Guest Editor's Foreword

*I know that I will fail and fail again.
I don't care. After all, what is the goal?
It is the end of an honourable contest.
The goal is death, but life consists of struggle.
The struggle in itself must be the goal.*

(Imre Madách: The Tragedy of Man)

Paul Erdős (1913–1996) was one of the great mathematical polymaths of our century. He contributed richly to prime number theory, the theory of additive and multiplicative functions, Diophantine approximation, probability theory, combinatorics, graph theory, statistical group theory, discrete geometry, approximation theory, set theory, etc., and in all these fields he established some fundamental results and opened up unexpected new paths. I am biased, of course, but I believe that discrete and combinatorial geometry was not just one among many areas of Erdős's interest, but it was one especially close to his heart.

He fell in love with elementary geometry already as a child. According to his recollections [E3], in the summer of 1930, at his first meeting with his lifelong friend Tibor Gallai, they discussed a geometric result: Steiner's proof of the isoperimetric property of the circle. Somewhat later Erdős read the beautiful book of Hilbert and Cohn-Vossen: *Anschauliche Geometrie* [HC], and the following problem occurred to him [E2]. Given n points in the plane, not all collinear, is it true that there is always a line passing through precisely two of them? It turned out that Sylvester had posed the same question some 40 years earlier, but it was first answered by Gallai, in the affirmative. This discovery led to a series of exciting problems even Euclid would appreciate, which involved the most elementary concepts of geometry. Incidence relations between points and lines came to new life as objects of combinatorial study. Many of these problems were strongly related to deep questions in additive number theory, as is illustrated by the papers of György Elekes and of Yuri Bilu, Vsevolod Lev, and Imre Ruzsa, published in the present issue of *Discrete & Computational Geometry*.

One of Erdős's first papers, written jointly with another lifelong friend, György Szekeres [ES], also dealt with a geometric problem. They rediscovered and applied Ramsey's theorem to prove the following result. For any $r \geq 3$, there exists a smallest $n = n(r)$ with the property that any set of n points in general position in the plane contains the vertex set of a convex r -gon. This seminal paper was the starting point of important new

trends in extremal graph theory and in combinatorial geometry, and it was also a decisive moment in the development of Erdős's mathematical thinking. Erdős and Szekeres proved that $n(r) \leq \binom{2r-4}{r-2} + 1$, and this bound stood unchanged for over 60 years. The paper of Fan Chung and Ron Graham in this special issue has now broken the ice by reducing this upper bound by 1. Based on this paper, Daniel Kleitman and Lior Pachter improved the bound by $2r - 7$, and shortly afterward Géza Tóth and Pavel Valtr made a further improvement by roughly a factor of 2. Imre Bárány and Pavel Valtr establish the following version of the Erdős–Szekeres theorem. There exists $c_r > 0$ such that any set of n points in general position in the plane has r pairwise disjoint $\lfloor c_r n \rfloor$ -element subsets with the property that no matter how we pick a point from each subset, they always form the vertex set of a convex r -gon. János Pach and József Solymosi give an alternative proof of this result, which yields a better bound for c_r , while Pach and Tóth discuss a generalization of the Erdős–Szekeres theorem to convex sets. All of the above-mentioned papers appear in this issue.

“My most striking contribution to geometry”—wrote Erdős in [E4]—“is, no doubt, my problem on the number of distinct distances.” In a paper published in the *American Mathematical Monthly* in 1946, Erdős raised two general questions about the distribution of distances determined by a finite set of points in a metric space [E1]. (1) What is the minimum number of distinct distances determined by a set of n points? (2) At most how many times can the unit distance occur among n points? Because of the many failed attempts to give reasonable bounds for these functions, even in the case of planar sets, one had to realize that these questions are not merely “gems” in recreational mathematics. Erdős strongly believed that for both questions the optimum is attained for lattice-like configurations. In Peter Brass' paper that appears in this special issue, it is shown that the number of point pairs in an n -element set, which determine a unit vector belonging to a fixed set of k directions, is maximized essentially for certain sections of lattices. Peter Fishburn's contribution is devoted to a similar problem.

In 1951–1952 Erdős worked at University College London with Harold Davenport and with C. Ambrose Rogers, and he became interested in the geometry of numbers. Erdős and Rogers [ER] applied probabilistic techniques to obtain thin coverings of Euclidean spaces with spheres. This approach goes back to Minkowski, and proved to be perhaps the most powerful method in the theory of higher-dimensional packings and coverings, and in coding theory. István Talata's paper, which appears in this special issue, addresses a related problem. He proves that there is a constant $c > 0$ such that any d -dimensional convex body K can be simultaneously touched by at least 2^{cd} non-overlapping translates of K . Noga Alon's contribution concentrates on a covering (piercing) problem for d -dimensional boxes, while Miklós Laczkovich's paper gives a classification of tilings of polygons with similar triangles.

Many problems and results of Erdős turned out to be crucially important in computational geometry, especially for the analysis of geometric algorithms. A typical example is the k -set problem [ELSS], which can be formulated as follows. Given n points in the plane in general position, connect two of them by a straight-line segment (by an “edge”) if there are precisely k points on one side of the line determined by them, $0 \leq k \leq n - 2$. What is the maximum number of edges of such a graph? Tamal Dey's paper appearing in this issue represents the most significant development in this area during the past 25 years. He proves the upper bound $O(n(k + 1)^{1/3})$. This result was inspired by another

paper in this issue, written by Pankaj Agarwal, Boris Aronov, Timothy Chan, and Micha Sharir, discussing various generalizations of the k -set problem. Our special issue includes two other beautiful contributions, one by Pavel Valtr and one by Meir Katchalski and Hagit Last, dealing with extremal problems for geometric graphs, i.e., with graphs whose edges are represented by straight-line segments.

Paul Erdős devoted all of his life to mathematics. As a child, he was already a “bit of a prodigy.” He was only 20 years old when Issai Schur called him “the wizard from Budapest,” and for more than 60 years he kept producing brilliant theorems and conjectures with remarkable ease. In the last 10 years of his life he also demonstrated that he was an unflinching fighter in the “honourable contest” for mathematical truth. Hundreds of his friends, colleagues, and admirers continue the “struggle.”

It was an easy task to collect 17 recent papers in discrete and combinatorial geometry directly inspired by Erdős's work. We dedicate this collection to the memory of Paul Erdős, who was an editor of *Discrete & Computational Geometry* and one of the greatest figures of our field.

JÁNOS PACH

References

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