Two Finiteness Theorems for Periodic Tilings of d -Dimensional Euclidean Space*

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Abstract. Consider the d -dimensional euclidean space $\mathbb{E}^d$. Two main results are presented: First, for any $N \in \mathbb{N}$, the number of types of periodic equivariant tilings $(\mathcal{T}, \Gamma)$ that have precisely N orbits of $(2, 4, 6, \dots)$ -flags with respect to the symmetry group Γ , is finite. Second, for any $N \in \mathbb{N}$, the number of types of convex, periodic equivariant tilings $(\mathcal{T}, \Gamma)$ that have precisely N orbits of tiles with respect to the symmetry group Γ , is finite. The former result (and some generalizations) is proved combinatorially, using Delaney symbols, whereas the proof of the latter result is based on both geometric arguments and Delaney symbols.

1. Introduction

In this note we want to prove some finiteness results concerning the number of types of various sorts of periodic tilings. What is a tiling? As there are many definitions in use and as proofs—as opposed to (equally important) communications of ideas—depend crucially on clearly delineated concepts (as well as their various specifications), it is imperative that we spell out the definitions we adopt in this paper:

Throughout this paper, for any $d \in \mathbb{N}$ let $I(d) := \{0, 1, \dots, d\}$. Given a (connected) d -dimensional manifold $\mathcal{X}$, a tiling $\mathcal{T}$ of $\mathcal{X}$ is a *regular CW-structure* for $\mathcal{X}$, that is, it consists of a nested sequence $\mathcal{T}^{-1}(\mathcal{X}) := \emptyset \subsetneq \mathcal{T}^0(\mathcal{X}) \subsetneq \mathcal{T}^1(\mathcal{X}) \subsetneq \dots \subsetneq \mathcal{T}^d(\mathcal{X}) := \mathcal{X}$

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of closed subsets such that the following hold:

- (T1) for every connected component $f = f_i$ of $T^1(\mathcal{X}) \setminus T^{i-1}(\mathcal{X})$ ($i \in I(d)$) there exists a homeomorphism $\varphi_f: \mathbb{D}_i \rightarrow \bar{f}$ of the i -dimensional disk $\mathbb{D}_i := \{(x_1, \dots, x_i) \in \mathbb{R}^i \mid \sum_{j=1}^i x_j^2 \leq 1\}$ onto the closure $\bar{f}$ of f which induces a homeomorphism between the interior $\mathring{\mathbb{D}}_i := \{(x_1, \dots, x_i) \in \mathbb{D}_i \mid \sum_{j=1}^i x_j^2 < 1\}$ of $\mathbb{D}_i$ and f , and
- (T2) a subset T of $T^i(\mathcal{X})$ is closed if and only if $T \cap \bar{f}$ is a closed subset of $\bar{f}$ for every connected component f of $T^i(\mathcal{X}) \setminus T^{i-1}(\mathcal{X})$.

The connected components f_i of $T^i(\mathcal{X}) \setminus T^{i-1}(\mathcal{X})$ are called the i -cells of T , while their closures $F_i := \bar{f}_i$ are called the i -dimensional tiles of T , or, if $i = d$, simply the tiles of T . The set of i -dimensional tiles of T is denoted by $T_i(\mathcal{X})$, or, if $i = d$, simply by T or $T(\mathcal{X})$.

It follows easily from (T2) that T is *locally finite*, that is, any closed subset A of $\mathcal{X}$ is compact if and only if the number of tiles $\bar{f} \in T$ with $\bar{f} \cap A \neq \emptyset$ is finite. In this case the number of i -dimensional tiles $\bar{f}_i \in T_i(\mathcal{X})$ with $\bar{f}_i \cap A \neq \emptyset$ is necessarily finite, too, for all $i \in I(d)$. It follows also that, for any i -dimensional tile $\bar{f}_i$ and any j -cell f_j with $f_j \cap \bar{f}_i \neq \emptyset$, one has $f_j \subseteq \bar{f}_i$ (and, hence, $j \leq i$, too) and, moreover, $f_j \subset \partial \mathcal{X}$, if $f_j \cap \partial \mathcal{X} \neq \emptyset$ (where $\partial \mathcal{X}$ denotes the boundary of $\mathcal{X}$). In particular, any intersection of (full- or lower-dimensional) tiles is a union of tiles and the same holds for the intersection of any full- or lower-dimensional tile with $\partial \mathcal{X}$.

A tiling T of $\mathcal{X}$ is called *face-to-face* if for any two distinct (full- or lower-dimensional) i -dimensional (resp. i' -dimensional) tiles $\bar{f}$ and $\bar{f}'$ the intersections $\bar{f} \cap \bar{f}'$ and $\bar{f} \cap \partial \mathcal{X}$ are either empty or a lower-dimensional tile. It follows that also, vice versa, every lower-dimensional tile in $\mathcal{X} \setminus \partial \mathcal{X}$ is the intersection of all (full-dimensional) tiles containing it, while any lower-dimensional tile in $\partial \mathcal{X}$ coincides with the intersection of $\partial \mathcal{X}$ with the tiles containing it.

If $\mathcal{X} := \mathbb{E}^d$, then a tiling T of $\mathcal{X}$ is called *convex*, if

- (CT1) every tile $\bar{f}$ is a convex d -polytope, and
- (CT2) the (convex) faces of such a tile $\bar{f}$ coincide with the lower-dimensional tiles contained in $\bar{f}$.

Equivalently, a tiling T of $\mathbb{E}^d$ is convex if it satisfies condition (CT1) and if, in addition, $T^i(\mathbb{E}^d)$ coincides with the union of the i -dimensional faces of the polytopes. Using the fact that any convex subset of a convex polytope P which is a union of faces of P is necessarily a face of P itself, it follows easily that any convex tiling of $\mathbb{E}^d$ is face-to-face.

Next, we define an *equivariant tiling* of a d -manifold $\mathcal{X}$ to be a pair (T, Γ) , consisting of a tiling T of $\mathcal{X}$ and a properly discontinuous group Γ of homeomorphisms of $\mathcal{X}$ such that

$$(ET) \quad \gamma T^i(\mathcal{X}) = T^i(\mathcal{X})$$

holds for all $\gamma \in \Gamma$ and all $i \in I(d)$.

Two equivariant tilings (T, Γ) and (T', Γ') of two d -manifolds $\mathcal{X}$ and $\mathcal{X}'$, respectively, are said to be of the same *type* if there exists a homeomorphism $\varphi: \mathcal{X} \rightarrow \mathcal{X}'$ satisfying the conditions $\varphi(T^i(\mathcal{X})) = T'^i(\mathcal{X}')$ ($i \in I(d)$) and $\Gamma' = \{\varphi \tau \varphi^{-1} \mid \tau \in \Gamma\}$.

Note that for $i \in I(d)$ every homeomorphism $\tau \in \Gamma$ induces a permutation, also

denoted by τ , of $\mathcal{T}_i(\mathcal{X})$, and that a group Γ of homeomorphisms of $\mathcal{X}$ satisfying condition (ET) for all $\tau \in \Gamma$ and all $i \in I(d)$ is properly discontinuous if and only if the only element in Γ which induces the identity permutation on $\mathcal{T}_i(\mathcal{X})$ for all $i \in I(d)$ is the identity element. Note also that if $\mathcal{T}$ is face-to-face, then a homeomorphism τ of $\mathcal{X}$ satisfying condition (ET) for all $i \in I(d)$ induces the identity permutation on $\mathcal{T}_i(\mathcal{X})$ for all $i \in I(d)$, if and only if it induces the identity on $\mathcal{T}_{i_0}(\mathcal{X})$ for some fixed $i_0 \in I(d)$.

An equivariant tiling $(\mathcal{T}, \Gamma)$ is called *tile-transitive* if Γ acts transitively on $\mathcal{T} = \mathcal{T}_d(\mathcal{X})$, and, more generally, *tile- N -transitive* if the number $\#(\Gamma \backslash \mathcal{T})$ of Γ -orbits of tiles of $\mathcal{T}$ equals N . The reader is cautioned that this usage of the term *N -transitive* differs from that in permutation group theory.

Finally, we define an equivariant tiling $(\mathcal{T}, \Gamma)$ of $\mathcal{X} := \mathbb{E}^d$ to be *periodic* if Γ is a d -dimensional crystallographic group, that is, if Γ is (affinely equivalent to) a cocompact discrete group of isometries of $\mathbb{E}^d$, or, equivalently, if the subgroup $\bar{\Gamma}$ of Γ consisting of all translations in Γ has a finite index $h = |\Gamma : \bar{\Gamma}|$ in Γ and is generated by exactly d linearly independent translations in $\mathbb{E}^d$.

Now, assume that $(\mathcal{T}, \Gamma)$ is a convex periodic tiling of $\mathbb{E}^d$. Minkowski [1] proved that if $(\mathcal{T}, \Gamma)$ is tile-transitive and if Γ coincides with its translational subgroup $\bar{\Gamma}$, then the number of *facets* f of any tile (that is, of its $(d - 1)$ -dimensional subtiles) of $\mathcal{T}$ is bounded (from above) by $2(2^d - 1)$. Together with results by Voronoi [2], this implies that there exist only a finite number of types of convex tile-transitive periodic tilings $(\mathcal{T}, \Gamma)$ of $\mathbb{E}^d$ with a *translational* symmetry group $\Gamma = \bar{\Gamma}$.

Later, this was generalized by Delone and Sandakova [3] as follows: for an arbitrary tile-transitive periodic tiling $(\mathcal{T}, \Gamma)$ of $\mathbb{E}^d$, the number of facets of any given tile $f \in \mathcal{T}$ is bounded by $2(2^d - 1) + (h - 1)2^d$, with $h = |\Gamma : \bar{\Gamma}|$, the index of the translational subgroup $\bar{\Gamma}$ of Γ in Γ . In the same paper it was also shown that the number of tiles meeting at any given vertex is bounded by a function depending only on d and $|\Gamma : \bar{\Gamma}|$ (the latter number is in turn bounded by some function of d) and it was indicated how these two results imply that the number of types of convex, periodic, and tile-transitive tilings of $\mathbb{E}^d$ is finite.

In Section 4 of this paper we use ideas from [1] and [3] to obtain the following more general result (Theorem 4.4): For any number $N \in \mathbb{N}$, there exists only a finite number of types of convex, periodic, and tile- N -transitive tilings $(\mathcal{T}, \Gamma)$ of euclidean d -space.

To be precise, we first prove that if the symmetry group Γ is translational and if Γ partitions the set of tiles $\mathcal{T}$ into exactly N different Γ -orbits, then the number of facets of any tile $P \in \mathcal{T}$ is bounded by $2(2^d - 1) + (N - 1)2^d$. Clearly, this result implies that the number of facets is also bounded if Γ is not purely translational. Combining this fact with a basic result concerning euclidean “Delaney symbols” yields the theorem.

Note that—for $d > 2$ —none of the above results holds when the tiling is *not* convex. A construction due to Heesch yields an infinite family of three-dimensional, periodic, and tile-transitive tilings satisfying condition (CT1), though not condition (CT2), as well as another such family where each tile is a union of finitely many convex polytopes with translational symmetry groups [4].

In [5] it was shown that only finitely many different types of periodic tilings $(\mathcal{T}, \Gamma)$ of euclidean 3-space exist that are *face- N -transitive*, i.e., that have precisely $N \in \mathbb{N}$ different Γ -orbits of two-dimensional faces. It should be emphasized that this result does not assume convexity of the tiles and, hence, stands in contrast—though, of course, not in contradiction—to the situation observed by Heesch.

In Section 3 of this paper we generalize this result to d -dimensional tilings, proving Theorem 3.3 (and some generalizations): For any number N , there exist only finitely many different types of periodic face-to-face tilings of euclidean d -space, which are $(2, 4, 6, \dots)$ - N -transitive, that is, with the property that there are precisely N different Γ -orbits of flags $(\bar{f}_2, \bar{f}_4, \bar{f}_6, \dots)$ of $\mathcal{T}$, i.e., of sequences $(\bar{f}_2, \bar{f}_4, \bar{f}_6, \dots)$ of 2-, 4-, 6-, $\dots$ -dimensional tiles of $\mathcal{T}$ satisfying the condition $\bar{f}_2 \subset \bar{f}_4 \subset \bar{f}_6 \dots$.

In Section 2 we recall the definition of a Delaney symbol and the fact that we can associate with any equivariant tiling its Delaney symbol in such a way that two periodic tilings (of two simply connected manifolds) are of the same type, if and only if their associated Delaney symbols are isomorphic. Moreover, we show that any d -dimensional Delaney set can give rise to only finitely many different euclidean Delaney symbols.

Consequently, to prove Theorems 3.3 and 4.4, we only need to determine a bound for the size of the Delaney set of the involved tilings. For Theorem 3.3, this is done in a purely combinatorial way, by considering the properties of the associated Delaney symbols. In the case of Theorem 4.4, we need to apply additional geometric arguments that make use of the convexity of the tilings involved, to establish the existence of a bound.

2. Delaney Symbols

In this section we recall the definition of d -dimensional Delaney symbols and some important results that we will need in Sections 3 and 4. For a more comprehensive introduction to the theory of Delaney symbols, see, e.g., [6]–[8].

Let $\Sigma_d := \langle \sigma_0, \sigma_1, \sigma_2, \dots, \sigma_d \mid \sigma_i^2 = 1 \rangle$ denote the free Coxeter group of rank d , that is, the group that is generated by $d + 1$ involutions. A d -dimensional Delaney symbol is a system $(\mathcal{D}, M)$ consisting of a transitive (right) Σ_d -set $\mathcal{D}$, together with a mapping

$$\begin{aligned} M: \mathcal{D} &\rightarrow \mathbb{N}^{(d+1) \times (d+1)} \\ D &\mapsto (m_{ij}(D))_{0 \leq i, j \leq d}, \end{aligned}$$

satisfying the following (slightly redundantly formulated) conditions:

- (DS1) $m_{ij}(D\sigma_i) = m_{ij}(D\sigma_j) = m_{ij}(D) = m_{ji}(D)$,
- (DS2) $D(\sigma_i\sigma_j)^{m_{ij}(D)} = D(\sigma_j\sigma_i)^{m_{ij}(D)} = D$,
- (DS3) $m_{ii}(D) = 1$,
- (DS4) $m_{ij}(D) = 2$, if $|i - j| > 2$,

for all $D \in \mathcal{D}$ and all $i, j \in I(d)$. We call $\mathcal{D}$ the *Delaney set* associated with the Delaney symbol.

To every equivariant tiling $(\mathcal{T}, \Gamma)$ of a d -manifold $\mathcal{X}$, there exists a Delaney symbol $(\mathcal{D}, M) := (\mathcal{D}_{(\mathcal{T}, \Gamma)}, M_{(\mathcal{T}, \Gamma)})$ which is defined as follows [6]–[8]:

The set $\mathcal{D} = \mathcal{D}_{(\mathcal{T}, \Gamma)}$, also called the *Delaney set* of $(\mathcal{T}, \Gamma)$, is defined to be the set of Γ -orbits of maximal flags $F = (F_0, F_1, \dots, F_d)$ of $\mathcal{T}$, that is, of sequences $F = (F_0, F_1, \dots, F_d) \in \mathcal{T}_0(\mathcal{X}) \times \mathcal{T}_1(\mathcal{X}) \times \dots \times \mathcal{T}_d(\mathcal{X})$ satisfying the condition $F_0 \subset F_1 \subset \dots \subset F_d$. Obviously, the orbit space $\Gamma \backslash \mathcal{X}$ is compact if and only if $\mathcal{D}$ is finite. In addition, appealing to some basic results in PL-topology, it is easy to show that, for any (maximal) flag F of $\mathcal{T}$ as above and any $i \in I(d)$, there exists at most (and in case

$i \neq d$ or $i = d$ and $F_{d-1} \not\subset \partial \mathcal{X}$ exactly) one i -neighbor, i.e., maximal flag F' of $\mathcal{T}$ with $F'_i \neq F_i$ and $F'_j = F_j$ for all $j \in I(d) \setminus \{i\}$. This allows us to define a canonical action of Σ_d on the set $\mathcal{F}(\mathcal{T})$ of all maximal flags of $\mathcal{T}$: for $i \in I(d)$ and $F \in \mathcal{F}(\mathcal{T})$, define $F\sigma_i := F'$, if F' is the i -neighbor of F , and $F\sigma_i := F$, if no such i -neighbor exists. This action of the generators of Σ_d extends to an action of Σ_d . As, for obvious reasons, the right action of Σ_d on $\mathcal{F}(\mathcal{T})$ commutes with the induced left action of Γ on $\mathcal{F}(\mathcal{T})$, a canonical action of Σ_d on the set $\mathcal{D}_{(\mathcal{T}, \Gamma)} := \Gamma \backslash \mathcal{F}(\mathcal{T})$ results.

Next, define $m_{ij}(D)$ for $0 \leq i, j \leq d$ and $D = \Gamma \cdot F \in \mathcal{D}$ by $m_{ij}(D) := \inf\{m \in \mathbb{N} \mid F(\sigma_i \sigma_j)^m = F\}$. It is easily checked that this way a well-defined natural number is associated with any i, j , and D as above and that the resulting system of numbers satisfies conditions (DS1)–(DS4). Moreover, if $\mathcal{T}$ is face-to-face, then also

$$(DS5) \quad m_{ij}(D) \geq 3, \text{ if } |i - j| = 1,$$

must hold. Otherwise, i - and $(i + 1)$ -dimensional tiles $F_i, F'_i, F_{i+1}, F'_{i+1}$ could be found with $F_i \subseteq F_{i+1}, F'_i \subseteq F_{i+1}, F_i \subseteq F'_{i+1}, F'_i \subseteq F'_{i+1}$, and $F_i \neq F'_i$ as well as $F_{i+1} \neq F'_{i+1}$, or $i = d - 1$ and $F_{d-1} \subseteq \partial \mathcal{X}$. In this case the intersection $F_{i+1} \cap F'_{i+1}$ (or $F_d \cap \partial \mathcal{X}$, respectively) would contain the two distinct tiles F_i and F'_i and, hence, could not be a lower-dimensional tile itself. More generally, we have $m_{i,i+1}(D) \geq 3$ for some $i \in I(d - 1)$ and all $D \in \mathcal{D}$, if for any two pairs of i - and $(i + 1)$ -dimensional tiles $F_i, F'_i, F_{i+1}, F'_{i+1}$ the relation $F_i \cup F'_i \subseteq F_{i+1} \cap F'_{i+1}$ implies $F_i = F'_i$ or $F_{i+1} = F'_{i+1}$.

Two Delaney symbols $(\mathcal{D}, M)$ and $(\mathcal{D}', M')$ are called *isomorphic* if there exists a bijection $\pi: \mathcal{D} \rightarrow \mathcal{D}'$ with $\pi(D\sigma_k) = (\pi D)\sigma_k$ and $m_{ij}(D) = m'_{ij}(\pi D)$ for all $i, j, k \in I(d)$ and all $D \in \mathcal{D}$. A d -dimensional Delaney symbol $(\mathcal{D}, M)$ is called *locally (i, j, k) -spherical* if

$$\sum_{D \in \mathcal{O}} \left(\frac{1}{m_{ij}(D)} + \frac{1}{m_{jk}(D)} + \frac{1}{m_{ik}(D)} - 1 \right) > 0$$

holds for every $\langle \sigma_i, \sigma_j, \sigma_k \rangle$ -orbit $\mathcal{O} \subset \mathcal{D}$ ($i, j, k \in I(d)$ with $i < j < k$). It can be shown [5] that the Delaney symbol of any periodic tiling of dimension $d \geq 3$ is necessarily locally (i, j, k) -spherical for all $i, j, k \in I(d)$ with $i < j < k$.

The following two results are used in the proofs of Theorems 3.3 and 4.4:

Lemma 2.1. *Let $(\mathcal{T}, \Gamma)$, and $(\mathcal{T}', \Gamma')$ be two equivariant tilings of two simply connected d -manifolds $\mathcal{X}$ and $\mathcal{X}'$ with corresponding Delaney symbols $(\mathcal{D}, M)$ and $(\mathcal{D}', M')$, respectively. Then the tilings $(\mathcal{T}, \Gamma)$ and $(\mathcal{T}', \Gamma')$ are of the same type if and only if $(\mathcal{D}, M)$ and $(\mathcal{D}', M')$ are isomorphic.*

It is easy to see that equivalent tilings must give rise to isomorphic Delaney symbols. The converse makes use of the fact that we have restricted ourselves to simply connected manifolds. It is false for nonsimply connected manifolds [6].

For a right Σ_d -set $\mathcal{D}$ and subgroup $\Delta \leq \Sigma_d$ let $\mathcal{D} \backslash \Delta$ denote the set of all Δ -orbits $D \cdot \Delta$ ($D \in \mathcal{D}$) of Δ in $\mathcal{D}$.

Lemma 2.2. *Given any subset J of the index set $I(d)$, let $\bar{J} = I(d) \setminus J$. Then the number $\#(\mathcal{D} / \langle \sigma_i \mid i \in J \rangle)$ of $\langle \sigma_i \mid i \in J \rangle$ -orbits in the Delaney-set $\mathcal{D} = \mathcal{D}_{(\mathcal{T}, \Gamma)}$ of some*

equivariant tiling (T, Γ) of a d -manifold $\mathcal{X}$ coincides with the number of Γ -orbits in the set

$$F_I(T) := \left\{ (F_j)_{j \in \bar{J}} \in \prod_{j \in \bar{J}} \mathcal{T}_j(\mathcal{X}) \mid F_{j_1} \subset F_{j_2} \text{ for any two indices } j_1 < j_2 \text{ in } \bar{J} \right\}.$$

Proof. This follows immediately from our definitions and the well-known fact that for any two maximal flags $F, F' \in F(T) = \mathcal{F}_\emptyset(T)$ one has $F \cdot \langle \sigma_i \mid i \in J \rangle = F' \cdot \langle \sigma_i \mid i \in J \rangle$ if and only if $F_j = F'_j$ for all $j \in \bar{J}$. $\square$

The following result states that any given Delaney set gives rise to only a finite number of different Delaney symbols that correspond to periodic tilings.

Lemma 2.3. *For any given finite, transitive Delaney set $\mathcal{D}$, there exist only a finite number of Delaney symbols of the form $(\mathcal{D}, M)$ that correspond to some periodic tiling (T, Γ) of euclidean space.*

Proof. Let $(\mathcal{D}, M)$ be a d -dimensional Delaney symbol. Define $r_{ij}(D) := \min\{r \in \mathbb{N} \mid D(\sigma_i \sigma_j)^r = D\}$ for all $D \in \mathcal{D}$ and $0 \leq i < j \leq d$. By definition, $r_{ij}(D) \leq \#\mathcal{D}$ and $r_{ij}(D)$ divides $m_{ij}(D)$ and we can define the *branching number* $v_{ij}(D) := m_{ij}(D)/r_{ij}(D)$ for all $D \in \mathcal{D}$ and $0 \leq i < j \leq d$. From (DS1) and (DS2) it follows that the maps m_{ij} and r_{ij} are constant on $\langle \sigma_i, \sigma_j \rangle$ -orbits and, hence, so is v_{ij} . It suffices to prove that there are only a finite number of different choices for the value of the branching numbers on each orbit. It is not difficult to see that, for any $\langle \sigma_i, \sigma_j \rangle$ -orbit $\mathcal{O}$, the value of $v_{ij}(\mathcal{O}) := v_{ij}(D)$, with $D \in \mathcal{O}$, is always equal to the order of some “rotation”—that is, an isometry with a fixed-point space of codimension 2—appearing in the group Γ . Hence, the well-known *crystallographic restriction* which states that the only rotational orders that can appear in a crystallographic space group (of any dimension $d \geq 2$) are 1, 2, 3, 4, and 6, completes the proof. $\square$

3. The First Finiteness Theorem

We begin with some simple observations.

Lemma 3.1. *Let $(\mathcal{D}, M)$ be a d -dimensional Delaney symbol that is locally (i, j, k) -spherical for some distinct indices $i, j, k \in I(d)$, satisfying in addition the condition $m_{jk}(D) \geq 3$ for all $D \in \mathcal{D}$, (and, therefore, $|j - k| = 1$). Then*

$$\#\mathcal{D} < 12 \cdot \#(\mathcal{D}/\langle \sigma_i, \sigma_j \rangle).$$

Proof. If $|i - j| > 1$, we have $\#\langle \sigma_i, \sigma_j \rangle = 4$ and, therefore,

$$\#\mathcal{D} \leq 4 \cdot \#(\mathcal{D}/\langle \sigma_i, \sigma_j \rangle) < 12 \cdot \#(\mathcal{D}/\langle \sigma_i, \sigma_j \rangle).$$

Otherwise, $i \neq k$ together with $|i - j| = |j - k| = 1$ implies $i = j - \varepsilon$ and $k = j + \varepsilon$ for some $\varepsilon \in \{-1, 1\}$ and, therefore, $m_{ik}(D) = 2$ for all $D \in \mathcal{D}$. In addition, local (i, j, k) -sphericity implies

$$\sum_{D \in \mathcal{O}} \left(\frac{1}{m_{ij}(D)} + \frac{1}{m_{jk}(D)} + \frac{1}{m_{ik}(D)} - 1 \right) > 0$$

for every $\langle \sigma_i, \sigma_j, \sigma_k \rangle$ -orbit $\mathcal{O} \subseteq \mathcal{D}$ and therefore

$$\begin{aligned} 0 &< \sum_{D \in \mathcal{D}} \left(\frac{1}{m_{ij}(D)} + \frac{1}{3} + \frac{1}{2} - 1 \right) \\ &= \sum_{D \in \mathcal{D}} \frac{1}{m_{ij}(D)} - \frac{1}{6} \# \mathcal{D}. \end{aligned}$$

As, moreover, $\#(D \cdot \langle \sigma_i, \sigma_j \rangle) \in \{r_{ij}(D), 2r_{ij}(D)\}$ and, therefore, $1/m_{ij}(D) \leq 2/\#(D \cdot \langle \sigma_i, \sigma_j \rangle)$ for all $D \in \mathcal{D}$, we have

$$\sum_{D \in \mathcal{D}} \frac{1}{m_{ij}(D)} \leq 2 \cdot \#(\mathcal{D}/\langle \sigma_i, \sigma_j \rangle)$$

and, hence, $\# \mathcal{D} < 12 \cdot \#(\mathcal{D}/\langle \sigma_i, \sigma_j \rangle)$. $\square$

Lemma 3.2. *Given some $i \in I(d-1)$, define*

$$H(i) := \{j \in I(d) \mid j \leq i \text{ and } j \equiv i(2) \text{ or } j \geq i+1 \text{ and } j \equiv i+1(2)\}.$$

Then, for any finite Σ_d -set $\mathcal{D}$ and any $i \in I(d-1)$, the number $\#(\mathcal{D}/\langle \sigma_i, \sigma_{i+1} \rangle)$ of $\langle \sigma_i, \sigma_{i+1} \rangle$ -orbits in $\mathcal{D}$ is bounded from above by

$$2^{\#H(i)-2} \cdot \#(\mathcal{D}/\langle \sigma_k \mid k \in H(i) \rangle).$$

Proof. Set $\Delta(i) := \langle \sigma_k \mid k \in H(i) \rangle$. Now, $|i - j| > 1$ and $|(i+1) - j| > 1$ for all $j \in H(i) \setminus \{i, i+1\}$ implies that σ_i and σ_j commute on $\mathcal{D}$, and thus $\Delta(i)$ is the direct product $\prod_{j \in H(i) \setminus \{i, i+1\}} \langle \sigma_j \rangle \times \langle \sigma_i, \sigma_{i+1} \rangle$. Hence, every $\Delta(i)$ -orbit splits into at most $2^{\#(H(i) \setminus \{i, i+1\})} = \# \prod_{j \in H(i) \setminus \{i, i+1\}} \langle \sigma_j \rangle$ different $\langle \sigma_i, \sigma_{i+1} \rangle$ -orbits. $\square$

Now we can state our first finiteness result:

Theorem 3.3 (First Finiteness Theorem). *For any dimension $d \geq 2$ and any $N \in \mathbb{N}$, there exist only finitely many distinct types of periodic face-to-face tilings (T, Γ) of $\mathbb{E}^d$ that are $(2, 4, 6, \dots)$ - N -transitive, i.e., for which $\#(\Gamma \setminus \mathcal{F}_{H(0)}(T)) \leq N$ holds.*

This is a special case ($i = 0$) of the following more general result:

Theorem 3.4. *For any dimension $d \geq 2$, any $N \in \mathbb{N}$, and any $i \in \{0, 1, \dots, d-1\}$, there exist only finitely many distinct types of periodic face-to-face tilings (T, Γ) of $\mathbb{E}^d$ with $\#(\Gamma \setminus \mathcal{F}_{H(i)}(T)) \leq N$.*

In view of Lemma 2.2, the result follows immediately from the following more general result which in turn follows immediately from Lemmas 2.1, 3.1, and 3.2:

Theorem 3.5. *For any dimension d and any $i \in \{0, 1, \dots, d-1\}$, the cardinality of the Delaney set $\mathcal{D} = \mathcal{D}_{(\mathcal{T}, \Gamma)}$ of any equivariant tiling $(\mathcal{T}, \Gamma)$ with*

- (1) $m_{i-1,i}(D) \geq 3$ for all $D \in \mathcal{D}$ and $(\mathcal{D}, M)$ is locally $(i-1, i, i+1)$ -spherical ($i > 0$), or with
- (2) $m_{i+1,i+2}(D) \geq 3$ for all $D \in \mathcal{D}$ and $(\mathcal{D}, M)$ is locally $(i, i+1, i+2)$ -spherical ($i < d-1$),

is bounded by

$$3 \cdot 2^{\#H(i)} \cdot \#(\Gamma \backslash \mathcal{F}_{H(i)}(\mathcal{T})).$$

Yet another result of this kind, which follows along the same line of reasoning, is:

Theorem 3.6. *For any dimension d and any $N \in \mathbb{N}$, there exist only finitely many types of periodic tilings $(\mathcal{T}, \Gamma)$ of $\mathbb{E}^d$ with $m_{i+1,i+2}(D) \geq 3$ for some fixed $i \in \{0, 1, \dots, d-2\}$ and all $D \in \mathcal{D}_{(\mathcal{T}, \Gamma)}$ and with $\#(\Gamma \backslash \mathcal{F}_{H(i)}(\mathcal{T})) \leq N$.*

4. The Second Finiteness Theorem

In this section we show that for any dimension d and any number N there exist only a finite number of types of convex, periodic, and tile- N -transitive tilings of euclidean d -space. We first prove a number of necessary results, using geometric arguments in the spirit of Minkowski and Delone.

Lemma 4.1. *Let $(\mathcal{T}, \Gamma)$ be a convex, periodic, and tile- N -transitive tiling with $N \geq 1$. Assume that Γ consists of translations only. Let $f(P)$ denote the number of facets of an arbitrary tile $P \in \mathcal{T}$. Then*

$$f(P) < c(d, N) := 2(2^d - 1) + (N - 1)2^d.$$

Proof. The group Γ can be partitioned into 2^d cosets:

$$\Gamma = \dot{\bigcup}_{i=1}^{2^d} \Gamma_i,$$

where two translations $\tau' \in \Gamma$ and $\tau'' \in \Gamma$ are defined to lie in the same coset Γ_i , for some $i \in \{1, \dots, 2^d\}$, if and only if $\tau' - \tau'' \equiv 0 \pmod{2}$ or, in other words, $\tau' - \tau'' = 2 \cdot \tau$, $\tau \in \Gamma$ (considered as vectors). Furthermore, by the definition of a tile- N -transitive tiling, we have

$$\mathcal{T} = \dot{\bigcup}_{i=1}^N \Gamma \cdot P_i,$$

where the N distinct tiles $P_1, P_2, \dots, P_k$ are all pairwise nonequivalent with respect to the symmetry group Γ .

Fig. 1. The situation described in the first part of the proof of Lemma 4.1.

Without loss of generality, we now prove the claim for all tiles equivalent to P_1 , i.e., for all tiles contained in $\Gamma \cdot P_1$. Consider $P \in \Gamma \cdot P_1$.

1. First, we determine a bound for the number of tiles that share a facet with P , but are *not* Γ -equivalent to P . We claim that, for each $j \in \{2, 3, \dots, N\}$, the number of tiles in $\Gamma \cdot P_j$ that share a facet with P cannot exceed 2^d . For any $j \in \{1, 2, \dots, N\}$, let

$$\mathcal{G}_j(P) := \{\tau \in \Gamma \mid \tau(P_j) \text{ and } P \text{ share a facet}\}.$$

Choose some fixed number $j \in \{2, 3, \dots, N\}$. To prove the claim, we assume that the number $|\mathcal{G}_j(P)|$ of tiles in $\Gamma \cdot P_j$ that share a facet with P , is larger than 2^d . In this case there must exist two symmetries τ' and $\tau'' \in \mathcal{G}_j(P)$, which are equivalent to each other modulo 2, that is, $\tau'' - \tau' = 2 \cdot \tau$, for some $\tau \in \Gamma$.

Let $P' = \tau'(P_j) := \tau' + P_j$ and $P'' = \tau''(P_j) := \tau'' + P_j$ be two tiles that each share a facet with P , say, f' and f'' . Let A' be an interior point of the common facet of P and P' , and let B'' be an interior point of the common facet of P and P'' . Furthermore, define $A'' := (2 \cdot \tau)(A) = 2 \cdot \tau + A$, and $B' := (-2 \cdot \tau)(B'') = -2 \cdot \tau + B''$ (see Fig. 1). It is clear that A'' is an interior point of some facet of P'' and, quite analogously, that B' is an interior point of some facet of P' . Therefore, convexity implies that the midpoints $C' := (A' + B')/2$ and $C'' := (A'' + B'')/2$ must lie in the interior of the tiles P' and P'' , respectively. Since $C'' = 2 \cdot \tau + C'$ and $\tau \in \Gamma$, the midpoint $O := (C' + C'')/2 = \tau + C'$ is an interior point of the tile $\tau + P' = \tau + \tau' + P_j$, which must be different from P , because $\tau + \tau' + P_j \in \Gamma \cdot P_j$, $j \neq 1$.

On the other hand, point O is the center of the parallelogram $A'A''B''B'$, hence, as $f' \neq f''$, $O = (A' + B'')/2$ must be contained in the interior of P . This, however, contradicts the fact that the tiles P and $\tau' + \tau + P_j$ do not have interior points in common.

2. Now we show that the number of tiles in $\Gamma \cdot P_1$ that share a facet with the tile P cannot exceed $2(2^d - 1)$. For all $i = 1, 2, \dots, 2^d$, define $G_i := \{\tau \in \Gamma_i \mid \tau + P \text{ has a facet in common with } P\}$. Without loss of generality, we assume that the above partitioning of Γ into 2^d cosets is indexed in such a way that the coset Γ_1 is the “even” coset, i.e., that

$$\Gamma_1 := \{\tau \in \Gamma \mid \tau = 2\tau', \tau' \in \Gamma\}.$$

Then, for $i \in \{2, 3, \dots\}$, we have $|G_i| \leq 2$. Indeed, let $\tau_1, \tau_2 \in G_i$. Then, by definition of Γ_i , there exists a symmetry $\tau \in \Gamma$ with $\tau := (\tau_1 + \tau_2)/2$. As above, this implies that

the tiles P and $\tau + P$ have to have interior points in common and, as a consequence, must coincide. Hence, $\tau = 0$ and $\tau_1 = -\tau_2$. In other words, any pair τ_1, τ_2 of distinct translations (or vectors) taken from G_i consists of mutually opposite translations, and so set G_i contains at most two elements.

Next, note that $G_1 = \emptyset$, i.e., no tile in $\Gamma_1 \cdot P$ except P itself can intersect with the tile P : Indeed, we now prove that no two distinct tiles in $\Gamma_1 \cdot P$ can intersect. Assume $\tau' \in \Gamma \setminus \{0\}$ and set $\tau := 2\tau' \in \Gamma_1$. As both P and $\tau'(P)$ are two distinct convex polytopes, they do not have interior points in common and, hence, there exists a hyperplane α that separates $\text{Int}(P)$ and $\text{Int}(\tau' + P)$. Furthermore, the strip which is contained between the hyperplanes α and $\tau' + \alpha$ completely separates P and $\tau + P$. $\square$

Lemma 4.1 immediately implies:

Corollary 4.2. *Given an arbitrary crystallographic group Γ . Let (T, Γ) be a convex, periodic, and tile- N -transitive tiling with $N \geq 1$ and let $\bar{\Gamma} \leq \Gamma$ denote the group of all translations in group Γ . Then the equation in Lemma 4.1 holds with $h = N \cdot |\Gamma : \bar{\Gamma}|$.*

To prepare for the proof of Theorem 4.4, we establish a link between the geometric results of this section and the combinatorial considerations in Section 2.

Lemma 4.3. *For any dimension d and any number N , there exists a number $b(d, N)$ such that the following holds: If (T, Γ) is a convex, periodic, and tile- N -transitive tiling of d -dimensional euclidean space, then the number of distinct maximal flags, whose d -dimensional component is some fixed tile $P \in T$, is bounded by $b(d, N)$.*

Proof. Let P be a convex d -dimensional polytope with precisely f facets. Each $(d-2)$ -dimensional face F of P is contained in precisely two facets F_1 and F_2 of P , and for those we have $F = F_1 \cap F_2$ [9, p. 34]. Hence, each facet of P contains at most $f-1$ different $(d-2)$ -dimensional faces. By induction, each $(i+1)$ -dimensional face of P contains at most $f-i$ different i -dimensional faces, for each $i \in I(d-1)$. In view of Lemma 4.1, choose $b(d, N) := \prod_{i=0}^{d-1} (c(d, N) - 1)$. Then $b(d, N) > \prod_{i=0}^{d-1} (f-i)$. $\square$

Theorem 4.4 (Second Finiteness Theorem). *For any dimension $d \in \mathbb{N}$ and any number $N \in \mathbb{N}$, there exist only a finite number of types of convex, periodic, and tile- N -transitive tilings (T, Γ) of euclidean d -space.*

It is possible to derive this result from Lemmas 4.1 and 4.3, using additional geometric arguments. It is more convenient, however, to base the proof on Delaney symbols, as follows:

Proof. Given $d, N \in \mathbb{N}$, Lemma 4.3 implies that there exists a bound for the size of any Delaney set corresponding to a convex, periodic, and tile- N -transitive tiling of euclidean d -space (see Section 2), because, by definition, the Delaney set of a tiling is the set of all Γ -orbits of maximal flags of T . This implies, as shown in Section 2, that there exist only a finite number of different Delaney sets corresponding to such tilings. These, in turn,

give rise to only a finite number of nonisomorphic Delaney symbols, by Lemma 2.3, for euclidean tilings. Finally, Lemma 2.1 ensures that each different Delaney symbol gives rise to precisely one equivariant type of tiling. $\square$

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