Intersection of Curves and Crossing Number of $C_m \times C_n$ on Surfaces*

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Abstract. Let $(\mathcal{K}_1, \mathcal{K}_2)$ be two families of closed curves on a surface $\mathcal{S}$, such that $|\mathcal{K}_1| = m$, $|\mathcal{K}_2| = n$, $m_0 \leq m \leq n$, each curve in $\mathcal{K}_1$ intersects each curve in $\mathcal{K}_2$, and no point of $\mathcal{S}$ is covered three times. When $\mathcal{S}$ is the plane, the projective plane, or the Klein bottle, we prove that the total number of intersections in $\mathcal{K}_1 \cup \mathcal{K}_2$ is at least $10mn/9$, $12mn/11$, and $mn + 10^{-13}m^2$, respectively. Moreover, when m is close to n , the constants are improved. For instance, the constant for the plane, $10/9$, is improved to $8/5$, for $n \leq 5(m-1)/4$. Consequently, we prove lower bounds on the crossing number of the Cartesian product of two cycles, in the plane, projective plane, and the Klein bottle. All lower bounds are within small multiplicative factors from easily derived upper bounds. No general lower bound has been previously known, even on the plane.

1. Introduction and Summary

Following the ideas of Richter and Thomassen [13], let an (m, n) -mesh be a pair $(\mathcal{K}_1, \mathcal{K}_2)$ of families of closed curves $\mathcal{K}_1, \mathcal{K}_2$, with $|\mathcal{K}_1| = m$, $|\mathcal{K}_2| = n$, on a surface $\mathcal{S}$, such that any curve in $\mathcal{K}_1$ intersects any curve in $\mathcal{K}_2$, and each point of $\mathcal{S}$ is covered by the curves in $\mathcal{K}_1 \cup \mathcal{K}_2$ at most twice. Let $\mathcal{C} \in \mathcal{K}_1 \cup \mathcal{K}_2$ and let x be a point of $\mathcal{S}$. We say that $\mathcal{C}$ has a *self-intersection* at x , if it covers x twice. Let $i_{\mathcal{S}}^*(\mathcal{K}_1, \mathcal{K}_2)$ denote the total

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number of intersections in $\mathcal{K}_1 \cup \mathcal{K}_2$ on $\mathcal{S}$. Note that any two distinct curves $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathcal{K}_1 \cup \mathcal{K}_2$ contribute to $i_{\mathcal{S}}^*(\mathcal{K}_1, \mathcal{K}_2)$ by $|\mathcal{C}_1 \cap \mathcal{C}_2|$, and any curve $\mathcal{C} \in \mathcal{K}_1 \cup \mathcal{K}_2$ contributes to $i_{\mathcal{S}}^*(\mathcal{K}_1, \mathcal{K}_2)$ by the number of points it covers twice. Let $\mathcal{S}_i, \mathcal{N}_i$ denote the orientable and the nonorientable compact surface of genus i , respectively. This paper is concerned with the minimum of $i_{\mathcal{S}}^*(\mathcal{K}_1, \mathcal{K}_2)$ for curves on $\mathcal{N}_1$, the real projective plane, on $\mathcal{N}_2$, the Klein bottle, and on $\mathcal{S}_0$, the sphere. (All the problems that we consider are equivalent for the sphere and the plane.)

Let $i_0^*(m, n), i_1^*(m, n)$, and $i_2^*(m, n)$ denote the minimum $i_{\mathcal{S}}^*(\mathcal{K}_1, \mathcal{K}_2)$ over all (m, n) -meshes $(\mathcal{K}_1, \mathcal{K}_2)$ when $\mathcal{S}$ is $\mathcal{S}_0, \mathcal{N}_1$, or $\mathcal{N}_2$, respectively, and note that $i_j^*(m, n) \geq mn$, $j = 0, 1, 2$. An (m, n) -mesh $(\mathcal{K}_1, \mathcal{K}_2)$ on $\mathcal{S}$ is *optimal* if $i_{\mathcal{S}}^*(\mathcal{K}_1, \mathcal{K}_2)$ is minimal over all (m, n) -meshes $(\mathcal{K}'_1, \mathcal{K}'_2)$.

Richter and Thomassen [13] studied $i_0^*(m, n)$ for specific values of m or n . In particular, they showed

$$i_0^*(3, n) = \begin{cases} 12, & n = 3 \\ 3n + \lceil (n+3)/4 \rceil + \lceil (n+4)/4 \rceil, & n > 4. \end{cases}$$

They also verified that $i_0^*(4, 4) = 24$, $i_0^*(4, 5) = 30$, $i_0^*(5, 5) = 40$, and $i_0^*(4, n) = 5n + o(n)$, for $n \geq 4$. In this paper we derive lower bounds for $i_j^*(m, n)$, for $j = 0, 1, 2$, for general values of m and n . Our lower bounds have the form $mn + \text{second term}$, where our *second terms* are within a small constant multiplicative factor from similar *second terms* appearing in our upper bounds. Our main result is the following.

Theorem 1.1. *For the plane and the projective plane we have:*

(i) *For $5 \leq m \leq n$,*

$$i_0^*(m, n) \geq \begin{cases} 8mn/5 & \text{if } n \leq 5(m-1)/4, \\ 10mn/9 & \text{otherwise.} \end{cases}$$

(ii) *For $4 \leq m \leq n$,*

$$i_1^*(m, n) \geq \begin{cases} 9mn/8 & \text{if } n \leq 4(m-1)/3, \\ 12mn/11 & \text{if } 6 \leq m \text{ and } n > 4(m-1)/3. \end{cases}$$

Moreover, using a different method in Theorem 2.1(iii), we prove that $i_2^*(m, n) \geq mn + cm^2$, for some positive constant c and sufficiently large m and n . We indicate that the method used to prove Theorem 1.1 combines the dual of Dilworth's theorem with induction. The applications of Dilworth's theorem on partially ordered sets to discrete geometry were previously observed by Larman *et al.* [10], and Pach and Töröcsik [12].

One motivation behind studying an (m, n) -mesh is estimating the crossing number of the Cartesian product of 2-cycles. This was first observed by Richter and Thomassen [13]. A *drawing* $\mathcal{D}_G^{\mathcal{S}}$ of a graph G on a surface $\mathcal{S}$ consists of placing the vertices of G on $\mathcal{S}$ and drawing the edges of G using continuous curves of $\mathcal{S}$ between the corresponding vertices, such that no curve has a vertex as an internal point, and no point is an internal point of three curves. Two edges of G *cross* in $\mathcal{D}_G^{\mathcal{S}}$ iff the corresponding curves intersect, where we assume with no loss of generality that incident edges of G do not cross in $\mathcal{D}_G^{\mathcal{S}}$. Let $cr(\mathcal{D}_G^{\mathcal{S}})$ denote the total number of crossings in $\mathcal{D}_G^{\mathcal{S}}$. Let $cr_{\mathcal{S}}(G)$ denote the

crossing number of G on S , that is, the minimum number of edge crossings over all drawings of $\mathcal{D}_G^S$ of G on S . Let $cr_j(G)$, for $j = 0, 1, 2$, denote the crossing number of the graph $G = (V, E)$ on $\mathcal{S}_0$, $\mathcal{N}_1$, and $\mathcal{N}_2$, respectively. Computing $cr_0(G)$ is NP-hard, and there have been only few results concerning the exact value of the crossing numbers, for very special and restricted classes of graphs, see, e.g., [8]. For $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, and let $G_1 \times G_2$ denote the Cartesian product of G_1 and G_2 . Thus $G_1 \times G_2$ is a graph with the vertex set $V_1 \times V_2$ in which two vertices (i, j) and (r, s) are adjacent iff either $i = r$ and $js \in E_2$ or $j = s$ and $ir \in E_1$. Let $C_m \times C_n$ denote the Cartesian product of an m -cycle with an n -cycle. Harary *et al.* [4] provided a simple drawing of $C_m \times C_n$, $m \leq n$ in the plane with $(m-2)n$ crossings, proved $cr_0(C_m \times C_n) \geq m$, and conjectured that $cr_0(C_m \times C_n) = (m-2)n$ for $3 \leq m \leq n$. Proving their conjecture means deriving a lower bound which matches the $(m-2)n$ upper bound. Ringeisen and Beineke [2], [14] proved the conjecture for $m \leq 4$. Richter and Thomassen [13] observed that

$$cr_0(C_m \times C_n) \geq i_0^*(m, n) - mn,$$

and consequently $i_0^*(5, 5) = 40$, which implies that $cr_0(C_5 \times C_5) = 15$ [13]. The most general result regarding $cr_0(C_m \times C_n)$ has been proved very recently by Klešč *et al.* [9]: $cr_0(C_5 \times C_n) = 3n$. We emphasize that for general values of m and n , no nontrivial lower bound on $cr_0(C_m \times C_n)$ has been known previously, probably due to the fact that $C_m \times C_n$ has genus 1 and very much resembles the planar two-dimensional grid. In particular, two very powerful methods developed by the VLSI community [11]—the bisection method and the embedding method—give trivial lower bounds for $cr_0(C_m \times C_n)$; see our survey [18]. Consequently, even an asymptotic value of $cr_0(C_m \times C_n)$ for arbitrary values of m and n has not been established. Concerning $cr_j(C_m \times C_n)$, $j = 1, 2$, the only known result is due to Riskin [15] who proved $cr_1(C_3 \times C_n) = n-1$ for $n \geq 5$, and $cr_1(C_3 \times C_4) = 2$. We note here that $C_m \times C_n$ can be drawn without any crossing on $\mathcal{S}_i$ for $i \geq 1$, and therefore on $\mathcal{N}_i$ for $i \geq 3$, and hence the minimum number of intersections for any (m, n) -mesh $(\mathcal{K}_1, \mathcal{K}_2)$ on these surfaces is exactly mn . Therefore, we only need to study our problems on $\mathcal{S}_0$, $\mathcal{N}_1$, and $\mathcal{N}_2$.

In [17], which is the conference version of this paper we showed that $cr_0(C_m \times C_n) \geq mn/90$ and improved the constant 90 to 6 for $n = m$ or $n = m + 1$. In this paper, we easily show using Theorem 1.1:

Theorem 1.2. *For the crossing number of $C_m \times C_n$ on surfaces, we have:*

(i) For $5 \leq m \leq n$,

$$cr_0(C_m \times C_n) \geq \begin{cases} 3mn/5 & \text{if } n \leq 5(m-1)/4, \\ mn/9 & \text{otherwise.} \end{cases}$$

(ii) For $4 \leq m \leq n$,

$$cr_1(C_m \times C_n) \geq \begin{cases} mn/8 & \text{if } n \leq 4(m-1)/3, \\ mn/11 & \text{if } 6 \leq m \text{ and } n > 4(m-1)/3. \end{cases}$$

(iii) For $544321 \leq m \leq n$,

$$cr_2(C_m \times C_n) \geq m^2/(544321)^2.$$

Simple drawings of $C_m \times C_n$ (a particular (m, n) -mesh) on S_0 with $(m-2)n$ crossings ($mn + (m-2)n$ intersections) provide for upper bounds which are within small multiplicative factors of the lower bounds of parts (i) and (ii) in Theorem 1.2 (Theorem 1.1) for $cr_j(C_m \times C_n) (i_j^*(m, n))$, $j = 0, 1$. Moreover, it is not hard to draw $C_m \times C_n$ with exactly $\binom{m}{2}$ crossings and thus to draw an (m, n) -mesh with $mn + \binom{m}{2}$ intersections, on $\mathcal{N}_2$. Consequently, our lower bound in Theorem 1.2(iii) is within a multiplicative constant of the upper bound, and the lower bound in Theorem 2.1(iii), differs from the optimal value by a multiplicative constant in the second-order term only.

2. Lower Bounds

We refer the reader to [3] and [19] for topological concepts. Adding a *crosscap* to the sphere means that a circular hole is cut out, and the opposite pairs of points on the circular hole are identified. The sphere with a crosscap comes with the quotient topology. The surface $\mathcal{N}_i$, $i \geq 1$, is obtained from the sphere by adding i crosscaps. Any simple closed curve on $\mathcal{N}_1$ which passes through the crosscap an even(odd) number of times is called an *even(odd) curve*. We start with a lemma which summarizes some important facts.

Lemma 2.1.

- (i) *An even curve $\mathcal{Z}$ on $\mathcal{N}_1$ divides $\mathcal{N}_1$ into two components, one contractible and homeomorphic to a disc, and one noncontractible.*
- (ii) *Any two odd curves in $\mathcal{N}_1$ must intersect at least one point.*
- (iii) *Any optimal (m, n) -mesh $(\mathcal{K}_1, \mathcal{K}_2)$ on $\mathcal{S}$ consists of simple curves.*

Proof. For topologists, (i) and (ii) are obvious facts. A helpful reference is [3]. Here, we will sketch the proofs. Identifying the crosscap with the infinite line in the projective geometry, recall that in $\mathcal{N}_1$ only odd curves are homotopic to a projective line and that even curves are contractible to a disc. Thus odd curves do not cut $\mathcal{N}_1$ into pieces, but even curve do. For (i), the other component cannot be contractible, since then $\mathcal{N}_1$ would be homeomorphic to a sphere. For the proof of (ii), one would argue that the intersection form over $\mathbb{Z}_2$ is nontrivial. (See pp. 30–33 of [3].) For (iii), eliminate self-crossings one after the other. $\square$

Remark. Throughout this paper when we derive lower bounds for a number of intersections of an (m, n) -mesh on a surface, we will assume without loss of generality that we deal with an optimal mesh. Then by Lemma 2.1(iii) all curves must be simple, and consequently any curve is either even or odd. We can then use parts (ii) and (i) in Lemma 2.1. We will make use of these facts, often without an explicit reference to Lemma 2.1.

For an even curve $\mathcal{Z}$ on $\mathcal{N}_1$, define its *body* $B(\mathcal{Z})$ as the contractible component of $\mathcal{N}_1 \setminus \mathcal{Z}$. We define the exterior of $\mathcal{Z}$, denoted by $X(\mathcal{Z})$, as the complement of $B(\mathcal{Z})$. Let

$\mathcal{Q}$ be a set of even curves in $\mathcal{N}_1$. (Thus all curves in $\mathcal{Q}$ pass through the crosscap an even number of times.) For $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathcal{Q}$, $\mathcal{Z}_1 < \mathcal{Z}_2$, mean that $\mathcal{Z}_1$ lies in $B(\mathcal{Z}_2)$. Let $\mathcal{A} \subseteq \mathcal{Q}$, $\mathcal{A} = \{\mathcal{Z}_i | i = 1, 2, \dots, t\}$. $\mathcal{A}$ is called a *chain*, if $\mathcal{Z}_1 < \mathcal{Z}_2 < \dots < \mathcal{Z}_t$. $\mathcal{A}$ is called an *antichain*, if neither $\mathcal{Z}_i < \mathcal{Z}_j$, nor $\mathcal{Z}_j < \mathcal{Z}_i$ for any $i, j = 1, 2, \dots, t, i \neq j$. It is easily seen that $<$ is a partial order relation on $\mathcal{Q}$ and thus by the dual of Dilworth's theorem [16], the minimum number of antichains that $\mathcal{Q}$ can be decomposed into equals the cardinality of the largest chain.

Given an (m, n) -mesh $(\mathcal{K}_1, \mathcal{K}_2)$ on $\mathcal{S}$, we construct a graph G which we call a *crossing graph* of $(\mathcal{K}_1, \mathcal{K}_2)$, as follows. For any $\mathcal{C}_1 \in \mathcal{K}_1$ and $\mathcal{C}_2 \in \mathcal{K}_2$, pick exactly one intersection of $\mathcal{C}_1$ and $\mathcal{C}_2$ and view it as a vertex of G . The set of edges of G are determined by the portions of curves of $(\mathcal{K}_1, \mathcal{K}_2)$ which are trapped between the selected intersections. Note that our original $(\mathcal{K}_1, \mathcal{K}_2)$ provides a drawing $\mathcal{D}_G^{\mathcal{S}}$ of G on the corresponding surface. Moreover, any crossing graph G of an (m, n) -mesh $(\mathcal{K}_1, \mathcal{K}_2)$ with $m \geq 4$ on $\mathcal{S}$ is simple, has mn vertices, $2mn$ edges, and satisfies $\text{girth}(G) \geq 4$ and $i_{\mathcal{S}}^*(\mathcal{K}_1, \mathcal{K}_2) = mn + cr(\mathcal{D}_G^{\mathcal{S}})$.

Lemma 2.2 [6], [7]. *Let $H = (V, E)$ be any simple graph, then for $j = 0, 1, 2$,*

$$cr_j(H) \geq |E| - \frac{\text{girth}(H)}{\text{girth}(H) - 2}(|V| + j - 2).$$

Our goal is to obtain a good lower bound for $i_j^*(m, n)$, $j = 0, 1, 2$, and first we obtain it for n which is not much larger than m .

Theorem 2.1. *For (m, n) -meshes on a surface $\mathcal{S}$:*

(i) *For $5 \leq m \leq n$,*

$$i_0^*(m, n) \geq \begin{cases} 8mn/5, & \text{if } n \leq 5(m-1)/4, \\ 2m(m-1) & \text{otherwise.} \end{cases}$$

(ii) *For $4 \leq m \leq n$,*

$$i_1^*(m, n) \geq \begin{cases} 9mn/8, & \text{if } n \leq 4(m-1)/3, \\ 2m(m-1) & \text{otherwise.} \end{cases}$$

(iii) *For $3 \cdot 9!/2 + 1 \leq m \leq n$*

$$i_2^*(m, n) \geq mn + m^2/(544321)^2.$$

Proof. For (i), consider an optimal (m, n) -mesh $(\mathcal{K}_1, \mathcal{K}_2)$. Hence all curves are simple by Lemma 2.1(iii). Note that we may select any five distinct curves in $\mathcal{K}_1$ and any five distinct curves $\mathcal{K}_2$ to form a $(5, 5)$ -mesh. Since it is known that $i^*(5, 5) = 40$ [13], it follows that each of the $\binom{m}{5}\binom{n}{5}$ possible $(5, 5)$ -meshes contains at least 40 intersections.

No intersection in the original (m, n) -mesh is contained by more than $A_{mn}(5, 5)$ -meshes where

$$A_{mn} = \max \left\{ \binom{m-1}{4} \binom{n-1}{4}, \binom{m}{5} \binom{n-2}{3}, \binom{m-2}{3} \binom{n}{5} \right\} \\ = \begin{cases} \binom{m-1}{4} \binom{n-1}{4} & \text{if } n \leq 5/4(m-1), \\ \binom{m-2}{3} \binom{n}{5} & \text{otherwise,} \end{cases}$$

which finishes the proof of (i).

To prove (ii), we essentially follow the same argument as for (i), but concentrate on $(4, 4)$ -meshes. We can select any four distinct curves in $\mathcal{K}_1$ and any four distinct curves in $\mathcal{K}_2$ to form a $(4, 4)$ -mesh. Note that Lemma 2.2 applied to any crossing graph G of this $(4, 4)$ -mesh gives $cr_1(G) \geq 2$. It follows that any one of the $\binom{m}{4} \binom{n}{4}$ possible $(4, 4)$ -meshes contains at least $16 + 2 = 18$ intersections. Now follow the same counting argument as in (i).

To prove (iii), set $m_0 = 10$, $n_0 = 3 \cdot 9!/2 + 1 = 544321$, and consider an (m_0, n_0) -mesh $(\mathcal{K}_1, \mathcal{K}_2)$ on $\mathcal{N}_2$. First we prove that $i_2^*(m_0, n_0) \geq m_0 n_0 + 1$. Assume for contradiction that $i_2^*(m_0, n_0) = m_0 n_0$. Any curve $\mathcal{C} \in \mathcal{K}_2$ defines a cyclic order on $\mathcal{K}_1$ by the cyclic order of the intersections made by the elements of $\mathcal{K}_1$ with $\mathcal{C}$. There are $(m_0 - 1)!/2 = 9!/2$ cyclic orders of $\mathcal{K}_1$; hence by the pigeonhole principle, some four curves from $\mathcal{K}_2$ define the same cyclic order. Now consider those four curves of $\mathcal{K}_2$, then any $\mathcal{C}' \in \mathcal{K}_1$ defines a cyclic order on them. There are three possible cyclic orders, and hence by the same pigeonhole principle some four curves from $\mathcal{K}_1$ define the same cyclic order. Now we have four curves in $\mathcal{K}_1$ and four curves in $\mathcal{K}_2$, so that their crossing graph is isomorphic to $C_4 \times C_4$. Further, observe that the nonorientable genus of $C_4 \times C_4$ is three [5]. Therefore, $cr_2(C_4 \times C_4) \geq 1$, which implies $i_2^*(\mathcal{K}_1, \mathcal{K}_2) \geq m_0 n_0 + 1$, a contradiction.

Hence, $i_2^*(n_0, n_0) \geq n_0^2 + 1$. Using this lower bound and applying the same counting argument as in (i) and (ii), we get for $m \geq n_0$:

$$i_2^*(m, m) \geq m^2(1 + 1/n_0^2)$$

and, finally, for $n \geq m \geq n_0$

$$i_2^*(m, n) \geq m(n - m) + i_2^*(m, m) \geq mn + m^2/n_0^2. \quad \square$$

It is not difficult to provide an (m, n) -mesh on $\mathcal{N}_2$ for any $n \geq m$ with precisely $mn + \binom{m}{2}$ intersections. Thus our lower bound in part (iii) of Theorem 2.1 differs from the optimal value only in the second-order term. In view of this observation, our next goal will be to improve our lower bounds for general values of m and n in parts (i) and (ii) of Theorem 2.1. We need some additional concepts. For $A \subset \mathcal{N}_1$, let $cl(A)$ denote the closure of A for the standard topology.

Lemma 2.3. *Let $(\mathcal{K}_1, \mathcal{K}_2)$ be an optimal (m, n) -mesh in $\mathcal{N}_1$.*

- (i) *Let $\mathcal{C}_1, \mathcal{C}_2$ be even curves in $\mathcal{K}_2$ such that $\mathcal{C}_1 < \mathcal{C}_2$, and $\mathcal{F} \in \mathcal{K}_1$. If $\mathcal{F} \not\subset cl(B(\mathcal{C}_2))$, then $|\mathcal{F} \cap \mathcal{C}_2| \geq 2$.*

- (ii) Let N_C denote the number of intersections of $\mathcal{C} \in \mathcal{K}_2$ with curves of $\mathcal{K}_1$. Then $i_1^*(\mathcal{K}_1, \mathcal{K}_2) \geq N_C + i_1^*(m, n - 1)$.
- (iii) Let $\mathcal{C}_1, \mathcal{C}_2$ be even curves in $\mathcal{K}_2$ so that neither $\mathcal{C}_1 < \mathcal{C}_2$ nor $\mathcal{C}_2 < \mathcal{C}_1$. Let $\mathcal{F} \in \mathcal{K}_1$ be such that $\mathcal{F} \subset cl(B(\mathcal{C}_1))$. Then $|\mathcal{C}_1 \cap \mathcal{C}_2| \geq 2$.

Proof. Recall that our curves are simple. For (i), note that $\mathcal{C}_2$ separates $\mathcal{N}_1$ into two components. $\mathcal{F}$ has a point in the contractible component, since $\mathcal{F}$ has to intersect $\mathcal{C}_1$. $\mathcal{F}$ has a point in the other component, since $\mathcal{F} \not\subset cl(B(\mathcal{C}_2))$. Therefore $\mathcal{F}$ intersects $\mathcal{C}_2$. Eliminate the intersection point with a small neighborhood from $\mathcal{F}$. The remainder of $\mathcal{F}$ is still connected, has points in both components, and therefore still intersects $\mathcal{C}_2$.

For (ii), just use the definitions. For (iii), note that $\mathcal{C}_2$ intersects $\mathcal{F}$, but cannot intersect it in $\mathcal{F} \cap \mathcal{C}_1$, since we do not have points triple covered. Hence $\mathcal{C}_2$ has points in $B(\mathcal{C}_1)$. If $\mathcal{C}_2 \subset cl(B(\mathcal{C}_1)) = \mathcal{C}_1 \cup B(\mathcal{C}_1)$, then $\mathcal{C}_2 < \mathcal{C}_1$, unless $|\mathcal{C}_1 \cap \mathcal{C}_2| \geq 1$. However, if in this case we have $|\mathcal{C}_1 \cap \mathcal{C}_2| \geq 1$, then the intersections are easily eliminated while keeping the mesh property, and we have a contradiction with the optimality of the mesh. Therefore $\mathcal{C}_2 \not\subset cl(B(\mathcal{C}_1))$. We are in a situation as in part (i), but $\mathcal{C}_2$ plays the role that $\mathcal{F}$ played there. $\square$

The following important result unifies the solution to our problems for $\mathcal{S}_0$, and $\mathcal{N}_1$ and is crucial in deriving our main result.

Theorem 2.2. *Let $(\mathcal{K}_1, \mathcal{K}_2)$ be an optimal (m, n) -mesh on $\mathcal{N}_1$ among those (m, n) -meshes, in which all curves in $\mathcal{K}_2$ are even. Then for $6 \leq m \leq n$,*

$$i_1^*(\mathcal{K}_1, \mathcal{K}_2) \geq 10mn/9.$$

Proof. We proceed by induction on n . The base of induction is provided by Theorem 2.1, for $n \leq 4(m - 1)/3$. Assume that $n > 4(m - 1)/3$. If k is the size of a largest chain in $\mathcal{K}_2$, then, using the dual of Dilworth's theorem, one can decompose $\mathcal{K}_2$ into k antichains $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_k$, of sizes $a_1, a_2, \dots, a_k$, so that $\sum_{i=1}^k a_i = n$.

Note that if $k \geq 3$, then we must have a chain $\mathcal{C}_1 < \mathcal{C}_2 < \mathcal{C}_3$ in $\mathcal{K}_2$. Since any $\mathcal{F} \in \mathcal{K}_1$ intersects $\mathcal{C}_3$, we must have $\mathcal{F} \not\subset cl(B(\mathcal{C}_2))$, and thus by Lemma 2.3(i) $N_{\mathcal{C}_2} \geq 2m$, and the claim follows by induction, using Lemma 2.3(ii).

So we can assume $k \leq 2$ and thus $\mathcal{K}_2$ can be decomposed into at most two antichains $\mathcal{Q}_1$ and $\mathcal{Q}_2$ of sizes a_1 and a_2 , $a_1 \geq a_2$. Note that $\mathcal{Q}_i$ contains exactly a_i curves $i = 1, 2$, such that any two curves in $\mathcal{Q}_i$ either intersect in at least two points, or are in the exterior of each other. For any $\mathcal{C} \in \mathcal{K}_2$, let $a(\mathcal{C})$ denote the number of curves $\mathcal{C}' \in \mathcal{K}_1$ so that $\mathcal{C}' \subseteq cl(B(\mathcal{C}))$.

Claim 1. *Either there are at least $7n/9$ curves $\mathcal{C} \in \mathcal{K}_2$ with $a(\mathcal{C}) = 0$, or $i_1^*(\mathcal{K}_1, \mathcal{K}_2) \geq 10mn/9$.*

Proof of Claim 1. Note that since $a_1 \geq a_2$, we must have $a_1 \geq n/2$, and thus, if $a(\mathcal{C}) > 0$ for some $\mathcal{C} \in \mathcal{Q}_1$, then by Lemma 2.3(iii), for any $\mathcal{C}' \in \mathcal{Q}_1$ it holds $|\mathcal{C}' \cap \mathcal{C}| \geq 2$, and we must have $N_{\mathcal{C}} \geq n - 2 + m$. Then by Lemma 2.3(ii) and induction we will be done. Thus we can assume that $a(\mathcal{C}) = 0$ for all $\mathcal{C} \in \mathcal{Q}_1$. Now if $a_2 \leq 2n/9$, we are done,

since $a_1 \geq n - a_2 \geq 7n/9$. Thus let us assume that $a_2 > 2n/9$ and that $a(\mathcal{C}) = 0$ for all $\mathcal{C} \in \mathcal{Q}_1$. Note that in this case, if $a(\mathcal{C}) > 0$ for any $\mathcal{C} \in \mathcal{Q}_2$, then by Lemma 2.3(iii), $N_{\mathcal{C}} \geq 2(2n/9 - 1) + m$ and thus using Lemma 2.3(ii) and induction we are done. So we may assume that $a(\mathcal{C}) = 0$ for all $\mathcal{C} \in \mathcal{Q}_2$, which implies $a(\mathcal{C}) = 0$ for $\mathcal{C} \in \mathcal{Q}$ and we are done. $\square$

With no loss of generality, we can assume that for any $\mathcal{C} \in \mathcal{Q}_1 \cup \mathcal{Q}_2$, we have $N_{\mathcal{C}} \leq 10m/9$, since otherwise, using Lemma 2.3 and induction we obtain the result. Thus in view of Claim 1 to finish the proof, we will show the following.

Claim 2. *Assume that for any $\mathcal{C} \in \mathcal{K}_2$, $N_{\mathcal{C}} \leq 10m/9$. Moreover, assume that $a(\mathcal{C}) = 0$ for at least $7n/9$ curves $\mathcal{C} \in \mathcal{K}_2$. Then $i_1^*(\mathcal{K}_1, \mathcal{K}_2) \geq 10mn/9$.*

Proof of Claim 2. Consider a crossing graph G of $(\mathcal{K}_1, \mathcal{K}_2)$. Let $\mathcal{C} \in \mathcal{K}_1$ ($\mathcal{C} \in \mathcal{K}_2$). Then there is a cycle in the crossing graph G on n vertices (m vertices) associated with $\mathcal{C}$ which we call a c-cycle (an r-cycle) and denote it by C . (Thus we distinguish between a curve and the cycle associated with it in the crossing graph by using different fonts.)

The idea behind the proof is the following. To each $\mathcal{C} \in \mathcal{K}_2$ with $a(\mathcal{C}) = 0$ we add some new edges as chords to the corresponding cycle C in the crossing graph G to obtain a supergraph G' of G . We then obtain $\mathcal{D}_{G'}^{\mathcal{N}_1}$, a drawing of G' , using $\mathcal{D}_G^{\mathcal{N}_1}$, and show that $cr(\mathcal{D}_{G'}^{\mathcal{N}_1}) = cr(\mathcal{D}_G^{\mathcal{N}_1})$ (recall that $cr(\mathcal{D}_G^{\mathcal{N}_1})$ and $cr(\mathcal{D}_{G'}^{\mathcal{N}_1})$ denote the number of crossings in the drawings $\mathcal{D}_G^{\mathcal{N}_1}$ and $\mathcal{D}_{G'}^{\mathcal{N}_1}$, respectively) and that $cr_1(G') \geq mn/9$.

Let $\mathcal{C} \in \mathcal{K}_2$ so that $a(\mathcal{C}) = 0$. Then, for any $\mathcal{F} \in \mathcal{K}_1$ either we must have $\mathcal{F} \subset X(\mathcal{C})$, in which case the optimality of $(\mathcal{K}_1, \mathcal{K}_2)$ implies that $\mathcal{F}$ and $\mathcal{C}$ must have only one point in common, or otherwise $|\mathcal{F} \cap \mathcal{C}| \geq 2$. Let U denote the set of all c-cycles F in G so that $|\mathcal{F} \cap \mathcal{C}| \geq 2$. Since $N_{\mathcal{C}} \leq \lfloor 10m/9 \rfloor$, and $\mathcal{C}$ is simple, there are at most $N_{\mathcal{C}} - m$ edges in C which are crossed in $\mathcal{D}_G^{\mathcal{N}_1}$, and thus we must have $|U| \leq N_{\mathcal{C}} - m \leq \lfloor m/9 \rfloor$.

We first describe the construction of G' for $m > 9$. By deleting all edges of C that are crossed in $\mathcal{D}_G^{\mathcal{N}_1}$ we divide C to at most $\lfloor m/9 \rfloor$ vertex-disjoint paths. On these paths, there are at most $\lfloor m/9 \rfloor$ vertices of G which are common to c-cycles in U and C , since $|U| \leq \lfloor m/9 \rfloor$. These vertices divide the $\lfloor m/9 \rfloor$ vertex-disjoint paths into at most $r \leq 2\lfloor m/9 \rfloor$ edge-disjoint paths p_i of length (number of edges) d_i , $1 \leq i \leq r$, where $\sum_{i=1}^r d_i \geq m - \lfloor m/9 \rfloor$. Let $x_1^i, x_2^i, \dots, x_{d_i+1}^i$ be the vertices of the path p_i , $1 \leq i \leq r$. To obtain G' , we add at least $\lfloor d_i/3 \rfloor$ new edges to p_i , $1 \leq i \leq r$, by joining the vertex x_{3j+1}^i to the vertex x_{3j+4}^i for $j = 0, 1, \dots, \lfloor d_i/3 \rfloor - 1$. We then draw each new edge inside of $B(\mathcal{C})$ using a curve very close to and parallel to the curve associated with p_i in $\mathcal{D}_G^{\mathcal{N}_1}$, $1 \leq i \leq r$. It is easy to see that the new edges can be drawn so that they do not cross each other. Moreover, our construction implies that the new edges do not cross the edges of any c-cycle $F \in U$. Finally, note that the new edges do not cross the edges of any c-cycle $F \notin U$ (i.e., $\mathcal{F} \subset X(\mathcal{C})$), or any edge of an r-cycle in $\mathcal{D}_G^{\mathcal{N}_1}$. $\square$

Figure 1 shows details of our drawing with one r-cycle (solid line) and two c-cycles (dashed lines). The new edges are drawn by dotted lines.

Fig. 1

Thus for $m > 9$ the number of new edges added to C is

$$\sum_{i=1}^r \left\lfloor \frac{d_i}{3} \right\rfloor \geq \sum_{i=1}^r \frac{d_i - 2}{3} \geq \frac{1}{3} \left(m - \left\lfloor \frac{m}{9} - 2r \right\rfloor \right) \geq \frac{4m}{27},$$

where the last inequality is obtained, since $r \leq 2\lfloor m/9 \rfloor$. To describe the construction for $m \leq 9$, note that if $6 \leq m \leq 8$, no edge of any r -cycle C is crossed in $\mathcal{D}_G^{\mathcal{N}_1}$, since at most $N_C - m = m/9$ edges in C are crossed in $\mathcal{D}_G^{\mathcal{N}_1}$, and if $m = 9$, then $N_C - m \leq 1$. When $m = 6, 7$ we join only one vertex pair at distance 3 by adding one new edge, and when $m = 8, 9$, we join two vertex pairs at distance 3 by adding exactly two new edges to each of the $7n/8$ r -cycles C . Consequently, for $6 \leq m \leq 9$ we have also added at least $4m/27$ new edges to any of the $7n/8$ r -cycles. (We remark that an important property of our construction is not to create any triangles. We also remark that the fact that $a(C) = 0$ is crucial to obtaining the lower bound, for otherwise the nontrivial upper bound on r cannot be obtained, and the lower bound on the number of added edges will tend to 0.) By repeating this process for each r -cycle C with $a(C) = 0$ we will obtain the drawing $\mathcal{D}_{G'}^{\mathcal{N}_1}$ of a graph G' with mn vertices, at least $2mn + (4m/27)(7n/9) = 2mn + 28mn/243$ edges. Due to our construction $cr(\mathcal{D}_{G'}^{\mathcal{N}_1}) = cr(\mathcal{D}_G^{\mathcal{N}_1})$. Moreover, it is easy to see that $\text{girth}(G') \geq 4$, and thus by Kainen and White's lower bound [7] in Lemma 2.2: $cr_1(G') \geq mn/9$. Thus we immediately obtain $i_1^*(\mathcal{K}_1, \mathcal{K}_2) \geq cr(\mathcal{D}_G^{\mathcal{N}_1}) + mn \geq 10mn/9$. $\square$

Proof of Theorem 1.1. For $n \leq 5(m-1)/4$, (i) is proved by Theorem 2.1(i). For $5 \leq m \leq 6$, and arbitrary $n \geq m$, the result of Richter and Thomassen [13]: $i_0^*(4, n) \geq 5n$ implies $i_0^*(m, n) \geq (m-4)n + i_0^*(4, n) \geq mn + n \geq 10mn/9$ if $m \leq 6$. For $n > 5(m-1)/4$ and $m \geq 6$, (i) is a direct consequence of Theorem 2.2.

For $n \leq 4(m-1)/3$, (ii) follows from Theorem 2.1(ii). Also observe that for $4(m-1)/3 < n \leq 11(m-1)/6$, (ii) is proved in Theorem 2.1(ii), case "otherwise." Hence, assume $n > 11(m-1)/6$ and consider an optimal (m, n) -mesh on $\mathcal{N}_1$ and let x denote the number of odd curves in $\mathcal{K}_2$.

Assume $n - x \geq 6$. Then applying Theorem 2.2 and Lemma 2.1(ii), we get

$$i_1^*(m, n) \geq \frac{10}{9}m(n - x) + mx + \binom{x}{2}.$$

If $x = 1$, then $i_1^*(m, n) \geq 10m(n - 1)/9 + m \geq 12mn/11$. If $x > 1$, then a straightforward computation shows that the discriminant of the quadratic polynomial $10m(n - x)/9 + mx + \binom{x}{2} - 12mn/11$ in the argument x is nonpositive, and thus the claim follows.

So we may assume that $n - x < 6$. Then

$$i_1^*(m, n) \geq mn + \binom{x}{2} \geq mn + \binom{n - 5}{2} \geq \frac{12mn}{11},$$

for $n \geq 11(m - 1)/6$, which finishes the proof.

A simple consequence of Theorems 1.1 and 2.1 is Theorem 1.2.

Proof of Theorem 1.2. The results (i) and (ii) follow immediately from Theorem 1.1, and (iii) from Theorem 2.1. $\square$

3. Conclusions

Recently Anderson *et al.* [1] have proved that $i_0^*(6, 6) = 60$. This result allows us to improve the constant $3/5$ to $2/3$ in Theorem 1.2(i) and the constant $8/5$ to $5/3$ in Theorem 1.1(i). Our general results in Theorems 1.1(ii), 1.2(ii), and 2.1(iii) do not cover specific cases for $m = 3, 4, 5$, when $n > m$, and small m, n , respectively. It may be possible to find good lower bound for these cases using different (*ad hoc*) methods. We leave these questions to the curious reader.

Consider G , the Cartesian product of $k \geq 3$ cycles drawn in the plane. If the cycles are of sizes $n_1, n_2, \dots, n_k$, $n_1 \leq n_2 \leq \dots \leq n_k$, then a careful application of a standard VLSI embedding method gives the lower bound of $4/5(\prod_{i=1}^{k-1} n_i)^2 - 2(k^2 + 2) \prod_{i=1}^k n_i$ on $cr_0(G)$. (A similar lower bound with much smaller constant than $4/5$ can be obtained using the bisection method.) Moreover, recursive drawings with $16n_k n_{k-1} (\prod_{i=1}^{k-2} n_i)^2 + 8k^2 \prod_{i=1}^k n_i$ paper crossings can be provided. The details are not difficult and can be found in our conference paper [17]. When $k \geq 3$ and $n_1 = n_2 = n_3 = \dots = n_k$, the embedding or bisection methods give lower bounds which are within a constant multiplicative factor of the upper bound. As a final remark we emphasize that when $k = 2$, both the bisection and the embedding method fail and give the trivial lower bound of 0 for $cr_0(C_m \times C_n)$.

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