Exponential Lower Bound for the Translative Kissing Numbers of d -Dimensional Convex Bodies*

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Abstract. The translative kissing number $H(K)$ of a d -dimensional convex body K is the maximum number of mutually nonoverlapping translates of K which touch K . In this paper we show that there exists an absolute constant $c > 0$ such that $H(K) \geq 2^{cd}$ for every positive integer d and every d -dimensional convex body K . We also prove a generalization of this result for pairs of centrally symmetric convex bodies.

1. Introduction and Statement of Results

Let K be a d -dimensional convex body (i.e., a compact convex subset of R^d with nonempty interior). Denote by $H(K)$ the *translative kissing number* of K , which is defined as the maximum number of mutually nonoverlapping translates of K which touch K . (Two convex bodies in R^d are *nonoverlapping* if they have no common interior point, and we say that they *touch each other* if they are nonoverlapping but their intersection is nonempty.) $H(K)$ is often called the *Hadwiger number* of K as well.

It was shown by Swinnerton-Dyer [15] that $H(K) \geq d^2 + d$ holds for every d -dimensional convex body K ($d \geq 1$). The main purpose of this paper is to improve on this bound for sufficiently large values of d , proving the following:

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Theorem 1. *There exists an absolute constant $c > 0$, such that*

$$H(K) \geq 2^{cd} \quad (1)$$

holds for every positive integer d and for every d -dimensional convex body K .

Combining this result with the inequality $H(K) \leq 3^d - 1$, proved by Hadwiger [6], it turns out that the order of magnitude of $H(K)$ is exponential in the dimension of K for every convex body K .

Although the previously mentioned bound of Swinnerton-Dyer is sharp for $d \leq 3$ (e.g., $H(B^d) = d^2 + d$ for the d -dimensional Euclidean unit ball B^d if $d \leq 3$), we conjecture that this is not the case when $d \geq 4$.

For additional related results and references on this topic, see [1], [2], [5], [7], [14], [16], and the survey paper by Fejes Tóth and Kuperberg [4].

Let $A \subset R^d$. We use the notation $\alpha A + v$ for the set $\{\alpha a + v \mid a \in A\}$ for any $\alpha \in R$ and $v \in R^d$. A subset A' of R^d is a *homothetic copy* of A if $A' = \alpha A + v$ for a suitable $\alpha > 0$ and $v \in R^d$. We denote by $A - B$ the set $\{a - b \mid a \in A, b \in B\}$. It is known that if A and B are d -dimensional convex bodies, then $A - B$ is a d -dimensional convex body as well. We denote the number of elements of a set S by $|S|$.

If K_1 and K_2 are d -dimensional convex bodies, then let us denote by $H(K_1, K_2)$ (resp., by $H^*(K_1, K_2)$) the maximum number of mutually nonoverlapping (resp., pairwise disjoint) translates of K_2 which touch K_1 . It is clear that $H(K_1, K_2)$ and $H^*(K_1, K_2)$ are invariant quantities for simultaneous affine transformation of K_1 and K_2 . We introduce the notation $H^*(K) = H^*(K, K)$. Then $H(K) = H(K, K)$ and

$$H(K_1, K_2) \geq H^*(K_1, K_2)$$

hold trivially. In particular, $H(K) \geq H^*(K)$. Sometimes $H^*(K)$ can be much smaller than $H(K)$. For example, if C is a d -dimensional cube, then $H^*(C) = 2^d$ but $H(C) = 3^d - 1$.

Shannon [13] and Wyner [17] proved that for a fixed positive $\varphi < \frac{1}{2}\pi$ it is possible to arrange at least $(1/\sin \varphi + o_d(1))^d$ congruent and nonoverlapping caps of angular (i.e., geodesic) diameter φ on the boundary of the Euclidean unit ball $B^d = \{x \in R^d \mid \|x\| \leq 1\}$. This implies directly that if $1 \leq r < \sqrt{2} + 1$, then

$$H^*(B^d, rB^d) \geq 2^{c_0(r)d} \quad (2)$$

holds for a suitable $c_0(r) > 0$, whose value depends on r only.

The restriction $r < \sqrt{2} + 1$ is necessary in (2). Davenport and Hajós [3] proved that it is not possible to choose more than $2d$ points on the boundary of B^d so that the angular distance between any two of them is at least $\frac{1}{2}\pi$. From this result follows that $H(B^d, (\sqrt{2} + 1)B^d) = 2d$.

We are able to prove a stronger version of Theorem 1, showing an inequality similar to (1) for $H^*(K)$. We get this result as a simple consequence of the following theorem, which is a generalization of (2) for pairs of centrally symmetric d -dimensional convex bodies.

Theorem 2. *Let r be a real number, $1 \leq r < \sqrt{2} + 1$. Then there exists a positive number $c(r)$ depending on r only, such that*

$$\max(H^*(K_1, rK_2), H^*(K_2, rK_1)) \geq 2^{c(r)d} \quad (3)$$

holds for every positive integer d and for every pair (K_1, K_2) of centrally symmetric d -dimensional convex bodies.

In the proofs of Theorems 1 and 2 we need to use a fundamental result of the local theory of Banach spaces, namely, the QS-theorem of Milman [9]. To prove Theorem 2, we establish a refinement of this theorem (Lemma 2). For further geometric aspects of the local theory of Banach spaces, see the book by Milman and Schechtman [10], and the survey paper by Lindenstrauss and Milman [8].

There are two interesting special cases of Theorem 2. In the first case, when $K_1 = K_2$, we get a direct generalization of (2) for centrally symmetric convex bodies. In the second case, when $r = 1$, we obtain a geometric phenomenon concerning the arrangements of translates of K_1 and K_2 .

Corollary 3. *For every ε , $0 \leq \varepsilon < \sqrt{2}$, there exists a real number $c(\varepsilon) > 0$, such that*

$$H^*(K, (1 + \varepsilon)K) \geq 2^{c(\varepsilon)d} \quad (4)$$

holds for every positive integer d and for every centrally symmetric d -dimensional convex body K .

Corollary 4. *There exists an absolute constant $c > 0$, such that*

$$\max(H^*(K_1, K_2), H^*(K_2, K_1)) \geq 2^{cd} \quad (5)$$

holds for every positive integer d and for every pair (K_1, K_2) of centrally symmetric d -dimensional convex bodies.

Considering Theorem 2 with $K_1 = K_2$ and $r = 1$ simultaneously, it is easy to see that the condition of central symmetry can be dropped (cf. the difference body method of Minkowski in the next section). This way we get a stronger version of Theorem 1.

Corollary 5. *There exists an absolute constant $c > 0$, such that*

$$H^*(K) \geq 2^{cd} \quad (6)$$

holds for every positive integer d and for every d -dimensional convex body K .

Now we turn our interest toward a possible further generalization of Theorem 1. But first we introduce some terminology that is needed to state the problem.

Let $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ be a finite collection of d -dimensional convex bodies. A collection $\mathcal{C}'$ is called a translate of $\mathcal{C}$ if $|\mathcal{C}'| = m$ and $\mathcal{C}' = \{C_i + v_i \mid 1 \leq i \leq m\}$ for a suitable set $\{v_1, v_2, \dots, v_m\} \subset \mathbb{R}^d$. We say that $\mathcal{C}$ is a packing if its elements are

mutually nonoverlapping. The nerve graph $n(\mathcal{C})$ of $\mathcal{C}$ is an abstract graph with vertices $\{p_1, p_2, \dots, p_m\}$, where $p_i p_j$ is an edge if and only if $C_i \cap C_j \neq \emptyset$. We call $\mathcal{C}$ *starlike* if $n(\mathcal{C})$ has $m - 1$ edges and a vertex which is contained in each edge (i.e., when each geometric realization of $n(\mathcal{C})$ with line-segment edges is starlike). We conjecture that Corollary 4 can be generalized in the following way:

Conjecture. *There exists an absolute constant $c > 0$ such that for every positive integer d and for every collection $\mathcal{C}$ of d -dimensional convex bodies, the following holds:*

If $|\mathcal{C}| \leq 2^{cd}$, then there exists a starlike packing $\mathcal{C}'$ which is a translate of $\mathcal{C}$.

We can confirm this conjecture only in a very special case. If the elements of $\mathcal{C}$ are homothetic copies of the same convex body and $|\mathcal{C}| \leq 2^{cd}$, then by Corollary 5, there exists a starlike packing $\mathcal{C}''$ with $|\mathcal{C}|$ translates of the largest copy. Then we can arrange a translate $\mathcal{C}'$ of $\mathcal{C}$ in such a way that each element of $\mathcal{C}''$ contains exactly one element of $\mathcal{C}'$, and $\mathcal{C}'$ is starlike. This proves the following:

Corollary 6. *There exists an absolute constant $c > 0$ with the following property: For every d -dimensional convex body K , every positive integer $m \leq 2^{cd}$, and every m -element collection $\mathcal{C}$ of homothetic copies of K , there exists a starlike packing $\mathcal{C}'$ which is a translate of $\mathcal{C}$.*

2. Proofs

First we recall some notion and facts. Then we state several lemmas from which we derive Theorem 1 in a very simple way. After that we prove Theorem 2, using a somewhat more elaborate argument. We remark that although it is easy to see that Theorem 1 is a consequence of Theorem 2, we prefer to give a direct proof for Theorem 1 because of its simplicity.

We begin with a well-known observation: the so-called “difference body method” of Minkowski [11] (also see [12] for a description of the method). Let $K + v_1$ and $K + v_2$ be two translates of a convex body K . They are nonoverlapping (resp., touching) if and only if $\frac{1}{2}(K - K) + v_1$ and $\frac{1}{2}(K - K) + v_2$ are nonoverlapping (resp., touching). From this fact follows that $H(K) = H(K - K)$ and $H^*(K) = H^*(K - K)$, thus it suffices to prove Theorem 1 for centrally symmetric convex bodies.

Let K and L be a pair of d -dimensional convex bodies. Denote by $M(K, L)$ the maximum number n for which there exist n points $v_1, v_2, \dots, v_n$ on the boundary of K such that $\|v_i - v_j\|_{L-L} \geq 1$ ($i \neq j$, $1 \leq i, j \leq n$). Denote by $M^*(K, L)$ the similar quantity, when we require $\|v_i - v_j\|_{L-L} > 1$ for each $i \neq j$. Here $\|\cdot\|_{L-L}$ denotes the Minkowski norm with the unit ball $L - L$, i.e., if $v \in \mathbb{R}^d$, then $\|v\|_{L-L} = \min\{t > 0 \mid v \in t(L - L)\}$. Thus we have $\|v\|_{L-L} = \|v\|/\|w\|$, where $\|\cdot\|$ denotes the usual Euclidean norm, and w is a vector parallel to v and having endpoint on the boundary of $L - L$. An easy computation shows that $H(K, L) = M(K - L, L)$, and $H^*(K, L) = M^*(K - L, L)$. Also, it is easy to see that

$$M^*(K, L) = \max_{\varepsilon > 0} M(K, (1 + \varepsilon)L) \leq M(K, L). \quad (7)$$

Let S be a centrally symmetric d -dimensional convex body, and let ε be a positive real number. A d -dimensional convex body K is said to be ε -close to S if $S \subseteq K \subseteq (1 + \varepsilon)S - \varepsilon v$, where v is the center of S . If v is the origin o of R^d , then we say that S is an o -symmetric convex body. It is clear that the translate $S - v$ of S is always o -symmetric.

Lemma 1. *Let S be a centrally symmetric d -dimensional convex body. Then for any positive number $\varepsilon < \frac{1}{2}$ and for any pair (K, L) of d -dimensional convex bodies which are ε -close to S ,*

$$M\left(K, \frac{t - 2\varepsilon}{1 + \varepsilon}L\right) \geq M(S, tS) \quad \text{for any } t > 2\varepsilon, \quad (8)$$

and

$$H(K, rL) \geq H\left(S, \frac{r + (3r + 2)\varepsilon}{1 - (3r + 2)\varepsilon}S\right) \quad \text{for any } 0 < r < \frac{1 - 2\varepsilon}{3\varepsilon}. \quad (9)$$

For $K = L$ and $r = 1$ the above bound can be improved to

$$H(K) \geq H(S, (1 + 4\varepsilon)S). \quad (10)$$

Proof of Lemma 1. We may assume that S is an o -symmetric convex body, $S \subseteq K \subseteq (1 + \varepsilon)S$, and $S \subseteq L \subseteq (1 + \varepsilon)S$.

To prove (8), it is enough to show that if p_1 and p_2 are boundary points of S such that $\|p_1 - p_2\|_{S-S} \geq t$ for some $t > 2\varepsilon$, and we denote by q_i the boundary point of K for which $p_i \in [o, q_i]$ ($i = 1, 2$), then $\|q_1 - q_2\|_{L-L} \geq (t - 2\varepsilon)/(1 + \varepsilon)$ holds.

We recall that $\|q_1 - q_2\|_{L-L} = \|q_1 - q_2\|/\|v\|$ and $\|q_1 - q_2\|_{S-S} = \|q_1 - q_2\|/\|u\|$, where v and u are vectors parallel to $q_1 - q_2$, and their endpoints are on the boundary of $L - L$ and $S - S$, respectively. Therefore, using the triangle inequality, we obtain that

$$\|q_1 - q_2\|_{L-L} \geq \frac{1}{1 + \varepsilon} (\|p_1 - p_2\|_{S-S} - \|q_1 - p_1\|_{S-S} - \|q_2 - p_2\|_{S-S})$$

holds. It is easy to see that the right side of this inequality is at least $(t - 2\varepsilon)/(1 + \varepsilon)$. Thus we get (8).

To prove (9), we choose $t = r + (3r + 2)\varepsilon$. Then we apply (8) for the convex bodies $K' = [1/(1 + r)](K - L)$ and $L' = L$, and we get

$$H(K, rL) = M(K - rL, rL) = M\left(K', \frac{r}{r + 1}L'\right) \geq M\left(S, \frac{t}{r + 1}S\right).$$

A simple calculation shows that

$$M\left(S, \frac{t}{r + 1}S\right) \geq H\left(S, \frac{r + (3r + 2)\varepsilon}{1 - (3r + 2)\varepsilon}S\right),$$

which completes the proof of (9).

To prove (10), we use the identity $S - S = (1 - \varepsilon)S - (1 + \varepsilon)S$. Then it is enough to show that $M(K - K, K) \geq M(S - S, (1 + \varepsilon)S)$. Let p_i and q_i be boundary points

of $S - S$ and $K - K$, respectively, such that $\|p_1 - p_2\|_{S-S} \geq (1 + \varepsilon)$ and $p_i \in [o, q_i]$ ($i = 1, 2$). It is clear that $p_1 = \alpha q_1$ and $p_2 = \beta q_2$ for suitable positive reals α and β ($\alpha, \beta \leq 1$). We may assume that $\alpha \geq \beta$. Then

$$1 + \varepsilon \leq \|\alpha q_1 - \beta q_2\|_{S-S} \leq (\alpha - \beta)\|q_1\|_{S-S} + \beta\|q_1 - q_2\|_{S-S} \leq \frac{\alpha - \beta}{\alpha} + \beta\|q_1 - q_2\|_{S-S}.$$

Therefore

$$\|q_1 - q_2\|_{K-K} \geq \frac{\|q_1 - q_2\|_{S-S}}{1 + \varepsilon} \geq \frac{\varepsilon}{\beta} + \frac{1}{\alpha} \geq 1.$$

This means that (10) holds. $\square$

Let E be a k -dimensional linear subspace of R^d ($0 \leq k \leq d$). We denote the orthogonal projection of a d -dimensional convex body K into E by $\pi_E(K)$. On the other hand, we introduce the notation $i_E(K) = K \cap E$ in the case when o is an interior point of K . It is clear that E can be identified with R^k , so we can consider $\pi_E(K)$ and $i_E(K)$ as k -dimensional convex bodies.

We now prove a refinement of the quotient of subspace theorem (QS-theorem for short) of Milman [9]. We remark that by a Euclidean ball B in R^d we mean a homothetic copy of the unit ball B^d in R^d .

Lemma 2. *For every integer $n \geq 1$ and for every positive $\varepsilon < 1$ there exists a positive number $c_n(\varepsilon)$ with the following property:*

For every positive integer d and every n -tuple $K_1, K_2, \dots, K_n$ of o -symmetric d -dimensional convex bodies, there exist linear subspaces E and F of R^d such that $E \subseteq F$, $\dim E \geq c_n(\varepsilon)d$, and the convex body $K'_j = \pi_E(i_F(K_j))$ is ε -close to an o -centered Euclidean ball B_j of E , for every j ($1 \leq j \leq n$).

Proof of Lemma 2. Let $\varepsilon > 0$ be a fixed number. We use induction on n to prove the existence of $c_n(\varepsilon)$. For $n = 1$, Lemma 2 is exactly the QS-theorem of Milman [9]. More precisely, the QS-theorem says that K'_1 is ε -close to an o -centered ellipsoid, not to a Euclidean ball. But it is easy to see, that if l is a positive integer, then any l -dimensional ellipsoid has an at least $\lfloor l/2 \rfloor$ -dimensional orthogonal projection which is a Euclidean ball. Combining this result with the QS-theorem we get Lemma 2 for $n = 1$.

Assume that Lemma 2 holds for $n - 1$. Let $K_1, K_2, \dots, K_n$ be d -dimensional convex bodies. Then there exist linear subspaces E and F of R^d , such that $E \subseteq F$, $k = \dim E \geq c_{n-1}(\varepsilon)d$, and $K'_j = \pi_E(i_F(K_j))$ is ε -close to a Euclidean ball B_j , for every j , $2 \leq j \leq n$. Let $K'_1 = \pi_E(i_F(K_1))$. Applying the QS-theorem (i.e., Lemma 2 for $n = 1$) for K'_1 , we get that there exist linear subspaces E' and F' of $\pi_E(i_F(R^d)) \cong R^k$ such that $E' \subseteq F'$, $k' = \dim E' \geq c_1(\varepsilon)k$, and $K''_1 = \pi_{E'}(i_{F'}(K'_1))$ is ε -close to a Euclidean ball B''_1 . It is clear that then $K''_j = \pi_{E'}(i_{F'}(K'_j))$ is ε -close to a Euclidean ball B''_j for every j , $1 \leq j \leq n$. On the other hand, it is easy to see that there exist linear subspaces E'' and F'' of R^d , such that $E'' \subseteq F''$, $\dim E'' = k'$, and $\pi_{E''}(i_{F''}(K_j)) = K''_j$, for every j , $1 \leq j \leq n$. This completes the proof of Lemma 2. $\square$

The next two lemmas express simple observations which play important roles in the proofs of the theorems. We leave their proofs to the reader (to prove Lemma 3, observe

that $\|x\|_{L-L} \geq \|\pi_E(x)\|_{\pi_E(L)-\pi_E(L)}$ for any $x \in R^d$). The analogous statements can be proved for $M^*(K, rL)$ and $H^*(K, rL)$ as well.

Lemma 3. *Let K and L be d -dimensional convex bodies. Let E be a linear subspace of R^d . Then $M(K, rL) \geq M(\pi_E(K), r\pi_E(L))$ and $H(K, rL) \geq H(\pi_E(K), r\pi_E(L))$ hold for every $r > 0$. In particular, $H(K) \geq H(\pi_E(K))$.*

Lemma 4. *Let K and L be o -symmetric d -dimensional convex bodies and let E be a linear subspace of R^d . Then $M(K, rL) \geq M(i_E(K), ri_E(L))$ and $H(K, rK) \geq H(i_E(K), ri_E(K))$ hold for every $r > 0$. In particular, $H(K) \geq H(i_E(K))$.*

Proof of Theorem 1. By the previously mentioned difference body method of Minkowski, it is enough to prove Theorem 1 when K is centrally symmetric. Moreover, we may assume that K is o -symmetric. Let $0 < \varepsilon < \sqrt{2}/4$. Then, by the QS-theorem (Lemma 2 for $n = 1$), there exist linear subspaces E and F of R^d such that $E \subseteq F$, $k = \dim E \geq c_1(\varepsilon)d$, and $\pi_E(i_F(K))$ is ε -close to a Euclidean ball B . Let $L = \pi_E(i_F(K))$. Then, by (2) and by the Lemmas 1 to 4,

$$H(K) \geq H(L) \geq H(B, (1 + 4\varepsilon)B) \geq 2^{c_0(1+4\varepsilon)k} \geq 2^{c_0(1+4\varepsilon)c_1(\varepsilon)d}.$$

Let $c = c_0(1 + 4\varepsilon)c_1(\varepsilon)$. Then $H(K) \geq 2^{cd}$. □

Proof of Theorem 2. Without loss of generality we may assume that K_1 and K_2 are o -symmetric convex bodies. Let $1 \leq r < 1 + \sqrt{2}$. Then there exist positive numbers $\delta = \delta(r)$ and $\varepsilon = \varepsilon(r) < \frac{1}{2}$ such that

$$t = t(r, \varepsilon, \delta) = \frac{r + (3r + 2)\varepsilon + r\delta(1 + \varepsilon)}{1 - (3r + 2)\varepsilon + r\delta(1 - \varepsilon)} < 1 + \sqrt{2}.$$

We now use Lemma 2 for $n = 4$, $K_3 = K_1 - rK_2$, and $K_4 = K_2 - rK_1$. Thus we get linear subspaces E and F of R^d such that $E \subseteq F$, $k = \dim E \geq c_4(\varepsilon)d$, and $L_i = \pi_E(i_F(K_j))$ is ε -close to a Euclidean ball B_j ($1 \leq j \leq 4$). Denote by r_j the radius of B_j for each j ($1 \leq j \leq 4$). Then either $r_1 \geq r_2$ or $r_1 < r_2$.

First, we assume that $r_1 \geq r_2$. It is easily seen that $B_1 + rB_2 \subseteq L_3$, so we may assume that $B_1 + rB_2 \subseteq B_3$. Then $r_1 + rr_2 \leq r_3$. Using (7), we get

$$H^*(K_1, rK_2) = M^*(K_3, rK_2) \geq M(K_3, (1 + \delta)rK_2).$$

By Lemmas 3 and 4,

$$M(K_3, (1 + \delta)rK_2) \geq M(L_3, (1 + \delta)rL_2) = M\left(\frac{1}{r_3}L_3, (1 + \delta)\frac{r}{r_3}L_2\right).$$

But $(1/r_3)L_3$ is ε -close to the unit ball B^k of E , $[(1 + \delta)r/r_3]L_2$ is ε -close to $[(1 + \delta)rr_2/r_3]B^k$, and

$$(1 + \delta)\frac{rr_2}{r_3} \leq \frac{r(1 + \delta)}{r + 1} = \frac{t/(t + 1) - 2\varepsilon}{1 + \varepsilon}.$$

Therefore, applying Lemma 1, we have

$$M\left(\frac{1}{r_3}L_3, (1+\delta)\frac{r}{r_3}L_2\right) \geq M\left(B^k, \frac{t}{t+1}B^k\right) = H(B^k, tB^k).$$

Thus we get

$$H^*(K_1, rK_2) \geq H(B^k, tB^k) \geq 2^{c_0(t)k} \geq 2^{c_0(t)c_4(\varepsilon)d}.$$

Since ε and δ depend on r , t depends on r , ε , and δ , hence $c_0(t)c_4(\varepsilon)$ depends on r only. Let $c(r) = c_0(t)c_4(\varepsilon)$. Then $H^*(K_1, rK_2) \geq 2^{c(r)d}$.

The other case, when $r_1 < r_2$, can be handled similarly, and we get that $H^*(K_2, rK_1) = M^*(K_4, rK_1) \geq 2^{c(r)d}$. $\square$

3. Concluding Remarks

A natural question is that whether there exists an expression similar to (7) for $H^*(K, L)$. A simple argument shows that

$$H^*(K, L) \leq \max_{\varepsilon > 0} H(K, (1+\varepsilon)L),$$

and if L is either a strictly convex body or it is a homothetic copy of a centrally symmetric convex body K , then equality holds (if $L = K$, then the requirement of central symmetry is not needed).

We conjecture that Theorem 2 remains valid for nonsymmetric convex bodies as well. Unfortunately there is no analogue of Lemma 4 for an arbitrary d -dimensional convex body. But the generalized version of Theorem 2 would be an easy consequence of an affirmative answer to the following:

Problem. Decide whether it is true that for every $\varepsilon > 0$ there exists a real number $c(\varepsilon) > 0$ with the following property:

For every positive integer d and every d -dimensional convex body K , there exists a linear subspace E of $\mathbb{R}^d$, such that $k = \dim E \geq c(\varepsilon)d$ and $\pi_E(K)$ is ε -close to a centrally symmetric k -dimensional convex body in E .

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