Piercing d -Intervals*

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Communicated by János Pach

A (*homogeneous*) d -interval is a union of d closed intervals in the line. Let $\mathcal{H}$ be a finite collection of d -intervals. The *transversal number* $\tau(\mathcal{H})$ of $\mathcal{H}$ is the minimum number of points that intersect every member of $\mathcal{H}$. The *matching number* $\nu(\mathcal{H})$ of $\mathcal{H}$ is the maximum number of pairwise disjoint members of $\mathcal{H}$. Gyárfás and Lehel [3] proved that $\tau \leq O(\nu^{d!})$ and Kaiser [4] proved that $\tau \leq O(d^2\nu)$. His proof is topological, applies the Borsuk–Ulam theorem, and extends and simplifies a result of Tardos [5]. Here we give a very short, elementary proof of a similar estimate, using the method of [2].

Theorem 1. *Let $\mathcal{H}$ be a finite family of d -intervals containing no $k+1$ pairwise disjoint members. Then $\tau(\mathcal{H}) \leq 2d^2k$.*

Proof. Let $\mathcal{H}'$ be any family of d -intervals obtained from $\mathcal{H}$ by possibly duplicating some of its members, and let n denote the cardinality of $\mathcal{H}'$. Note that $\mathcal{H}'$ contains no $k+1$ pairwise disjoint members. Therefore, by Turán's theorem, there are at least $n(n-k)/(2k)$ unordered intersecting pairs of members of $\mathcal{H}'$. Each such intersecting pair supplies at least two ordered pairs (p, I) , where p is an endpoint of one of the intervals in a member of $\mathcal{H}'$, I is a different member of $\mathcal{H}'$, and p lies in I . Since there are altogether at most $2dn$ possible choices for p , there is such a point that lies in at least $n(n-k)/(k2dn)$ members of $\mathcal{H}'$ besides the one in which it is an endpoint of an interval, showing that there is a point that lies in at least $n/(2dk)$ of the members of $\mathcal{H}'$. This implies that for any rational weights on the members of $\mathcal{H}$ there is a point that lies in at least a fraction $1/(2dk)$ of the total weight. By the min-max theorem it follows that

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* This research was supported in part by the Minerva Center for Geometry in Tel Aviv University.

there is a collection of m points so that each member of $\mathcal{H}$ contains at least $m/(2dk)$ of them, and thus contains an interval that contains at least $m/(2d^2k)$ of the points. Order the points from left to right, and take the set of all points whose rank in this ordering is divisible by $\lceil m/(2d^2k) \rceil$. This is a set of at most $2d^2k$ points that intersects each member of $\mathcal{H}$, completing the proof. $\square$

Remarks. It may be possible to improve the constant factor in the above proof. Kaiser's estimate is indeed better by roughly a factor of 2; $\tau(\mathcal{H}) \leq (d^2 - d + 1)v(\mathcal{H})$. It will be interesting to decide if the quadratic dependence on d is indeed best possible. Higher-dimensional extensions are possible, using the techniques in [2] and [1].

References

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Received August 15, 1996, and in revised form March 7, 1997.