Tiling Groups: New Applications in the Triangular Lattice

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Abstract. We first give a new presentation of an algorithm, from W. Thurston, to tile polygons with “calissons” (i.e., lozenges formed from two cells of the triangular lattice Λ). Afterward, we use a similar method to get a linear algorithm to tile polygons with m -leaning bars (parallelograms of length m formed from $2m$ cells of Λ) and equilateral triangles (whose sides have length m) and we produce a quadratic algorithm to tile polygons with m -leaning bars.

1. Introduction

Given a finite set of basic tiles (which are called prototiles), how can a “good” algorithm of tiling (i.e., an algorithm in polynomial time which, given a figure of the plane, gives a tiling of this figure with copies of the prototiles or indicates that the figure cannot be tiled) be exhibited? Conway and Lagarias [CL] defined tiling groups as a tool to study tiling problems. Their method gives a lot of necessary conditions for a simply connected figure to be tileable (see [LR] and [P]). Their work has been prolonged by Thurston [T] who introduced (in some particular cases) the notion of height function associated with a tiling. Using this new notion, algorithms to tile a simply connected figure with dominoes, or with “calissons,” have been produced. Thurston’s ideas have again been taken by C. and R. Kenyon [KK], which obtained an algorithm to tile a simply connected figure with m -bars (rectangles of length m and unit width).

In Section 2 we recall the notions introduced by J. H. Conway, J. C. Lagarias, and W. P. Thurston. We point out the notions of “overtilings” and “tiling projections” which make the method of tiling more natural. In Section 3 these notions are illustrated by an example of W. P. Thurston: the tiling with calissons. We give an algebraic presentation when W. P. Thurston used geometrical arguments.

Our purely algebraic framework permits us to extend the notion of a height function to other examples: in Section 4 we produce a linear algorithm of tiling of a simply connected figure with “leaning bars” of length m (i.e., parallelograms with large sides of length m and small sides of unit length) and equilateral triangles with sides of length m . In Section 5 we produce a quadratic algorithm of tiling with leaning bars, simultaneously using four height functions. These new algorithms are the main results of this paper. They generalize and completely prove the algorithms announced in a previous paper [R].

2. Tiling Groups and Tiling Graphs

2.1. Tilings

Let Λ be the triangular lattice of the Euclidean plane. A (finite) *figure* F of Λ is a (finite) union of (closed) cells of Λ . A figure F is *simply connected* if F and its complement $\mathbb{R}^2 - F$ are both connected. A finite simply connected figure F is called a *polygon* of Λ . The boundary of a polygon P canonically induces a cycle in Λ called the *boundary cycle* of P .

A set S of *prototiles* is a fixed finite set of polygons of Λ . A *tile* is a *translated* copy of a prototile. A *tiling* T of a figure F is a set of tiles included in F , with pairwise disjoint interiors, such that the union of the tiles of T equals F .

2.2. Cayley Graphs

Let Σ be the set $\{a, b, c, a^{-1}, b^{-1}, c^{-1}\}$, let $F_{a,b,c}$ be the free group generated by $A = \{a, b, c\}$ and let s denote the canonical surjection from the language Σ^* to $F_{a,b,c}$.

Let R be a finite set of words of Σ^* . Group N_R denotes the *normal* group of $F_{a,b,c}$ generated by the elements of $s(R)$ and G_R denotes the quotient group $F_{a,b,c}/N_R$. The *Cayley graph* C_R is the directed graph with labeled edges such that:

- (i) the vertices of C_R are the elements of G_R ;
- (ii) the set of labels is $\{a, b, c\}$; and
- (iii) for each pair (g, g') of elements of $(G_R)^2$ and each element u of A , there exists an edge of C_R from g to g' , labeled by u , if and only if $gu = g'$.

The underlying graph of Λ can be seen as the Cayley graph $C_{\text{triangle}} = C_R$, with $R = \{abc, acb\}$ (see Fig. 1), each element of A being associated to a unit step. Hence, each vertex of Λ is identified to an element of G_R and has three incoming edges, one of each label, and three outgoing edges, one of each label.

Let $p = (v_0, v_1, \dots, v_p)$ be a path of Λ , i.e., a sequence of vertices such that, for each integer i , with $0 \leq i < p$, there exists an element of Σ such that $v_i u_i = v_{i+1}$. The *path word* $w(p)$ is the word $u_0 u_1, \dots, u_{p-1}$. Moreover, if p is a boundary cycle of a polygon P , we say that $w(p)$ is a *contour word* of P . Let $S = \{t_1, t_2, \dots, t_p\}$ be a set of prototiles. The *tiling group* of S is the group $G_{\text{tile}} = G_{R_S}$, where $R_S = \{w_1, w_2, \dots, w_p\}$ and for each integer i such that $1 \leq i \leq p$, w_i is a contour word of t_i . The *tiling Cayley graph* of S is the graph $C_{\text{tile}} = C_{R_S}$.

Fig. 1. Cayley graph C_{triangle} .

2.3. Overtilings and Tiling Projections

Let T be a tiling of a figure F . The graph $G_T = (V_T, E_T)$ of T is the subgraph of C_{triangle} such that E_T is the set of the edges which are included on the boundary of a tile of T , and V_T is the set of the extremities of the edges of E_T .

Definition 2.1. Let S be a set of prototiles, F a figure of Λ , T a tiling of F generated by the prototiles of S , and let G_{tile} denote the tiling group associated to the set S of prototiles. An *overtiling* constructed over T is a mapping f_T from the set V_T of the vertices of G_T to G_{tile} such that, for each pair (v, u) of $V_T \times E_T$, the equality $f_T(vu) = f_T(v)u$ holds.

Proposition 2.2 (J. H. Conway). Let F be a figure of Λ , T a tiling of F , v_0 a vertex of G_T , and v'_0 a vertex of G_{tile} .

If F is connected (resp., simply connected), then there exists at most one (resp., exactly one) overtiling f_T constructed over T such that $f_T(v_0) = v'_0$.

Sketch of the Proof. Function f_T can easily be constructed successively exploring the contour of each tile: function f_T is first defined on the vertices of the boundary of a tile t_0 , which contains v_0 . Afterwards, f_T is defined on the vertices of a tile t_1 which has a common vertex with t_0 and so on (i.e., f_T is defined on a tile t_i which has a common vertex with the figure tiled by $\{t_0, t_1, \dots, t_{i-1}\}$). This method proves that there exists at most one overtiling f_T for F being connected.

If F has some holes, a conflict (i.e., a vertex v such that two distinct values of $f_T(v)$ are obtained) can arise, which yields that there is no overtiling. Such a conflict cannot arise when F is simply connected, since the sequence $(t_0, t_1, \dots, t_p)$ of tiles can be chosen in such a way that, for each integer i , such that $1 \leq i \leq p$, the intersection of the boundary of tile t_i with the boundary of the subfigure of F tiled by $\{t_0, t_1, \dots, t_{i-1}\}$ is connected. $\square$

Remark 2.3. Let T and T' be tilings of a same polygon P . If f_T and $f_{T'}$ are overtilings such that, for a vertex v_0 of the boundary of P , $f_T(v_0) = f_{T'}(v_0)$, then for each vertex v of the boundary of P , $f_T(v) = f_{T'}(v)$.

The use of the overtilings is a main method to study the tilings. Interesting examples (although the notion of overtiling is not always clearly presented) are developed in [CL], [T], [P], [KK], and [LR].

Nevertheless, generally, the use of the overtilings is not easy, since the tiling group is complex. Thus, quotient groups are used, in order to have a group that can be easily described. To obtain such a quotient group, it suffices to exhibit a set R of words such that the contour words of the prototiles are null in the group G_R . In this case, we have a natural surjection s from G_{tile} to G_R .

Let T be a tiling of a polygon P , and let f_T be an overtiling over T . Function g_T , from the vertices of F to G_R , defined by $g_T = s \circ f_T$, is called a *tiling projection* of tiling T .

3. Tiling with Calissons

We now study the case when the set of prototiles $S_{\text{cal}} = \{\text{cal}_1, \text{cal}_2, \text{cal}_3\}$ (the set of the *calissons*), each calisson being a figure formed from two cells with a common edge, called the *central edge* of the calisson. A set of contour words of the calissons is $R_{\text{cal}} = \{aba^{-1}b^{-1}, ac^{-1}a^{-1}c, c^{-1}bcb^{-1}\}$. The tiling group of the calissons (the *calisson group*) is denoted by G_{calisson} .

This instance was treated by Thurston [T] using geometrical arguments. We give a purely algebraic presentation of its method, using the notions introduced in the previous section. The main point is the use of a height function to get an order on the tilings, which is specific from the set S_{cal} of prototiles.

3.1. Order on the Tilings of a Polygon

For the calissons, the group G_R , with $R = \{ab^{-1}, ac^{-1}\}$, is used. If $ab^{-1} = ac^{-1} = 1$, then $a = b = c$, which yields that $aba^{-1}b^{-1} = ac^{-1}a^{-1}c = c^{-1}bcb^{-1} = 1$. Thus the group G_R can be seen as a quotient group of the calisson group. Let P be a polygon and fix a vertex v_0 of the boundary of P . For each tiling T of P , g_T denotes the only tiling projection of T such that $g_T(v_0) = 1$.

Notice that for each element g of G_R , there exists a unique integer m such that $g = a^m$. Integer m is called the *height* of g (denoted by $h(g)$). For each tiling T of P , the height $h_T(v)$ of a vertex v of V_T according to T is defined by $h_T(v) = h(g_T(v))$.

The set of the tilings of P can be ordered as follows: $T \leq T'$ if, for each vertex v of P , $h_T(v) \leq h_{T'}(v)$. This ordered set has been studied completely in [STCR].

3.2. Local Rotations and Key-Lemma

Let T be a tiling of a polygon P such that three tiles of T form a tiling of a hexagon H_x centered in a vertex x . A *local rotation* (see Fig. 2) is the substitution of these tiles by the three tiles which form the other tiling of H_x . A new tiling T' is obtained. For each vertex v of P , except x , $h_T(v) = h_{T'}(v)$ and $|h_T(x) - h_{T'}(x)| = 3$.

Lemma 3.1 (Key-Lemma). *Let T be a tiling of a polygon P , let M denote the height (according to T) of a highest vertex of G_T , let M' denote the height of a highest vertex of the boundary of P , and let x be a vertex such that $h_T(x) = M$.*

Fig. 2. The Cayley graph of group G_R .

If $M > M'$, then the hexagon H_x formed from the six cells having x for common vertex is covered by three tiles of T .

Proof. The edge of C_{triangle} from x to xa (resp., xb, xc) cannot be an edge of G_T , since in this case, $h_T(xa)$ (resp., $h_T(xb), h_T(xc)$) is equal to $M + 1$, which contradicts the maximality of M . Thus this edge is the central edge of a tile of T . $\square$

Corollary 3.2. Let T be a minimal (for the order defined in Lemma 3.1) tiling of a polygon P and let M be the maximal height of a vertex of P according to T . There exists a vertex v of the boundary of P such that $h_T(v) = M$.

Proof. If it was false, the hypothesis of the key-lemma would hold, and consequently a tiling T' , which contradicts the minimality of T , could be obtained from T using a local rotation. $\square$

3.3. Algorithm of Tiling

From Corollary 3.2, we deduce an algorithm which constructs a minimal tiling T of the polygon given as input (if such a tiling exists).

Algorithm of Tiling with Calissons

Input: A polygon P of the triangular lattice formed from n squares.

Initialization: Arbitrarily choose a vertex v_0 of the boundary of P . For each vertex v of the boundary of P , compute $g_T(v)$ and $h_T(v)$, where g_T is a tiling projection such that $g_T(v_0) = 1$ and h_T is a height deduced from g_T .

Main loop: Let v be a highest vertex of the boundary. Place a tile included in P whose central edge (seen as an edge of G_{triangle}) is incoming in v . Remove the two covered cells from P , and update P , the boundary of P , and functions g_T and h_T .

Repeat these instructions until either P is empty (i.e., a tiling is constructed) or a contradiction appears in the definition of g_T (i.e., there is no tiling).

Fig. 3. A rotation of a tiling with calissons.

Fig. 6. A part of the Cayley graph of group $G_{\text{tricolored}}$ (with $m = 4$) with an associated height.

An integer value (which will be called the height) can be associated to each element of $G_{\text{tricolored}}$ as follows: we first attribute a height value to the elementary cycles of $G_{\text{tricolored}}$, according to the following rules: for each cycle C of height h , there exists a unique cycle C' of height $h - 1$ which share an element g_0 with C , the height of all the other cycles which share g_0 with C is h and all the cycles which are joined with C by an element g_1 such that $g_1 \neq g_0$ are cycles of height $h + 1$. Thus, for each element g of $G_{\text{tricolored}}$, two of the three elementary cycles containing g have the same height, say h , and the height of the third cycle containing g is $h - 1$. Value $h - 1$ becomes the height of the element g .

Such a height function is neither unique nor canonical, but it can easily be constructed step by step, applying the above rules.

Notice that:

- (a) the height of all the elements of an elementary cycle of height is h is also h , except for one element (the lowest element of the cycle) whose height is $h - 1$;
- (b) each element of $G_{\text{tricolored}}$ is the lowest element of exactly two cycles which include this element; and
- (c) if we take two neighbors of a fixed element g of $G_{\text{tricolored}}$ which are not on the same elementary cycle, then at least one of the neighbors is (strictly) higher than g .

As in Section 3, given a tiling T of a polygon P with a fixed vertex v_0 of the boundary, the height $h_T(v)$ of each vertex v of a cell of P is defined saying that $h_T(v)$ is the height of $g_T(v) = s \circ f_T(v)$, where f_T denotes the overtiling of T such that $f_T(v_0) = 1$ and s denotes the canonical surjection from the tiling group to $G_{\text{tricolored}}$. Notice that h_T is not defined for the vertices which are elements of the interiors of the triangles of T .

Fig. 9. The lozenge rotations.

such that a subset L_T of tiles of T is a tiling of L . Notice that lozenge L admits three tilings. Replacing the tiles of L_T by other tiles which cover L (see Fig. 9) is our first type of local rotations (the *lozenge rotations*).

- (b) Assume that there exists an isosceles trapeze R , formed from $m(m+1)$ triangular cells of P , such that a triangle and a leaning bar of T form a tiling of R . Replacing these two tiles of T which cover R by the two other tiles which can cover R is our second type of local rotations (the *trapeze rotations*) (see Fig. 10).

Lemma 4.2 (Key-Lemma). *Let T be a tiling of a polygon P , let M denote the height of a highest vertex of V_T , let M' denote the height of a highest vertex of the boundary of P , and let x be a vertex such that $h_T(x) = M$.*

If $M > M'$, then there exists a lozenge L_x with sides of length m (or an isosceles trapeze R_x whose little basis has length at most $m-1$ and whose nonparallel sides have length m) tiled by tiles of T , such that x is an interior vertex of L_x (or R_x), and the height of all the interior vertices of L_x (or R_x) which are in V_T is equal to M .

Proof. From Remark 4.1, if x is on the boundary of a tile ti of T and $h_T(x) = M$, then x cannot be a corner of ti . Moreover, the height of the extremities of the side of ti containing x is $M-1$, and all the other vertices of this side have height M .

The hypothesis: $M > M'$ yields that x is on the boundary of two tiles, say ti and ti' . These tiles share a complete side since otherwise, the height the corner of one tile would be M , and, consequently, this tile would have vertices of height $M+1$.

Thus if at least one of the tiles ti or ti' is a triangle, the proof is finished. If tiles ti and ti' both are leaning bars, then ti and ti' are translated copies of the same prototile, since otherwise a corner of a tile of height M appears.

Let P_2 be the parallelogram tiled by ti and ti' . If $m = 2$, we state $L_x = P_2$. For $m > 2$, remark that, necessarily, the height of vertices of a large side of P_2 (except the extremities) is M , since at least one of the tiles, ti or ti' , is on the case (a) of Remark 4.1.

Fig. 10. The trapeze rotations.

Since $M > M'$, those vertices are in the interior of P , and consequently are on the large side of another tile ti'' of T . If ti'' is a triangle, a trapeze R_x is obtained which finishes the proof. Otherwise, we can repeat the argument for the parallelogram P_3 tiled by ti , ti' , and ti'' , and so on, until either a trapeze is obtained or a parallelogram P_m is obtained, and we state $L_x = P_m$. $\square$

Corollary 4.3. *Let T be a minimal tiling of a polygon P and let M be the maximal height of a vertex of P according to T . There exists a vertex v of the boundary of P such that $h_T(v) = M$.*

Proof. Assume that T is a minimal tiling of P which does not satisfy this corollary. Let x be a vertex of P such that $h_T(x) = M$. We can apply the key-lemma.

If x is an element of the interior of a lozenge L_x satisfying this lemma, then a lozenge rotation on L_x easily gives a tiling T' such that T' is lower than T , which is a contradiction.

If x is an element of the interior of an isocetes trapeze R_x satisfying the lemma, then the height of the vertices of a side of R_x of length m is M (except the height of the extremities of this side). Let x' denote such a vertex. If x' is an element of the interior of a lozenge $L_{x'}$ satisfying the lemma, a contradiction is obtained as above. Otherwise, x' is an element of the interior of an isocetes trapeze $R_{x'}$ satisfying the lemma. Let $P_{x,x'}$ be the reunion of R_x and $R_{x'}$. Polygon $P_{x,x'}$ is a parallelogram with small sides of length m and larges sides of length m' , with $m' > m$. We easily verify that the tiling T'' , obtained from T replacing the two triangles and the $m' - m$ leaning bars which cover $P_{x,x'}$ by the m bars which are translated copies of the same prototiles and form a tiling of $P_{x,x'}$, is lower than T , which is a contradiction. $\square$

Let T be a minimal tiling of minimal height of a polygon P and let v denote a highest vertex of the boundary of P . Corollary 4.3 gives at most two possible alternatives (a triangular tile tr_0 or a leaning bar b_0) for the tile of T a side of which contains vertex v . How can the good tile be chosen?

If this tile is tr_0 , then the height of a vertex v' of another side of tr_0 is maximal. If v' is on the boundary of P , we easily see that b_0 is not included in P , thus tile tr_0 is forced. Otherwise, vertex v' is on the boundary of another tile t' of T , which no contradiction with Corollary 4.3. The tiles tr_0 and t' cover either a lozenge L (if t' is a triangle) or an isocetes trapeze R (if t' is a translated copy of b_0). Using a local rotation, a tiling T' containing the bar b_0 is obtained.

That means that when two possible alternatives arise, we can always choose to put the leaning bar. This fact guarantees that the following algorithm constructs a tiling T of minimal height, when the polygon can be tiled.

Algorithm of Tiling with Leaning Bars and Triangles

Input: A polygon P of the triangular lattice formed from n triangular cells.

Initialization: Arbitrarily choose a vertex v_0 of the boundary of P . For each vertex v of the boundary of P , compute $g_T(v)$ and $h_T(v)$, where g_T is a tiling projection on $G_{\text{tricolored}}$ such that $g_T(v_0) = 1$ and h_T denotes the height induced by an arbitrarily fixed height of $G_{\text{tricolored}}$ (for instance, the height of Fig. 6).

Main loop: Let v be a highest vertex of the boundary. If it is possible, place a leaning bar in P with v on its boundary and no contradiction with Corollary 4.3. Otherwise place a triangle in P with v on its boundary and no contradiction with Corollary 4.3.

Remove the covered cells from P , and update the boundary of P and functions g_T and h_T .

These instructions are repeated until either P is empty (in this case, a tiling is constructed) or a contradiction or an impossibility appears (in this case, there is no tiling).

Since all the updates are local operations, this algorithm can be implemented so as to have a linear running time. We can claim the following theorem:

Theorem 4.4. *A polygon P of the triangular lattice formed from n cells can be tiled with leaning bars and triangles formed from four cells, or proved not to be tileable in time linear in n .*

5. Tiling with Leaning Bars

In this section, the set of prototiles is $\{b_1, b_2, b_3, b_4, b_5, b_6\}$. The method is a little different from that used for the other sets of prototiles, since the vertices of maximal height of a minimal tiling (using height and order of Section 4) can all be interior. This situation has to be studied carefully.

We need a new notation. Let x and y be distinct elements of $\{a, b, c\}$. We say that a tile ti is a tile of type $x^m y$ if ti admits $x^m y x^{-m} y^{-1}$ for contour word. Hence, we have six possible types: $a^m b$, $a^m c$, $b^m a$, $b^m c$, $c^m a$, and $c^m b$.

We need two kinds of rotations: the lozenge rotations of Section 4 which do not use triangular tiles and local rotations of a new type (called the *cubic rotations*) involving $m + 2$ tiles and $2m(m + 2)$ cells. A cubic rotation is presented in Fig. 11.

Lemma 5.1. *Let T be a minimal tiling of a polygon P , let M denote the height of a highest vertex of P , and let M' denote the height of a highest vertex of the boundary of P . Assume that $M \neq M'$ and let x be a vertex of P such that $h_T(x) = M$.*

Then there exists a lozenge L_x with sides of length m , tiled by tiles of T , such that x is an interior vertex of L_x , the height of all the interior vertices of L_x is equal to M , the height of the corners of L_x is equal to $M - 2$, and the height of the other vertices of boundary of L_x is equal to $M - 1$.

Fig. 11. A cubic rotation.

Proof. The hypothesis: $M > M'$ yields that x is on the boundary of two tiles, say ti and ti' . These tiles share a complete side since otherwise, the height of the corner of one tile would be M , and, consequently, this tile would have vertices of height $M + 1$. If tiles ti and ti' both are leaning bars, then ti and ti' are translated copies of the same prototile, since otherwise a corner of a tile of height M also appears.

Let P_2 be the parallelogram tiled by ti and ti' . If $m = 2$, we state $L_x = P_2$. For $m > 2$, remark that, necessarily, the height of vertices of a large side of P_2 (except the extremities) is M , since at least one of the tiles, ti or ti' , is on the case (a) of Remark 4.1.

Since $M > M'$, those vertices are in the interior of P , and consequently are on the large side of another tile ti'' of T . We can repeat the argument for the parallelogram P_3 tiled by ti , ti' , and ti'' , and so on, until a parallelogram P_m is obtained, and we state $L_x = P_m$.

Since T is minimal, the tiling T' , obtained from T by a lozenge rotation on L_x , is not lower than T , which yields the conditions announced for the heights on the boundary of L_x . $\square$

Lemma 5.2. *Let T be a minimal tiling of a polygon P , let M denote the height of a highest vertex of P , and let M' denote the height of a highest vertex of the boundary of P .*

Assume that $M > M'$ and there exists a lozenge L , tiled by tiles of T , a contour word of which is $a^m b^m a^{-m} b^{-m}$, such that the height of all the interior vertices of L is equal to M .

Then there exists a sequence $(L_1, L_2, \dots, L_p)$ (called a zigzag) of translated copies of L such that (see Fig. 12):

- (i) *there exists an integer i such that $1 \leq i \leq p$ and $L_i = L$;*
- (ii) *the left side or the lower side of L_1 is included in the boundary of P ;*
- (iii) *the right side or the upper side of L_p is included in the boundary of P ;*
- (iv) *for each integer i such that $1 \leq i < p$, L_i and L_{i+1} share exactly one side, which is either the right side of L_i and the left side of L_{i+1} or the upper side of L_i and the lower side of L_{i+1} ; and*
- (v) *for each integer i such that $1 \leq i < p$, L_i is tiled by m tiles of T .*

Proof. Since T is minimal, the height of the vertices of the sides of L (except the extremities) is $M - 1$ and the height of the corners of L is $M - 2$, since otherwise a tiling T' , which is lower than T , would be created, using a lozenge rotation.

Fig. 12. A zigzag of lozenges.

If the right side or the upper side of L is included in the boundary of P , then state $L = L_p$. Otherwise, let v be any vertex of the right side or the upper side of L , except the upper right corner of L , and let v' be one of the two vertices of P which are not in L . If the edge whose extremities are v and v' is an edge of G_T (i.e., this edge crosses no tile of T), then $h_T(v') = M$, thus there exists a lozenge $L' = L_v$ satisfying the previous lemma. We easily see that L' is necessarily the translated copy of L whose left side is the right side of L . Thus, lozenge L' can be taken as the successor of L in the zigzag.

Thus, now, we only have to treat the case when each edge as above crosses a tile of T . This yields that there exists a tile ti of T whose left side is the right side of L , and there exists a tile ti' of T whose lower side is the upper side of L . If ti is a tile of type $b^m a$, then the height of the vertices of the right side of ti (except the extremities) is M , thus there exists a lozenge L' satisfying the previous lemma, which is necessarily the translated copy of L whose left side is the right side of L and, consequently can be taken as the successor of L in the zigzag. The same argument can be used if ti' is a tile of type $a^m b$.

Thus, a difficulty arises only when ti is a tile of type $b^m c$, and ti' is a tile of type $a^m c$. But, in such as case, a tiling T' , which is lower than T , would be created, using a cubic rotation replacing tiles ti , ti' , and the tiles which cover L . This contradicts our hypothesis.

This argument can be repeated to construct the successor of L' , and so on until the right side or the upper side of a lozenge is on the boundary of P . Afterwards, the beginning of the zigzag is constructed by the same way. $\square$

Corollary 5.3. *For each minimal tiling of P , $M \leq M' + 1$.*

Proof. Obvious. $\square$

Let $R_a = \{b, c\}$. A tiling projection can be defined on $G_a = G_{R_a}$, since all the contour words of the leaning dominoes are null in G_a . Group G_a is isomorphic to the additive group $\mathbb{Z}$: since $b = c = 1$, for each element g of G_a , there exists a unique integer m such that $g = a^m$. Integer m is called the a -height of g (denoted by $ha(g)$). The height obtained by the tiling projection of a tiling T on C_a is denoted by ha_T . The b -height hb_T and the c -height hc_T are defined by the same way.

Lemma 5.4. *Let T be a tiling of a polygon P and let v and v' be two vertices of P . If there exists an edge of the triangular lattice from v to v' whose label is a or b , then $hc_T(v) \geq hc_T(v')$.*

Fig. 13. The Cayley graph of G_a .

Proof. Assume that the edge from v to v' is labeled by a . If this edge crosses no tile of T , then $hc_T(v') = hc_T(v)$. Otherwise, this edge crosses a tile ti of type $b^m a$, $c^m a$, $b^m c$, or cb . If ti is of type $b^m a$ or $c^m a$, then $hc_T(v') = hc_T(v)$. If ti is of type $b^m c$ or $c^m b$, then $hc_T(v') = hc_T(v) - 1$.

The case when the edge from v to v' is labeled by b is treated in the same way. $\square$

Lemma 5.5. *Let P be a polygon such that the heights h_T , ha_T , hb_T , hc_T are constructed on the boundary of P with no contradiction. Let M' be the maximal height (for h_T) of a vertex of the boundary of P , and let w and w' be two vertices of the boundary of P such that $h_T(w) = h_T(w') = M'$ and $hc_T(w) = hc_T(w')$.*

Assume we can tile, with no contradiction for the heights, a zigzag $(L_1, L_2, \dots, L_p)$ such that:

- (i) *for each integer i such that $1 \leq i \leq p$, $a^m b^m a^{-m} b^{-m}$ is a contour word of L_i ;*
- (ii) *for each integer i such that $1 \leq i \leq p$, each interior vertex of L_i has height $M' + 1$;*
- (iii) *vertex w is a vertex of either the left side or the lower side of L_0 ; and*
- (iv) *vertex w' is a vertex of either the right side or the upper side of L_p .*

Then, for each minimal tiling T of P , for each integer i , with $1 \leq i \leq p$, T contains m tiles which tile L_i .

Proof. Obviously, there exists a path $(v_0, v_1, \dots, v_q)$ of vertices of P such that:

- (i) $v_0 = w$ and $v_q = w$;
- (ii) for each integer i such that $0 < i < p$, vertex v_i is an interior vertex of polygon $L_1 \cup L_2 \cup \dots \cup L_p$; and
- (iii) for each integer i such that $0 \leq i < p$, the edge from v_i to v_{i+1} is labeled by a or b .

From Lemma 4.4, we have $hc_T(v_{i+1}) \leq hc_T(v_i)$, thus, since $hc_T(v_0) = hc_T(v_q)$, all the values $hc_T(v_i)$, for $0 \leq i \leq q$, are equal. Thus, if the edge from v_i to v_{i+1} is labeled by a , either this edge crosses not tile or this edge crosses a tile of type $b^m a$ or $c^m a$.

Now let k be an integer such that $0 \leq k < p$ and assume that each lozenge L_i , with $i \leq k$, is covered by tiles of T . Let r be the largest integer such that v_r is element of L_i . Assume that the edge from v_r to v_{r+1} is labeled by a (the case when the edge from v_r to v_{r+1} is labeled by b is treated by a symmetric way). Our hypothesis yields that $h_T(v_r) = M'$ and the edge from v_r to v_{r+1} cannot cross a tile of type $c^m a$.

Thus, $h_T(v_{r+1}) = M' + 1$, which is the maximal height of a vertex of P . This yields, from Lemma 5.1, that L_{k+1} is covered by tiles of T . This finishes the proof, by induction on integer k . $\square$

The above lemma and Corollary 5.3 guarantee the correctness of the following algorithm.

Algorithm of Tiling with Leaning Bars

Input: A polygon P of the triangular lattice formed from n triangular cells.

Initialization: Arbitrarily choose a vertex v_0 of the boundary of P . For each vertex v of the boundary of P , compute g_T , $h_T(v)$, $ha_T(v)$, $hb_T(v)$, $hc_T(v)$ step by step. Those functions are initialized giving value 1 to v_0 .

Main loop: Let w be a highest vertex of the boundary of P . If the hypothesis of Lemma 5.3 (or a symmetric version of this lemma) can be satisfied, place some tiles which cover the lozenges of the zigzag found.

Otherwise, place the only tile ti such that w is on a large side of ti and w is a highest vertex of ti .

Remove the tiled cells from P , and update the boundary of P and functions g_T , h_T , ha_T , hb_T , hc_T .

These instructions are repeated until either P is empty (in this case, a tiling is constructed) or a contradiction appears in the definition of g_T (in this case, there is no tiling).

The search of a zigzag costs at most $O(n)$ time units since it is very similar to the search of a special leave of a binary tree with at most n vertices (since a lozenge has at most two potential successors for a zigzag, and the number of potential lozenges to explore is lower than n). Thus this algorithm can be implemented so as to have a quadratic running time. We can claim the following theorem:

Theorem 5.6. *A polygon P of the triangular lattice formed from n cells can be tiled with leaning bars, or proved not to be tileable in time quadratic in n .*

6. Conclusion

This paper gives a partial unification of the methods used to get an algorithm of tiling. Thurston [T] and C. and R. Kenyon [KK] have obtained similar results, which can be explained using height functions on Cayley graphs on quotient groups, with “dominoes” and “bars” of the square lattice. Chaboud [C], using a very neighboring height function, gave an algorithm of tiling with “dominoes” on lattices which cannot be seen as supports of Cayley graphs. Underlying algebraic structures do not seem to explain its result.

Is there a better unification? Why can our method not be applied in the hexagonal lattice, where only very partial results have been obtained (see [T], [KR], and [LR])? For which sets of prototiles is there a height function which permits us to give an algorithm of tiling? How can the quotient group be chosen?

All these questions have not received a clear answer yet.

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