On the Independence Number of Minimum Distance Graphs*

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Abstract. Let $F = F(n)$ denote the largest number such that any set of n points in the plane with minimum distance 1 has at least F elements, no two of which are at unit distance. Improving a result by Pollack, we show that $F(n) \geq \frac{9}{35}n$. We also give an $O(n \log n)$ time algorithm for selecting at least $\frac{9}{35}n$ elements in a set of n points with minimum distance 1 so that no two selected points are at distance 1.

1. Introduction

Let X be a set of n points in the plane with minimum distance 1. The *minimum distance graph* associated with X is a graph with vertex set X , in which two vertices are connected by an edge if and only if their distance is 1.

Let G be a graph with n vertices. A subset I of the vertex set of G is said to be *independent*, if no two elements of I are connected by an edge. The *independence number* $\alpha(G)$ of G is defined as the size of the largest independent set of vertices in G . Let

$$F(n) = \min \alpha(G),$$

where the minimum is taken over all minimum distance graphs on n vertices.

In 1983, Erdős asked for bounds on $F(n)$. The vertex set of $\lfloor n/3 \rfloor$ widely spaced unit triangles shows that $F(n) \leq \lceil \frac{1}{3}n \rceil$. In 1996, Pach and Tóth [3] gave a better construction which yields $F(n) \leq \lceil \frac{5}{16}n \rceil$ for large n .

In 1985, Pollack [4] proved $F(n) \geq \lceil \frac{1}{4}n \rceil$ by observing that the minimum distance graph is planar, therefore 4-colorable. Hence, the largest color class is an independent set of size at least $\frac{1}{4}n$. For planar graphs, in general, the bound $\alpha(G) \geq \frac{1}{4}n$ cannot be improved. However, for minimum distance graphs, we can establish a better bound.

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Theorem. *Given n points in the plane with minimum distance 1, we can always choose at least $\frac{9}{35}n$ of them so that their minimum distance is greater than 1.*

Using the above terminology, the theorem states that $F(n) \geq \frac{9}{35}n$.

2. Proof of the Theorem

Let X be a set of n points in the plane with minimum distance 1. Let $G = G(X)$ denote the minimum distance graph associated with X . We will prove that $\alpha(G) \geq \frac{9}{35}n$. It is sufficient to prove the following lemma:

Lemma 1. *Every minimum distance graph G has an independent set P of $k \leq 9$ vertices such that if m is the number of vertices in $G - P$, incident to at least one element of P , then $k/(k + m) \geq \frac{9}{35}$.*

The main theorem follows from Lemma 1, since we can repeatedly apply the lemma to remove an independent set and its neighbors with the given property. The union of these independent sets is an independent set with at least $\frac{9}{35}n$ elements.

The proof uses two auxiliary results (Lemmas 2 and 3), whose proofs are postponed to the last section.

The main idea of the proof is the following. First we can suppose that all vertices have degree at least 3. Then if you walk counterclockwise around the boundary of G , then the direction you are facing must rotate counter-clockwise around 360° . Only vertices of degree 3 contribute to this counter-clockwise rotation so the boundary must contain some such vertices. Moreover, the counterclockwise rotations at degree 3 vertices must outweigh the clockwise rotations at degree 4 and 5 vertices. A careful case analysis of the possible positions of the vertices and their neighbors gives the desired independent set P .

Proof of Lemma 1. We can make the following simple observations:

- (a) If a vertex has degree 2, then it forms an independent set satisfying the conditions of Lemma 1;
- (b) if two nonadjacent vertices of degree 3 share a neighbor, then they form such an independent set; and
- (c) if a vertex of degree 3 shares two neighbors with a nonadjacent vertex of degree 4, then they form such a set; etc.

Let $d(p)$ denote the degree of a vertex p of G . By the above observations, we can suppose that $d(p) \geq 3$ for all $p \in G$.

Define the boundary graph C' as follows. Let r_1 denote a vertex of G with minimum y -coordinate. Draw a half-line from r_1 pointing downward, and turn it counterclockwise around r_1 until it hits a neighbor of r_1 . Call that neighbor r_2 . Then turn the segment $r_1 r_2$ counterclockwise around r_2 until it hits another neighbor of r_2 . Call that neighbor r_3 . Continue this procedure until we hit a point r_i that has been encountered before. Then $r_i = r_{i+t}$, so we have a cycle $C' = \{r_{i+1}, r_{i+2}, \dots, r_{i+t}\}$.

Fig. 1. Irregular p_i .

Let C denote the path obtained from C' by the removal of r_{i+t} . For simplicity, denote the vertices of C (in order) by $p_1, \dots, p_{t-1}$. In the sequel $d(p)$ will denote the degree of p in the *original* graph G .

Clearly, $3 \leq d(p_i) \leq 5$ for all $p_i \in C$. Let $a(p_i) := 180^\circ - \angle p_{i-1}p_i p_{i+1}$ (the angle is understood in clockwise direction, facing the inner part of the cycle). Then $\sum_{i=1}^{t-1} a(p_i) \geq 240^\circ$ (since $a(p_t) \leq 120^\circ$ and $a(p_i) \leq 240^\circ - d(p_i) \cdot 60^\circ$).

Definition. Let p_{i-1}, q_1 , and p_{i+1} denote the neighbors of a point $p_i \in C'$ with $d(p_i) = 3$ (in clockwise order as is shown in Fig. 1). If $\angle p_{i-1}p_i q_1 = \angle q_1 p_i p_{i+1} = 60^\circ$, then p_i is called *regular*. A vertex of degree 3 is called *irregular* if it is not regular.

If p_i is irregular, then at least one of its neighbors, say p_{i+1} , does not have a common neighbor with p_i . If $d(p_{i+1}) \geq 4$, then call the (p_i, p_{i+1}) pair a *special pair*. Note that an irregular p_i may be the member of two special pairs.

Lemma 2. Let $p_i \in C$ be irregular, $d(p_{i+1}) \geq 4$, and $\angle q_1 p_i p_{i+1} = \alpha > 60^\circ$. Then $a(p_i) + a(p_{i+1}) \leq 0$ (see Fig. 1).

Thus, counterclockwise (positive) turns at irregular vertices are followed by clockwise (negative) turns of at least the same degree, unless the irregular vertex is followed by another vertex of degree 3. In other words, for special pairs (p_i, p_{i+1}) we have $a(p_i) + a(p_{i+1}) \leq 0$.

Corollary. If p_i is irregular and $\sum_{k=i-1}^{i+1} a(p_k) > 0$, then $d(p_{i-1}) = 3$ or $d(p_{i+1}) = 3$. Furthermore, if $\sum_{k=i-1}^{i+1} a(p_k) > 60^\circ$, then $d(p_{i-1}) = d(p_{i+1}) = 3$.

Fig. 2. Regular p_i .

We may suppose that there are no two vertices $p_j, p_{j+2} \in C$ with $d(p_j) = d(p_{j+2}) = 3$, for otherwise p_j and p_{j+2} would have five neighbors altogether, and Lemma 1 would follow.

Next we choose a family of pairwise disjoint *blocks* (strings of consecutive vertices) in C . This procedure is the heart of the proof.

Going through the vertices of a negative block in counterclockwise direction yields a negative turn of at least -60° , and all positive turns take place in a positive block. Therefore, since going around the boundary is a 360° turn, there must be more positive blocks than negative ones.

- (i) (a) Assume that a regular vertex r_i has a neighbor of degree 5. Then fix such a neighbor s_i , and let $\{r_i, s_i\}$ be called a *neutral block*. (Note that no two neutral blocks have nonempty intersection, because $s_i \neq s_j$ unless $r_i = r_j$.)
- (b) Assume that an irregular vertex r_i is an element of a special pair. For each such r_i fix an s_i such that (r_i, s_i) forms a special pair. Then $\{r_i, s_i\}$ is called a *neutral block*.
- (ii) (a) If a regular vertex r_i has no neighbor s_i with $d(s_i) = 5$, then it forms a *positive block*.
- (b) Each pair of irregular vertices (p_i, p_{i+1}) forms a *positive block*.
- (iii) (a) Each s_i with $d(s_i) = 5$, which does not belong to any neutral block, forms a *negative block*.
- (b) Each set of l consecutive vertices $p_{k+1}, \dots, p_{k+l}$, is called a *negative block*, whenever $d(p_i) = 4$ ($k+1 \leq i \leq k+l$), $d(p_k) \neq 4$, $d(p_{k+l+1}) \neq 4$, $\sum_{i=k+1}^{k+l} a(p_i) \leq -60^\circ$, and neither p_{k+1} nor p_{k+l} is a member of a neutral block.

Note that some vertices $p_{i_1}, \dots, p_{i_l}$ may not belong to any of the blocks defined above, but for them $\sum_{j=1}^l a(p_{i_j}) \leq 0$ holds.

Clearly, each negative block contributes -60° (or less) to $\sum_{p_i \in C} a(p_i)$. Each positive turn not neutralized by the negative turn, at a following vertex of degree 4 or degree 5, is now represented by a positive block. Moreover, by the corollary, a positive block contributes at most 60° to $\sum_{p_i \in C} a(p_i)$. So $\sum_{p_i \in C} a(p_i) \geq 240^\circ$ implies

$$|\{P: P \text{ is a positive block}\}| \geq |\{N: N \text{ is a negative block}\}| + 4.$$

Let $B_1, \dots, B_s$ denote the disjoint blocks in the order as they appear in C . There exists a positive block B_j for which B_{j+1} is either positive or neutral.

- (i) If B_j consists of a single regular vertex, then suppose it is p_2 . It follows that $d(p_1) \leq 4$. If B_j consists of two irregular vertices, then denote them by p_1 and p_2 .
- (ii) If B_{j+1} is a positive block, and it is a regular vertex, then we can suppose that it is p_m . If B_{j+1} consists of two irregular vertices, then denote them by p_m and p_{m+1} .
- (iii) If B_{j+1} is a neutral block, then let $p_m \in B_{j+1}$ be the vertex in B_{j+1} with degree 3. In this case, either $d(p_{m-1}) \geq 4$ or $d(p_{m+1}) \geq 4$.

Thus there are eight possible types of (B_j, B_{j+1}) pairs, but in any case, the degrees of $p_1, p_2, \dots, p_m$ are either:

- (a) 4, 3, 3; or
- (b) 4, 3, $\leq 4, \leq 4, \dots, \leq 4, \leq 5, 3$; or
- (c) 3, 3, $\leq 4, \leq 4, \dots, \leq 4, \leq 5, 3$;

and in cases (a) and (b), p_2 is regular and so the first and third vertices share two neighbors. ≤ 4 and ≤ 5 are denoting vertices of degree at most 4 and at most 5, respectively.

This is all we need for the rest of the proof. We distinguish two cases:

- (a) Suppose that there exists an h with $3 \leq h \leq 15$, $h \leq m$, such that $d(p_h) = 3$. There are no negative blocks between B_j and B_{j+1} , therefore $d(p_i) \leq 4$ for all i where $3 \leq i \leq m - 2$. Then:
 - (i) if h is even, set $P = \{p_h, p_{h-2}, \dots, p_2\}$;
 - (ii) if h is odd, set $P = \{p_h, p_{h-2}, \dots, p_1\}$;
 and Lemma 1 follows.
- (b) Let $m \geq 16$ and $d(p_i) = 4$ ($3 \leq i \leq 14$). There was no negative block between B_j and B_{j+1} , therefore $\sum_3^{14} a(p_i) > -60^\circ$.

Lemma 3. *Suppose that there is a sequence $p_1, p_2, \dots, p_{14}$ in C such that $d(p_2) = 3$, $d(p_i) = 4$ ($3 \leq i \leq 14$), $\sum_3^{14} a(p_i) > -60^\circ$, and p_1 and p_3 have at most six neighbors altogether.*

Then we can find $k \leq 9$ independent vertices of G such that the number of vertices incident to at least one of them is at most $3k - 1$.

Thus, Lemma 1 follows. □

3. Proofs of the Lemmas

Proof of Lemma 2. We may suppose that $d(p_{i+1}) = 4$. Denote the neighbors of p_{i+1} by p_i, q_2, q_3 , and p_{i+2} (see Fig. 1). Clearly, $\angle q_1 p_i p_{i+1} = \alpha$ implies $\angle p_i p_{i+1} q_2 \geq 180^\circ - \alpha$. Therefore,

$$\begin{aligned} 180^\circ - a(p_i) + 180^\circ - a(p_{i+1}) &= \angle p_{i-1} p_i p_{i+1} + \angle p_i p_{i+1} p_{i+2} \\ &\geq 60^\circ + \alpha + (180^\circ - \alpha) + 60^\circ + 60^\circ = 360^\circ. \end{aligned}$$

Hence $a(p_i) + a(p_{i+1}) \leq 0$. □

For the proof of Lemma 3, we need the following claim.

Claim. *Let $p_0, \dots, p_6$ be consecutive vertices of C such that $d(p_i) = 4$ ($1 \leq i \leq 5$) and $a(p_i) + a(p_{i+1}) > -60^\circ$ ($1 \leq i \leq 4$). Then:*

- (i) p_k and p_{k+1} ($k = 1, 2, 3, 4$) have a common neighbor r_k (see Fig. 3).

Fig. 3

Let r_0 and r_5 denote the remaining unlabeled neighbors of p_1 and p_5 , respectively. Then:

- (ii) r_k ($k = 1, 2, 3, 4$) have at most two neighbors (s_{k-1}, s'_k) different from p_k, p_{k+1}, r_{k-1} , and r_{k+1} .

Suppose that both r_2 and r_3 have exactly two such neighbors and $\sum_2^4 a(p_i) > -30^\circ$. Then:

- (iii) r_2 and r_3 have a common neighbor $s_2 = s'_2$ ($s_2 \neq p_3$); and
 (iv) s_2 has at most two neighbors different from s_1, r_2, r_3 , and s'_3 (see Fig. 7).

Proof. (i) Let $k = 1$ and suppose for contradiction that p_1 and p_2 have no common neighbor. Let us use the notation of Fig. 4. Clearly, $\angle bp_1p_2 + \angle p_1p_2c \geq 180^\circ$, $\angle p_0p_1b \geq 120^\circ$, and $\angle cp_2p_3 \geq 120^\circ$. So

$$\begin{aligned} a(p_1) + a(p_2) &= 180^\circ - \angle p_0p_1p_2 + 180^\circ - \angle p_1p_2p_3 \\ &\leq 360^\circ - 180^\circ - 120^\circ - 120^\circ = -60^\circ. \end{aligned}$$

This is a contradiction, so Claim (i) is true for $k = 1$. For $k = 2, 3, 4$, the proof is similar.

(ii) Suppose $k = 1$ and that p_1 and p_2 have the common neighbor r_1 , and r_1 has three neighbors s_1, s_2, s_3 as is shown in Fig. 5.

Let $\alpha = \angle p_0p_1r_1$ and $\beta = \angle r_1p_2r_2$. Then $\alpha + \beta < 180^\circ$. We also know that $\angle p_1r_1s_1 \geq 180^\circ - \alpha$, $\angle s_3r_1p_2 \geq 180^\circ - \beta$. But

$$\begin{aligned} 360^\circ &= \angle p_1r_1s_1 + \angle s_1r_1s_2 + \angle s_2r_1s_3 + \angle s_3r_1p_2 + \angle p_2r_1p_1 \\ &\geq (180^\circ - \alpha) + 60^\circ + 60^\circ + (180^\circ - \beta) + 60^\circ = 540^\circ - (\alpha + \beta) > 360^\circ, \end{aligned}$$

a contradiction. For $k = 2, 3, 4$, the proof is similar.

Fig. 4

Fig. 5

(iii) Suppose that $s_2 \neq s'_2$ (see Fig. 6). Let $|r_2 r_3| = t$ and $|s'_2 s_2| = s$. Let $\alpha = \angle r_1 p_2 r_2$, $\beta = \angle r_2 p_3 r_3$, and $\gamma = \angle r_3 p_4 r_4$. Then $\alpha + \beta + \gamma < 210^\circ$. Clearly, $t = 2 \sin(\beta/2)$ and

$$\begin{aligned} \delta &= \angle s'_2 r_2 r_3 \leq 360^\circ - (\angle p_2 r_2 s_1 + \angle s_1 r_2 s'_2 + \angle r_3 r_2 p_3 + \angle p_3 r_2 p_2) \\ &\leq 360^\circ - \left[(180^\circ - \alpha) + 60^\circ + \left(90^\circ - \frac{\beta}{2} \right) + 60^\circ \right] = \alpha + \frac{\beta}{2} - 30^\circ \\ &< 210^\circ - \gamma - \frac{\beta}{2} - 30^\circ = 180^\circ - \gamma - \frac{\beta}{2} \leq 120^\circ - \frac{\beta}{2}. \end{aligned}$$

Similarly, $\delta' = \angle r_2 r_3 s_2 < 120^\circ - \beta/2$. This implies

$$\frac{t-s}{2} \geq \sin(90^\circ - \max\{\delta, \delta'\}) > \sin\left[90^\circ - \left(120^\circ - \frac{\beta}{2}\right)\right] = \sin\left(\frac{\beta}{2} - 30^\circ\right).$$

So

$$s < t - 2 \sin\left(\frac{\beta}{2} - 30^\circ\right) = 2 \left[\sin \frac{\beta}{2} - \sin\left(\frac{\beta}{2} - 30^\circ\right) \right] \leq 1,$$

because $30^\circ \leq \beta/2 < 45^\circ$. This is a contradiction, so $s_2 = s'_2$.

(iv) In part (iii), we proved $s_2 = s'_2$. Let α, β, γ be as before, and let $\varepsilon = \angle s_1 r_2 s_2$ and $\varepsilon' = \angle s_2 r_3 s'_3$. We get that $\varepsilon \leq \alpha + \beta - 60^\circ$ and $\varepsilon' \leq \gamma + \beta - 60^\circ$ (see Fig. 7). We also

Fig. 6

Fig. 7

know that $\angle r_3 s_2 r_2 = \beta$. Thus if s_2 had three neighbors t_1, t_2, t_3 , then we would get

$$\begin{aligned} 360^\circ &\geq \beta + (180^\circ - \varepsilon) + 60^\circ + 60^\circ + (180^\circ - \varepsilon') \\ &\geq \beta + 480^\circ - (\alpha + \beta - 60^\circ) - (\beta + \gamma - 60^\circ) \\ &= 600^\circ - (\alpha + \beta + \gamma) > 600^\circ - 210^\circ = 390^\circ, \end{aligned}$$

which is a contradiction. $\square$

Now we are ready to prove Lemma 3.

Proof of Lemma 3. Claim (i) implies that all p_i, p_{i+1} pairs ($3 \leq i \leq 13$) have a common neighbor r_i (see Fig. 8). Suppose that there is an r_j ($4 \leq j \leq 12$) which has only one neighbor different from r_{j-1}, r_{j+1}, p_j , and p_{j+1} .

Now:

- (a) if j is odd, then pick $p_{j+2}, r_j, p_{j-1}, p_{j-3}, \dots, p_2$; and
- (b) if j is even, then pick $p_{j+2}, r_j, p_{j-1}, p_{j-3}, \dots, p_1$.

These $\lfloor j/2 \rfloor + 2$ vertices have altogether $3(\lfloor j/2 \rfloor + 2) - 1$ neighbors.

Thus, we can suppose by Claim (ii) that all r_j ($4 \leq j \leq 12$) has two other neighbors. $\sum_3^{14} a(p_i) > -60^\circ$ implies that either $\sum_4^8 a(p_i) > -30^\circ$ or $\sum_9^{13} a(p_i) > -30^\circ$. Without loss of generality suppose $\sum_9^{13} a(p_i) > -30^\circ$.

Claim (iii) implies that there exists $s_9 = s'_9$, $s_{10} = s'_{10}$, and $s_{11} = s'_{11}$ (see Fig. 8). By Claim (iv), s_{10} has at most two neighbors different from s_9, s_{11}, r_{10} , or r_{11} . Let

Fig. 8

$P = \{s_{10}, r_{12}, r_9, p_{14}, p_{11}, p_8, p_6, p_4, p_2\}$. Then the number of points incident to at least one element of P is at most $26 = 3 \cdot 9 - 1$, completing the proof of Lemma 3. $\square$

Remark. The above proof gives us an algorithm for selecting at least $\frac{9}{35}n$ elements in a set of n points with minimum distance 1 so that no two selected points are at distance 1. The algorithm finds at most 9 and at least 1, new independent points in each step.

The minimum distance graph can be constructed in $O(n \log n)$ time, and after that, each step takes at most C time, where C is a large constant. We have to make at most n steps, therefore the algorithm can be completed in $O(n \log n)$ time.

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