obtained in $O(1)$ time. (Recall that the graphs $G(r)$ depend on s and t .)

The value of $r(e)$ is the largest r such that $T_r(e) \cap P \neq \emptyset$. Thus, using the hierarchical decomposition of P , it is now possible to compute, in $O(\log n)$ time, the two intersection points x_1, x_2 between ∂P and the rays induced by $T_r(p)$ and $T_r(q)$ as r varies. Moreover, by locally inspecting $\partial P \cap H$ near x_1, x_2 , it can be decided whether any of these points is the last intersection point of $T_r(e)$ with ∂P , where H is the plane spanned by the origin and e . If not, the required intersection point lies between x_1 and x_2 on $\partial(P \cap H)$. We can find this intersection point by performing an extreme-point query on $P \cap H$. Namely, we are looking for the maximal r for which $l(r) \cap P \neq \emptyset$, where $l(r)$ is the line passing through $T_r(e)$. See Fig. 5.

This can be easily done in $O(\log n)$ time if we are given the hierarchical decomposition of the planar polygon $P \cap H$. This two-dimensional hierarchical decomposition (of $P \cap H$) is stored implicitly in the three-dimensional hierarchical decomposition of P , and the query can be performed in $O(\log n)$ time. See [8] for details.

Thus, we can compute $r(e)$ in $O(\log n)$ time, for each edge e of G . Hence, as already noted, we can calculate r^* in $O(\log n)$ time.

We return $16r^*$ as our approximation value $\Delta_P(s, t)$. By Lemma 3.5, $2r^* \leq d_P(s, t) \leq 16r^*$, from which lemma follows. $\square$

Remark 3.8. A constant factor approximation algorithm, similar to the algorithm of Lemma 3.7, was developed independently by Hershberger and Suri [10], [11].

Fig. 5. Computing the minimal value for which edge e appears in $G(r)$.

3.2. An $(1 + \varepsilon)$ -Approximation Algorithm

Given a convex polytope P with n edges, two points $s, t \in \partial P$, and a parameter $0 < \varepsilon \leq 1$, the algorithm of [2] computes an outer path of P connecting s and t and approximating the shortest path, as follows (see [2] for more details):

- Compute a crude approximation d satisfying $d_P(s, t) \leq d \leq 2d_P(s, t)$, using the algorithm of [11]. This stage takes $O(n)$ time.
- Compute, in $O(n)$ time, the Dobkin–Kirkpatrick hierarchical decomposition of P [7], [8].
- Using d and the hierarchical decomposition of P , compute, in

$$O((\log n + \log(1/\varepsilon))/\varepsilon^{1.5})$$

time, an approximation polytope $Q \supseteq P$ having $O(1/\varepsilon^{1.5})$ faces, such that $s, t \in \partial Q$ and $d_P(s, t) \leq d_Q(s, t) \leq (1 + \varepsilon)d_P(s, t)$.

- Using the algorithm of [4], compute, in $O(1/\varepsilon^3)$ time, the shortest path on ∂Q between s and t . This is the desired approximating outer path.

A more detailed inspection of the algorithm of [2] reveals that any approximation d , satisfying $d_P(s, t) \leq d \leq c \cdot d_P(s, t)$ for some prescribed constant $c > 1$, can be used in the first stage of the algorithm. Thus, by moving the computation of the Dobkin–Kirkpatrick decomposition of P into the preprocessing stage, and by using the algorithm of Lemma 3.7 instead of the algorithm of [11], we get:

Theorem 3.9. *A given convex polytope P with n edges can be preprocessed in $O(n)$ time, such that given any two points s and t on ∂P , and a parameter $\varepsilon > 0$, one can construct a polygonal outer path of P between s and t , whose length is at most $(1 + \varepsilon)d_P(s, t)$, and which consists of $O(1/\varepsilon^{1.5})$ segments, in time $O((\log n)/\varepsilon^{1.5} + 1/\varepsilon^3)$.*

Remark 3.10. The outer path σ of P generated by the algorithm of Theorem 3.9 can be projected onto ∂P in $O(n \log(1/\varepsilon) + 1/\varepsilon^3)$ time, resulting in a path σ' on ∂P , such that $|\sigma'| \leq |\sigma|$. See [2]. Since there are (easy) cases where any path on ∂P between s and t has $\Omega(n)$ edges, there is no hope of constructing an approximate shortest path on ∂P itself in worst-case sublinear query time.

Remark 3.11. Given a convex polytope P with n edges, the fastest known algorithm for computing a data-structure that supports queries of computing the length of the exact shortest path between any pair of points on ∂P is due to [1]; it requires $O(n^6 m^{1+\delta})$ space and preprocessing time, and answers a query in $O((\sqrt{n}/m^{1/4}) \log m)$ time, where $1 \leq m \leq n^2$, $\delta > 0$ (and the constants of proportionality depends on δ).

4. Approximating the Geodesic Diameter of a Convex Polytope

In this section we present an efficient algorithm that approximates the geodesic diameter of a convex polytope in $\mathbb{R}^3$.

Definition 4.1.

- (i) Given a convex polytope P in $\mathbb{R}^3$, we define the *geodesic diameter* of P to be

$$\mathcal{D}_P = \max_{s,t \in \partial P} d_P(s, t).$$

- (ii) Given a subset $S \subseteq \partial P$, we denote by $\Delta_P(S)$ the geodesic diameter of S on the boundary of P , namely

$$\Delta_P(S) = \max_{s,t \in S} d_P(s, t).$$

- (iii) Let $C(P)$ denote the smallest axis-parallel cube bounding P , and define the *outer diameter* of P to be $\delta(P) = 6d$, where d is the side length of $C(P)$.

Lemma 4.2. *Given a convex polytope P with n edges in $\mathbb{R}^3$, the outer diameter $\delta(P)$ of P can be calculated in linear time. Moreover, $\delta(P)/6 \leq \mathcal{D}_P \leq \delta(P)$.*

Proof. Clearly, $C(P)$, and thus also $\delta(P)$, can be computed in linear time. Let $d = \delta(P)/6$ be the edge length of $C(P)$. Since ∂P must contain two points s, t on opposite facets of $C(P)$, the Euclidean distance between s and t is $\geq d$, implying that $\mathcal{D}_P \geq d_P(s, t) \geq d$. It is also easy to verify that any pair of points on the boundary of P can be connected by an outer path of P of length at most $\delta(P) = 6d$. It follows that $\delta(P)/6 = d \leq \mathcal{D}_P \leq \delta(P)$. $\square$

The approximation algorithm is presented in Fig. 6. The following lemma proves the correctness of the algorithm, and describes it in more detail.

Algorithm APPROXIMATE-GEODESIC-DIAMETER(P, ε)

Input: A convex polytope P , approximation factor ε

Output: Two points $\mathfrak{s}, \mathfrak{t}$ on the boundary of P , such that $d_P(\mathfrak{s}, \mathfrak{t}) \geq (1 - \varepsilon)\mathcal{D}_P$

begin

Compute the outer diameter of P : $\delta = \delta(P)$

$\mathcal{P} \leftarrow T_{1/\delta}(P)$, where $T_{1/\delta}(\cdot)$ is the linear scaling of $\mathbb{R}^3$ by the factor $1/\delta$

$S \leftarrow \text{DenseSet}(\mathcal{P}, \varepsilon/100)$, $S' \leftarrow \text{DenseSet}(\mathcal{P}, \varepsilon^{3/4}/100)$

$Q = P(S') = \bigcap_{(p, \eta_p) \in S'} H^+(p, \eta_p)$

Project the points of S to the boundary of Q :

$\mu_Q(S) \leftarrow \{\mu_Q(s) \mid s \in S\}$, where $\mu_Q(s) = \text{ray}(s, \eta_s) \cap \partial Q$

for each $s \in \mu_Q(S)$ **do**

 Compute the distance $d_Q(s, t)$, for all $t \in \mu_Q(S)$

end for

Let $\mathfrak{s}, \mathfrak{t}$ be the pair of points of S for which $d_Q(\mu_Q(\mathfrak{s}), \mu_Q(\mathfrak{t}))$ is maximal

return $T_\delta(\mathfrak{s}), T_\delta(\mathfrak{t})$

end APPROXIMATE-GEODESIC-DIAMETER

Fig. 6. Algorithm for computing an approximated geodesic diameter of a polytope.

Lemma 4.3. *Given a convex polytope $\mathcal{P}$ with n edges in $\mathbb{R}^3$ such that $\frac{1}{6} \leq \mathcal{D}_{\mathcal{P}} \leq 1$, and a parameter $0 < \varepsilon \leq 1$, then a pair of points $\mathfrak{s}, \mathfrak{t} \in \partial \mathcal{P}$ can be calculated, in $O(n + 1/\varepsilon^5)$ time, such that $d_{\mathcal{P}}(\mathfrak{s}, \mathfrak{t}) \geq (1 - \varepsilon)\mathcal{D}_{\mathcal{P}}$.*

Proof. Let S be an $(\varepsilon/100)$ -dense set of $\mathcal{P}$, and let S' be an $(\varepsilon^{3/4}/100)$ -dense set of $\mathcal{P}$. By Lemma 2.4, such sets can be constructed in $O(n + (1/\varepsilon^2) \log(1/\varepsilon))$ time. Let s, t be two points on $\partial \mathcal{P}$ realizing the geodesic diameter of $\mathcal{P}$. Thus, there are two points $(s', \eta_{s'}), (t', \eta_{t'}) \in S$ such that $|ss'|, |tt'| \leq \varepsilon/100$ and $e(\eta_s, \eta_{s'}), e(\eta_t, \eta_{t'}) \leq \varepsilon/100$, where η_s, η_t are vectors normal to $\mathcal{P}$ at s, t , respectively, and $e(a, b)$ denotes the angle between the vectors a and b .

We claim that $d_{\mathcal{P}}(s, s'), d_{\mathcal{P}}(t, t') \leq \varepsilon/50$. Indeed, put $x = |ss'|$ and $\psi = e(\eta_s, \eta_{s'})$. Draw the planes H, H' that support $\mathcal{P}$ at s, s' , and have normals $\eta_s, \eta_{s'}$, respectively. Let W be the wedge bounded by H and H' and containing $\mathcal{P}$. Let σ denote the shortest path on ∂W between s and s' . This path consists of two segments su, us' , with a common endpoint u lying on $H \cap H'$. Moreover, the angle $\alpha = \angle sus'$ is at least the obtuse dihedral angle between H and H' . That is $\alpha \geq \pi - \psi$. See Fig. 7.

Put $y = |su|$, $z = |us'|$. By the cosine law,

$$\begin{aligned} x^2 &= y^2 + z^2 - 2yz \cos \alpha \\ &= y^2 + z^2 + 2yz \cos(\pi - \alpha) \geq y^2 + z^2 + 2yz \cos \psi \geq (y + z)^2 \cos \psi. \end{aligned}$$

Since $\psi \leq \varepsilon/100 \leq \frac{1}{100}$, we have $\cos \psi \geq \frac{1}{4}$, so $y + z \leq 2x$. Since $\sigma = sus'$ is an outer path of $\mathcal{P}$ between s and s' , we have $d_{\mathcal{P}}(s, s') \leq 2|ss'| \leq \varepsilon/50$. Thus,

$$\mathcal{D}_{\mathcal{P}} = d_{\mathcal{P}}(s, t) \leq d_{\mathcal{P}}(s, s') + d_{\mathcal{P}}(s', t') + d_{\mathcal{P}}(t', t) \leq \varepsilon/25 + \Delta_{\mathcal{P}}(S).$$

Fig. 7. $y + z \leq 2x$.

It follows that $\mathcal{D}_{\mathcal{P}} \leq \varepsilon/25 + \Delta_{\mathcal{P}}(S) \leq (\varepsilon/2)\mathcal{D}_{\mathcal{P}} + \Delta_{\mathcal{P}}(S)$, or

$$\Delta_{\mathcal{P}}(S) \geq (1 - \varepsilon/2)\mathcal{D}_{\mathcal{P}}. \quad (1)$$

Let $Q = P(S') = \bigcap_{(p, \eta_p) \in S'} H^+(p, \eta_p)$. The complexity of Q is $O(1/\varepsilon^{3/2})$, and, by Lemma 2.4, $Q \subset \mathcal{P}_r$, for $r = \varepsilon^{3/2}/5000$.

For $s \in S$, let $\mu_Q(s)$ denote the point $\text{ray}(s, \eta_s) \cap \partial Q$, where η_s denotes the outward normal to $\mathcal{P}$ at s . The mapping $\mu_Q(\cdot)$ can be computed in $O((1/\varepsilon^2) \log(1/\varepsilon))$ time, by preprocessing Q for ray shooting by computing, in $O(1/\varepsilon^{3/2})$ time, the Dobkin–Kirkpatrick hierarchical decomposition of Q [7], [8]. For $p \in S$, this decomposition can be used to compute, in $O(\log(1/\varepsilon))$ time, the intersection point between $\text{ray}(p, \eta_p)$ and ∂Q .

Applying Lemma 2.3, with this value of r and with $\varepsilon/2$, we obtain

$$\begin{aligned} d_{\mathcal{P}}(s, t) &\leq d_Q(\mu_Q(s), \mu_Q(t)) + 2r \leq (1 + \varepsilon/4)d_{\mathcal{P}}(s, t) + \frac{1}{400}\varepsilon^{3/2} + (\sqrt{2}/50)\varepsilon \\ &\leq d_{\mathcal{P}}(s, t) + \left(\frac{1}{4}\varepsilon + \frac{6}{400}\varepsilon^{3/2} + (6\sqrt{2}/50)\varepsilon\right)\mathcal{D}_{\mathcal{P}} \leq d_{\mathcal{P}}(s, t) + (\varepsilon/2)\mathcal{D}_{\mathcal{P}}, \end{aligned}$$

for any $s, t \in S$.

Let $\mathfrak{s}, \mathfrak{t}$ be two points in S for which $d = d_Q(\mu_Q(\mathfrak{s}), \mu_Q(\mathfrak{t}))$ is maximal. The above inequality implies

$$\Delta_{\mathcal{P}}(S) \leq d_Q(\mu_Q(\mathfrak{s}), \mu_Q(\mathfrak{t})) + 2r \leq d_{\mathcal{P}}(\mathfrak{s}, \mathfrak{t}) + (\varepsilon/2)\mathcal{D}_{\mathcal{P}}.$$

Hence, by (1),

$$d_{\mathcal{P}}(\mathfrak{s}, \mathfrak{t}) \geq \Delta_{\mathcal{P}}(S) - (\varepsilon/2)\mathcal{D}_{\mathcal{P}} \geq (1 - \varepsilon/2)\mathcal{D}_{\mathcal{P}} - (\varepsilon/2)\mathcal{D}_{\mathcal{P}} = (1 - \varepsilon)\mathcal{D}_{\mathcal{P}}.$$

As for the time complexity, the distances from a fixed point $\mu_Q(p)$ to all other points of $\mu_Q(S)$ along Q , can be calculated, in $O(1/\varepsilon^3)$ time, by the algorithm of [4]. Thus, the pair of points realizing d can be calculated in $O(1/\varepsilon^5)$ time. $\square$

Theorem 4.4. *Given a convex polytope P in $\mathbb{R}^3$ with n edges, and a parameter $0 < \varepsilon \leq 1$, one can compute, in $O(n + 1/\varepsilon^5)$ time, a pair of points $\mathfrak{s}, \mathfrak{t} \in \partial P$ and a value Δ , such that $d_P(\mathfrak{s}, \mathfrak{t}) \geq (1 - \varepsilon)\mathcal{D}_P$ and $(1 - \varepsilon)\mathcal{D}_P \leq \Delta \leq (1 + \varepsilon)\mathcal{D}_P$.*

Proof. Compute in linear time the outer diameter $\delta = \delta(P)$. Let $\mathcal{P} = T_{1/\delta}(P)$, where $T_{1/\delta}(\cdot)$ is the linear scaling of $\mathbb{R}^3$ by the factor $1/\delta$. By Lemma 4.2, $\frac{1}{6} \leq \mathcal{D}_{\mathcal{P}} \leq 1$.

By Lemma 4.3, a pair of points $\mathfrak{s}', \mathfrak{t}'$ of $\partial \mathcal{P}$ can be calculated in $O(n + 1/\varepsilon^5)$ time, such that

$$d_{\mathcal{P}}(\mathfrak{s}', \mathfrak{t}') \geq (1 - \varepsilon)\mathcal{D}_{\mathcal{P}}. \quad (2)$$

Let $\mathfrak{s}, \mathfrak{t}$ be the points on ∂P corresponding to $\mathfrak{s}', \mathfrak{t}'$, respectively. Then, by multiplying (2) by δ , it follows that $d_P(\mathfrak{s}, \mathfrak{t}) \geq (1 - \varepsilon)\mathcal{D}_P$.

Using the algorithm of [2], we can compute in $O(n + 1/\varepsilon^3)$ time an ε -approximation Δ to $d_P(\mathfrak{s}, \mathfrak{t})$, such that $(1 - \varepsilon)\mathcal{D}_P \leq \Delta \leq (1 + \varepsilon)\mathcal{D}_P$. $\square$

Remark 4.5. The fastest exact algorithm for computing the geodesic diameter of a convex polytope in $\mathbb{R}^3$ is due to Agarwal et al. [1], and runs in $O(n^8 \log n)$ time.

5. Conclusions

In this paper we have presented two simple and efficient algorithms for computing approximate solutions to problems involving shortest paths on the surface of a convex polytope in 3-space. We conclude by mentioning a few open problems.

- Can an ε -approximate shortest path between two points on a polyhedral terrain, or on the surface of a nonconvex polyhedron, be computed in time that is near-linear in the number of edges? A recent subquadratic solution has been announced by Varadarajan and Agarwal [18].
- Can the exact shortest path between two points on a convex polyhedron be computed in near-linear time? in subquadratic time?
- A geodesic path on a convex polytope P is a path which is locally optimal (i.e., it can not be made shorter by a “local” shortcut; in other words, it is a polygonal path with vertices only on edges of P , so that the angles between two consecutive segments and the edge of P to which they are adjacent are equal). Clearly, a shortest path is geodesic. Can one compute a geodesic path between two points on a convex polyhedron in subquadratic time?

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