Containment and Circumscribing Simplices*

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Abstract. The following containment theorem is presented: If K and L are convex bodies such that every simplex that contains L also contains some translate of K , then in fact the body L must contain a translate of the body K . One immediate consequence of this theorem is a strengthened version of Weil's mixed-volume characterization of containment.

The aim of this note is to prove, and present some simple consequences of, the following:

Theorem. Suppose K and L are convex bodies. If every simplex that contains L contains a translate of K , then the body L contains a translate of K .

It is helpful to begin by standardizing terminology and notation. It is the author's hope that the reader will find this article to be both self-contained and elementary. Some simple mixed volumes are used, but all the necessary definitions are given, and all the required properties of mixed volumes are established. For general reference, the reader may wish to consult the standard reference by Schneider on the subject of convex bodies and mixed volumes [6].

A *convex body* is a compact convex subset of Euclidean n -space, $\mathbb{R}^n$, whose interior is nonempty. A *closed half-space* in $\mathbb{R}^n$ is a set of the form $\{x \in \mathbb{R}^n: u \cdot x \leq c\}$, where u is a unit vector, $c \in \mathbb{R}$, and $u \cdot x$ denotes the usual inner product of x and u in $\mathbb{R}^n$. A convex body that is the intersection of $n + 1$ closed half-spaces is called a *simplex*. Thus a simplex has $n + 1$ *faces* and associated with each simplex is the set of $n + 1$ outer unit normals to these faces.

A simplex is said to be *circumscribed* about a convex body if it contains the body and each face of the simplex meets the body. If Δ is a simplex and Q is a convex body, then $\Delta(Q)$ denotes the simplex circumscribed about Q whose faces have the same set

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of outer normals as those of Δ . The simplex $\Delta(Q)$ is called the simplex circumscribed about Q corresponding to Δ .

For a convex body Q , an $x_0 \in \mathbb{R}^n$, a real $c > 0$, and a linear transformation $\varphi \in GL(n)$, the convex body $x_0 + c\varphi Q$ is defined by

$$x_0 + c\varphi Q = \{x_0 + c\varphi x : x \in Q\}.$$

The *support function*, $h(Q, \cdot) : S^{n-1} \rightarrow \mathbb{R}$, of the convex body Q is defined by

$$h(Q, u) = \max\{u \cdot x : x \in Q\}.$$

The Hausdorff distance, $\delta(K, L)$, between the convex bodies K and L can be defined by

$$\delta(K, L) = \max\{h(K, u) - h(L, u) : u \in S^{n-1}\}.$$

The unit ball centered at the origin is denoted by B . The *inradius*, $r(K)$, of a convex body K is the largest $r > 0$ for which there exists an $x \in \mathbb{R}^n$ such that $x + rB \subset K$.

A *polytope* is a convex body that is the intersection of a finite number of closed half-spaces. If P is a polytope with faces that have outer normals $u_0, u_1, \dots, u_s \in S^{n-1}$ ($s \geq n$) and areas $a_0, a_1, \dots, a_s > 0$, then

$$\sum_{i=0}^s a_i u_i = 0. \quad (1)$$

While (1) is easily seen to hold for arbitrary polytopes, it will be used only for the special case where P is a simplex. The *mixed volume* $V_1(P, Q)$, of the polytope P and an arbitrary convex body Q , can be defined by

$$V_1(P, Q) = \frac{1}{n} \sum_{i=0}^s a_i h(Q, u_i).$$

If K is an arbitrary convex body, then the mixed volume $V_1(K, Q)$ can be defined by approximating K with polytopes. The only mixed volumes that are used in this article are those in which a polytope, in fact a simplex, is the first argument.

A fact regarding mixed volumes that will be needed is that, for a simplex Δ ,

$$V_1(\Delta, \Delta) = V(\Delta), \quad (2)$$

where $V(\Delta)$ denotes the n -dimensional volume of Δ . It is easily seen that this is the case for an arbitrary polytope (rather than just a simplex), however, only the case of the simplex will be used.

Proposition. Suppose Δ is a simplex, $c > 0$, and K is a convex body. If a translate of $c\Delta$ circumscribes K , then

$$c = \frac{r(\Delta(K))}{r(\Delta)} = \frac{V_1(\Delta, K)}{V(\Delta)}.$$

In particular,

$$\Delta(K) = \frac{V_1(\Delta, K)}{V(\Delta)} \Delta,$$

up to translation.

Proof. Let $u_0, u_1, \dots, u_n \in S^{n-1}$ denote the outer normals, and let $a_0, a_1, \dots, a_n > 0$ denote the corresponding areas, of the faces of Δ .

Let $H_i = \{x \in \mathbb{R}^n: x \cdot u_i \leq h(K, u_i)\}$ and let

$$\Delta' = \bigcap_{i=0}^n H_i.$$

Clearly, Δ' is a simplex circumscribed about K with the same set of outer unit normals as Δ . Recall that B is the unit ball centered at the origin. Suppose $x, x' \in \mathbb{R}^n$ and $r, r' > 0$ are such that $-x + rB \subset \Delta$ is the largest ball contained in Δ and $-x' + r'B \subset \Delta'$ is the largest ball contained in Δ' . Then B is the largest ball contained in the simplices $(1/r)x + (1/r)\Delta$ and $(1/r')x' + (1/r')\Delta'$. Thus $(1/r)x + (1/r)\Delta$ and $(1/r')x' + (1/r')\Delta'$ are simplices with the same set of outer unit normals circumscribed about the unit ball centered at the origin. Hence $\Delta' = y + c\Delta$ where $y = (r'/r)x - x'$ and $c = (r'/r)$.

Now if $y + c\Delta$ is circumscribed about K , then

$$h(K, u_i) = h(y + c\Delta, u_i) = y \cdot u_i + ch(\Delta, u_i),$$

for all i . From the definition of the mixed volume V_1 and (1), we have

$$\begin{aligned} nV_1(\Delta, K) &= \sum_{i=0}^n a_i h(K, u_i) \\ &= \sum_{i=0}^n a_i [y \cdot u_i + ch(\Delta, u_i)] \\ &= y \cdot \sum_{i=0}^n a_i u_i + c \sum_{i=0}^n a_i h(\Delta, u_i) \\ &= ncV_1(\Delta, \Delta) \\ &= ncV(\Delta). \end{aligned}$$

This completes the proof of the proposition. $\square$

Suppose Δ is a simplex, K is a convex body, and $\varphi \in SL(n)$. The proposition shows that a translate of $[V_1(\Delta, K)/V(\Delta)] \Delta$ circumscribes K . This implies that a translate of $[V_1(\Delta, K)/V(\Delta)] \varphi\Delta$ circumscribes φK . However, by the proposition, a translate of $[V_1(\varphi\Delta, \varphi K)/V(\varphi\Delta)] \varphi\Delta$ circumscribes φK . The uniqueness part of the proposition, and the fact that $V(\varphi\Delta) = V(\Delta)$, now yields

$$V_1(\varphi\Delta, \varphi K) = V_1(\Delta, K). \quad (3)$$

The invariance of mixed volumes under simultaneous $SL(n)$ transformations is well known to hold for arbitrary convex bodies. However, (3) is sufficient for the purposes of this article.

Proof of the Theorem. First, note that it is only necessary to prove the theorem for a set of bodies L that is dense in the space of all convex bodies. To see this assume that

the theorem has been established for all $L \in \mathcal{D}$, where $\mathcal{D}$ is a dense subset of the space of convex bodies. Given an arbitrary convex body L , choose $L_i \in \mathcal{D}$ such that

$$L \subset L_i \subset L + \frac{1}{i}B = \left\{ x + \frac{1}{i}y : x \in L \text{ and } y \in B \right\}.$$

If the hypothesis (of the theorem) holds for L , then *a fortiori* it holds for each L_i . Hence, for each i , there exists an $x_i \in \mathbb{R}^n$ such that

$$x_i + K \subset L_i \subset L + \frac{1}{i}B.$$

If x_o is an accumulation point of the x_i , then it is easily seen that

$$x_o + K \subset L.$$

The dense subset $\mathcal{D}$ will be the set of those polytopes, in $\mathbb{R}^n$, no n of whose faces have outer normals that are linearly dependent. Note that if a polytope does not satisfy these conditions, then, by a finite number of small perturbations of its vertices, it can be approximated by polytopes that do meet the conditions.

The theorem is proven by using a standard argument involving Helly's theorem. Recall that Helly's theorem (see, e.g., p. 4 of [6]) states that if $H_1, \dots, H_s$ ($s > n$) are convex sets in $\mathbb{R}^n$ with the property that the intersection of any $n + 1$ of these sets is nonempty, then the intersection of all s sets is nonempty.

Now suppose $L \in \mathcal{D}$ is a polytope whose faces have outer unit normals $u_1, \dots, u_s \in S^{n-1}$. For each i , let

$$H_i = \{x \in \mathbb{R}^n : x \cdot u_i \leq h(L, u_i)\}$$

and

$$K_i = \{x \in \mathbb{R}^n : x + K \subset H_i\}.$$

Obviously, the K_i are convex and closed.

Suppose $K_{i_0}, \dots, K_{i_n}$ are $n + 1$ members of the collection $\{K_1, \dots, K_s\}$. Since no n of the vectors $u_{i_0}, \dots, u_{i_n}$ are linearly dependent, there are two possibilities: either that $\bigcap_{j=0}^n H_{i_j}$ contains translates of rB for all $r > 0$, or that $\bigcap_{j=0}^n H_{i_j}$ is a simplex (which obviously contains L). In the first case there exists an $x_o \in \mathbb{R}^n$ such that $x_o + K \subset \bigcap_{j=0}^n H_{i_j}$. In the second case $\bigcap_{j=0}^n H_{i_j}$ is a simplex which contains L , so an x_o with the property that $x_o + K \subset \bigcap_{j=0}^n H_{i_j}$ is guaranteed by the hypothesis of the theorem. Thus in either case $x_o \in \bigcap_{j=0}^n K_{i_j}$, and Helly's theorem shows that there exists an $x \in \mathbb{R}^n$ such that

$$x \in \bigcap_{i=1}^s K_i.$$

Thus $x + K \subset H_i$, for $i = 1, \dots, s$, and therefore

$$x + K \subset \bigcap_{i=1}^s H_i = L.$$

This completes the proof. □

An immediate consequence of the theorem is:

Corollary 1. *Suppose K and L are convex bodies. If, for each simplex Δ , the simplex $\Delta(L)$ contains a translate of $\Delta(K)$, the L contains a translate of K .*

An immediate consequence of Corollary 1 is:

Corollary 2. *Suppose K and L are convex bodies. If the volume of each circumscribing simplex to K is no greater than that of the corresponding circumscribing simplex to L , then some translate of K is contained in L .*

Corollary 1 can be combined with the proposition to yield immediately:

Corollary 3. *Suppose K and L are convex bodies. If*

$$V_1(\Delta, K) \leq V_1(\Delta, L), \quad \text{for every simplex } \Delta,$$

then some translate of K is contained in L .

The converse of Corollary 3 is an immediate consequence of (1) and the definition of the mixed volume V_1 .

Recently, Scholtes [7] and Zhang [9] strengthened an important (see, e.g., pp. 283–284 of [6], p. 151 of [1], and p. 66 of [5]) mixed-volume characterization of containment due to Weil [8]. Scholtes and Zhang proved that Weil’s characterization of containment can be strengthened to read: If K and L are convex bodies and if

$$V_1(Q, K) \leq V_1(Q, L), \quad \text{for every convex body } Q,$$

then a translate of K is contained in L . Zhang’s proof uses Weil’s containment theorem and “mixed bodies” (see p. 396 of [6] for a definition).

Weil’s proof of his mixed-volume containment theorem is not elementary. In addition to considerable ingenuity, the proof uses information regarding the completion of the space of Aleksandrov-Fenchel-Jessen measures of convex bodies (as well as the fact that every C^2 -function may be represented as the difference of support functions of convex bodies). Clearly, Corollary 3 strengthens Scholtes’ and Zhang’s result. Clearly also, the proof presented is elementary.

Write $S(Q)$ for the surface area of the convex body Q . A question of Fáy and Makai (see p. 22 of [3]) asks: If K and L are convex bodies, and if

$$S(\varphi K) \leq S(\varphi L), \quad \text{for every } \varphi \in SL(n),$$

does it follow that $V(K) \leq V(L)$?

It was shown in [2] (see also Theorem 4.2.11 of [4]) that the Fáy and Makai question has a negative answer, even if K and L are symmetric about the origin.

Suppose that instead of being told that for each $\varphi \in SL(n)$ the surface area of φK is no greater than that of φL , it is given instead that the surface area of a particular simplex circumscribed about φK is no greater than that of the corresponding simplex

circumscribed about φL . Does it follow $V(K) \leq V(L)$? Specifically, suppose Δ_o is an arbitrary but fixed simplex. If K and L are convex bodies, and if

$$S(\Delta_o(\varphi K)) \leq S(\Delta_o(\varphi L)), \quad \text{for all } \varphi \in SL(n),$$

does it follow that $V(K) \leq V(L)$?

Even though the given data appears to be weaker here than in the Fáry and Makai question, it will be shown that this variant of the Fáry and Makai question has a positive answer. No assumptions regarding symmetry need be made. In fact, a much stronger conclusion than $V(K) \leq V(L)$ can be drawn. From the given data, it follows that some translate of K is contained in L . Furthermore, the “surface area” in the question can be replaced by “volume” or any other elementary mixed volume (such as “mean width”). Even replacing “surface area” in the question by a functional such as “total edge length,” will still lead to the conclusion that a translate of K is contained in L . It should be emphasized again that no symmetry assumptions about the bodies involved is needed or will be made.

A translation invariant functional, Φ , defined on the set of simplices, is said to be *increasing* if, for each pair of positive numbers c_1, c_2 ,

$$c_1 < c_2 \quad \text{implies} \quad \Phi(c_1 \Delta) < \Phi(c_2 \Delta),$$

for each simplex Δ .

Corollary 4. *Suppose Φ is an increasing, translation invariant functional defined on the set of simplices. Suppose that Δ_o is an arbitrary but fixed simplex. If K and L are convex bodies, and if*

$$\Phi(\Delta_o(\varphi K)) \leq \Phi(\Delta_o(\varphi L)), \quad \text{for all } \varphi \in SL(n),$$

then some translate of K is contained in L .

To prove Corollary 4, note that the definition of an increasing functional together with the proposition give

$$V_1(\Delta_o, \varphi K) \leq V_1(\Delta_o, \varphi L), \quad \text{for all } \varphi \in SL(n).$$

Thus from (3) it follows that

$$V_1(\varphi^{-1} \Delta_o, K) \leq V_1(\varphi^{-1} \Delta_o, L), \quad \text{for all } \varphi \in SL(n).$$

However, this, and the definition of mixed volumes, shows that $V_1(\Delta, K) \leq V_1(\Delta, L)$, for every simplex Δ . Corollary 3 now gives the desired result.

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