On Galleries with No Bad Points*

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Abstract. For any k we construct a simply connected compact set (art gallery) in $\mathbb{R}^3$ whose every point sees a positive fraction (in fact, more than $\frac{5}{9}$) of the gallery, but the whole gallery cannot be guarded by k guards. This disproves a conjecture of Kavraki et al.

1. Introduction

Consider an art gallery (i.e., compact set) X of Lebesgue measure 1 in $\mathbb{R}^d$, $d \geq 2$, such that a guard placed anywhere in the gallery sees a region of Lebesgue measure at least ε . Kavraki et al. [3] conjectured that if X has at most h holes, then it can be guarded by at most $f_d(h, \varepsilon)$ guards, for some function f_d polynomial in h and $1/\varepsilon$.

Kalai and Matoušek [2] proved a weaker form of the planar version of the conjecture (their function f_2 was not polynomial in h). Following some ideas of Kalai and Matoušek, the author [5] proved the planar version of the conjecture with $f_2(h, \varepsilon) = (2 + o(1))(1/\varepsilon) \log(1/\varepsilon) \log_2(h + 2)$. Kalai and Matoušek [2] gave an example showing that $f_2(h, \varepsilon_0) = \Omega(\log h)$ for some fixed $\varepsilon_0 > 0$. Considering $\lfloor \varepsilon_0/\varepsilon \rfloor$ shifted copies of the set in their example, an example showing the lower bound $f_2(h, \varepsilon) = \Omega((1/\varepsilon) \log h)$ for $\varepsilon \in (0, \varepsilon_0)$, which almost matches the above upper bound, can be obtained.

In this paper we disprove the conjecture of Kavraki et al. for $d \geq 3$. For any $k \geq 1$, we construct a simply connected gallery in $\mathbb{R}^3$, in which each point sees more than $\frac{5}{9}$ of

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the gallery, but the whole gallery cannot be guarded by k guards. A motivation for the problem coming from robot path planning can be found in [3].

Let X be a set in $\mathbb{R}^d$, $d \geq 2$. We say that a point $x \in X$ *sees* a point $y \in X$ in X if the line segment xy is fully contained in X . For $x \in X$, let $V(x) = \{y \in X : x \text{ sees } y\}$. We say that a subset $G \subseteq X$ *guards* X if each point of X sees at least one point of G . We say that a point $x \in X$ is ε -*good*, if $\lambda_d(V(x)) \geq \varepsilon \lambda_d(X)$, where $\varepsilon > 0$ and λ_d denotes Lebesgue measure in $\mathbb{R}^d$. We say that the set X is ε -*good*, if each point of X is ε -good.

Here is the main result:

Theorem 1. *For any $k \geq 1$, there is a $\frac{5}{9}$ -good simply connected compact set in $\mathbb{R}^3$ which cannot be guarded by k points (guards).*

In the proof of Theorem 1 we show that the constant $\frac{5}{9}$ can be slightly improved. It is easy to modify our construction so that it gives a $\frac{5}{9}$ -good simply connected compact set in $\mathbb{R}^3$ which cannot be guarded by finitely many points (guards). An easy generalization of our construction also gives, for any $d \geq 3$, a simply connected compact $(1 - ((d - 1)/d)^{d-1})$ -good gallery in $\mathbb{R}^d$ which cannot be guarded by a finite number of guards.

On the other hand, a nontrivial upper bound in any dimension follows from a well-known theorem of Krasnosel'skij. Krasnosel'skij's theorem [4] says that if $X \subset \mathbb{R}^d$ is a connected compact set in $\mathbb{R}^d$ with the property that every $d + 1$ points in X can be seen from a single point in X , then the whole set X can be seen from a single point, i.e., X is star-shaped. Since the assumptions of Krasnosel'skij's theorem are obviously satisfied for any $(d/(d + 1) + \omega)$ -good compact set X ($\omega > 0$), the following upper bound follows:

Theorem 2. *For $d \geq 2$ and $\omega > 0$, any $(d/(d + 1) + \omega)$ -good compact gallery in $\mathbb{R}^d$ can be guarded by one guard (in other words, it is star-shaped).*

There is a significant gap between the lower bound $1 - ((d - 1)/d)^{d-1} \approx 1 - 1/e$ in the d -dimensional version of Theorem 1 and the upper bound $d/(d + 1)$ in Theorem 2. In particular, we do not have an answer to the following question.

Problem 1. Is there an $\varepsilon < 1$ such that for any $d \geq 2$ any (simply connected) ε -good compact gallery in $\mathbb{R}^d$ can be guarded by at most $k = k(d)$ guards?

We can partially solve Problem 1 if other Borel measures than the uniform (Lebesgue) one are considered. This is shown in Section 3. Section 2 is devoted to the proof of Theorem 1. Throughout the paper, $B(x, r)$ denotes the r -neighborhood of x .

2. The Construction

In this section we prove Theorem 1. Let S be a regular tetrahedron of edge length 1 with vertices A, B, C, D , let U be the center of S , and let V be the center of the triangle

Fig. 3. One of the regions $S_{i2} \subset S$ of measure $\frac{5}{9}\lambda_3(S)$.

with sides parallel to those of T . Let V' be the center of T' , and let E be a point inside S such that the angle between the line $V'E$ and the plane containing T is bigger than $\pi/2 - \alpha$. Then the set $S \setminus S'$, where $S' = \text{conv}(T' \cup \{E\})$, is ε -good.

Proof. Let $\varrho_1, \varrho_2, \varrho_3$ be the three planes containing the points V, D and intersecting the triangle $T = ABC$ in a line segment parallel to a side of T . Then each ϱ_i cuts the tetrahedron S into two parts S_{i1} and S_{i2} of measure $\frac{4}{9}\lambda_3(S)$ and $\frac{5}{9}\lambda_3(S)$, respectively (see Fig. 3). If $\eta > 0$ and $\alpha > 0$ are small, then every point in $S \setminus S'$ sees in $S \setminus S'$ almost entirely one of the three regions $S_{i2} \cap (S \setminus S')$ of measure $\approx \frac{5}{9}\lambda_3(S \setminus S')$. The observation follows. $\square$

Observation 1 yields the following weaker version of Lemma 1.

Lemma 2. For every $\eta, \delta \in (0, 0.01)$, $W \in (B(V, \eta) \cap T)$, $t > 0$, $P \in S \setminus B(W, 2t)$, there exist a tetrahedron $S' = S'(\eta, \delta, W, t, P) \subset S$ and a real number $t' = t'(\eta, \delta, W, t, P) > 0$ such that

- (i) $S' \subset B(W, 2t) \setminus B(W, 2t')$,
- (ii) $S \setminus S'$ is simply connected and $(\frac{5}{9} - g_1(\eta, \delta))$ -good, where $g_1(\eta, \delta)$ is independent of W, t, P and $g_1(\eta, \delta) \rightarrow 0$ as $\eta + \delta \rightarrow 0$,
- (iii) every point of $(S \setminus B(W, 2t)) \cup B(W, 2t')$ is $(1 - g_2(\delta))$ -good in $S \setminus S'$, where $g_2(\delta)$ is independent of W, t, P and $g_2(\delta) \rightarrow 0$ as $\delta \rightarrow 0$,
- (iv) W sees no point of $B(P, \delta^2 \cdot \text{dist}(P, W))$ in $S \setminus S'$.

Proof. Let η, δ, W, t, P satisfy the conditions of Lemma 2. Consider a plane ρ containing the points W, P and perpendicular to the plane τ containing T (ρ is uniquely determined unless the line WP is perpendicular to τ). Choose the Cartesian coordinate system in ρ so that the line $\rho \cap \tau$ coincides with the x -axis, W coincides with the origin, and each of the two coordinates of P is nonnegative. Denote $W' = (\delta t, 0)$, $E = (-\delta t, t)$,

Fig. 4. The set S'_2 consisting of two tetrahedra.

and return to $\mathbb{R}^3$. Let $T' \subset T$ be the equilateral triangle with center W' and with sides of length $10\delta^2 t$ parallel to those of T . Then $S' = \text{conv}(T' \cup \{E\})$ satisfies the lemma: (i) and (iii) hold with $t' = \delta^2 t$, (ii) follows from Observation 1, and (iv) follows from the construction. $\square$

Repeated applications of Lemma 2 give the desired Lemma 1.

Proof of Lemma 1. Given $k, \eta, \delta, W, t, \{P_1, \dots, P_k\}$, an application of Lemma 2 with parameters $\eta, \delta, W, t, P = P_1$ gives a set $S'(\eta, \delta, W, t, P_1) = S'_1$ and a number $t'(\eta, \delta, W, t, P_1) = t'_1 > 0$. Thereafter, an application of Lemma 2 with parameters $\eta, \delta, W, t = t'_1, P = P_2$ gives a set $S'(\eta, \delta, W, t'_1, P_2) = S'_2$ and a number $t'(\eta, \delta, W, t'_1, P_2) = t'_2 > 0$ (see Fig. 4). We then apply Lemma 2 an additional $k-2$ times. In the i th application ($i = 3, 4, \dots, k$), we use parameters $\eta, \delta, W, t = t'_{i-1}, P = P_i$, and obtain a set $S' = S'_i$ and a number $t' = t'_i > 0$. It is easily verified that the set $S'' = \bigcup_{i=1}^k S'_i$ satisfies Lemma 1. $\square$

3. Related Results

In this section we prove some related results for other Borel measures than the uniform (Lebesgue) one. A Borel measure μ on $\mathbb{R}^d$ is called *locally finite* if $\mu(K)$ is finite for any compact measurable set $K \subset \mathbb{R}^d$. Let X be a set in $\mathbb{R}^d$, $d \geq 2$, and let μ be a Borel measure on $\mathbb{R}^d$. We say that a point $x \in X$ is (μ, ε) -good, if $\mu(V(x)) \geq \varepsilon \mu(X)$. We say that the set X is (μ, ε) -good, if each point of X is (μ, ε) -good. We call a set in $\mathbb{R}^d$ *polyhedral* if it is a union of finitely many convex d -dimensional polytopes. An analogue of Theorem 2 for locally finite Borel measures gives almost a tight result.

Theorem 3. *Let $\omega > 0$. Then*

- (i) *for any locally finite Borel measure μ in $\mathbb{R}^d$, $d \geq 2$, any polyhedral $(\mu, d/(d+1) + \omega)$ -good set of positive μ -measure in $\mathbb{R}^d$ can be guarded by one guard,*

- (ii) for any $d \geq 3$, there is a locally finite Borel measure μ_d in $\mathbb{R}^d$ such that for any $k \geq 1$ there is a polyhedral $(\mu_d, (d-1)/d - \omega)$ -good simply connected gallery of positive μ_d -measure which cannot be guarded by k guards.

Thus, an analogue of the question in Problem 1 for locally finite Borel measures has a negative answer. We remark that in the plane the number of guards needed in a (μ, ε) -good gallery with h holes is bounded by $(2 + o(1))(1/\varepsilon) \log(1/\varepsilon) \log_2(h+2)$ for any locally finite Borel measure μ , like for the uniform measure. This is because the proof in [5] for the uniform (Lebesgue) measure does not use any specific properties of the uniform measure, and therefore can be applied for any locally finite Borel measure.

Proof of Theorem 3. (i) Let $d \geq 2$, $\omega > 0$, and let μ be a locally finite Borel measure in $\mathbb{R}^d$. Let X be a $(\mu, d/(d+1) + \omega)$ -good polyhedral gallery in $\mathbb{R}^d$ with $\mu(X) > 0$. Let P be a centerpoint¹ of X with respect to μ . We claim that P sees every point in X . Suppose to contrary that P does not see a point $Q \in X$. Let $Q' \in X$ be the point of X on the segment PQ closest to P with the property that it does not see P . Then there is a point Q'' of X in a small neighborhood of Q' which sees a part of X contained in a closed halfspace not containing P . From the centerpoint property of P , this implies that Q'' is not $(\mu, d/(d+1) + \omega)$ -good, a contradiction.

(ii) We outline the construction for $d = 3$. The construction is analogous for the other values of d . For $d = 3$, the gallery is similar to the gallery from the proof of Theorem 1. We again consider a regular tetrahedron $S = ABCD$ with edge length 1. Consider the Borel measure μ_3 which is the Lebesgue measure on the $\frac{1}{10}$ -neighborhood of each of the vertices A, B, C , and the zero measure on the rest of $\mathbb{R}^3$. The proof now goes literally as the proof of Theorem 1 in Section 2 (including Observation 1 and Lemmas 2 and 1) with the last paragraph omitted and with the exceptions that every occurrence of $\frac{5}{9}$ is replaced by $\frac{2}{3}$, and that the measure μ_3 is considered instead of the Lebesgue measure (thus, the notion ε -good is always replaced by (μ_3, ε) -good). $\square$

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¹ P is a centerpoint of a compact set $X \subset \mathbb{R}^d$ with respect to a locally finite Borel measure μ if every halfspace in $\mathbb{R}^d$ containing P contains a subset of X of measure $\geq (1/(d+1))\mu(X)$. The existence of a centerpoint for any compact set $X \subset \mathbb{R}^d$ with respect to a locally finite Borel measure follows easily from the existence of a centerpoint for any finite set in $\mathbb{R}^d$ with respect to Lebesgue measure (see [1]).

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