Covering the Plane with Copies of a Convex Disk

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Abstract. We prove that for every convex disk C in the plane there exists a double-lattice covering of the plane with copies of C with density $\vartheta \leq 1.2281772$. This improves the best previously known upper bound $\vartheta \leq 8/(3 + 2\sqrt{3}) \approx 1.2376043$, due to Kuperberg, but it is still far from the conjectured value $\vartheta = 2\pi/3\sqrt{3} \approx 1.2091993$.

1. Introduction

Given a convex *disk* (i.e., compact convex set) C in the plane, let $\vartheta(C)$ denote the minimum density of a covering of the plane with copies of C . Further, let $\vartheta = \max \vartheta(C)$, where the maximum is taken over all convex disks on the plane.

According to a classical conjecture $\vartheta = 2\pi/3\sqrt{3}$. It is known that $\vartheta(C) \leq 2\pi/3\sqrt{3}$ holds for every *centrally symmetric* convex disk C , with equality if and only if C is a circular disk or an ellipse (see [1], [2], [4], and [5]).

For general convex disks, the best previously known upper bound was established by Kuperberg [3]: $\vartheta \leq 8/(3 + 2\sqrt{3})$.

In Kuperberg's proof the following notion played a key role. A convex hexagon is called a *p-hexagon* if it has two parallel edges of the same length, opposite to each other (in the sense that they are separated by exactly two edges). It is easy to see that the plane can be tiled with congruent copies of any *p-hexagon* in a double-lattice fashion (see Fig. 1).

Thus, given a *p-hexagon* H inscribed in a convex disk C , we can generate a double-lattice covering of the plane with copies of C whose density is $\vartheta' = \text{area}(C)/\text{area}(H)$. Therefore, finding an inscribed *p-hexagon* with large area results in a low density (double-lattice) covering with congruent copies of C .

Let C be a convex disk and let θ be an arbitrary direction (see Fig. 2). Let t_θ and t'_θ be the tangents to C that are parallel to θ . These lines touch the boundary at A and A' (if $t_\theta \cap \text{bd}(C)$ contains more than one point—and this is the case when the boundary

Fig. 1

of C contains a segment parallel to θ —then let A denote an arbitrary point from the intersection). Consider now the tangents to C that are parallel to AA' and denote by B and B' the points where these lines touch the boundary. Now, for every $0 \leq l \leq |AA'|$ we can find points M , N , P , and Q on the boundary of C so that $MN \parallel PQ \parallel AA'$ and $|MN| = |PQ| = l$. We obtain two p -hexagons inscribed in C : $H_1(\theta, l) = BMQB'PN$ and $H_2(\theta, l) = MAQPA'N$. We say that these two p -hexagons belong to direction θ . Clearly, there are infinitely many p -hexagons belonging to the same direction because l can be chosen arbitrarily between 0 and $|AA'|$. Kuperberg's results can be formulated as follows.

Fig. 2

Fig. 3

Theorem A [3]. *For every convex disk C and for every direction θ , there exists a p -hexagon H_θ belonging to direction θ such that $\text{area}(C)/\text{area}(H_\theta) \leq 8/(3 + 2\sqrt{3})$.*

The above result is best possible in the following sense.

Theorem B [3]. *There exist a convex disk C_0 and a direction θ_0 such that every p -hexagon H belonging to the direction θ_0 satisfies $\text{area}(C_0)/\text{area}(H) \geq 8/(3 + 2\sqrt{3})$.*

This is the case when C_0 is a regular dodecagon and θ_0 is the direction of one of its edges (see Fig. 3).

The main result of this paper is the following.

Theorem. *For every planar convex disk C there exist a direction $\theta(C)$ and a p -hexagon H belonging to this direction that satisfies*

$$\vartheta = \frac{\text{area}(C)}{\text{area}(H)} \leq \frac{4(\sqrt{2} - 2 \sin x_0)}{4 + \sqrt{2} + 2 \sin x_0} + \frac{4x_0}{x_0 + \sqrt{2}} = 1.2281771 \dots,$$

where $x_0 = 0.2594296 \dots$ is the real root of the equation $\cos x - x = \sqrt{2}/2$.

2. The Main Construction

Let C be a convex disk and let $TWUV$ be the quadrilateral of maximum area inscribed in C (see Fig. 4). Let d_T denote the line passing through T and parallel to WV . Obviously, d_T is tangent to C .

We obtain (making the same constructions for points W , U , and V) a parallelogram circumscribed about C so that TU and VW are parallel to the edges of this parallelogram.

Fig. 4

Applying an appropriate affine transformation if necessary, we can assume that this parallelogram is a unit square.

Moreover, since $TWUV$ is the quadrilateral of maximum area inscribed in C , it follows that every quadrilateral inscribed in C has area at most $\frac{1}{2}$. This is used later to prove Lemma 3.

Let n be a positive integer and let $\alpha = \pi/4n$. For every $k \in \{1, 2, \dots, 2n-1\}$ take a pair of segments of length $L_k = \sin k\alpha$, that are parallel to TU , and whose ends are both on the boundary of C . Let these points be as indicated in Fig. 5. Hence, we have $A_k B_k \parallel C_k D_k \parallel TU$ and $|A_k B_k| = |C_k D_k| = L_k = \sin k\alpha$ for every $k \in \{1, 2, \dots, 2n-1\}$.

Denote by $d(A_k B_k, C_k D_k) = h_k$ the perpendicular distance between $A_k B_k$ and $C_k D_k$. Obviously, $1 \geq h_1 \geq h_2 \geq \dots \geq h_{2n-2} \geq h_{2n-1} \geq 0$. For convenience, let $L_0 = 0$, $L_{2n} = 1$; $h_0 = 1$, $h_{2n} = 0$.

3. Preliminary Lemmas

Lemma 1. *Using the notation introduced above, we have*

$$\text{area}(C) \leq \sin \alpha \cdot (1 + L_{2n-1}h_1 + L_{2n-2}h_2 + \dots + L_2h_{2n-2} + L_1h_{2n-1}). \quad (1)$$

Proof. We consider the area of the nonconvex polygon that contains C and has the edges parallel to TU and VW (see Fig. 5—this polygon is C plus the shaded small triangles). It follows that

$$\begin{aligned} \text{area}(C) &< \sum_{k=0}^{2n-1} L_{k+1}(h_k - h_{k+1}) \\ &= \sum_{k=0}^{2n-1} (L_{k+1} - L_k)h_k = 2 \sin \frac{\alpha}{2} \cdot \sum_{k=0}^{2n-1} h_k \cos(k + \frac{1}{2})\alpha \end{aligned}$$

Fig. 5

$$\begin{aligned}
 &= \sin \alpha \cdot \sum_{k=0}^{2n-1} h_k \cos k\alpha - 2 \sin^2 \frac{\alpha}{2} \cdot \sum_{k=0}^{2n-1} h_k \sin k\alpha \\
 &< \sin \alpha \cdot \sum_{k=0}^{2n-1} L_{2n-k} h_k.
 \end{aligned}$$

□

Lemma 2. For every $0 \leq i \leq j \leq k \leq 2n$ we have

$$\begin{vmatrix} 1 & 1 & 1 \\ L_i & L_j & L_k \\ h_i & h_j & h_k \end{vmatrix} \leq 0. \quad (2)$$

Proof. We consider a coordinate-system with origin O (the lower left corner of the circumscribed square and the positive axes as shown in Fig. 6). Triangle $A_i A_j A_k$ (resp. $B_i B_j B_k$) is oriented in the clockwise (resp. counterclockwise) direction, according to our system of coordinates. Therefore,

$$\begin{vmatrix} 1 & 1 & 1 \\ x_{B_i} & x_{B_j} & x_{B_k} \\ y_{B_i} & y_{B_j} & y_{B_k} \end{vmatrix} \geq 0 \quad \text{and} \quad \begin{vmatrix} 1 & 1 & 1 \\ x_{A_i} & x_{A_j} & x_{A_k} \\ y_{A_i} & y_{A_j} & y_{A_k} \end{vmatrix} \leq 0.$$

Fig. 6

However, $y_{A_m} = y_{B_m}$ for every $m \in \{i, j, k\}$, which implies that

$$\begin{vmatrix} 1 & 1 & 1 \\ x_{B_i} - x_{A_i} & x_{B_j} - x_{A_j} & x_{B_k} - x_{A_k} \\ y_{B_i} & y_{B_j} & y_{B_k} \end{vmatrix} \geq 0,$$

so

$$\begin{vmatrix} 1 & 1 & 1 \\ L_i & L_j & L_k \\ y_{B_i} & y_{B_j} & y_{B_k} \end{vmatrix} \geq 0. \quad (3)$$

Analogously, we obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ L_i & L_j & L_k \\ y_{C_i} & y_{C_j} & y_{C_k} \end{vmatrix} \leq 0. \quad (4)$$

Now subtracting inequality (3) from (4) and using $y_{C_m} - y_{B_m} = h_m$ for every $m \in \{i, j, k\}$, we obtain the required relation. $\square$

Lemma 3. For every $0 \leq i, j \leq 2n$ we have

$$L_i h_j + L_j h_i \leq 1. \quad (5)$$

Proof. Consider the same system of coordinates as in Lemma 2 (see Fig. 7). By sym-

Fig. 7

metry, we can take $i \leq j$. We have that

$$\begin{aligned} 2 \cdot \text{area}(A_i B_j C_i D_j) &= (x_{B_j} - x_{D_j})(y_{C_i} - y_{A_i}) + (x_{C_i} - x_{A_i})(y_{D_j} - y_{B_j}) \\ &= (x_{B_j} - x_{D_j})h_i + (x_{C_i} - x_{A_i})h_j. \end{aligned}$$

Analogously, we obtain

$$2 \text{ area}(A_j B_i C_j D_i) = (x_{C_j} - x_{A_j})h_i + (x_{B_i} - x_{D_i})h_j.$$

The last two relations imply that

$$\text{area}(A_i B_j C_i D_j) + \text{area}(A_j B_i C_j D_i) = L_i h_j + L_j h_i.$$

Since every quadrilateral inscribed in $\mathcal{C}$ has area at most $\frac{1}{2}$ (as we have seen in Section 2), the required relation follows. $\square$

Return to Figure 5. We are interested in p -hexagons belonging to direction TU . There are two types of such p -hexagons:

Type 1: $TA_i B_i UC_i D_i$, $i \in \{0, 1, \dots, 2n-1\}$,

$$\text{area}(TA_i B_i UC_i D_i) = \frac{1}{2} h_i (1 + L_i).$$

Type 2: $A_i WB_i C_i VD_i$, $i \in \{1, 2, \dots, 2n\}$,

$$\text{area}(A_i WB_i C_i VD_i) = \frac{1}{2} (1 + h_i) L_i.$$

Denote

$$h_p(1 + L_p) = \max_{0 \leq i \leq 2n-1} h_i(1 + L_i), \quad (6)$$

$$(h_q + 1)L_q = \max_{1 \leq i \leq 2n} (1 + h_i)L_i, \quad (7)$$

and

$$2S = \max\{h_p(1 + L_p), (1 + h_q)L_q\}. \quad (8)$$

Hence, if H denotes the p -hexagon of maximum area belonging to direction TU , we obtain $\text{area}(H) \geq S$. We need another lemma.

Lemma 4. For every $k \in \{1, 2, \dots, 2n - 1\}$,

$$h_k \leq \frac{2S(1 + 2L_p - L_k)}{(1 + L_p)^2} + O\left(\frac{1}{n}\right) \quad (9)$$

and

$$h_k \leq \frac{2S(2L_q - L_k)}{L_q^2} - 1 + O\left(\frac{1}{n}\right). \quad (10)$$

Proof. First suppose $k \leq p$. Using Lemma 2 for the triplet $(k, p - 1, p)$, it follows that

$$h_k(L_p - L_{p-1}) + h_p(L_{p-1} - L_k) \leq h_{p-1}(L_p - L_k).$$

However, (6) implies that $h_{p-1} \leq h_p(1 + L_p)/(1 + L_{p-1})$.

After some straightforward calculations we obtain

$$h_k \leq \frac{h_p(1 + L_p + L_{p-1} - L_k)}{1 + L_{p-1}} \leq \frac{2S(1 + L_p + L_{p-1} - L_k)}{(1 + L_{p-1})(1 + L_p)}.$$

We can easily check that

$$h_k \leq \frac{2S(1 + 2L_p - L_k)}{(1 + L_p)^2} + O\left(\frac{1}{n}\right).$$

Analogously, if we use Lemma 2 for the triplet $(p, p + 1, k)$, where $k \geq p + 1$, and $h_{p+1}(1 + L_{p+1}) \leq h_p(1 + L_p)$, the same type of calculations gives

$$h_k \leq \frac{2S(1 + L_p + L_{p+1} - L_k)}{(1 + L_{p+1})(1 + L_p)} = \frac{2S(1 + 2L_p - L_k)}{(1 + L_p)^2} + O\left(\frac{1}{n}\right).$$

To prove (10), it would be sufficient to repeat the above procedure for the triplets $(k, q, q + 1)$ if $k \leq q$, and $(q, q + 1, k)$ if $k \geq q + 1$. $\square$

4. Proof of the Theorem

Our goal is to estimate $\text{area}(C)$ in terms of L_p , L_q , and S . Consider $x_0 = 0.2594296\dots$, the real root of the equation $\cos x - x = \sqrt{2}/2$, and let $t = \lfloor x_0/\alpha \rfloor = \lfloor 4nx_0/\pi \rfloor$. Note that $t \leq n$. We use the following notation:

$$\begin{aligned} T_1 &= 1 + \sum_{k=1}^t (L_{2n-k}h_k + L_k h_{2n-k}), \\ T_2 &= \sum_{k=t+1}^n L_{2n-k}h_k, \\ T_3 &= \sum_{k=n+1}^{2n-t-1} L_{2n-k}h_k. \end{aligned}$$

Obviously,

$$1 + \sum_{k=1}^{2n-1} L_{2n-k}h_k = T_1 + T_2 + T_3.$$

Inequality (1) and these last relations imply that

$$\text{area}(C) \leq \sin \alpha \cdot (T_1 + T_2 + T_3). \quad (11)$$

Lemma 3 implies $L_{2n-k}h_k + L_k h_{2n-k} \leq 1$ for every $k \in \{1, 2, \dots, 2n-1\}$. Therefore,

$$\begin{aligned} \sin \alpha \cdot T_1 &\leq \sin \alpha \cdot \left(1 + \sum_{k=1}^t 1\right) = \sin \alpha \cdot (t+1) \\ &= \left[\alpha + O\left(\frac{1}{n^2}\right)\right] \cdot \left[\frac{x_0}{\alpha} + O\left(\frac{1}{n}\right)\right] \\ &= \left[\frac{\pi}{4n} + O\left(\frac{1}{n^2}\right)\right] \cdot \left[\frac{4n}{\pi}x_0 + O\left(\frac{1}{n}\right)\right] \\ &= x_0 + O\left(\frac{1}{n}\right). \end{aligned}$$

Thus,

$$\sin \alpha \cdot T_1 \leq x_0 + O\left(\frac{1}{n}\right). \quad (12)$$

In order to estimate the other two quantities from (11), we need the following.

Lemma 5.

$$\sin \alpha \cdot \sum_{k=t+1}^n L_{2n-k} = \frac{\sqrt{2}}{2} - \sin x_0 + O\left(\frac{1}{n}\right), \quad (13)$$

$$\sin \alpha \cdot \sum_{k=t+1}^n L_{2n-k}L_k = \cos \frac{2x_0}{4} + O\left(\frac{1}{n}\right), \quad (14)$$

$$\sin \alpha \cdot \sum_{k=n+1}^{2n-t-1} L_{2n-k} L_k = \frac{\cos 2x_0}{4} + O\left(\frac{1}{n}\right), \quad (15)$$

$$\sin \alpha \cdot \sum_{k=n+1}^{2n-t-1} L_{2n-k} = \cos x_0 - \frac{\sqrt{2}}{2} + O\left(\frac{1}{n}\right). \quad (16)$$

Proof. Some simple calculations give that

$$\begin{aligned} \sin \alpha \cdot \sum_{k=t+1}^n L_{2n-k} &= \frac{\cos(\pi/4) + \cos(\pi/4 - \alpha)}{2} - \frac{\sin(t+1)\alpha + \sin t\alpha}{2} \\ &= \frac{\sqrt{2}}{2} - \sin x_0 + O\left(\frac{1}{n}\right), \\ \sin \alpha \cdot \sum_{k=t+1}^n L_{2n-k} L_k &= \frac{\cos(2t+1)\alpha + \sin \alpha}{4} = \frac{\cos 2x_0}{4} + O\left(\frac{1}{n}\right), \\ \sin \alpha \cdot \sum_{k=n+1}^{2n-t-1} L_{2n-k} L_k &= \frac{\cos(2t+1)\alpha - \sin \alpha}{4} = \frac{\cos 2x_0}{4} + O\left(\frac{1}{n}\right). \end{aligned}$$

Finally,

$$\begin{aligned} \sin \alpha \cdot \sum_{k=n+1}^{2n-t-1} L_{2n-k} &= \frac{\cos(t+1)\alpha + \cos t\alpha}{2} - \frac{\sin(\pi/4) + \sin(\pi/4 - \alpha)}{2} \\ &= \cos x_0 - \frac{\sqrt{2}}{2} + O\left(\frac{1}{n}\right). \quad \square \end{aligned}$$

Now,

$$\sin \alpha \cdot T_2 = \sin \alpha \cdot \sum_{k=t+1}^n L_{2n-k} h_k.$$

Using relation (9), it follows that

$$\begin{aligned} \sin \alpha \cdot T_2 &\leq \sin \alpha \cdot \left[\sum_{k=t+1}^n L_{2n-k} \left(\frac{2S(1+2L_p - L_k)}{(1+L_p)^2} + O\left(\frac{1}{n}\right) \right) \right] \\ &= \sin \alpha \cdot \left[\frac{2S}{(1+L_p)^2} \cdot \sum_{k=t+1}^n L_{2n-k}(1+2L_p - L_k) + O(1) \right] \\ &= \frac{2S(1+2L_p)}{(1+L_p)^2} \cdot \sin \alpha \cdot \sum_{k=t+1}^n L_{2n-k} - \frac{2S}{(1+L_p)^2} \cdot \sin \alpha \\ &\quad \sum_{k=t+1}^n L_{2n-k} L_k + O\left(\frac{1}{n}\right). \end{aligned}$$

Now, using equalities (13), (14), and (15) we obtain

$$\begin{aligned} \sin \alpha \cdot T_2 &\leq \frac{2S(1+2L_p)}{(1+L_p)^2} \cdot \left(\frac{\sqrt{2}}{2} - \sin x_0 + O\left(\frac{1}{n}\right) \right) \\ &\quad - \frac{2S}{(1+L_p)^2} \cdot \left(\frac{\cos 2x_0}{4} + O\left(\frac{1}{n}\right) \right) + O\left(\frac{1}{n}\right). \end{aligned}$$

Hence,

$$\sin \alpha \cdot T_2 \leq \frac{2S(1+2L_p)}{(1+L_p)^2} \cdot \left(\frac{\sqrt{2}}{2} - \sin x_0 \right) - \frac{S \cos 2x_0}{2(1+L_p)^2} + O\left(\frac{1}{n}\right). \quad (17)$$

On the other hand, using (10) we obtain

$$\begin{aligned} \sin \alpha \cdot T_3 &= \sin \alpha \cdot \sum_{k=n+1}^{2n-t-1} L_{2n-k} h_k \\ &\leq \sin \alpha \cdot \left[\sum_{k=n+1}^{2n-t-1} L_{2n-k} \cdot \left(\frac{2S(2L_q - L_k)}{L_q^2} - 1 + O\left(\frac{1}{n}\right) \right) \right] \\ &= \sin \alpha \cdot \left[\frac{4S}{L_q} \cdot \sum_{k=n+1}^{2n-t-1} L_{2n-k} - \frac{2S}{L_q^2} \cdot \sum_{k=n+1}^{2n-t-1} L_{2n-k} L_k \right. \\ &\quad \left. - \sum_{k=n+1}^{2n-t-1} L_{2n-k} + O(1) \right]. \end{aligned}$$

Now (14), (15), and (16) imply that

$$\begin{aligned} \sin \alpha \cdot T_3 &\leq \frac{4S}{L_q} \cdot \left(\cos x_0 - \frac{\sqrt{2}}{2} + O\left(\frac{1}{n}\right) \right) - \frac{2S}{L_q^2} \cdot \left(\frac{\cos 2x_0}{4} + O\left(\frac{1}{n}\right) \right) \\ &\quad - \left(\cos x_0 - \frac{\sqrt{2}}{2} \right) + O\left(\frac{1}{n}\right). \end{aligned}$$

Therefore,

$$\sin \alpha \cdot T_3 \leq \frac{4S}{L_q} \cdot \left(\cos x_0 - \frac{\sqrt{2}}{2} \right) - \frac{S \cos 2x_0}{2L_q^2} - \left(\cos x_0 - \frac{\sqrt{2}}{2} \right) + O\left(\frac{1}{n}\right). \quad (18)$$

From (11), (12), (17), and (18) we obtain that

$$\begin{aligned} \text{area}(C) &\leq \left(x_0 + \frac{\sqrt{2}}{2} - \cos x_0 \right) + \frac{2S(1+2L_p)}{(1+L_p)^2} \cdot \left(\frac{\sqrt{2}}{2} - \sin x_0 \right) - \frac{S \cos 2x_0}{2(1+L_p)^2} \\ &\quad + \frac{4S}{L_q} \cdot \left(\cos x_0 - \frac{\sqrt{2}}{2} \right) - \frac{S \cos 2x_0}{2L_q^2} + O\left(\frac{1}{n}\right). \end{aligned}$$

Finally,

$$\begin{aligned} \frac{\text{area}(C)}{S} &\leq \frac{(1+2L_p)}{(1+L_p)^2} \cdot (\sqrt{2} - 2\sin x_0) - \frac{\cos 2x_0}{2(1+L_p)^2} \\ &\quad + \frac{2}{L_q} \cdot (2\cos x_0 - \sqrt{2}) - \frac{\cos 2x_0}{2L_q^2} + O\left(\frac{1}{n}\right). \end{aligned}$$

It is now easy to check that

$$\frac{(1+2L_p)}{(1+L_p)^2} \cdot (\sqrt{2} - 2\sin x_0) - \frac{\cos 2x_0}{2(1+L_p)^2} \leq \frac{4(\sqrt{2} - 2\sin x_0)}{4 + \sqrt{2} + 2\sin x_0}$$

and equality occurs iff $L_p = (\sin x_0)/2 + \sqrt{2}/4 = 0.481818\dots$ Also,

$$\frac{2}{L_q} \cdot (2\cos x_0 - \sqrt{2}) - \frac{\cos 2x_0}{2L_q^2} \leq \frac{4x_0}{x_0 + \sqrt{2}}$$

and equality occurs iff $L_q = (\cos x_0)/2 + \sqrt{2}/4 = 0.8368216\dots$ Hence,

$$\frac{\text{area}(C)}{S} \leq \frac{4(\sqrt{2} - 2\sin x_0)}{4 + \sqrt{2} + 2\sin x_0} + \frac{4x_0}{x_0 + \sqrt{2}} + O\left(\frac{1}{n}\right) = 1.2281771\dots$$

This final relation proves the theorem.

This result is nearly the best that can be obtained using only the ideas presented above. Consider C as shown in Fig. 8. TU and VW are two perpendicular diameters of the circle

Fig. 8

of radius $\frac{1}{2}$. Point A is the middle of the arc TW and the points B and C are chosen so that the measure of the small angles $\widehat{TOB}$ and $\widehat{COW}$ are equal to $x_0 = 0.2594296\dots$. Repeat this construction in the other three quadrants and then take $\mathcal{C}$ to be the circular disk from which we take off those eight shaded segments of the circle. Obviously, $TVUW$ is the quadrilateral of maximum area from all quadrilaterals inscribed in $\mathcal{C}$.

Some straightforward calculations show that

$$\text{area}(\mathcal{C}) = \left(\frac{\sqrt{2}}{2} + 1 \right) \cos x_0 - \frac{\sqrt{2}}{2} (1 + \sin x_0) = 0.7614804\dots$$

On the other hand, it is not difficult to check that all p -hexagons belonging to direction TU have area at most

$$\text{area}(H) \leq \frac{[(\sqrt{2} + 1) \cos x_0 - 1 - \sin x_0]^2}{4(2 \cos x_0 - \sqrt{2})(\sqrt{2} - 2 \sin x_0)} = 0.6200668\dots$$

Consequently, it follows that

$$\frac{\text{area}(\mathcal{C})}{\text{area}(H)} \geq \frac{0.7614804}{0.6200668} = 1.2280619\dots$$

This bound is not far from the one obtained in the theorem.

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Received June 1, 1996, and in revised form January 24, 1997.