An Upper and a Lower Bound Theorem for Combinatorial 4-Manifolds*

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Abstract. In this paper we prove two inequalities. The first one gives a lower bound for the Euler characteristic of a tight combinatorial 4-manifold M under the additional assumptions that $|M|$ is 1-connected, that M is a subcomplex of $\mathcal{H}(M)$, and that $\mathcal{H}(M)$ is a centrally symmetric and simplicial d -polytope. The second inequality relates the Euler characteristic with the number of vertices of a combinatorial 4-manifold admitting a fixed-point free involution. Furthermore, we construct a new and highly symmetric 12-vertex triangulation of $S^2 \times S^2$ realizing equality in each of these inequalities.

1. Introduction and Definitions

The notion of an Upper or Lower Bound Theorem refers to bounds for the numbers f_i of i -dimensional faces. The classical or the generalized Lower Bound Theorem for simplicial polytopes gives lower bounds for h_j or $h_j - h_{j-1}$, respectively. For triangulated 4-manifolds M in the skeleton of a simplicial polytope, there is a Lower Bound Theorem for $h_3 - h_2 = 10(\chi(M) - 2)$. In this paper we present a centrally symmetric version, based on Stanley's Lower Bound Theorem for centrally symmetric polytopes [St]. Similarly, an Upper Bound Theorem for triangulated 4-manifolds M gives an upper bound for $h_3 - h_2$, which for given $\chi(M)$ turns into a lower bound for the number of vertices. We give a "centrally symmetric" version for triangulations admitting a fixed-point free involution. As far as embeddings into Euclidean space are concerned, there is a close relationship with tight embeddings. The notion of tightness is a generalization of the concept of convexity. For a background of this theory we refer to the monograph [Kü3], which focuses on the theory of tight polyhedra, or to the monograph by Cecil and Ryan

* Research supported by a fellowship of the Landesgraduiertenförderung Baden-Württemberg.

[CR]. The latter not only covers aspects of tight polyhedra, introduced in mathematics by T. Banchoff, but also the theory of tight and taut (nonpolyhedral) sets in $\mathbb{E}^n$ and tight maps of topological spaces. We recall here that a compact and connected subset $X \subset \mathbb{E}^d$ is called *k-tight with respect to a field $\mathbb{F}$* if for every open or closed half-space $h \subset \mathbb{E}^d$ the induced homomorphisms

$$H_i(h \cap X; \mathbb{F}) \rightarrow H_i(X; \mathbb{F}), \quad 0 \leq i \leq k,$$

are injective, where H_i denotes an appropriate homology theory. For polyhedra it is convenient to use simplicial homology because $h \cap X$ has always the homotopy type of a finite polyhedron. X is called *tight with respect to $\mathbb{F}$* if it is *k-tight* with respect to $\mathbb{F}$ for all k . X is *(k-)tight* if it is *(k-)tight* with respect to at least one field, where in the literature the field $\mathbb{Z}_2$ is used most often. X is called *substantial* if it is not contained in any hyperplane and *centrally symmetric* if $X = -X$, i.e., if X equals its reflection at the origin $0 \in \mathbb{E}^d$.

A (polyhedral) complex M in $\mathbb{E}^d$ is called *(k-)tight* if its underlying set $|M| \subset \mathbb{E}^d$ is *(k-)tight*, where $|M|$ is defined to be the union of all cells of M . The convex hull of $|M|$ is denoted by $\mathcal{H}(M)$.

A special kind of polyhedral complexes is combinatorial manifolds defined as follows:

Definition 1.1. A simplicial complex M (not necessarily in some $\mathbb{E}^d$) is called a *combinatorial manifold of dimension n* if the dimension of any simplex of M is at most n and if the link $lk(\Delta^k)$ of any k -simplex Δ^k of M , $0 \leq k \leq n$, is a triangulated $(n-k-1)$ -sphere.

Now let P be a polytope. The complex $\mathcal{C}(P)$ is the complex of all faces of P . A subcomplex of P is a subcomplex of $\mathcal{C}(P)$. An example is the boundary complex $\mathcal{C}(\partial P)$ of P formed by all proper faces of P . Hence $|\mathcal{C}(\partial P)| = \partial P = P \setminus \text{relint}(P)$ with $\text{relint}(P)$ the relative interior of P . If all facets ($= (d-1)$ -faces) of P are simplices, P is called a simplicial d -polytope.

A connection between the concepts of tightness and the concept of subcomplexes of polytopes is given by the following theorem.

Theorem 1.2 [Kü3, Proposition 4.1]. *For every subcomplex M of a convex d -polytope $P \subset \mathbb{E}^d$ such that $P = \mathcal{H}(M)$ and such that $|M|$ is homeomorphic to a $(k-1)$ -connected $2k$ -manifold the tightness of M with respect to $\mathbb{Z}_2$ for $k = 1$ and with respect to any field for $k \geq 2$ is equivalent to the fact that M is k -Hamiltonian in $\mathcal{H}(M)$, i.e., M contains the k -skeleton $Sk_k(P)$ of P .*

The proof uses essentially the Poincaré duality that applies for $k = 1$ only for $\mathbb{Z}_2$ and for $k \geq 2$ for all fields $\mathbb{F}$ as any simply connected manifold is orientable.

Several examples of (tight) $(k-1)$ -connected $2k$ -manifolds can be found in [Kü1] for $k = 1$ and [Kü3] for general k . We consider in the first part of this note the special case of $(k-1)$ -tight combinatorial $2k$ -manifolds M in $\mathbb{E}^d$ such that:

- (i) $|M|$ is $(k-1)$ -connected,
- (ii) M is a subcomplex of $\mathcal{H}(M) \subset \mathbb{E}^d$.

That is, $|M| \subset \mathbb{E}^d$ is a $(k - 1)$ -tightly embedded $(k - 1)$ -connected polyhedron and therefore k -Hamiltonian in $\mathcal{H}(M)$ (see [Kü3, Proposition 3.4 or Corollary 3.8]). Consequently, an application of Theorem 1.2 reveals that M is not only $(k - 1)$ -tight but even tight.

We make one further assumption:

(iii) $\mathcal{H}(M)$ is centrally symmetric.

The two-dimensional case ($k = 1$) was already analyzed and, especially, a lower bound for $3(2 - \chi(M))$ was given by Kühnel [Kü4]; we would like to thank W. Kühnel for helpful discussions of this case. His methods are applicable (with slight modifications) also for $k = 2$ and even for general k . Section 2 focuses on the case $k = 2$, where we deduce a lower bound for the Euler characteristic of M . If for a combinatorial 4-manifold this bound is attained, we prove that the number of vertices must be at least one more than the “theoretical” minimum number of vertices according to the Upper Bound Theorem for combinatorial 4-manifolds.

In Section 3 we construct a triangulation of $S^2 \times S^2$ with $f_0 = 12$ vertices as an example for the sharpness of our lower bound theorem.

The last section deals with another inequality that gives an upper bound for the Euler characteristic of combinatorial 4-manifolds admitting a fixed-point free involution acting on the triangulation. According to this inequality our 12-vertex triangulation of $S^2 \times S^2$ has the minimum number of vertices.

Each of these new results will appear as the first theorem in its section.

2. A Lower Bound Theorem for the Euler Characteristic

Recently Kühnel [Kü4, Theorem 3.2] has shown the following bound for a tight polyhedral surface M in $\mathbb{E}^d$, whose convex hull is a centrally symmetric simplicial d -polytope under the assumption that M is a subcomplex of $\mathcal{H}(M)$:

$$4^2 \binom{\frac{1}{2}(d-1)}{2} = 2(d-1)(d-3) \leq 3(2 - \chi(M)). \quad (1)$$

As every polyhedral surface without boundary can be subdivided to get a combinatorial 2-manifold, and as in [Kü4] only 0-tightness is used to prove (1), our next theorem is a generalization of (1) to the four-dimensional case.

Lower Bound Theorem 2.1. *Assume M in $\mathbb{E}^d$ is a 1-tight combinatorial 4-manifold whose convex hull $\mathcal{H}(M)$ is—up to affine transformations—a centrally symmetric and simplicial d -polytope $P \subset \mathbb{E}^d$. Assume further that $|M|$ is 1-connected. If M is a subcomplex of the boundary complex of P , then the following statements hold:*

$$(i) \quad 4^3 \binom{\frac{1}{2}(d-1)}{3} \leq 10(\chi(M) - 2). \quad (2)$$

(ii) *For $d > 5$, the equality in (i) holds if and only if—up to affine transformations— P is the d -dimensional crosspolytope.*

Remark 2.2.

- (i) The d -dimensional crosspolytope (or d -octahedron) is the convex hull of the $2d$ points $(0, \dots, 0, \pm 1, 0, \dots, 0) \in \mathbb{E}^d$. It contains the complete d -partite graph with two vertices in each partite set. As it is the dual of the d -cube C_d we use the notation C_d^* for the d -crosspolytope. In terms of the Schläfli symbol we can also write $C_d^* = \{3^{d-2}, 4\}$ and $C_d = \{4, 3^{d-2}\}$.
- (ii) Here we explain the assumptions of Theorem 2.1 and describe the immediate consequences of them. Observe that only $\mathcal{H}(M)$ and not $|M|$ itself is required to be centrally symmetric (up to affine transformations).

Surely in Theorem 2.1 we must have $d \geq 5$. In the case of $d = 5$, the proof is trivial as then $|M|$ is PL -homeomorphic to a 4-sphere (thus $\chi(M) = 2$), because M is a combinatorial 4-manifold with $|M| \subset \partial P \simeq S^4$.

Theorem 2.1(ii) does not hold for $d = 5$ as can be seen from the boundary complex of all centrally symmetric simplicial 5-polytopes $P \neq C_5^*$. (In these cases, we have $|M| \simeq S^4$ and equality holds in (2) although M is not a subcomplex of C_5^* .)

The left-hand side of (2) can be written alternatively in terms of a falling factorial or a product as

$$4^3 \binom{\frac{1}{2}(d-1)}{3} = \frac{2^6}{6} \left(\frac{d-1}{2} \right)_3 = \frac{4}{3} \prod_{i=0}^2 (d-1-2i).$$

For a combinatorial 0-manifold M in $\mathbb{E}^d$ whose convex hull is a centrally symmetric simplicial d -polytope, we trivially have

$$2(d-1) = 4 \binom{\frac{1}{2}(d-1)}{1} \leq \chi(M) - 2.$$

Hence the zero-, two-, and four-dimensional cases give a bound for the Euler characteristic of combinatorial manifolds satisfying several conditions; these inequalities can also be interpreted as an upper bound for the substantial codimension of the underlying set of the combinatorial manifold, but we stick to the terminology “Lower Bound Theorem,” because the proof depends on a Lower Bound Theorem for $h_3(P) - h_2(P)$. Considering the zero-, two-, and four-dimensional cases gives rise to the following conjecture saying that for general $k \in \mathbb{N}$ there holds an analogous bound.

Lower Bound Conjecture 2.3. *Let M be a $(k-1)$ -tight combinatorial $2k$ -manifold in $\mathbb{E}^d$ such that $|M|$ is $(k-1)$ -connected and such that M is a subcomplex of $\mathcal{H}(M)$ where $\mathcal{H}(M)$ is—up to affine transformations—a centrally symmetric and simplicial d -polytope. Then it follows*

$$4^{k+1} \binom{\frac{1}{2}(d-1)}{k+1} \leq \text{const.} \cdot (\chi(M) - 2)$$

with equality for $d > 2k+1$ (or for $k = 0$ for all d) if and only if—up to affine transformations— $\mathcal{H}(M) = C_d^*$.

The constant turns out to be $(-1)^k \binom{2k+1}{k+1}$ as we prove in [Sp].

We now begin to prepare the proof of the above theorem. We recall the definition of the h -vector of a $\tilde{d}$ -dimensional simplicial complex K to be

$$h_j(K) := \sum_{i=-1}^{j-1} (-1)^{j-i-1} \binom{\tilde{d}-i}{j-i-1} f_i(K), \quad 0 \leq j \leq \tilde{d}+1,$$

where $f_i(K)$ denotes the number of i -simplices of K for $i \geq 0$ and $f_{-1}(K) := 1$. The boundary complex $\mathcal{C}(\partial P)$ of a simplicial d -polytope is a combinatorial $(d-1)$ -manifold and we write $h_j(P) = h_j(\mathcal{C}(\partial P))$ for all j . Stanley [St] showed in 1987, for a centrally symmetric simplicial d -polytope P , the inequality

$$h_j(P) - h_{j-1}(P) \geq \binom{d}{j} - \binom{d}{j-1}, \quad \forall j, \quad 1 \leq j \leq \left\lceil \frac{d}{2} \right\rceil,$$

including a discussion of equality. The definition of the h -vector implies

$$f_{j-1}(P) \geq \binom{d}{j} - \binom{d}{j-1} + \sum_{i=-1}^{j-2} (-1)^{j-i} \binom{d-i}{j-i-1} f_i(P), \quad \forall j, \quad 1 \leq j \leq \frac{d}{2}, \quad (\dagger)$$

or even shorter

$$f_{j-1}(P) \geq \sum_{i=0}^{j-2} (-1)^{j-i} \binom{d-i}{j-i-1} f_i(P) + \begin{cases} -2\binom{d}{j-1} & \text{for } j \in 2\mathbb{N}, \\ 2\binom{d}{j} & \text{for } j \in 2\mathbb{N}+1, \end{cases}$$

from the basic recurrence relation

$$\binom{d+1}{j} = \binom{d}{j} + \binom{d}{j-1}.$$

In the case of Theorem 2.1 we clearly have $f_0(P) \geq 2d$. Equation $(\dagger)$ gives a simple verification of this fact. With these prerequisites we are now in a position to prove Theorem 2.1 for $d \geq 6$.

Proof of Theorem 2.1(i). According to Theorem 1.2, M contains the 2-skeleton of its convex hull, i.e., $f_i(M) = f_i(P) := f_i$ for $-1 \leq i \leq 2$. Using the above corollary we therefore have

$$10(\chi(M) - 2)$$

$$\begin{aligned} &= h_3(M) - h_2(M) = \sum_{i=-1}^2 (-1)^{2-i} \binom{5-i}{3} f_i \\ &\geq \sum_{i=-1}^1 (-1)^{2-i} \binom{5-i}{3} f_i + \binom{d}{3} - \binom{d}{2} + \sum_{i=-1}^1 (-1)^{3-i} \binom{d-i}{2-i} f_i \end{aligned}$$

$$\begin{aligned}
&= \binom{d}{3} - \binom{d}{2} + \sum_{i=-1}^0 (-1)^{2-i} \left(\binom{5-i}{3} - \binom{d-i}{2-i} \right) f_i - \left(\binom{4}{3} - \binom{d-1}{1} \right) f_1 \\
&\geq \binom{d}{3} - \binom{d}{2} + (d-5) \left(\binom{d}{2} - \binom{d}{1} \right) \\
&\quad + \sum_{i=-1}^0 (-1)^{2-i} \left(\binom{5-i}{3} - \binom{d-i}{2-i} + (d-5) \binom{d-i}{1-i} \right) f_i \\
&= \binom{d}{3} - \binom{d}{2} + (d-5) \left(\binom{d}{2} - \binom{d}{1} \right) + \binom{d-4}{2} f_0 \\
&\quad + (-1)^3 \left(\binom{6}{3} - \binom{d+1}{3} + (d-5) \binom{d+1}{2} \right) f_{-1} \\
&\geq \binom{d}{3} - \binom{d}{2} + (d-5) \left(\binom{d}{2} - d \right) + \binom{d-4}{2} 2d - \binom{6}{3} + \binom{d+1}{3} \\
&\quad - (d-5) \binom{d+1}{2} \\
&= 4^3 \binom{\frac{1}{2}(d-1)}{3}.
\end{aligned}$$

The last step can be verified by a direct calculation, but it can also be carried out using a *generalized Vandermonde convolution formula* [R]. (This technique becomes essential for the higher-dimensional cases. For more details we refer to [Sp].)

For the proof of (ii) we need the following corollary to the Stanley bounds (see, e.g., [St] or [BL, p. 502]):

If K is a centrally symmetric simplicial d -polytope and if for some i , $1 \leq i \leq d-1$, $h_i(K) = \binom{d}{i}$, then $h_j(K) = \binom{d}{j}$, $\forall 0 \leq j \leq d+1$ and K is affinely equivalent to the d -dimensional crosspolytope C_d^ .*

(In fact, we just need the weaker statement that $h_1(K) = d$ implies the following: (i) $h_j(K) = \binom{d}{j}$, $\forall 0 \leq j \leq d+1$; and (ii) K is affinely equivalent to C_d^* . But this is obvious even without the Stanley bounds as $h_1(K) = d$ gives $f_0(K) = 2d$ and therefore K is (up to affine transformations) C_d^* . This in turn implies $f_j(K) = 2^{j+1} \binom{d}{j+1}$, $\forall 0 \leq j \leq d$, or $h_j(K) = \binom{d}{j}$, $\forall 0 \leq j \leq d+1$.)

Now let equality hold in (2) and let $d \geq 6$. Then we have $f_0 = 2d$ and therefore $h_1(P) = -d + f_0(P) = d = \binom{d}{1}$. Hence by the above-mentioned consequence it follows that P is affinely equivalent to C_d^* .

If vice versa $P = \mathcal{H}(M)$ can be affinely transformed into C_d^* , the number of i -simplices is $f_i(M) = f_i(C_d^*) = 2^{i+1} \binom{d}{i+1}$, $\forall 0 \leq i \leq 2$, but this gives equality in (2). $\square$

In the two-dimensional case equality in (1) for $d \geq 5$ implies that the induced triangulation has the minimum number of vertices [Kü3, Theorem 2.22]. We see in Proposition 2.5 and Corollary 2.6 that this is not true in the case of equality in (2). To show this result we need the following theorem.

Theorem 2.4 [Kü2, Proposition 4.1]. *If M is a combinatorial 4-manifold with n vertices, then the inequality*

$$\binom{n-4}{3} \geq 10(\chi(M) - 2)$$

holds with equality if and only if M is 3-neighborly, i.e., if any 3-tuple of vertices of M spans a 2-simplex of M .

Proposition 2.5. *Let M be a combinatorial 4-manifold in $\mathbb{E}^d$ whose convex hull is a d -polytope. If M fulfills equality in (2), its number of vertices is at least*

$$f_{0,d}^{\min}(M) := f_{0,d}^{\min} = \begin{cases} 2d - 1 & \text{for } d \geq 6, \\ 6 & \text{for } d = 5. \end{cases}$$

(Our notation does not assume that there exists a combinatorial 4-manifold M with $f_{0,d}^{\min}$ vertices and equality in (2) except for $d = 5$, where are easy examples.)

Before we prove this proposition we note as an immediate consequence of Proposition 2.5.

Corollary 2.6. *If M in $\mathbb{E}^d$ is a combinatorial 4-manifold which satisfies all assumptions of Theorem 2.1 and for which equality in (2) holds, then its number of vertices is at least one more than the “theoretical” minimum number given by Proposition 2.5, hence $f_0(M)$ is at least $1 + f_{0,d}^{\min}$.*

Proof of Proposition 2.5. Let $d \geq 8$. (For $d = 7$ equality in (2) cannot hold; see also Proposition 2.7.) First we show that we need at least $2d - 1$ vertices. Suppose we would only need $2d - 2$ vertices, then we have

$$\begin{aligned} & \binom{2d-2-4}{3} - 10(\chi(M) - 2) \\ &= \frac{1}{6}((2d-6)(2d-7)(2d-8) - 2^3(d-1)(d-3)(d-5)) \\ &= -2(d^2 - 9d + 18) < 0, \quad \forall d \geq 8, \end{aligned}$$

in contradiction to Theorem 2.4. Hence we need at least $2d - 1$ vertices.

Now we show that this bound cannot be improved with Theorem 2.4. To do this we demonstrate that according to Theorem 2.4, $2d - 1$ vertices are sufficient, i.e., we have to show

$$\binom{2d-5}{3} \geq \frac{4}{3} \prod_{i=0}^2 (d-1-2i).$$

This is equivalent to

$$\prod_{i=0}^2 (2d-5-i) \geq \prod_{i=0}^2 (2d-2-4i) \quad \Leftrightarrow \quad (2d-5)(2d-7) \geq (2d-2)(2d-10).$$

The last inequality is clearly true for all d .

For $d = 6$ we have

$$\binom{10-4}{3} = 10(\chi(M) - 2) = \frac{4}{3} \prod_{i=0}^2 (d - 1 - 2i) = 20 > \binom{9-4}{3},$$

so analogous arguments show that we need ten or more vertices to realize equality in (2), where M has to be 3-neighborly by Theorem 2.4 if it only contains ten vertices. But Kühnel and Lassmann [KL] showed with a computer-aided enumeration that there does not exist a 3-neighborly combinatorial 4-manifold with ten vertices. Hence we get $f_{0,6}^{\min} = 11$.

Finally, let $d = 5$. As any combinatorial 4-manifold has at least six vertices, it trivially follows $f_{0,5}^{\min} \geq 6$. But in this case all 6-vertex combinatorial 4-manifolds are an example for equality in (2), as any such manifold is the boundary complex of a 5-simplex. Thus we get $f_{0,5}^{\min} = 6$. $\square$

Proposition 2.7. *Only for $d \equiv 0, 1, 3 \pmod{5}$ equality in (2) can hold.*

Proof. It suffices to demonstrate that only in the cases $d \equiv 0, 1, 3 \pmod{5}$

$$\frac{4^3}{10} \binom{\frac{1}{2}(d-1)}{3} = \frac{2}{15} (d-1)(d-3)(d-5)$$

is an integer. For all d one of the factors $d-5$, $(d-5)+2 = d-3$, $(d-5)+4 = d-1$ is divisible by 3. Hence we have to show that for $d \equiv 0, 1, 3 \pmod{5}$ one of these factors is divisible by 5 and that for $d \equiv 2, 4 \pmod{5}$ this is not the case. But for $d \equiv 0 \pmod{5}$ $d-5$ clearly is divisible by 5, while for $d \equiv 1 \pmod{5}$ $d-1$ and for $d \equiv 3 \pmod{5}$ $d-3$ is divisible by 5. For $d \equiv 2 \pmod{5}$, i.e., $d = 5m + 2$, where $m \in \mathbb{N}$, we have the factors $5m+1$, $5m-1$, $5m-3$ and none of these is divisible by 5. An analogous argument for $d \equiv 4 \pmod{5}$ proves Proposition 2.7. $\square$

3. A 12-Vertex Triangulation of $S^2 \times S^2$

In what follows if there is no danger of confusion we always abbreviate the components of the f -vector of a combinatorial manifold by f_i .

Theorem 2.4 implies that all triangulations of $S^2 \times S^2$ must have at least ten vertices and that a triangulation with ten vertices must be 3-neighborly, which is not possible by [KL]. It is still unknown if eleven vertices are sufficient.

In this section we construct a triangulation of $S^2 \times S^2$ with $n = f_0 =$ twelve vertices. In this example all assumptions of Theorem 2.1 are satisfied with $\chi(M) = 4$ and $d = 6$, hence equality in (2) holds. Therefore inequality (2) is sharp.

Theorem 3.1. *There exists a 12-vertex combinatorial 4-manifold M in $\mathbb{E}^6$ that is a triangulation of $S^2 \times S^2$ and that fulfills the following conditions:*

- (i) $|M|$ is tightly embedded in $\mathbb{E}^6$,
- (ii) M is a subcomplex of the boundary complex of the six-dimensional crosspolytope C_6^* .

(That is, M satisfies all assumptions of Theorem 2.1.)

- (iii) *The combinatorial automorphism group $\mathfrak{A} := \text{Aut}(M)$ of M is isomorphic to $A_5 \times \mathbb{Z}_2$, where A_5 denotes the fifth alternating group.*

Proof. To calculate the f -vector of M we apply that the number of i -simplices of C_l^* is $2^{i+1} \binom{l}{i+1}$. Hence Theorem 1.2 says $f_i = f_i(M) = f_i(C_6^*) = 2^{i+1} \binom{6}{i+1}$ for $i = 0, 1, 2$. Therefore $f_1 = 60$ and $f_2 = 160$. Using the Dehn–Sommerville equations for triangulated 4-manifolds [Kü3]

$$2f_1 - 3f_2 + 4f_3 - 5f_4 = 0 = 2f_3 - 5f_4,$$

we can determine the f -vector of M to be $(1, 12, 60, 160, 180, 72)$.

We denote the vertices of C_6^* by $0, \dots, 5, \bar{0}, \dots, \bar{5}$, where we use the notation i instead of $\langle i \rangle$ for the vertices. (To shorten notation we will omit the symbols “ $\langle$ ” and “ $\rangle$ ” between which the vertices of a simplex usually are included, also in the list of all the 4-simplices below.) The vertices i and $\bar{i}$, $0 \leq i \leq 5$, shall be opposite vertices of C_6^* and can therefore never be joined by an edge of M .

To construct the triangulation we take the union of the two orbits of the four-dimensional simplices $\langle 01234 \rangle$ and $\langle 012\bar{3}\bar{5} \rangle$ under the action of the group $A_5 \times \mathbb{Z}_2$ on the set of vertices which is generated by the permutations

$$\begin{aligned} \alpha &= (12345)(\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}), \\ \beta &= (01)(\bar{0}\bar{1})(2\bar{2})(34)(\bar{3}\bar{4})(5\bar{5}), \\ \gamma &= (0\bar{0})(1\bar{1})(2\bar{2})(3\bar{3})(4\bar{4})(5\bar{5}). \end{aligned}$$

We have to show that α, β, γ generate $A_5 \times \mathbb{Z}_2$: First we note that

$$\beta^2 = z^3 = (\beta z)^5 = ((04\bar{5}\bar{2}3)(\bar{0}\bar{4}\bar{5}2\bar{3}))^5 = id,$$

with $z = (\beta\alpha^3)^2 = (031)(\bar{0}\bar{3}\bar{1})(2\bar{4}\bar{5})(\bar{2}45)$. However, by this set of relations A_5 is generated (see [CM, Table 5]). Hence α and β generate a group isomorphic to A_5 , while the group generated by γ clearly is isomorphic to $\mathbb{Z}_2$. As these groups have no common element except the identity, and as all elements of the first group commute with those of the second one, α, β , and γ generate (a group isomorphic to) the direct product $A_5 \times \mathbb{Z}_2$.

Under the mentioned action the orbit of $\langle 0123\bar{5} \rangle$ consists of 60 4-simplices, while that of $\langle 012\bar{3}\bar{5} \rangle$ consists of 12 4-simplices. The complete list of all the 72 4-simplices is

the following:

$012\bar{3}\bar{5}$	$0\bar{1}23\bar{4}$	$0\bar{2}34\bar{5}$	$0\bar{1}\bar{3}45$	$01\bar{2}\bar{4}5$
$\bar{0}13\bar{4}\bar{5}$	$\bar{0}\bar{1}24\bar{5}$	$\bar{0}\bar{1}\bar{2}35$	$\bar{0}\bar{1}\bar{2}\bar{3}4$	$\bar{0}\bar{2}\bar{3}\bar{4}5$
$12345 \quad \bar{1}\bar{2}\bar{3}\bar{4}\bar{5}$				
$0123\bar{5}$	$0\bar{1}\bar{2}34$	$0\bar{2}\bar{3}45$	$0\bar{1}\bar{3}45$	$01\bar{2}\bar{4}5$
$\bar{0}\bar{1}\bar{2}\bar{3}5$	$\bar{0}\bar{1}\bar{2}\bar{3}4$	$\bar{0}\bar{2}\bar{3}\bar{4}5$	$\bar{0}\bar{1}\bar{3}\bar{4}5$	$\bar{0}\bar{1}\bar{2}4\bar{5}$
01234	$0234\bar{5}$	$0\bar{1}345$	$01\bar{2}45$	$012\bar{3}5$
$\bar{0}\bar{1}\bar{2}\bar{3}4$	$\bar{0}\bar{2}\bar{3}\bar{4}5$	$\bar{0}\bar{1}\bar{3}\bar{4}5$	$\bar{0}\bar{1}\bar{2}4\bar{5}$	$\bar{0}\bar{1}\bar{2}\bar{3}5$
$0\bar{1}\bar{2}45$	$01\bar{2}\bar{3}5$	$0\bar{1}\bar{2}\bar{3}4$	$0\bar{2}\bar{3}\bar{4}5$	$0\bar{1}\bar{3}\bar{4}5$
$\bar{0}\bar{1}24\bar{5}$	$\bar{0}\bar{1}\bar{2}35$	$\bar{0}\bar{1}\bar{2}34$	$\bar{0}\bar{2}\bar{3}45$	$\bar{0}\bar{1}\bar{3}4\bar{5}$
$0\bar{1}\bar{2}\bar{3}4$	$0\bar{2}\bar{3}\bar{4}5$	$0\bar{1}\bar{3}\bar{4}5$	$01\bar{2}4\bar{5}$	$0\bar{1}\bar{2}\bar{3}5$
$\bar{0}\bar{1}\bar{2}\bar{3}5$	$\bar{0}\bar{1}\bar{2}\bar{3}4$	$\bar{0}\bar{2}\bar{3}\bar{4}5$	$\bar{0}\bar{1}\bar{3}\bar{4}5$	$\bar{0}\bar{1}\bar{2}4\bar{5}$
$013\bar{4}\bar{5}$	$0\bar{1}\bar{2}4\bar{5}$	$0\bar{1}\bar{2}35$	$01\bar{2}\bar{3}4$	$0\bar{2}\bar{3}\bar{4}5$
$\bar{0}\bar{1}\bar{2}\bar{3}4$	$\bar{0}\bar{2}\bar{3}\bar{4}5$	$\bar{0}\bar{1}\bar{3}\bar{4}5$	$\bar{0}\bar{1}\bar{2}45$	$\bar{0}\bar{1}\bar{2}\bar{3}5$
$1234\bar{5}$	12345	$1\bar{2}\bar{3}\bar{4}5$	$1\bar{2}\bar{3}45$	$1\bar{2}34\bar{5}$
$\bar{1}\bar{2}\bar{3}\bar{4}5$	$\bar{1}\bar{2}\bar{3}\bar{4}5$	$\bar{1}\bar{2}\bar{3}\bar{4}5$	$\bar{1}\bar{2}\bar{3}\bar{4}5$	$\bar{1}\bar{2}\bar{3}\bar{4}5$

As can be seen from this list the number of k -simplices, $0 \leq k \leq 4$, is the required number of k -simplices given by the f -vector above.

Before continuing the proof, we remark some important properties of the triangulation and its automorphism group:

- (1) The automorphism group $\mathfrak{A}$ of M , i.e., the symmetry group of $|M|$, is also isomorphic to the symmetry group of the icosahedron. It is transitive on the vertices but not on the edges of M . The edges of M separate under the group action into two collections each of 30 edges where one collection is generated by the edge $\langle 01 \rangle$ and the other by the edge $\langle 0\bar{1} \rangle$.
- (2) Properties (i) and (iii) of Theorem 3.1 are clear from the construction. Property (ii) follows as every vertex i is connected by an edge of the triangulation to any other vertex except vertex $\bar{i}$ (where, e.g., $\bar{1}$ denotes 1).
- (3) $|M|$ is centrally symmetric because if a simplex with the vertices $i_1, \dots, i_k$, $1 \leq k \leq 5$, belongs to the triangulation, then the “complementary” simplex consisting of the vertices $\bar{i}_1, \dots, \bar{i}_k$ also belongs to the triangulation.

If $lk(0)$ is a triangulated 3-sphere, the transitivity of $\mathfrak{A}$ on the vertices of M implies that $lk(i)$ is a triangulated 3-sphere for all vertices i . This in turn gives: $lk(\Delta^k)$ is a triangulated $(3 - k)$ -sphere for all k -simplices Δ^k , $0 \leq k \leq 4$, of the triangulation. Hence, for showing that the triangulation is a combinatorial 4-manifold, we check if $lk(0)$ is a triangulated 3-sphere. $lk(0)$ consists of the α -orbits of the tetrahedra $\langle 123\bar{4} \rangle$, $\langle 123\bar{5} \rangle$, $\langle \bar{1}\bar{2}4\bar{5} \rangle$, $\langle \bar{1}\bar{2}\bar{3}4 \rangle$, $\langle 12\bar{3}\bar{5} \rangle$, $\langle 13\bar{4}\bar{5} \rangle$, and all simplices contained in one of those tetrahedra. This collection of simplices is a triangulated 3-sphere. To see this, one has to show: (a) that $lk(i)$ in $lk(0)$ for all vertices i of $lk(0)$ is a triangulation of S^2 ; and (b) that $|lk(0)|$ is (PL) -homeomorphic to S^3 .

To (a): The link of 1 in $lk(0)$, i.e., the link of the edge $\langle 01 \rangle$ in M , contains the triangles $\langle \bar{2}\bar{3}4 \rangle$, $\langle \bar{2}4\bar{5} \rangle$, $\langle \bar{2}\bar{3}5 \rangle$, $\langle 23\bar{5} \rangle$, $\langle \bar{3}4\bar{5} \rangle$, $\langle 2\bar{4}\bar{5} \rangle$, $\langle \bar{2}\bar{3}\bar{5} \rangle$, $\langle \bar{2}\bar{4}\bar{5} \rangle$, $\langle \bar{2}\bar{3}5 \rangle$, $\langle \bar{2}\bar{4}\bar{5} \rangle$, $\langle \bar{2}\bar{3}4 \rangle$, $\langle 3\bar{4}\bar{5} \rangle$ which form a triangulated 2-sphere as can immediately be deduced from Fig. 1. Analogously,

Fig. 1. On the left-hand side $lk(1)$ in $lk(0)$, i.e., $lk(\langle 01 \rangle)$, is sketched. This link is a subdivision of the boundary of a 3-octahedron as can be seen from its three-dimensional realization on the right-hand side.

$lk(\bar{1})$ is a triangulated S^2 ; now by the transitivity of α on the vertices $1, \dots, 5$ and $\bar{1}, \dots, \bar{5}$, (a) follows.

To (b): Split the vertices of $lk(0)$ into the two groups $1, 2, 3, \bar{4}, \bar{5}$ and $\bar{1}, \bar{2}, \bar{3}, 4, 5$. Then the subcomplex of $lk(0)$ spanned by the first group consists of the tetrahedra $\langle 123\bar{4} \rangle, \langle 123\bar{5} \rangle, \langle 12\bar{4}\bar{5} \rangle$, and its faces. This subcomplex is a collapsible one. The subcomplex spanned by the second group of vertices, consisting of the tetrahedron $\langle \bar{1}\bar{3}45 \rangle$, the triangles $\langle \bar{1}\bar{2}5 \rangle, \langle \bar{2}45 \rangle, \langle \bar{2}\bar{3}4 \rangle$, and all faces of these simplices, is also collapsible. A well-known theorem (see [RS, Corollary 3.28]) implies that the underlying sets of these subcomplexes are PL -homeomorphic to a 3-ball. So we can establish $|lk(0)| \simeq_{PL} |S^3|$, where “ $\simeq_{PL}$ ” denotes “is PL -homeomorphic to.”

It remains to determine the topological type of $|M|$:
 $|M|$ is a connected polyhedron with finitely many vertices and therefore path-connected. As $|M|$ is also 2-Hamiltonian in C_6^* , M is simply connected (see [Kü3, Corollary 3.8]).

To see $|M| \simeq S^2 \times S^2$, we prove that the intersection form of $|M|$ equals the intersection form of $S^2 \times S^2$, namely $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is an even form, so we are done using Freedman’s theorem [F, Theorem 1.5], [Ki, Theorem 2.1]:

Given an even form, there exists exactly one simply connected, closed, topological 4-manifold representing that form.

The Euler–Poincaré equation says $\chi(M) = 4 = \sum_{i=0}^4 (-1)^i \beta_i(M)$, where $\beta_i(M)$ is the i th Betti number of M . From the simply connectedness of $|M|$ we get $\beta_0(M) = 1$ and $\beta_1(M) = 0$. So by Poincaré’s duality it follows that $\beta_2(M) = 2$. As there is no torsion in $H_2(M; \mathbb{Z})$ we have $H_2(M; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let $\alpha_1 := \partial \langle 0124 \rangle$ and $\alpha_2 := \partial \langle 0\bar{1}\bar{2}\bar{3} \rangle$. α_1 and α_2 generate $H_2(M; \mathbb{Z})$ as $\partial \alpha_1 = 0 = \partial \alpha_2$ and as there is no 3-chain α in M such that $\partial \alpha = \alpha_i$ for $i = 1$ or $i = 2$ (otherwise, α_i is homologous to zero for $i = 1$ or $i = 2$ and therefore we have as intersection number $\alpha_1 \cdot \alpha_2 = 0$ in contradiction to $\alpha_1 \cdot \alpha_2 = \pm 1$). Finally, α_1 and α_2 are not homologous, that is, $\alpha_1 \not\sim \alpha_2$, because $\alpha_1 \cdot \alpha_1 \neq \alpha_1 \cdot \alpha_2$ (see below).

Let $\Delta := \langle 0124\bar{5} \rangle$, then $|\partial \Delta \cap M| = \langle 012\bar{5} \rangle \cup \langle 024\bar{5} \rangle \cup \langle 124\bar{5} \rangle$, i.e., only the simplices $\alpha_1 \subset \Delta$ and $\langle 014\bar{5} \rangle \subset \Delta$ are not contained in $|\partial \Delta \cap M|$. Therefore $\partial(\partial \Delta \cap M) =$

Fig. 2. The Möbius strips $N_1 \cap lk(0)$ and $N_2 \cap lk(0)$.

$\alpha_1 + \partial\langle 014\bar{5} \rangle$ or $\alpha_1 \sim -\partial\langle 014\bar{5} \rangle$. In the same way, the following chain of homologous 2-chains can be shown:

$$\alpha_1 \sim -\partial\langle 014\bar{5} \rangle \sim \partial\langle 1\bar{2}4\bar{5} \rangle \sim \partial\langle \bar{0}1\bar{2}\bar{5} \rangle \sim -\partial\langle \bar{0}\bar{2}4\bar{5} \rangle \sim -\partial\langle \bar{0}\bar{1}\bar{2}\bar{4} \rangle := -\bar{\alpha}_1.$$

Hence the self-intersection number of α_1 is $\alpha_1 \cdot \alpha_1 = 0$ (because α_1 and $\bar{\alpha}_1$ are disjoint). Analogously $\alpha_2 \cdot \alpha_2 = 0$.

Regard the simplicial subcomplexes $N_1 := \text{span}\langle 0, 1, 2, 3, 4, 5 \rangle$ and $N_2 := \text{span}\langle 0, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5} \rangle$ of M . $N_1 \cap lk(0)$ and $N_2 \cap lk(0)$ are disjoint 5-vertex Möbius strips.

As

$$\text{cl}(lk(0) \setminus \text{span}\langle 1, 2, 3, 4, 5 \rangle) = \text{cl}(lk(0) \setminus (N_1 \cap lk(0))) \searrow N_2 \cap lk(0)$$

(where “ $X \searrow Y$ ” means “ X collapses on Y ”), the complement of $N_1 \cap lk(0)$ is homotopically equivalent to $N_2 \cap lk(0)$. Hence these Möbius strips are unknotted and linked in $lk(0)$ and therefore also the homotopy representatives $\partial\langle 124 \rangle$ and $\partial\langle \bar{1}\bar{2}\bar{3} \rangle$ of the two Möbius strips are linked. Consequently $\alpha_1 \cdot \alpha_2 = \pm 1$ and with appropriate orientations we have the intersection form of $S^2 \times S^2$.

We can even show $|M| \simeq_{PL} S^2 \times S^2$ without using Freedman’s theorem. Let $A, B \subset \partial C_6^*$ such that $A \cup B = \partial C_6^*$ and $|M| = A \cap B$. A and B are 5-manifolds each consisting of $32 = \frac{1}{2}f_5(C_6^*)$ 5-simplices. Their boundary is $\partial A = \partial B = |M|$ and they satisfy

$$A \searrow A_1, \quad B \searrow B_1 \quad \text{with} \quad A_1 \simeq_{PL} S^2 \simeq_{PL} B_1.$$

(The collapsing can be done such that $A_1 = \alpha_1$ and $B_1 = \alpha_2$.)

There is only one PL -structure on $C_6^* \simeq_{PL} S^5$. As A is a subset of C_6^* of codimension 0, A “inherits” this standard PL -structure.

In the following we use the notions of a regular neighborhood and a shelling as defined in [RS, Chapter 3]. $A_1 \subset A$ is a compact polyhedron and A is a regular neighborhood of A_1 in A . Therefore an application of [RS, Theorem 3.26] yields that A shells to a regular neighborhood of A_1 in A . But a regular neighborhood of A_1 necessarily has to be a 3-ball-bundle over A_1 . Hence A is PL -homeomorphic to a 3-ball-bundle over S^2 . There are only two not homeomorphic 3-ball-bundles over S^2 : the trivial bundle with boundary $S^2 \times S^2$ and a bundle whose boundary is $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$. $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$ has intersection form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, hence from the above calculation of the form of $|M|$ it follows $A \simeq_{PL} B^3 \times S^2$. This shows the assertion $|M| = \partial A \simeq_{PL} S^2 \times S^2$. $\square$

We go on with some interesting properties of $lk(0)$ of our triangulation.

Proposition 3.2.

- (i) The automorphism group $\mathfrak{B} := \text{Aut}(lk(0))$ of $lk(0)$ is isomorphic to the dihedral group D_5 . It is a subgroup of the group of Altshuler's sphere N_{425}^{10} .
- (ii) The Petersen graph is a subgraph of the 1-skeleton of $lk(0)$ containing the two boundaries or the interior edges of the Möbius bands above.

Proof. (i) The automorphism group $\mathfrak{B}$ has to be at least of order 10, as

$$|\text{Aut}(lk(0))| \geq \frac{|\text{Aut}(M)|}{f_0(M)} = 10.$$

One automorphism of $lk(0)$ is α . As in $lk(0)$ every vertex x is joined to any other vertex except vertex $\bar{x}$ by an edge, we know the following:

If under an automorphism of $lk(0)$ vertex x is mapped onto vertex y , then vertex $\bar{x}$ must be mapped onto vertex $\bar{y}$.

(This holds, because otherwise we get the contradiction that the image under the automorphism of at least one tetrahedron of $lk(0)$ contains the “diagonal” $\langle y\bar{y} \rangle$.) The last remark drastically reduces the number of possible automorphisms. It turns out that $\mathfrak{B}$ is generated by α and $\delta := \gamma\beta\alpha\beta\alpha^4\beta\alpha = (34)(\bar{3}\bar{4})(25)(\bar{2}\bar{5})$. The relations $\alpha^5 = \delta^2 = (\alpha\delta)^2 = id$ show that $\mathfrak{B}$ is isomorphic to D_5 .

Note, that δ maps the Möbius strip $N_1 \cap lk(0)$ onto itself and likewise for the other Möbius strip in $lk(0)$. Consider now a simplexwise linear function f on $lk(0)$ with $f(1) = f(2) = f(3) = f(4) = f(5) = 1 = -f(\bar{1}) = -f(\bar{2}) = -f(\bar{3}) = -f(\bar{4}) = -f(\bar{5})$. Figure 3 shows the 0-level of f , where $\frac{x}{y}$ denotes the center of the edge joining x and y . Figure 3 reveals that the 0-level of f is a 20-vertex torus. Automorphism δ can be recognized in this torus as a simultaneous reflection at the four points indicated in the figure by circles.

The most interesting case among neighborly combinatorial 3-manifolds with ten vertices, classified by Altshuler [A], is the combinatorial 3-sphere N_{425}^{10} . It is not polytopal [BG] and invariant under the action of a K -metacyclic group of order 20. Hence it contains D_5 as a subgroup.

Fig. 3. The 0-level of f .

Fig. 4. Petersen graphs in $lk(0)$.

Note. N_{425}^{10} also contains two disjoint Möbius strips, the ones in $lk(0)$. So it seems probable that $lk(0)$ can be “transformed” into N_{425}^{10} in a natural way by replacing some simplices by new simplices such that the diagonals become edges of some of the newly inserted simplices and such that the Möbius strips are preserved.

Question. *Is there such a connection between M and N_{425}^{10} ?*

(ii) The boundaries of the two Möbius strips together with five additional 2-simplices form a Petersen graph as shown in Fig. 4. The same holds for the interior edges of the Möbius strips (see the right-hand side of Fig. 4). $\square$

We conclude the discussion of the 12-vertex triangulation of $S^2 \times S^2$ with some remarks and conjectures: Although there is a torus in $\mathbb{E}^4$ with eight vertices that is tight as a subcomplex of C_4^* (call the vertices $0, \dots, 3, \bar{0}, \dots, \bar{3}$ and take the orbits of the triangles $\langle 013 \rangle$ and $\langle 023 \rangle$ under the $\mathbb{Z}_8$ -action $0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow \bar{0} \rightarrow \bar{1} \rightarrow \bar{2} \rightarrow \bar{3}$), i.e., equality in (1) is fulfilled, such a torus cannot be constructed in the same way as our example.

Regard $lk(0)$ in such a torus. Using the same construction principle as in Theorem 3.1, $lk(0)$ contains the vertices $1, 2, 3, \bar{1}, \bar{2},$ and $\bar{3}$ and consists of orbits of 2-simplices under the action $(123)(\bar{1}\bar{2}\bar{3})$. In this case, the 2-simplices $\langle 12 \rangle$, $\langle 13 \rangle$, and $\langle 23 \rangle$ form the first orbit, while $\langle \bar{1}\bar{2} \rangle$, $\langle \bar{1}\bar{3} \rangle$, and $\langle \bar{2}\bar{3} \rangle$ form the second orbit. The third orbit consists of the simplices $\langle \bar{1}\bar{2} \rangle$, $\langle \bar{1}\bar{3} \rangle$, and $\langle \bar{2}\bar{3} \rangle$, and the last orbit of $\langle \bar{1}\bar{2} \rangle$, $\langle \bar{1}\bar{3} \rangle$, and $\langle \bar{2}\bar{3} \rangle$. We cannot take the first two orbits as they are triangulations of S^1 with only three vertices out of the required six vertices. Hence $lk(0)$ must consist of the vertices $1, 2, 3, \bar{1}, \bar{2}, \bar{3}$ and the orbits of $\langle \bar{1}\bar{2} \rangle$ and $\langle \bar{1}\bar{3} \rangle$. Analogously, $lk(\bar{0})$ must consist of the same simplices. Therefore the underlying sets of the stars $st(0)$ and $st(\bar{0})$ have a common boundary and are PL -homeomorphic to a disk. Consequently, we would get a triangulation of a 2-sphere and not of $S^1 \times S^1$. The problem in this case is that we do not have a choice which orbits to choose for $lk(0)$, $lk(\bar{0})$, respectively.

In higher dimensions we always have this choice (as in Theorem 3.1 already seen for $k = 2$). This leads to:

Conjecture 3.3. *Using the same principle for arbitrary $k \geq 2$, $S^k \times S^k$ can be triangulated with $4k + 4$ vertices as a tight subcomplex of $C_{2k+2}^* \subset \mathbb{E}^{2k+2}$. More precisely, the principle says that $lk(0)$ consists of orbits of $(2k - 1)$ -simplices under the action $(12 \dots (2k + 2))(\bar{1}\bar{2} \dots \overline{(2k + 2)})$.*

These examples would give equality in our generalization Conjecture 2.3 of Theorem 2.1 as can be seen as follows.

We have

$$\chi(S^k \times S^k) = \begin{cases} 4 & \text{for even } k, \\ 0 & \text{for odd } k. \end{cases}$$

Therefore we must show

$$2 \binom{2k+1}{k+1} = 4^{k+1} \binom{\frac{1}{2}(2k+1)}{k+1}$$

or equivalently

$$\frac{(2k+1)!}{k!} = 2^k \prod_{i=0}^k (2k+1-2i). \quad (3)$$

Equation (3) is true as can be seen by a simple inductive proof.

Removing from our triangulation of $S^2 \times S^2$ the vertices 4, $\bar{4}$, 5, and $\bar{5}$, the torus mentioned above is embedded in the remaining complex. We conjecture that this process can be generalized.

Conjecture 3.4. *We can remove two pairs of opposite vertices from a triangulation of $S^k \times S^k$ given by Conjecture 3.3 such that in the remaining complex, which is a subset of C_{2k}^* , a triangulation of $S^{k-1} \times S^{k-1}$ with $4k$ vertices is embedded.*

4. An Upper Bound Theorem for Combinatorial 4-Manifolds

The Main Theorem 4.1 of this section is a counterpart of Theorem 2.4 and gives an upper bound for the Euler characteristic of combinatorial 4-manifolds that admit a fixed-point free involution, i.e., a fixed-point free endomorphism of order 2. If M is embedded in $\mathbb{E}^d$, then the involution induces a linear transformation f of $\mathbb{E}^d$ which maps $|M|$ onto itself, and the condition “fixed-point free” implies that f has no fixed points in $\mathbb{E}^d$ except possibly the origin or, equivalently, that f does not have 1 as an eigenvalue.

A fixed-point free involution acting on M always exists if M is a combinatorial manifold in some $\mathbb{E}^d$ with $(0, \dots, 0) \notin |M|$ and if $|M|$ is centrally symmetric. Hence Theorem 4.1 can be applied to show that our triangulation of $S^2 \times S^2$ has the minimum number of vertices in the sense that any triangulation of $S^2 \times S^2$ which possesses a fixed-point free involution cannot have less than twelve vertices. (In this special case the automorphism γ is a fixed-point free involution of the required type.)

Upper Bound Theorem 4.1. *Let M be a combinatorial 4-manifold with n vertices and with a fixed-point involution σ acting on the triangulation. Then $n = 2m$ is even and the inequality*

$$10(\chi(M) - 2) \leq \frac{4}{3}(m-1)(m-3)(m-5) = 4^3 \binom{\frac{1}{2}(m-1)}{3}. \quad (4)$$

holds with equality if and only if M can be embedded in C_m^* such that this embedding contains the 2-skeleton of C_m^* . (Hence, in the case of equality, M is 1-tight and 1-connected [Kü3, Corollary 3.8] and by the Poincaré duality it is tight.)

Proof. σ acts as a permutation of the vertices v_i , $1 \leq i \leq n$, with $\sigma(v_i) \neq v_i$ and $\langle v_i, \sigma(v_i) \rangle$ is not an edge of M , because otherwise the center of this edge would be a fixed point of M . This implies $n = 2m$ is even, $f_1 \leq \binom{n}{2} - m$ and $f_2 \leq \binom{n}{3} - m(2m - 2)$. We define $\bar{f}_1 := \binom{n}{2} - m - f_1$ and $\bar{f}_2 := \binom{n}{3} - 2m(m - 1) - f_2$.

First let $m \geq 7$. It suffices to prove $4\bar{f}_1 \leq \bar{f}_2$ with equality if and only if $\bar{f}_1 = 0 = \bar{f}_2$ (*), because (*) implies

$$\begin{aligned} 10(\chi(M) - 2) &= f_2 - 4f_1 + 10n - 20 \\ &= \binom{n}{3} - 2m(m - 1) - \bar{f}_2 - 4\binom{n}{2} + 4m + 4\bar{f}_1 + 20(m - 1) \\ &\leq \binom{2m}{3} - 2m(m - 1) - 4\binom{2m}{2} + 4m + 20(m - 1) \\ &= \frac{4}{3}(m - 1)(m - 3)(m - 5) \end{aligned}$$

with equality if and only if $\bar{f}_1 = 0 = \bar{f}_2$, i.e., M contains in the case of equality all possible edges except for the diagonals $\langle v_i, \sigma(v_i) \rangle$ and all possible triangles except those containing diagonals. This is the 2-skeleton of C_m^* . The embeddability of M into C_m^* follows because the m -crosspolytope contains all possible k -simplices, $0 \leq k \leq m$, except those containing diagonals.

Proof of ().* Let $f_0(i)$ denote the number of vertices belonging to the link of vertex i and let $\bar{f}_0(i)$ denote the number of vertices different from i and $\sigma(i)$ that do not belong to $lk(i)$. Analogously, we define $f_1(i)$ to be the number of edges contained in $lk(i)$ and $\bar{f}_1(i)$ to be the number of edges which do not belong to $lk(i)$, which are not diagonals and which do not contain one of the vertices i or $\sigma(i)$, i.e.,

$$\bar{f}_1(i) = \binom{n-2}{2} - \frac{n-2}{2} - f_1(i).$$

Because the link of every vertex is a 3-sphere with at least five vertices we get $\bar{f}_0(i) \leq n - 7$. This implies:

$$\begin{aligned} \bar{f}_1(i) &\geq f_0(i)\bar{f}_0(i) + \binom{\bar{f}_0(i)}{2} - \bar{f}_0(i) \\ &= \bar{f}_0(i) \left(2(m - 1) - \bar{f}_0(i) + \frac{\bar{f}_0(i) - 1}{2} \right) - \bar{f}_0(i) \\ &= \bar{f}_0(i)(2m - 3 - \frac{1}{2}(\bar{f}_0(i) + 1)) \end{aligned}$$

$$\begin{aligned} &\geq \bar{f}_0(i)(2m - 3 + \frac{1}{2}(6 - n)) \\ &= m\bar{f}_0(i) \geq 6\bar{f}_0(i), \end{aligned}$$

where equality holds if and only if $\bar{f}_0(i) = 0 = \bar{f}_1(i)$, as by assumption $m \geq 7$.

Summation over all vertices yields $3\bar{f}_2 \geq 6 \cdot 2\bar{f}_1$ with equality if and only if $\bar{f}_2 = 0 = \bar{f}_1$. This proves (*).

Now let us discuss the exceptional cases $m \leq 6$.

As M is a combinatorial 4-manifold we have $n \geq 6$ or $m \geq 3$.

Let $m \in \{3, 4\}$, then M must be homeomorphic to S^4 (compare [BK]), thus $\chi(M) = 2$.

For $m = 3$, M is the boundary complex of a 5-simplex, i. e. M contains all possible edges (also the “diagonals”). Hence M does not admit a fixed-point free involution. For $m = 4$, we have $f_1 \geq \binom{5}{1}f_0 - \binom{6}{2} = 25$ according to a special case of a Lower Bound Theorem proved by Kalai [Ka, Theorem 1.1]. As there are maximally $\binom{8}{2} = 28$ possible edges, M must contain at least one of the “diagonals” and consequently a fixed-point free involution is not possible.

For $m = 5$ we use the notation from the cases $m \geq 7$. With the same argumentation we conclude $\bar{f}_1(i) \geq m\bar{f}_0(i) = 5\bar{f}_0(i)$. Summation over all vertices gives $\bar{f}_2 \geq \frac{10}{3}\bar{f}_1$. Using $f_1 \geq \binom{5}{1}f_0 - \binom{6}{2} = 35$, we get

$$\bar{f}_1 = \binom{10}{2} - 5 - f_1 = 40 - f_1 \leq 5.$$

This implies

$$4\bar{f}_1 - \bar{f}_2 \leq \frac{2}{3}\bar{f}_1 \leq \frac{10}{3}.$$

Hence

$$\begin{aligned} 10(\chi(M) - 2) &= f_2 - 4f_1 + 10n - 20 \\ &= \binom{10}{3} - 10(5 - 1) - \bar{f}_2 - 4\binom{10}{2} + 20 + 4\bar{f}_1 + 20(5 - 1) \\ &= 0 + 4\bar{f}_1 - \bar{f}_2 \\ &\leq \frac{2}{3}\bar{f}_1 \leq \frac{10}{3}. \end{aligned}$$

As $\chi(M) \in \mathbb{Z}$ it follows that $\chi(M) \leq 2$ and inequality (4) is proved. If equality holds, then $\frac{2}{3}\bar{f}_1 = 0 = \bar{f}_2$ and M can be embedded in C_3^* .

For $m = 6$ the inequality we have to prove holds with the same arguments used for $m \geq 7$, but in this case equality could also hold if, for all vertices i , $\bar{f}_0(i) = n - 7 = 5$ and $\bar{f}_1(i) = 30$, i.e., $f_0(i) = 5$ and $f_1(i) = 10$.

We now show that these values do not give a combinatorial 4-manifold M .

First we notice that they imply $f_1 = 30$ and $f_2 = 40$. As equality can only hold if $\chi(M) = 4$, the f -vector of M can be calculated to be

$$(f_{-1}, f_0, f_1, f_2, f_3, f_4) = (1, 12, 30, 40, 30, 12).$$

in contradiction to Kalai’s Lower Bound Theorem $f_1 \geq 5f_0 - 15$. □

Remark 4.2.

- (i) The inequality of the above theorem is of the same structure as that in Theorem 2.1. We just have to replace in Theorem 2.1 “ d ” by “ m ” and “ $\geq$ ” by “ $\leq$ ” to get the inequality of Theorem 4.1. In [Kü4] it is shown that such a correspondence is also valid for combinatorial 2-manifolds (see inequality (1)). We conjecture that the same correspondence holds even for the general case (see the inequality in Conjecture 2.3).
- (ii) Theorem 4.1 gives in connection with Theorem 2.1 the following bounds for the Euler characteristic of a 1-tight combinatorial 4-manifold M in $\mathbb{E}^d$ with n vertices if:
1. M admits a fixed-point free involution acting on the triangulation;
 2. $|M|$ is 1-connected; and
 3. $\mathcal{H}(M)$ is a centrally symmetric and simplicial d -polytope such that M is a subcomplex of $\mathcal{H}(M)$

$$\frac{2}{15}(d-1)(d-3)(d-5)+2 \leq \chi(M) \leq \frac{2}{15}\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-3\right)\left(\frac{n}{2}-5\right)+2.$$

Corollary 4.3. *Let M be a combinatorial 4-manifold. If there exists a number $m \in \mathbb{N}$ such that the equality in (4) holds, then M has at least $2m-1$ vertices if $m \geq 6$ and six vertices if $m \leq 5$.*

Proof. This is, for $m \geq 5$, the same calculation as in Proposition 2.5 if we replace “ d ” by “ m .”

So let us regard the remaining the cases. As

$$0 = \binom{6-4}{3} \geq \frac{4}{3}(m-1)(m-3)(m-5)$$

for $m \leq 1$ and $m = 3$ we cannot (according to Theorem 2.4) improve the trivial statement that the number of vertices is at least six.

For $m = 4$ and $m = 2$ the equality in (4) cannot hold by Proposition 2.7. $\square$

We close the paper with an Upper Bound Conjecture for even-dimensional combinatorial manifolds with a fixed-point free involution acting on the triangulation. This conjecture is the analogue of Theorem 4.1 for arbitrary dimensions.

Upper Bound Conjecture 4.4. *For a combinatorial $2k$ -manifold admitting a fixed-point free involution acting on the triangulation, the number of vertices $n = 2m$ is even and there holds the inequality*

$$(-1)^k \binom{2k+1}{k+1} (\chi(M) - 2) \leq 4^{k+1} \binom{\frac{1}{2}(m-1)}{k+1}$$

with equality if and only if M can be embedded in C_m^ such that this embedding contains the k -skeleton of C_m^* (and is therefore tight and $(k-1)$ -connected).*

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Received January 23, 1996, and in revised form September 13, 1996.