On Helly's Theorem: Algorithms and Extensions*

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Abstract. This paper studies algorithmic Helly-type problems in the framework of the algorithmic theory of convex bodies developed by Grötschel, Lovász, and Schrijver. Various oracle-polynomial-time algorithms are presented that are complemented by $\mathbb{NP}$ -hardness results for polytopes. In addition, some new Helly-type theorems are derived.

1. Introduction

For $d \in \mathbb{N}$, let $\mathcal{C}^d$ (resp. $\mathcal{K}^d$) denote the family of all nonempty closed (resp. compact) convex subsets of $\mathbb{R}^d$; the members of $\mathcal{K}^d$ are called *convex bodies*. *Helly's theorem* discovered in 1913 [15], [12] states that:

A subfamily of $\mathcal{K}^d$ (or a finite subfamily of $\mathcal{C}^d$) (of cardinality at least $d + 1$) has nonempty intersection if and only if the intersection of each $d + 1$ of its members is nonempty.

Due to the importance of Helly's theorem there are numerous generalizations and ramifications to other *Helly-type* results of the form: there exists a natural number h (the smallest such being called the *Helly number*), such that a family of objects has a specific property (the *Helly property*) if and only if each subfamily of cardinality h has this property.

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Further, there is a natural algorithmic problem associated with any Helly-type result: either assert that a given family has the Helly property, or determine a short *certificate* for the lack of it by producing a subfamily of cardinality at most h which already fails to have the Helly property.

The main goal of this paper is to study algorithmic aspects of various geometric Helly-type theorems for polytopes and convex bodies with the emphasis on the latter. Hence the *oracle model* of Grötschel *et al.* [11] is employed. For the appropriate algorithmic variant of Helly's theorem the standard oracles are strong enough to yield oracle-polynomial-time algorithms. If one goes one step further to deal with a theorem of Vincensini [18] and Klee [14] it becomes apparent (by appropriate hardness results) that the model needs to be augmented by an oracle that allows certain subset relations to be checked. This, in turn, leads to oracle-polynomial-time algorithms even for a much more general new Helly-type problem.

Section 2 gives a short account of the algorithmic theory of convex bodies, specifies the *algorithmic Helly-type problems* precisely that are relevant for this paper, and states the main computational results. Their proofs are then contained in Section 3. The final section, Section 4, gives various new Helly-type theorems dealing with the “shape” of intersections.

For a survey of Helly's theorem and its relatives up to 1963 see [4], for a collection of newer results see [5]; some implications of Helly's theorem in optimization can be found in [17], and [1] contains some related concepts in computational geometry. For general background information we refer to [16] and [11].

2. Algorithmic Helly-Type Problems and Main Computational Results

The underlying model of computation is the binary Turing machine model, which will be augmented by certain oracles; see [11]. Polytopes can be presented in a finite and (if they are rational) algorithmically suitable manner: A string $(d, p; v_1, \dots, v_p)$ with $d, p \in \mathbb{N}$ and $v_1, \dots, v_p \in \mathbb{Q}^d$ is called a $\mathcal{V}$ -polytope in $\mathbb{R}^d$; it represents the geometric object $P = \text{conv}\{v_1, \dots, v_p\}$. A string $(d, q; Q, b)$ where $d, q \in \mathbb{N}$, Q is a rational $q \times d$ matrix, and $b \in \mathbb{Q}^q$ such that $P = \{x \in \mathbb{R}^d : Qx \leq b\}$ is bounded is called an $\mathcal{H}$ -polytope in $\mathbb{R}^d$; it is again identified with the geometric object P . The *binary size* (or *size* for short) of a $\mathcal{V}$ - or an $\mathcal{H}$ -polytope P is the number of binary digits needed to encode the data of the presentation; see [9].

A way to deal algorithmically with a general convex body K has been introduced and extensively studied in [11]: it is assumed that K is given by an algorithm (called an *oracle*) that answers certain sorts of questions about the body, and that some additional *a priori* information about K is available. In the following we generally assume that each body K is *circumscribed*, i.e., a rational number R is given explicitly such that $K \subset R\mathbb{B}^d$, where $\mathbb{B}^d$ denotes the Euclidean unit ball; the *size* of K is then defined by $\text{size}(K) = d + \text{size}(R)$. In some cases it will be necessary to assume that a body is *well-bounded*, i.e., circumscribed and, in addition, a positive rational number r is given such that K contains a ball (whose position may not be known) of radius r . Then, of course, $\text{size}(K) = d + \text{size}(r) + \text{size}(R)$. The most convenient oracle for our purposes is the weak separation oracle, where “weak” refers to the fact that we have to allow for

rounding errors since only finite precision is possible. Recall that for $\varepsilon \geq 0$ the *outer parallel body* and the *inner parallel body* of a convex body K are given by

$$K(\varepsilon) = K + \varepsilon \mathbb{B}^d \quad \text{and} \quad K(-\varepsilon) = K \setminus ((\mathbb{R}^d \setminus K) + \varepsilon \mathbb{B}^d).$$

Then a *weak separation oracle* for a convex body K of $\mathbb{R}^d$ solves the following problem.

Weak Separation Problem. Given $u \in \mathbb{Q}^d$, and a positive rational ε , assert that $u \in K(\varepsilon)$ or determine a vector $c \in \mathbb{Q}^d$ with $\|c\|_\infty = 1$ such that $c^T x \leq c^T u + \varepsilon$ for all $x \in K(-\varepsilon)$.

Naturally, it is assumed that the output is of a size that is bounded by a fixed polynomial in the input size. Note that an otherwise polynomial-time algorithm for a given task that is, however, allowed to access a specified oracle is turned into an ordinary polynomial-time algorithm if the oracle is replaced by a polynomial-time subroutine. This point will be stressed by using the term *oracle-polynomial-time algorithm*; for details see [11].

If we extend the definitions of parallel bodies to the empty set by setting $\emptyset(\varepsilon) = \emptyset(-\varepsilon) = \emptyset$, the WEAK SEPARATION PROBLEM extends to general compact convex sets. It should be noted however that the problem is actually defined with a view to *proper* (i.e., d -dimensional) convex bodies. In fact, if K is not full-dimensional or empty, any vector $c \in \mathbb{Q}^d$ with $\|c\|_\infty = 1$ induces a valid weakly separating hyperplane, but this information is rather weak then. For many purposes provisions can be made though to include improper convex bodies. To keep the exposition simple, we, however, do not emphasize this point beyond absolute necessity.

The variant of a weak problem (like WEAK SEPARATION) that is obtained by setting $\varepsilon = 0$ is referred to as *strong*. Of course, the strong version of a problem is only appropriate in our model of computation if the convex bodies can be specified precisely in finite precision. While this is the case for $\mathcal{V}$ - or $\mathcal{H}$ -polytopes it is not the case in general. For a $\mathcal{V}$ - or $\mathcal{H}$ -polytope P it is easy to construct a weak (or strong) separation oracle and to compute the “circumscribedness” parameter R ; if P is proper, a positive lower bound for the inradius can also be computed in polynomial time; see [11] or [9]. Hence oracle-polynomial-time algorithms that rely on a weak separation oracle extend to the polytopal case while the NP-hardness results in the latter usually indicate that (unless $\mathbb{P} = \mathbb{NP}$) no polynomial-time algorithm exists even when calls to a weak separation oracle are allowed.

In the realm of the algorithmic theory of convex bodies, Helly's theorem leads to the following problem.

Weak Helly Problem. For $m \in \mathbb{N}$, $m \geq d + 1$, circumscribed bodies $K_1, \dots, K_m \in \mathcal{K}^d$ given by weak separation oracles, and a positive rational ε , compute a vector $a \in \bigcap_{i=1}^m K_i$, or determine a subset I of $\{1, \dots, m\}$ of cardinality at most $d + 1$ such that $K_I(-\varepsilon) = \emptyset$, where $K_I = \bigcap_{i \in I} K_i$.

Note that the size of an instance of WEAK HELLY is $\text{size}(\varepsilon) + \sum_{i=1}^m \text{size}(K_i)$. Our first result shows that WEAK HELLY is algorithmically tractable.

Theorem 2.1. WEAK HELLY can be solved in oracle-polynomial time.

In view of this positive result, it is natural to wonder whether geometric extensions of Helly's theorem can be dealt with computationally in a similarly efficient way. As was shown by Vincensini [18] and Klee [14], a given subfamily $\mathcal{K}$ of $\mathcal{K}^d$ (of cardinality at least $d + 1$) has the property that its intersection contains the translate of another given convex body A if and only if each $(d + 1)$ -element subfamily of $\mathcal{K}$ has this property. Can the corresponding weak algorithmic problem be solved in oracle-polynomial time? It is easy to see that if the answer was "yes," then in particular it could be decided in polynomial time whether an $\mathcal{H}$ -polytope A is contained in a $\mathcal{V}$ -polytope K . However, this problem is known to be coNP -complete [6] even if A is the standard cube and K is the affine image of a cross-polytope [3], [8]; see also [9]. Hence, unless $\mathbb{P} = \text{coNP}$ there cannot be an oracle-polynomial-time algorithm solving the corresponding extension of WEAK HELLY. The appropriate way of modeling the requirement to be able to solve such containment problems in the algorithmic theory of convex bodies is to strengthen the separation oracle. So the following task could be introduced for a convex body K and a subfamily $\mathcal{A}$ of $\mathcal{K}^d$.

For a convex body $A \in \mathcal{A}$ and a positive rational ε , assert that $A \subset K(\varepsilon)$, or determine a vector $c \in \mathbb{Q}^d$ with $\|c\|_\infty = 1$ such that there exists a vector $u \in A$ with $c^T x \leq c^T u + \varepsilon$ for all $x \in K(-\varepsilon)$.

For $\mathcal{A}$ being the family of all singletons we obtain the ordinary weak separation problem. However, the input sizes are measured differently since $\text{size}(A) = d$, but if $A = \{a\}$, $\text{size}(a)$ is the binary length of a . Hence, since we are aiming at the weakest possible oracular requirements that suffice for our purposes we introduce our containment problems differently. Let $k \in \mathbb{N}$, $\mathcal{A} = \{A_1, \dots, A_k\} \subset \mathcal{K}^d$, set $T^k = \{z \in \mathbb{R}^k : z \geq 0 \wedge z^T e = 1\}$, where $e = (1, \dots, 1)^T \in \mathbb{R}^k$, and $A[z] = \sum_{i=1}^k \zeta_i A_i$, for $z = (\zeta_1, \dots, \zeta_k)^T \in T^k$. Now we define:

Weak $\mathcal{A}$ -Containment Problem. Given $a \in \mathbb{Q}^d$, $z \in T^k$, and a positive rational ε , assert that $a + A[z] \subset K(\varepsilon)$, or determine a vector $c \in \mathbb{Q}^d$ with $\|c\|_\infty = 1$ such that there exists a vector $u \in a + A[z]$ with $c^T x \leq c^T u + \varepsilon$ for all $x \in K(-\varepsilon)$.

As the input for WEAK $\mathcal{A}$ -CONTAINMENT is the triple $(a, z; \varepsilon)$ the input size is now $\text{size}(a) + \text{size}(z) + \text{size}(\varepsilon)$ (as it should be). If for each specified instance of WEAK $\mathcal{A}$ -CONTAINMENT a solution is available, we say that K is given by a *weak $\mathcal{A}$ -containment oracle*.

It turns out that these weak containment oracles are strong enough to yield an oracle-polynomial-time algorithm for the Vincensini/Klee extension of Helly's problem, and also for the following more general Helly-type problem.

Weak Fat Helly Problem. For $d, k, m \in \mathbb{N}$ with $m \geq d + k$, well-bounded convex bodies $A_1, \dots, A_k$ in $\mathbb{R}^d$ given by weak separation oracles, circumscribed convex bodies $K_1, \dots, K_m$ in $\mathbb{R}^d$ given by weak $\mathcal{A}$ -containment oracles, where $\mathcal{A} = \{A_1, \dots, A_k\}$, and a positive rational ε , compute rational vectors $a \in \mathbb{R}^d$ and $z \in T^k$ such that $a + A[z] \subset \bigcap_{i=1}^m K_i$, or determine a subset I of $\{1, \dots, m\}$ of cardinality at most $d + k$ such that $K_I(-\varepsilon)$ fails to contain any set $a + A[z]$ with $a \in \mathbb{R}^d$ and $z \in T^k$, where $K_I = \bigcap_{i \in I} K_i$.

Our main tractability result is the following.

Theorem 2.2. *WEAK FAT HELLY can be solved in oracle-polynomial time.*

The next theorem gives the complete picture of the complexity status (in the standard binary model of computation) of STRONG FAT HELLY for $\mathcal{V}$ - and $\mathcal{H}$ -polytopes. Its tractability results hold even in the case that the polytopes $A_1, \dots, A_k$ are lower-dimensional. Note further that, in particular, the algorithmic problem associated with the Vincensini/Klee extension of Helly's theorem can be solved in polynomial time whenever the “test body” A is a $\mathcal{V}$ -polytope.

Theorem 2.3. *Whenever the polytopes $A_1, \dots, A_k$ and $K_1, \dots, K_m$ are of the types indicated in the first and second column of the table below, the complexity of STRONG FAT HELLY is as indicated in the third column.*

$A_1, \dots, A_k$	$K_1, \dots, K_m$	Complexity
$\mathcal{V}$ -polytopes	$\mathcal{V}$ -polytopes	NP-hard
$\mathcal{V}$ -polytopes	$\mathcal{H}$ -polytopes	$\mathbb{P}$
$\mathcal{H}$ -polytopes	$\mathcal{V}$ -polytopes	NP-hard
$\mathcal{H}$ -polytopes	$\mathcal{H}$ -polytopes	$\mathbb{P}$

The NP-hardness result stated in the third row of the table persists if $k = 1$, while, for any constant k , STRONG FAT HELLY can be solved in polynomial time if the bodies $A_1, \dots, A_k$ and $K_1, \dots, K_m$ are all $\mathcal{V}$ -polytopes.

We point out that in general even in the tractable cases of the problem, contained polytopes cannot be presented in a $\mathcal{V}$ - or an $\mathcal{H}$ -manner since the numbers of vertices and facets of the Minkowski sum of $\mathcal{V}$ - or $\mathcal{H}$ -polytopes may grow exponentially; see [10].

The emphasis of the present paper is on the case of the dimension being part of the input; so we just remark in passing that STRONG FAT HELLY can be solved in polynomial time for $\mathcal{V}$ - or $\mathcal{H}$ -polytopes $A_1, \dots, A_k$ and $K_1, \dots, K_m$, if d is regarded a constant.

It is not too surprising that results analogous to Theorem 2.2 can be derived with “is contained in $K_1, \dots, K_m$ ” in the statement of WEAK FAT HELLY replaced by “contains $K_1, \dots, K_m$ ” or “intersects $K_1, \dots, K_m$.” Of course, the containment oracle then has to be modified but the changes are all straightforward and are hence omitted. We also mention that [2] studies the “inherently exponential” counterpart problem (in an exact real arithmetic model of computation) to devise a good strategy for producing a point in the intersection of all convex bodies provided a procedure is available for producing a point in the intersection of any $d + 1$ of the bodies.

To end this section it should be pointed out that there are more general versions of WEAK FAT HELLY to which Theorem 2.2 can be extended; we refrain from doing that but we state some related new Helly-type results in Section 4.

3. Algorithms for (WEAK) HELLY and (WEAK) FAT HELLY

We now prove Theorems 2.1–2.3. Throughout this section the following notation is used freely:

$$\begin{aligned} m, d &\in \mathbb{N} \text{ with } m - 1 \geq d \geq 2; \\ M &= \{1, \dots, m\}; \\ \text{when } I \subset M, K_1, \dots, K_m &\in \mathcal{K}^d: K_I = \bigcap_{i \in I} K_i; \\ e_1, \dots, e_d &\text{ the standard basis vectors of } \mathbb{R}^d; Y_d = \text{conv}\{\pm e_1, \dots, \pm e_d\}. \end{aligned}$$

Conceptually, the main algorithm works as follows:

First it is checked whether K_M has the Helly-property under consideration. If this is the case, the algorithm terminates. If not, it produces a proper subset $I \subset M$ that fails to have the desired property. This step is repeated until $|I|$ is reduced to the cardinality of the desired certificate.

Clearly, if this *reduction* algorithm (or an appropriate weak version of it) runs in oracle-polynomial time the whole procedure does.

The first lemma shows how weak separation oracles for $K_1, \dots, K_m$ can be used to produce a weak separation oracle for K_M ; it should be compared with Section 4.7 of [11].

Lemma 3.1. *There is an oracle-polynomial-time algorithm which accepts as input numbers $d, m \in \mathbb{N}$, compact convex sets $K_1, \dots, K_m$ in $\mathbb{R}^d$ given by weak separation oracles, and a positive rational ε , that solves the weak separation problem for K_M .*

Proof. Suppose $(u; \varepsilon)$ is an instance of the weak separation problem for K_M . Let $\delta = \varepsilon/(d+1)$ and call, for $i = 1, \dots, m$ and $j = 1, \dots, d$, the weak separation oracle for K_i for the inputs $(u + d\delta e_j; \delta)$ and $(u - d\delta e_j; \delta)$. If the answers are all affirmative, i.e., $u \pm d\delta e_j \in K_i(\delta)$ for $i = 1, \dots, m$ and $j = 1, \dots, d$, it follows that

$$u + \delta \mathbb{B}^d \subset u + d\delta Y_d \subset K_i(\delta) = K_i + \delta \mathbb{B}^d,$$

for $i = 1, \dots, m$, whence $u \in K_M$.

Otherwise, at least once a weakly separating hyperplane is produced—say for K_1 and $(u + d\delta e_1; \delta)$. So a rational d -vector c with $\|c\|_\infty = 1$ is output such that $c^T x \leq c^T u + d\delta c^T e_1 + \delta$ for all $x \in K_1(-\delta)$. Since $c^T e_1 \leq 1$ and $K_M(-\varepsilon) \subset K_1(-\varepsilon) \subset K_1(-\delta)$ this implies

$$c^T x \leq c^T u + \varepsilon \quad \text{for all } x \in K_M(-\varepsilon);$$

whence a weakly separating hyperplane is found. $\square$

The following result is a modification of Theorem 4.6.1 of [11].

Lemma 3.2. *There is an oracle-polynomial-time algorithm which accepts as input a positive integer d , a circumscribed compact convex set K in $\mathbb{R}^d$ given by a weak*

separation oracle and a positive rational ε , and which asserts that $K(-\varepsilon) = \emptyset$, or constructs a rational d -vector b and a nonsingular rational $(d \times d)$ -matrix Q such that

$$b + Q(\mathbb{B}^d) \subset K \subset b + \sqrt{d}(d+1)Q(\mathbb{B}^d).$$

Proof. The algorithm that is to be constructed is similar to that underlying Theorem 4.6.1 of [11] which proves an analogous result for convex bodies that are well-bounded. The latter which is based on the *shallow cut ellipsoid algorithm* is sketched first, and then the modifications that are needed in the more general situation are outlined.

For a convex body K with given well-boundedness parameters r, R , the algorithm underlying Theorem 4.6.1 of [11] starts with $R\mathbb{B}^d$, and $j = 0$ and carries out the following steps iteratively. (The description concentrates on the basics of the algorithm; simple additional details involving the approximation of reals by appropriate rationals, Cholesky factorization, etc., are omitted.) Given an ellipsoid $b_j + \sqrt{d}(d+1)Q_j(\mathbb{B}^d)$ which contains K , the weak separation oracle for K is used to decide whether the endpoints of the axes of the smaller ellipsoid $b_j + \sqrt{d}Q_j(\mathbb{B}^d)$ are contained in K . If all answers are affirmative it follows that $b_j + Q_j(\mathbb{B}^d) \subset K$, and the algorithm stops with the required pair of ellipsoids. Otherwise a weakly separating hyperplane is obtained. Using this hyperplane as a shallow cut, a new smaller ellipsoid can be produced that still contains K . Clearly, the algorithm terminates before the volume of the containing ellipsoid becomes smaller than $(r/d)^d$, and it is shown that this must happen after a polynomial number of iterations.

Now suppose that no lower bound for the inradius of K is known; in fact, K may be lower-dimensional or even empty. The algorithm must then be able to detect whether $K(-\varepsilon) = \emptyset$. Suppose first that, when applied to K , the algorithm outlined before still produces an ellipsoid E that is contained in K . Then a lower bound r for the inradius of K can be computed from E in polynomial time. Adapting the error parameter for the calls to the separation oracle for K if necessary to guarantee that the corresponding inner parallel body is not empty, and rerunning the whole algorithm, it follows again that K is contained in the ellipsoids $b_j + \sqrt{d}(d+1)Q_j(\mathbb{B}^d)$ produced in each step. So the algorithm finally outputs a correct choice of b and Q .

If ε is such that $K(-\varepsilon) = \emptyset$, then a call to the weak separation oracle still produces a shallow cut, but it is no longer guaranteed that K is contained in the corresponding ellipsoid $b_j + \sqrt{d}(d+1)Q_j(\mathbb{B}^d)$. However, it is always true that $K(-\varepsilon) \subset b_j + \sqrt{d}(d+1)Q_j(\mathbb{B}^d)$. So the algorithm can continue essentially as before. Adapting the error parameter for the calls to the separation oracle appropriately, the algorithm then constructs, after a polynomial number of steps, an ellipsoid E' such that $K(-\varepsilon/2) \subset E'$, and $E'(-\varepsilon/2) = \emptyset$. This shows that $K(-\varepsilon) = \emptyset$, and this information is output. $\square$

Next the basic weak reduction for WEAK HELLY is described.

Lemma 3.3. *There is an oracle-polynomial-time algorithm which accepts as input $d, m \in \mathbb{N}$, with $m \geq d+2$, a positive rational ε , and circumscribed compact convex sets $K_1, \dots, K_m$ in $\mathbb{R}^d$ with $K_M(-\rho) = \emptyset$, where $\rho = \varepsilon/(d(d+1))$, given by weak separation oracles, that determines an $(m-1)$ -element subset I of M such that $K_I(-\varepsilon) = \emptyset$.*

Proof. Let $\mathcal{I}$ denote the family of all $(m-1)$ -element subsets of M . First the algorithm of Lemma 3.2 is called for each compact convex set K_I with $I \in \mathcal{I}$ and error bound ε . This is possible in view of Lemma 3.1. If for one of these m runs the output is the assertion that $K_I(-\varepsilon) = \emptyset$ we are done. So suppose, in each case the algorithm produces a vector $b_I \in \mathbb{Q}^d$ and a nonsingular rational $(d \times d)$ -matrix Q_I such that $b_I + Q_I(\mathbb{B}^d) \subset K_I \subset b_I + \sqrt{d}(d+1)Q_I(\mathbb{B}^d)$. Clearly, $Q_I^T Q_I$ is positive definite, whence the smallest eigenvalue λ_I of $Q_I^T Q_I$ is positive. Further note that $\sqrt{\lambda_I}$ is a lower bound for the inradius of K_I , while $\sqrt{d}(d+1)\sqrt{\lambda_I}$ is an upper bound. Of course, the smallest eigenvalue of $Q_I^T Q_I$ is not directly available in our binary model of computation but it can be approximated efficiently. The same is true for the square root, so approximations α_I, β_I of $\sqrt{\lambda_I}$ can be computed in polynomial time such that

$$\alpha_I \leq \sqrt{\lambda_I} < \beta_I \quad \text{and} \quad \beta_I - \alpha_I \leq \frac{\rho}{3}.$$

Then, in particular,

$$K_I(-\alpha_I) \neq \emptyset, \quad \text{but} \quad K_I(-\sqrt{d}(d+1)\beta_I) = \emptyset.$$

Now it is checked whether, for any one of the index sets $I \in \mathcal{I}$, the inequality

$$\varepsilon^2 \geq d(d+1)^2 \beta_I^2$$

holds. If this is the case—say for I_0 —then $K_{I_0}(-\varepsilon) = \emptyset$, and the algorithm outputs I_0 .

Otherwise, $\varepsilon < \sqrt{d}(d+1)\beta_I$ for all $I \in \mathcal{I}$. Hence,

$$\rho = \frac{\varepsilon}{d(d+1)} < \frac{\sqrt{d}(d+1)}{d(d+1)} \beta_I = \frac{\beta_I}{\sqrt{d}} < \frac{3}{4} \beta_I, \quad \text{whence} \quad \rho \leq \beta_I - \frac{\rho}{3} \leq \alpha_I$$

and $K_I(-\rho) \neq \emptyset$ for all $I \in \mathcal{I}$. However, then the family $\{K_i(-\rho) : i \in M\}$ has the property that each $(m-1)$ -element subfamily admits a common point. Since $m-1 \geq d+1$, Helly's theorem implies that $K_M(-\rho) = \bigcap_{i \in M} K_i(-\rho) \neq \emptyset$, a contradiction to the assumption that $K_M(-\rho) = \emptyset$. This concludes the proof of the weak reduction lemma. $\square$

Proof of Theorem 2.1. Let ε denote the error bound in the given instance of WEAK HELLY. The algorithm begins by using Lemma 3.1 and applying the procedure of Lemma 3.2 to K_M and error bound $\varepsilon_0 = (1/(d(d+1)))^{m-d-1} \varepsilon$. If the algorithm terminates with a point b (and a matrix Q) we are done since $b \in K_M$. Otherwise, $K_M(-\varepsilon_0) = \emptyset$, and the algorithm sets $\rho_1 = \varepsilon_0$ and $\varepsilon_1 = d(d+1)\varepsilon_0$ and applies Lemma 3.3 to $K_1, \dots, K_m$ with error bound ε_1 . This subroutine produces an $(m-1)$ -element subset I_1 of M such that $K_{I_1}(-\varepsilon_1) = \emptyset$. Now this procedure is repeated successively for $j = 1, \dots, m-d-2$ with $\{K_i : i \in I_j\}$, $\rho_{j+1} = \varepsilon_j$ and $\varepsilon_{j+1} = d(d+1)\varepsilon_j$, to obtain an $(m-j-1)$ -element subset I_{j+1} of I_j with $K_{I_{j+1}}(-\varepsilon_{j+1}) = \emptyset$. Hence the algorithm produces the required certificate in polynomial time. $\square$

We now turn to the proof of Theorem 2.2. We use the additional notation:

$$\begin{aligned}
 k &\in \mathbb{N}, n = d + k; \\
 T^k &= \{z = (\zeta_1, \dots, \zeta_k)^T \in \mathbb{R}^k : z \geq 0 \wedge \sum_{i=1}^k \zeta_i = 1\}; \\
 A_1, \dots, A_k &\in \mathcal{K}^d; \mathcal{A} = \{A_1, \dots, A_k\}; \\
 \text{for } z &= (\zeta_1, \dots, \zeta_k)^T \in T^k, A[z] = \sum_{i=1}^k \zeta_i A_i; \\
 H &= \text{aff}(\mathbb{R}^d \times T^k); \\
 \text{for } K &\in \mathcal{K}^d, S_K = \{(a, z) \in \mathbb{R}^d \times T^k : a + A[z] \subset K\}.
 \end{aligned}$$

Further, we use the convention that whenever vectors y or z are used, then their coordinates will be denoted by η_i, ζ_i , respectively, i.e., $y = (\eta_1, \dots, \eta_k)^T$ and $z = (\zeta_1, \dots, \zeta_k)^T$.

Note that the set S_K is closed and convex but, of course, it may be empty. In any case, it is contained in the hyperplane H , and it will be necessary to identify this hyperplane with the affine space $\mathbb{R}^{n-1}$.

Lemma 3.4. *Let $0 < \varepsilon \leq 1/n^2$, and $1 \leq R$, and suppose that $A_1, \dots, A_k \subset R\mathbb{B}^d$. Further, let $(b, y) \in \mathbb{R}^d \times T^k$ such that $b + A[y] \subset K(-\varepsilon)$, and set $B = ((b, y) + (\varepsilon/(\sqrt{n}R))\mathbb{B}^n) \cap (\mathbb{R}^d \times T^k)$. Then $a + A[z] \subset K$ for each point $(a, z) \in B$, and $\rho(S_K) \geq \varepsilon/(\sqrt{2nn}R)$, where $\rho(S_K)$ denotes the inradius of S_K taken in H .*

Proof. Let $(a, z) \in B$; then $\|a - b\|_1 + \|z - y\|_1 \leq \varepsilon/R$, and it follows that

$$\begin{aligned}
 a + A[z] &= b + (a - b) + \sum_{i=1}^k (\eta_i + (\zeta_i - \eta_i)) A_i \\
 &\subset b + A[y] + \|a - b\|_1 \mathbb{B}^d + R \sum_{i=1}^k |\zeta_i - \eta_i| \mathbb{B}^d \\
 &\subset b + A[y] + R(\|z - y\|_1 + \|a - b\|_1) \mathbb{B}^d \\
 &\subset b + A[y] + \varepsilon \mathbb{B}^d \subset K(-\varepsilon) + \varepsilon \mathbb{B}^d \subset K.
 \end{aligned}$$

This proves the first assertion, and also the second if $k = 1$. For $k \geq 2$ just verify that $B \cap \text{aff}(T^k)$ contains a regular $(k - 1)$ -dimensional simplex of edge-length $\varepsilon/(\sqrt{n}R)$, whence B contains an $(n - 1)$ -dimensional ball of radius $\varepsilon/(\sqrt{2nn}R)$. $\square$

Lemma 3.5. *Let $0 \leq R$, and suppose that $K, A_1, \dots, A_k \subset R\mathbb{B}^d$. Then $S_K \subset (2R + 1)\mathbb{B}^n$.*

Proof. Let $(a, z) \in S_K$. Then, by definition, $(a, z) \in \mathbb{R}^d \times T^k$ and $a + A[z] \subset K$. Since $A_1, \dots, A_k \subset R\mathbb{B}^d$, it follows from convexity that $A[z] \subset R\mathbb{B}^d$, whence $a \in (-y + K) \subset 2R\mathbb{B}^d$ for every point $y \in A[z]$. Since $T^k \subset \mathbb{B}^k$ this implies that $S_K \subset 2R\mathbb{B}^d \times \mathbb{B}^k \subset (2R + 1)\mathbb{B}^n$. $\square$

The following proposition is Corollary 4.2.7 of [11].

Proposition 3.6. *There is an oracle-polynomial-time algorithm which accepts as input $d \in \mathbb{N}$, a circumscribed convex body K in $\mathbb{R}^d$ given by a weak separation oracle, a*

rational d -vector c and a positive rational ε , and asserts that $K(-\varepsilon) = \emptyset$ or produces a rational d -vector w such that $w \in K(\varepsilon)$ and $c^T x \leq c^T w + \varepsilon$ for all $x \in K(-\varepsilon)$.

Note that the point w is a “weak optimizer” of the linear functional associated with c over the convex body K .

The next lemma derives a separation oracle for a set S_K . Since $S_K \subset H$ such oracles would not be particularly powerful if defined with respect to $\mathbb{R}^n$. Hence the construction will be with respect to H . In particular, parallel bodies are formed with respect to H , specifically $S_K(-\varepsilon) = S_K \setminus ((H \setminus S_K) + \varepsilon \mathbb{B}^n)$, and the normals of separating hyperplanes must be parallel to H . (For clarity of exposition we work in $\mathbb{R}^n$ as the containing space rather than directly in the $\mathbb{R}^{n-1}$ equivalent to H .)

Lemma 3.7. *There is an oracle-polynomial-time algorithm which solves the weak separation problem for S_K with respect to H for well-bounded convex bodies $A_1, \dots, A_k$ given by weak separation oracles and a circumscribed convex body K given by a weak $\mathcal{A}$ -containment oracle.*

Proof. Let (a, z) be the query point and let ε be the error bound in a given instance of the weak separation problem for S_K . Let $(r_1, R_1), \dots, (r_k, R_k)$ denote the well-boundedness parameters for $A_1, \dots, A_k$, respectively, that are given with these bodies, and set $r = \min_{i=1, \dots, k} r_i$, $R = \max_{i=1, \dots, k} R_i$, and $\delta = \min\{r, \varepsilon r / (7Rd)\}$.

It is easy to check whether $z \in \text{relint}(T^k)$ and—if not—to produce a (strongly) separating hyperplane. Otherwise call the weak $\mathcal{A}$ -containment oracle for the $2d$ inputs $(a \pm d\delta e_i, z; \delta)$, where $i = 1, \dots, d$. Suppose first that all assertions are affirmative, i.e., $a \pm d\delta e_i + A[z] \subset K(\delta)$ for all i , hence $K(\delta) \supset a + A[z] + d\delta Y_d \supset a + A[z] + \delta \mathbb{B}^d$. This implies that $a + A[z] \subset K$, whence $(a, z) \in S_K$, and the algorithm outputs $(a, z) \in S_K \subset S_K(\delta)$.

Suppose, now, one of the $2d$ calls to the weak $\mathcal{A}$ -containment oracle for K —say the one for the input $(a + d\delta e_1, z; \delta)$ —produces a rational d -vector c with $\|c\|_\infty = 1$ such that there exists a vector $u \in a + d\delta e_1 + A[z]$ with

$$c^T x \leq c^T u + \delta \quad \text{for all } x \in K(-\delta).$$

Now Proposition 3.6 is applied to the inputs $(d; A_i, c; \delta)$ for $i = 1, \dots, k$. Since $\delta \leq r$, the underlying algorithm will produce weak optimizers. So the k calls to the algorithm underlying Proposition 3.6 lead to the following information:

$$w_i \in A_i(\delta) : \quad c^T x \leq c^T w_i + \delta \quad \text{for all } x \in A_i(-\delta) \quad (i = 1, \dots, k).$$

It is easy to verify that

$$A_i \subset A_i(-\delta) + \tau \delta \mathbb{B}^d \quad (i = 1, \dots, k),$$

where $\tau = 2R/r$, and then it follows that

$$c^T w_i - (\tau + 1)\delta d \leq \max_{x \in A_i(-\delta)} c^T x.$$

Now let Q denote the $d \times k$ matrix with columns $w_1, \dots, w_k$. Suppose first that $S_K \neq \emptyset$. Then $K(-\delta) \neq \emptyset$, and we have, for every $(b, y) \in S_K$ and hence for every $(b, y) \in S_K(-\varepsilon)$,

$$c^T b + c^T Q y - (\tau + 1)\delta d \leq \max_{v \in K(-\delta)} c^T v \leq c^T u + \delta \leq c^T a + c^T Q z + (2 + (\tau + 1)d)\delta,$$

whence

$$c^T b + c^T Q y \leq c^T a + c^T Q z + \varepsilon.$$

Let $v = Q^T c - (e^T Q^T c/k)e$, where $e = (1, \dots, 1)^T \in \mathbb{R}^k$. Then $(c^T, v^T)^T$ is the orthogonal projection of $(c^T, c^T Q)^T$ on the linear subspace parallel to H . Hence

$$c^T b + v^T y \leq c^T a + v^T Q z + \varepsilon.$$

Since $\|(c^T, v^T)^T\|_\infty = \max\{\|c\|_\infty, \|v\|_\infty\} \geq 1$, the vector $(c^T, v^T)^T / \|(c^T, v^T)^T\|_\infty$ is the required normal vector of a weakly separating hyperplane. Of course, this remains true if $S_K = \emptyset$.

Clearly, the reduction works in polynomial time; this concludes the construction of the asserted weak separation oracle for S_K . $\square$

It is now easy to complete the proof of Theorem 2.2.

Proof of Theorem 2.2. By Lemma 3.5 the sets S_{K_i} are circumscribed, and by Lemma 3.7 weak separation oracles with respect to H are available.

Let ε be the error bound in the given instance of WEAK FAT HELLY. By Lemma 3.2 it can be determined in polynomial time whether the inradius (taken in H) of any of the sets S_{K_i} is less than $\varepsilon/(\sqrt{2nn}R)$. If that is the case for some i_0 , it follows from Lemma 3.4 that there does not exist a point $(b, y) \in \mathbb{R}^d \times T^k$ such that $b + A[y] \subset K_{i_0}(-\varepsilon)$, and the algorithm outputs $I = \{i_0\}$.

Otherwise the assertion follows from Theorem 2.1 applied to $S_{K_1}, \dots, S_{K_m}$ with error bound $\varepsilon/(\sqrt{2nn}R)$. $\square$

Finally we turn to the polytopal case; we begin with two lemmas.

Lemma 3.8. *There is a polynomial-time algorithm which accepts as input $d, m, k \in \mathbb{N}$, nonempty $\mathcal{V}$ -polytopes (resp. $\mathcal{H}$ -polytopes) $A_1, \dots, A_k$, and an $\mathcal{H}$ -polytope K in $\mathbb{R}^d$, and that produces an $\mathcal{H}$ -presentation of S_K .*

Proof. Let K denote the $\mathcal{H}$ -polytope $(d, s; Q, b)$, and let $Q^T = (q_1, \dots, q_s)$, $b = (\beta_1, \dots, \beta_s)^T$. For $i = 1, \dots, k$ and $j = 1, \dots, s$, let

$$\gamma_{i,j} = \max_{x \in A_i} q_j^T x.$$

Note that, for a $\mathcal{V}$ -polytope A_i , the maximum can be computed simply by evaluation of the linear functional at the given points of the presentation and subsequent comparison, while, for an $\mathcal{H}$ -polytope, the task is just linear programming.

The set S_K is the set of solutions of the following linear feasibility problem in the variables a and z :

$$q_j^T a + \sum_{i=1}^k \zeta_i \gamma_{i,j} \leq \beta_j \quad (j = 1, \dots, s),$$

$$\sum_{i=1}^k \zeta_i = 1, \quad z \geq 0.$$

The linear description comprises an equality constraint and $s + k$ inequality constraints in n variables; the coefficients are all of sizes that are bounded in the size of the input. It can easily be converted into a standard $\mathcal{H}$ -presentation of S_K . $\square$

Lemma 3.9. *The variants of STRONG FAT HELLY where the bodies $K_1, \dots, K_m$ are all $\mathcal{V}$ -polytopes and the bodies $A_1, \dots, A_k$ are all $\mathcal{V}$ -polytopes or all $\mathcal{H}$ -polytopes are $\mathbb{NP}$ -hard.*

Proof. We begin with the latter assertion. Let A be the standard unit cube $[-1, 1]^d$ and let K be the image of the regular cross-polytope Y_d under a nonsingular linear mapping Q . By Theorem 1.1 of [7], K contains a translate $a + A$ if and only if $A \subset K$. Now take k copies of A and m copies of K to form an instance of STRONG FAT HELLY. Clearly, the solution for this instance determines whether $A \subset K$. However, checking this inclusion is a coNP -complete task by [3]; see also [9]. Note that it is easy to produce an $\mathcal{H}$ -presentation for A and a $\mathcal{V}$ -presentation for K .

We now turn to the case that all bodies involved are $\mathcal{V}$ -polytopes. Let

$$K_1 = \text{conv} \left\{ 0, e_1, \dots, e_d, \frac{2}{2d-1} e \right\},$$

where $e = (1, \dots, 1)^T \in \mathbb{R}^d$, and, for $i = 1, \dots, d$,

$$A_i = \frac{2d}{2d-1} [0, 1] e_i = \text{conv} \left\{ 0, \frac{2d}{2d-1} e_i \right\}.$$

Note that the facet half-spaces of K_1 are given by the equations

$$e_j^T x \geq 0, \quad c_j^T x \leq 1 \quad (j = 1, \dots, d),$$

where $c_j = (\gamma_{j,1}, \dots, \gamma_{j,d})$ with $\gamma_{j,j} = \frac{1}{2}$ and $\gamma_{j,s} = 1$ otherwise. Now, let $(a, z) \in S_{K_1}$. Then the conditions that $e_j^T x \geq 0$ for every $x \in a + A[z]$ imply $a \geq 0$. Further, when evaluated for the nonzero endpoints of the A_i , the condition $c_j^T x \leq 1$ becomes

$$1 \geq c_j^T a + \frac{2d}{2d-1} \sum_{i=1}^d \zeta_i c_j^T e_i \geq \frac{2d}{2d-1} \left(1 - \frac{\zeta_j}{2} \right),$$

for $j = 1, \dots, d$. Hence, $z \geq (1/d)e$, with equality in the j th component only if $c_j^T a = 0$. Since $e^T z = 1$, it follows that $(0, (1/d)e)$ is the unique point in S_{K_1} . The

corresponding set $a + A[z]$ is just the cube $P = (2d/(2d - 1))[0, 1]^d$. Now let K_2 be an affine image of Y_d centered at $(d/(2d - 1))e$. Then $S_{K_1} \cap S_{K_2} \neq \emptyset$ if and only if $P \subset K_2$, a problem that is clearly equivalent to checking the containment of $[-1, 1]^d$ in polytopes of the form $Q(Y_d)$. From here it is easy to finish the transformation. $\square$

Proof of Theorem 2.3. The hardness results are contained in Lemma 3.9, and the tractability results indicated in the table of Theorem 2.3 follow easily from Lemma 3.8 in conjunction with the conceptual algorithm outlined at the beginning of this section. So we deal with the remaining tractability result for k fixed.

Let $K \in \{K_1, \dots, K_m\}$, $K = \text{conv}\{v_1, \dots, v_{p_0}\}$, and for each $i = 1, \dots, k$ let $A_i = \text{conv}\{w_{i,1}, \dots, w_{i,p_i}\}$. Further, let $\Pi = \{1, \dots, p_1\} \times \dots \times \{1, \dots, p_k\}$, and let P_K be the set of feasible points of the following linear feasibility problem in the variables a, z and $y^s = (\eta_1^s, \dots, \eta_{p_0}^s)$ with $s = (\sigma_1, \dots, \sigma_k)^T \in \Pi$:

$$\begin{aligned} a + \sum_{i=1}^k \zeta_i w_{i,\sigma_i} &= \sum_{j=1}^{p_0} \eta_j^s v_j & (s \in \Pi), \\ \sum_{i=1}^k \zeta_i &= 1, & z \geq 0, \\ \sum_{j=1}^{p_0} \eta_j^s &= 1, & y^s \geq 0 & (s \in \Pi). \end{aligned}$$

Clearly, $P_K \subset \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^p$, where $p = p_0 \cdot p_1 \cdot \dots \cdot p_k$, and S_K is just the orthogonal projection of P_K on $\mathbb{R}^d \times \mathbb{R}^k$. Further, the polytope P_K is given by at most $(d + 1)p + 1$ equality constraints and $k + p$ inequality constraints. Since k is fixed this is a polynomial $\mathcal{H}$ -presentation of P_K .

We now consider $\bigcap_{i \in I} S_{K_i}$ for $I \subset M$. If it can be decided in polynomial time whether this intersection is empty, we can again proceed as in the conceptual algorithm outlined at the beginning of this section. We may suppose (without loss of generality) that for each $\mathcal{V}$ -polytope K_i the same number p_0 of points is specified in the input string. Now consider the following linear feasibility problem:

$$v \in \mathbb{R}^n \times \{0\}^p, \quad x_i \in P_{K_i}, \quad y_i \in \{0\}^n \times \mathbb{R}^p, \quad x_i + y_i = v \quad (i \in I)$$

in the variables x_i, y_i ($i \in I$) and v . Clearly, the problem is feasible if and only if $\bigcap_{i \in I} S_{K_i} \neq \emptyset$, and each vector v in a solution is contained in this intersection. Hence the assertion follows from the fact that the linear feasibility problem can be solved in polynomial time. $\square$

4. Extending Helly's Theorem

As we have seen in the previous section, the underlying idea for deriving the new Helly-type theorem associated with STRONG FAT HELLY was to consider suitable mappings of $\mathbb{R}^n$ into certain convexity spaces so as to parametrize more general objects by convex

bodies in $\mathbb{R}^n$. This approach could now be axiomatized to derive further Helly-type theorems. We, however, pursue a more concrete way of deriving extensions of Helly's theorem by only considering finite-dimensional sections of the product $\mathcal{K}^d \times \cdots \times \mathcal{K}^d$ of cones of convex bodies. We begin with a rather general result along these lines; a few special cases follow to exemplify the spirit of the present generalization.

For $A_1, \dots, A_k \in \mathcal{K}^d$, let $\text{conv}_{\mathcal{K}^d}(A_1, \dots, A_k)$ denote the convex hull of $A_1, \dots, A_k$ in $\mathcal{K}^d$ (with respect to scalar multiplication and Minkowski addition). We identify $\mathbb{R}^d$ with the subset of $\mathcal{K}^d$ of singletons. Further, for $\mathbf{B} = (B_1, \dots, B_s), \mathbf{C} = (C_1, \dots, C_s) \in \mathcal{K}^d \times \cdots \times \mathcal{K}^d = (\mathcal{K}^d)^s$, and $\xi \geq 0$, the operations $\xi \mathbf{B}$, and $\mathbf{B} + \mathbf{C}$ are defined componentwise, i.e.,

$$\xi \mathbf{B} = (\xi B_1, \dots, \xi B_s), \quad \mathbf{B} + \mathbf{C} = (B_1 + C_1, \dots, B_s + C_s).$$

So in this sense $\mathbf{B}$ and $\mathbf{C}$ are just regarded as vectors. It will, however, also be necessary to regard the objects as s -tuples of convex bodies, and to apply some of the usual set theoretic relations and operations. They are denoted by the square versions $\sqsubset, \sqsupset, \sqcup$, etc., of the familiar symbols, and are to be understood componentwise. For instance, $\mathbf{B} \sqsubset \mathbf{C}$ means $B_i \subset C_i$ for all $i = 1, \dots, s$.

Theorem 4.1. *Let $s \in \mathbb{N}$, $n \in \mathbb{N}_0$, and let $\mathcal{K} \subset (\mathcal{K}^d)^s$ and $\mathcal{C} \subset (\mathcal{C}^d)^s$, $\mathcal{C}$ finite, each of cardinality at least $n + 1$. For $t \in \mathbb{N}$, $k_0, \dots, k_t \in \mathbb{N}_0$, with $n = \sum_{j=0}^t k_j$, and $j = 1, \dots, t$, let $\mathbf{A}_{j,0}, \dots, \mathbf{A}_{j,k_j} \in (\mathcal{K}^d)^s$ and $\mathbf{a}_0, \dots, \mathbf{a}_{k_0} \in (\mathbb{R}^d)^s$. Further, let $\mathcal{A} \subset \text{aff}\{\mathbf{a}_0, \dots, \mathbf{a}_{k_0}\} + \sum_{j=1}^t \text{conv}_{\mathcal{K}^d}(\mathbf{A}_{j,0}, \dots, \mathbf{A}_{j,k_j})$ be closed and convex. Then the following hold:*

- (i) *There exists a body $\mathbf{A} \in \mathcal{A}$ that is contained in each member of $\mathcal{K}$ (resp. $\mathcal{C}$) if and only if the same is true for each $n + 1$ members of $\mathcal{K}$ (resp. $\mathcal{C}$).*
- (ii) *There exists a body $\mathbf{A} \in \mathcal{A}$ that contains each member of $\mathcal{K}$ if and only if the same is true for each $n + 1$ members of $\mathcal{K}$.*
- (iii) *There exists a body $\mathbf{A} \in \mathcal{A}$ that intersects each member of $\mathcal{K}$ (resp. $\mathcal{C}$) if and only if the same is true for each $n + 1$ members of $\mathcal{K}$ (resp. $\mathcal{C}$).*

Proof. We begin with (i). Since the “only if” part of the assertion is trivial, suppose that the intersection of each $n + 1$ members of $\mathcal{K}$ (resp. $\mathcal{C}$) contains a body $\mathbf{A}$ of $\mathcal{A}$. Without loss of generality we may assume that $\mathbf{a}_0, \dots, \mathbf{a}_{k_0}$ are affinely independent. For $j = 0, \dots, k_0$ let $\mathbf{a}_j = (a_{j,1}^T, \dots, a_{j,s}^T)^T$, and set $\mathbf{A}_j = (\{a_{j,1}\}, \dots, \{a_{j,s}\})$. For simplicity of notation we collect all sets in the ordered $(n + t + 1)$ -tuple

$$(\mathbf{A}_0, \dots, \mathbf{A}_{k_0}, \mathbf{A}_{1,0}, \dots, \mathbf{A}_{1,k_1}, \dots, \mathbf{A}_{t,0}, \dots, \mathbf{A}_{t,k_t})$$

and then rename them $(\mathbf{A}_1, \dots, \mathbf{A}_p)$, where $p = n + t + 1$. Then, let

$$T = \left\{ x = (\xi_1, \dots, \xi_p)^T : \xi_{k_0+2}, \dots, \xi_p \geq 0 \wedge \sum_{i=1}^{k_0+1} \xi_i = 1 \right. \\ \left. \wedge \sum_{i=k_0+2}^{k_0+k_1+2} \xi_i = 1, \dots, \sum_{i=p-k_t}^p \xi_i = 1 \right\}.$$

Note that T is a subset of $\mathbb{R}^p$ of dimension n , hence $\text{aff}(T)$ can be identified with the affine space $\mathbb{R}^n$. For $x \in T$ set $\mathbf{A}[x] = \sum_{i=1}^p \xi_i \mathbf{A}_i$. Now let, for $\mathbf{C} \in \mathcal{C}$,

$$S_{\mathbf{C}} = \{x \in T : \mathbf{A}[x] \in \mathcal{A} \wedge \mathbf{A}[x] \sqsubset \mathbf{C}\}.$$

Clearly, $S_{\mathbf{C}}$ is a closed convex subset of $\mathbb{R}^p$ of dimension at most n . Further, the family $S_{\mathcal{K}} = \{S_{\mathbf{C}} : \mathbf{C} \in \mathcal{K}\}$ (resp. $S_{\mathcal{C}} = \{S_{\mathbf{C}} : \mathbf{C} \in \mathcal{C}\}$) has the property that each $n+1$ members intersect. Hence the result for $\mathcal{C}$ follows from Helly's theorem, and so does the statement for $\mathcal{K}$ once the boundedness of the sets $S_{\mathbf{C}}$ is established.

So, suppose that $\mathbf{K} \in \mathcal{K}$ but $S_{\mathbf{K}}$ is not bounded. Then there exists a sequence $(x_i)_{i \in \mathbb{N}}$ in T with $\|x_i\| \rightarrow \infty$ as $i \rightarrow \infty$ such that $\mathbf{A}[x_i] \sqsubset \mathbf{K}$. By Blaschke's selection theorem (applied componentwise) there is a subsequence of $(\mathbf{A}[x_i])_{i \in \mathbb{N}}$ that converges (componentwise in the Hausdorff metric δ_H) to an element $\mathbf{A}$ of $(\mathcal{K}^d)^s$. We may assume that the whole sequence converges. Let I_1, I_2 be a partition of $\{1, \dots, p\}$ such that the coordinate subsequences $(\xi_{ij})_{j \in \mathbb{N}}$ of $(x_j)_{j \in \mathbb{N}}$ are unbounded for $i \in I_1$ and bounded for $i \in I_2$. Then, of course, $I_1 \subset \{1, \dots, k_0 + 1\}$. Using the standard diagonal argument we may assume that the sequences $(\xi_{ij})_{j \in \mathbb{N}}$ are actually convergent for $i \in I_2$; let ξ_i denote the limits and set $\hat{\mathbf{A}} = \sum_{i \in I_2} \xi_i \mathbf{A}_i$. It follows that

$$\delta_H \left(\sum_{i \in I_1} \xi_{ij} \mathbf{a}_{i-1} + \hat{\mathbf{A}}, \mathbf{A} \right) \rightarrow 0 \quad (\text{as } j \rightarrow \infty),$$

whence $\sum_{i \in I_1} \xi_{ij} \mathbf{a}_{i-1}$ is convergent. This implies that the set $\{\mathbf{a}_{i-1} : i \in I_1\}$ is affinely dependent, contradicting the assumption that $\mathbf{a}_0, \dots, \mathbf{a}_{k_0}$ are affinely independent. This shows that the members of $S_{\mathcal{K}}$ are convex bodies in $\mathbb{R}^n$ and concludes the proof of the first assertion.

In order to prove (ii) and (iii) replace the sets $S_{\mathbf{C}}$ by

$$S'_{\mathbf{C}} = \{x \in T : \mathbf{A}[x] \in \mathcal{A} \wedge \mathbf{A}[x] \sqsupset \mathbf{C}\}$$

and

$$S''_{\mathbf{C}} = \{x \in T : \mathbf{A}[x] \in \mathcal{A} \wedge \mathbf{A}[x] \sqcap \mathbf{C} \neq \emptyset\},$$

respectively. Then the previous reasoning carries over to $S'_{\mathbf{C}}$ and $S''_{\mathbf{C}}$ with all but trivial changes; for the part dealing with the boundedness of the sets $S'_{\mathbf{C}}$ and $S''_{\mathbf{C}}$ for $\mathbf{C} \in \mathcal{K}$ just use again that $\bigsqcup (\sum_{j=1}^t \text{conv}_{\mathcal{K}^d}(\mathbf{A}_{j,0}, \dots, \mathbf{A}_{j,k_j}))$ is compact and apply the same reasoning as before. $\square$

Note that the number $n+1$ in Theorem 4.1 is in general only an upper bound for the Helly number since the parametrization may not be one-to-one. However, even if it is, T has still to be intersected with $Z_{\mathcal{A}} = \{x : \mathbf{A}[x] \in \mathcal{A}\}$. This leads to the following remark.

Remark 4.2. The Helly number of the family $\mathcal{K}$ or $\mathcal{C}$ in Theorem 4.1 is bounded above by $\dim(T \cap \text{aff}(Z_{\mathcal{A}})) + 1$.

Note that Helly's classical theorem is recovered from Theorem 4.1 by the choice of $s = t = 1$, $k_0 = d$, $\mathbf{a}_0, \dots, \mathbf{a}_d$ affinely independent, $k_1 = 0$, $\mathbf{A}_{1,0} = A = \{0\}$, and $\mathcal{A} = \mathbb{R}^d$. Further, the same choice for $s, t, k_0, k_1, \mathbf{a}_0, \dots, \mathbf{a}_d$, and $\mathcal{A}$ but with an arbitrary $A \in \mathcal{K}^d$ yields the Vincensini/Klee extension, while the choice $s = t = 1$, $k_0 = d$, $\mathbf{a}_0, \dots, \mathbf{a}_d$ affinely independent, $k_1 = k - 1$, $\mathbf{A}_{1,0} = A_1, \dots, \mathbf{A}_{1,k_1} = A_k$, and $\mathcal{A} = \mathbb{R}^d + \text{conv}_{\mathcal{K}^d}(\mathbf{A}_{1,0}, \dots, \mathbf{A}_{1,k_1})$ yields the Helly-type result underlying FAT HELLY.

A few additional examples are now given to indicate further what kind of flavor the extension has. The first result deals again with the case $s = t = 1$.

For a set $U = \{u_0, \dots, u_k\}$ of mutually noncollinear vectors of the Euclidean unit sphere $\mathbb{S}^{d-1}$, a U -zonotope P is a (not necessarily full-dimensional) zonotope whose edge-directions are contained in U . Equivalently, there exist $x \in \mathbb{R}^d$ and $\lambda_0, \dots, \lambda_k \geq 0$ such that $P = x + \sum_{i=0}^k [0, \lambda_i]u_i$. Note that λ_i is just the length of each edge in direction u_i . It is not hard to see that the above presentation of P is unique, hence $\beta_i(P) = \lambda_i$ for $i = 0, \dots, k$ and $\alpha(P) = (\sum_{i=0}^k \lambda_i)/(k+1)$ can be defined. Note that $\beta_i(P)$ is the length of each edge parallel to u_i whence, if U is in general position, $\alpha(P)$ is just the average edge-length of P .

Example 4.3. *Let k be a nonnegative integer, let $u_0, \dots, u_k \in \mathbb{S}^{d-1}$ be in general position, set $U = \{u_0, \dots, u_k\}$, let $\mathcal{K} \subset \mathcal{K}^d$ of cardinality at least $d+k+1$, and let $\alpha, \beta_0, \dots, \beta_k \geq 0$. Then there exists a U -zonotope P with $\beta_i(P) \geq \beta_i$ for $i = 0, \dots, k$, and of average edge-length α , that is contained in (resp. contains) all members of $\mathcal{K}$ if and only if the same property holds for each $d+k+1$ members of $\mathcal{K}$.*

Proof. The assertion follows from Theorem 4.1 with the following choice of parameters: $s = t = 1$, $k_0 = d$, $\mathbf{a}_0, \dots, \mathbf{a}_d$ affinely independent, $k_1 = k$, $\mathbf{A}_{1,0} = [0, (k+1)\alpha]u_0, \dots, \mathbf{A}_{1,k_1} = [0, (k+1)\alpha]u_k$, and $\mathcal{A}$ is chosen to be the subset of $\mathbb{R}^d + \text{conv}_{\mathcal{K}^d}(\mathbf{A}_{1,0}, \dots, \mathbf{A}_{1,k_1})$ that consists of sets of the form $\mathbf{A}[x]$, where $x = (\xi_1, \dots, \xi_{d+k+2})^T$ belongs to

$$\left\{ x: \xi_{d+2}, \dots, \xi_{d+k+2} \geq 0 \wedge \sum_{i=1}^{d+1} \xi_i = 1 \wedge \sum_{i=d+2}^{d+k+2} \xi_i = 1 \wedge \xi_{d+2} \geq \beta_0, \dots, \xi_{d+k+2} \geq \beta_k \right\}.$$

In fact, Theorem 4.1 shows that there exists a translate $x + P \in \mathcal{A}$ of a convex combination $P = \sum_{i=0}^k \zeta_i \mathbf{A}_{1,i}$ of the sets $\mathbf{A}_{1,i}$ such that $x + P$ is contained in (resp. contains) all members of $\mathcal{K}$ if and only if the same property holds for each $d+k+1$ members of $\mathcal{K}$. Now note that P is a U -zonotope with $\alpha(P) = 1/(k+1) \sum_{i=0}^k \zeta_i (k+1)\alpha = \alpha$, and since $x + P \in \mathcal{A}$ the edge-lengths are appropriately bounded from below. $\square$

To give an example for $t = 2$, recall that the standard parallel body of a convex body $K \in \mathcal{K}^d$ (with respect to the Euclidean unit ball $\mathbb{B}^d$) is the body $K + \mathbb{B}^d$. Then with the additional choice of $t = 2$, $k_2 = 0$, and $\mathbf{A}_{2,0} = \mathbb{B}^d$ the “ U -zonotope” in Example 4.3 can be replaced by the “parallel body of a U -zonotope.”

The next example involves a higher value of s ; it is a “tomographic” Helly-type theorem.

Example 4.4. Let k be a nonnegative integer, let $\ell_1, \dots, \ell_k$ be lines in $\mathbb{R}^d$ through the origin, and let $\mathcal{K} \subset \mathcal{K}^d$ be of cardinality at least $d + k + 1$. For $a \in \mathbb{R}^d$ and $K \in \mathcal{K}^d$, let $v(a; K) = \sum_{i=1}^k V_1((a + \ell_i) \cap K)$, where V_1 denotes the one-dimensional volume, and let $\tau > 0$. Then there exists a point $a \in \mathbb{R}^d$ such that $v(a; \bigcap_{K \in \mathcal{K}} K) \geq \tau$ if and only if this is true for the intersection of each $d + 2k + 1$ members of $\mathcal{K}$.

Proof. For $i = 1, \dots, k$ let $u_i \in \ell_i \cap \mathbb{S}^{d-1}$. Then the following choices for the parameters in Theorem 4.1 are made: $s = k$, $t = k$, $k_0 = d$, $b_0, \dots, b_d \in \mathbb{R}^d$ affinely independent, $\mathbf{a}_i = (b_i, \dots, b_i)$ for $i = 1, \dots, d$, $k_1 = \dots = k_t = 2$, $\mathbf{A}_{j,0} = ([-\tau, 0]u_j, \{0\}, \dots, \{0\})$, $\mathbf{A}_{j,1} = (\{0, \tau\}u_j, \{0\}, \dots, \{0\})$, $\mathbf{A}_{j,2} = (\{0\}, \dots, \{0\})$ for $j = 1, \dots, t$, and $\mathcal{A}$ is chosen to be the subset of $(\mathbb{R}^d)^s + \sum_{i=1}^k \text{conv}_{\mathcal{K}^d}(\mathbf{A}_{i,0}, \mathbf{A}_{i,1}, \mathbf{A}_{i,2})$ that consists of sets of the form $\mathbf{A}[x]$, where $x = (\xi_1, \dots, \xi_{d+3k+1})^T$ satisfies $\sum_{i=1}^{d+1} \xi_i = 1$, $\xi_{d+i+1} \geq 0$ for $i = 1, \dots, 3k$, $\xi_{d+3i-1} + \xi_{d+3i} + \xi_{d+3i+1} = 1$ for $i = 1, \dots, k$, and $\sum_{i=1}^k (\xi_{d+3i-1} + \xi_{d+3i}) \geq 1$. Further, the statement involves the family $\{(K, \dots, K) \in (\mathcal{K}^d)^k : K \in \mathcal{K}\}$ and it follows directly from Theorem 4.1 in conjunction with Remark 4.2. $\square$

It is easy to derive a variant of Example 4.4 that deals with U -cross-polytopes and measures the size of the cross. To be more specific, let $U = \{u_1, \dots, u_d\} \subset \mathbb{S}^{d-1}$ be linearly independent. Then a polytope P is called a U -cross-polytope if there exist reals $\lambda_1, \dots, \lambda_d \geq 0$ such that $P = \text{conv}\{\pm\lambda_1 u_1, \dots, \pm\lambda_d u_d\}$. The number $\mu(P) = 2 \sum_{i=1}^d \lambda_i$ is the *cross-length* of P .

Example 4.5. Let $U = \{u_1, \dots, u_d\} \subset \mathbb{S}^{d-1}$ be a basis of $\mathbb{R}^d$, $\tau > 0$, and let $\mathcal{K} \subset \mathcal{K}^d$ be of cardinality at least $2d + 1$. Then $\bigcap_{K \in \mathcal{K}} K$ contains a U -cross-polytope P of cross-length τ if and only if this is true for the intersection of each $2d + 1$ members of $\mathcal{K}$.

To close this section we point out that in view of the one-to-one correspondence between convex bodies and sublinear functions (see, e.g., Theorem 1.7.1 of [16]), Theorem 4.1 could be formulated in terms of sublinear functions, and would then turn into a statement involving the existence of a sublinear function h that is majorized by all given sublinear functions c (or with $h \geq c$, or such that there exists a linear functional l with $l \leq h, c$). We refrained from doing that but we like to point out that the convex hull operation with respect to $\mathcal{K}^d$ could then be replaced by an affine hull operation, allowing differences of sublinear functions to be included. This may be relevant in view of the fact that the real vector space spanned by the difference of two sublinear functionals (restricted to $\mathbb{S}^{d-1}$) is dense in $C(\mathbb{S}^{d-1})$ (see Lemma 1.7.9 of [16]), a property that is important in global optimization; see [13].

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