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Branches of positive solutions for some semilinear Schrödinger equations

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1 Introduction

Semilinear elliptic problems on all of $\mathbb{R}^N$ of the form

$$- \Delta u + q(x)u = \lambda u + g(x)h(u)u, \quad x \in \mathbb{R}^N , \tag{P}$$

have been widely investigated under various assumptions on q, g and h, *see*, for example, [6, 7, 13] and references therein.

In particular, the results of [7] deal with the case in which $q \in L^\infty$, $g > 0$, $h(0) = 0$, $h(s)$ is strictly decreasing and $h(s) \to -\infty$ as $s \to +\infty$ and yield the existence of a bifurcation branch of positive solutions of (P) that, roughly, blows up (in a suitable Lebesgue norm) as λ tends to a, the infimum of the essential spectrum of the linear Schrödinger operator $-\Delta + q$.

The main purpose of the present paper is to consider that case in which h has, roughly, the same asymptotic behaviour but is not necessarily decreasing and $g(x)$ can possibly vanish in a bounded domain $\Omega_0 \subset \mathbb{R}^N$.

These specific features of h and g are motivated also by some problems arising in Nonlinear Optics.[1] Actually, the study of nonlinear modes in a layered structure leads to a Schrödinger equation of the form (see [2, 8, 9 and 13])

$$u'' + \varepsilon(x, u^2)u = k^2 u, \quad x \in \mathbb{R} , \tag{1.1}$$

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[1] When this work was in progress, references [2] and [9] were brought to the attention of the first author by Tassilo Küpper. We wish to thank him for the useful information.

where, whenever the material out of the layer is assumed to be defocusing,

$$\varepsilon(x,u^2) = \begin{cases} \alpha_1 & \text{for } |x| \leqq d \\ \alpha_2 - u^2 & \text{for } |x| > d \ . \end{cases}$$

Setting $\lambda = -k^2$,

$$q(x) = \begin{cases} -\alpha_1 & \text{for } |x| \leqq d \\ -\alpha_2 & \text{for } |x| > d \end{cases} \qquad g(x) = \begin{cases} 0 & \text{for } |x| \leqq d \\ 1 & \text{for } |x| > d \end{cases}$$

and $h(u) = -u^2$, (1.1) becomes of the form (P) with a nonlinear term like that discussed in the present paper.

In order to find bifurcation branches of positive solutions of (P) we approximate (P) with Dirichlet boundary value problems on balls. Actually, we deal in Sect. 2 with an elliptic eigenvalue problem such as

$$\begin{cases} -\Delta u + q(x)u = \lambda u + g(x)h(u)u, & x \in \Omega \ , \\ u(x) = 0, & x \in \partial\Omega \ , \end{cases} \tag{P_Ω}$$

where Ω is a general bounded domain in $\mathbb{R}^N$. When g vanishes on a subset Ω_0 of Ω, problem (P_Ω) has been studied by Alama and Tarantello in [1] by variational methods, see also [10]. Unlike [1] we use here bifurcation theory and improve those results by showing that the branch of positive solutions of (P_Ω) bifurcating from the trivial solution at $\lambda = \lambda_\Omega$ (the first eigenvalue of $-\Delta + q$ on $W_0^{1,2}(\Omega)$), blows up in L^p, $p \geqq 1$, as $\lambda \to \lambda_{\Omega_0}$; moreover (P_Ω) has no positive solutions for $\lambda \geqq \lambda_{\Omega_0}$, see Theorem 2.6.

In Sect. 3 we turn to problem (P) and prove a general global bifurcation result, see Theorem 3.4, under an uniform a-priori estimate on the branches of the approximated problems. Taking $\Omega = B_R$ and letting $R \to \infty$, this a-priori estimate allows us to show that the branches of the approximated problems converge, in an appropriate sense, to a branch of positive solutions of (P). As a first application, we handle a problem studied by Brézis and Kamin in [5], see Theorem 3.6.

In Sect. 4 we still deal with (P) and show that the bound above can be actually found provided that, roughly, the principal eigenvalue Λ of $-\Delta + q$ on $W^{1,2}(\mathbb{R}^N)$ is smaller than q, see Theorem 4.4 for the precise statement. In particular this result applies when $g(x) > 0$ on $\mathbb{R}^N$ yielding an improvement of the existence results of Edelson and Stuart [7].

Finally, in Sect. 5 we assume that $\Omega_0 \neq \emptyset$ and, roughly, $\lambda_{\Omega_0} < a$, and show, by an appropriate comparison with problems with decreasing nonlinearities, that the branch of positive solutions of (P) blows up in L^p, $p \geqq 1$, iff $\lambda \uparrow \lambda_{\Omega_0}$, see Theorem 5.2. Such a result is in striking contrast with that in [7], see Remark 5.3.

Notation. In the sequel Ω denotes a bounded domain of $\mathbb{R}^N$ with (smooth) boundary $\partial\Omega$.

$W_0^{1,2}(\Omega)$ or $W_0^{1,2}(\mathbb{R}^N)$ denote Sobolev spaces and $L^p = L^p(\Omega)$ or $L^p = L^p(\mathbb{R}^N)$ denote Lebesgue spaces. For brevity and whenever unambiguous, the indication of Ω or $\mathbb{R}^N$ will be omitted. The standard norm in L^p will be denoted by $|u|_p$.

In the rest of the paper we will often extend a function z outside its support by setting $z(x) \equiv 0$ therein. To keep the notation as light as possible, this extended function will still be denoted by z.

2 Problems on bounded domains

In this section we deal with problem (P_Ω). We set $E = W_0^{1,2}(\Omega)$, endowed with norm $\|u\|^2 := \int |\nabla u|^2$. The first eigenvalue of the linear problem

$$\begin{cases} -\Delta u + q(x)u = \lambda u, & x \in \Omega\,, \\ u(x) = 0, & x \in \partial\Omega\,, \end{cases}$$

will be denoted by $\lambda_\Omega[q]$, or simply by λ_Ω. We also denote by ϕ_Ω, the eigenfunction corresponding to λ_Ω, with $\phi_\Omega(x) > 0$ and $|\phi_\Omega|_2 = 1$. We shall use the variational characterization of λ_Ω, namely:

$$\lambda_\Omega = \inf_{u \in E,\ |u|_2 = 1} \int_\Omega [|\nabla u|^2 + qu^2]\,,$$

We consider problem (P_Ω) and assume

(Q) $\quad q \in L^\infty(\Omega)$;

(A_1) $\quad g \in L^\infty(\Omega)$, $g \geqq 0$, *and there exists a (bounded) domain* Ω_0 *with smooth (say* $C^{1,\nu}$*) boundary* $\partial\Omega_0$ *such that* $\Omega_0 \subset \Omega$ *and* $g(x) = 0$ *iff* $x \in \Omega_0$;

(A_2) $\quad h \in C(\mathbb{R}^+)$, $h(0) = 0$, $\exists \kappa > 0$, $c_0 > 0$ *such that* $h(s) \leqq c_0 s^\kappa$, $\forall s > 0$ *and* $h(s) \to -\infty$ *as* $s \to +\infty$.

In particular, we explicitly point out that (A_2) implies there exists $C_0 \geqq 0$ such that

$$h(s) \leqq C_0\,. \tag{2.1}$$

In the sequel it is understood that assumptions (Q) and $(A_1$–$A_2)$ hold true.

a) Some preliminary Lemmas. Although the following lemma is perhaps well known, we give an outline of the proof for the reader's convenience.

Hereafter, by a positive solution of (P_Ω) we mean an $u \in E$, $u > 0$, which solves (P_Ω) weakly. Actually, in our case, weak solutions belong to $C^{1,\nu}$. Of course, if q and g are Hölder continuous, u will become a classical solution.

Lemma 2.1 *From* $(\lambda_\Omega, 0)$ *bifurcates an unbounded branch of positive solutions of problem* (P_Ω).

Proof. Let $M \geqq 0$ be such that $q(x)+M \geqq 0$ in Ω, and consider the problem:

$$\begin{cases} -\Delta u + [q(x)+M]u = \mu u + g(x)h(u)u & \text{in } \Omega , \\ u(x) = 0 & \text{on } \partial\Omega . \end{cases} \tag{2.2}$$

Trivially, (λ, u) is a solution of problem (P_Ω) iff $(\mu, u) = (\lambda+M, u)$ is a solution of problem (2.2).

Letting $K = (-\Delta + [q+M])^{-1}$, equation (2.2) can be written as

$$u = \mu K u + K(gh(u)u) . \tag{2.3}$$

It is immediate to check that problem (2.3) satisfies the hypotheses of the Global Bifurcation Theorem of Rabinowitz (see [11]). Then a branch of positive solutions (μ, u) of (2.2) bifurcates from $\lambda_\Omega[q+M] = \lambda_\Omega[q]+M$ and yields a branch of solutions of (P_Ω) bifurcating from λ_Ω. Moreover, by standard arguments it can be proved that this branch remains in the interior of the cone of positive functions of $C_0^1(\Omega)$. Since λ_Ω is the only eigenvalue with corresponding positive eigenfunction, the branch cannot meet another eigenvalue different from λ_Ω, and thus it is unbounded. □

Let $\lambda_0 = \lambda_{\Omega_0}$ and $\phi_0 = \phi_{\Omega_0}$.

Lemma 2.2 *There exists* $\underline{\lambda} \in \mathbb{R}$ *such that for every positive solutions* (λ, u) *of* (P_Ω) *one has*

$$\underline{\lambda} < \lambda < \lambda_0 .$$

Proof. We set $\underline{q} = \inf_\Omega q(x)$ and $\overline{g} = \sup_\Omega g(x)$. Let (λ, u) be a positive solution of (P_Ω). Then

$$\|u\|^2 + \int_\Omega q(x)u^2 = \lambda |u|_2^2 + \int_\Omega g(x)h(u)u^2 ,$$

and one has:

$$\lambda = \frac{\|u\|^2 + \int_\Omega q(x)u^2 - \int_\Omega g(x)h(u)u^2}{|u|_2^2} \geqq \underline{\lambda} := \underline{q} - \overline{g}C_0 ,$$

where C_0 is given in (2.1).

Next, from (P_Ω) it follows that

$$-\int_\Omega \Delta u \phi_0 + \int_\Omega q(x)u\phi_0 = \lambda \int_\Omega u\phi_0 + \int_\Omega g(x)h(u)u\phi_0 .$$

Since $g(x) \equiv 0$ on Ω_0 one infers that

$$-\int_{\Omega_0} \Delta u \phi_0 - \int_{\Omega_0} q(x)u\phi_0 = \lambda \int_{\Omega_0} u\phi_0 . \tag{2.4}$$

Since $u > 0$ and $\partial\phi_0/\partial\mathbf{n} < 0$ on $\partial\Omega_0$ ($\mathbf{n}$ denotes the outer unit normal at Ω_0), an integration by parts yields

$$-\int_{\Omega_0} \Delta u\phi_0 < -\int_{\Omega_0} u\Delta\phi_0 = \int_{\Omega_0} u[\lambda_0 - q(x)]\phi_0 .$$

This and (2.4) imply that $\lambda < \lambda_0$. □

Remark 2.3 Let us explicitely point out that the lower bound $\underline{\lambda}$ does not depend upon Ω. □

b) Blow up as $\lambda \uparrow \lambda_0$. We begin with the following Lemma.

Lemma 2.4 *Let* (λ_n, u_n) *be a sequence of positive solutions of* (P_Ω) *such that* $|u_n|_2 \to \infty$. *Then* $\lambda_n \to \lambda_0$, $\lambda_n < \lambda_0$.

Proof. By Lemma 2.2 we can assume that $\lambda_n \to \widehat{\lambda} \in [\underline{\lambda}, \lambda_0]$. Setting $z_n = u_n/|u_n|_2$ one has

$$\begin{cases} -\Delta z_n + q(x) z_n = \lambda_n z_n + g(x) h(u_n) z_n, & x \in \Omega\ , \\ z_n(x) = 0 & x \in \partial\Omega\ . \end{cases} \tag{2.5}$$

Hence

$$\int_\Omega |\nabla z_n|^2 + \int_\Omega q(x) z_n^2 = \lambda_n \int_\Omega z_n^2 + \int_\Omega g(x) h(u_n) z_n^2\ .$$

Using (2.1) and since $|z_n|_2 = 1$ we infer from (2.5)

$$\|z_n\|^2 = \int_\Omega |\nabla z_n|^2 \leqq \int_\Omega (\lambda_n - q(x) + g(x) C_0) z_n^2 \leqq c_1\ .$$

Thus, up to a subsequence, $z_n \to z$ strongly in $L^2(\Omega)$, and $|z|_2 = 1$.

Let D be any domain such that $\overline{D} \subset \Omega \backslash \overline{\Omega}_0$ and let $\phi \in C_0^\infty(D)$. From (2.5) it follows

$$-\int_D z_n \Delta\phi + \int_D q(x) z_n \phi - \lambda_n \int_D z_n \phi = \int_D g(x) h(u_n) z_n \phi\ .$$

If $z(x) > 0$ for a.e. $x \in D$, one has $u_n(x) = z_n(x)|u_n|_2 \to \infty$ a.e. in D. Since $g > 0$ in $\overline{D}$, (A_2) implies

$$\int_D g(x) h(u_n) z_n \phi \to -\infty\ .$$

On the other hand

$$\int_D [-z_n \Delta\phi + q(x) z_n \phi - \lambda_n z_n \phi] \to \int_D [-z\Delta\phi + q(x) z\phi - \widehat{\lambda} z\phi] > -\infty\ ,$$

a contradiction. This shows that $z(x) = 0$ a.e. in $\Omega \backslash \Omega_0$. Recalling that (see [4], Proposition IX.18)

$$W_0^{1,2}(\Omega_0) = \{u \in W_0^{1,2}(\Omega) \colon u = 0 \text{ a.e. in } \Omega \backslash \Omega_0\}\ ,$$

it follows that $z \in W_0^{1,2}(\Omega_0)$.

Now, let $\varphi \in C_0^\infty(\Omega_0)$. Since $g(x) \equiv 0$ in Ω_0, one has

$$\int_\Omega g(x) h(u_n) z_n \varphi = 0\ ,$$

for all n. Then, multiplying (2.5) by φ and integrating by parts, one finds

$$-\int_{\Omega_0} z_n \Delta\varphi + \int_{\Omega_0} q(x) z_n \varphi = \lambda_n \int_{\Omega_0} z_n \varphi\ .$$

Passing to the limit one has

$$-\int_{\Omega_0} z\Delta\varphi + \int_{\Omega_0} q(x)z\varphi = \widehat{\lambda}\int_{\Omega_0} z\varphi, \quad \forall\varphi\in C_0^\infty(\Omega_0)\ .$$

This shows that z satisfies

$$-\Delta z + q(x)z = \widehat{\lambda}z \quad \text{in } \Omega_0\ .$$

Since $z \in W_0^{1,2}(\Omega_0)$, $z \geqq 0$ in Ω_0 and $|z|_2 = 1$, it follows that $\widehat{\lambda} = \lambda_0$ and $z = \phi_0$. □

Remarks 2.5 (i) The preceding proof shows that $u_n \simeq |u_n|_2\phi_0$, for n large.

(ii) When $\Omega_0 = \emptyset$, or else when Ω_0 has zero Lebesgue measure, the preceding arguments prove that $\lambda_n \to +\infty$. Actually, if not, the first part of the proof shows that $z(x)=0$ a.e. in Ω, which is in contradiction with the fact that $|z|_2 = 1$. □

c) Branches of positive solutions. We are in position to prove the main result of this section. In the sequel we shall denote by Π the canonical projection of $\mathbb{R}\times E$ onto $\mathbb{R}$.

Theorem 2.6 *Assume* (Q) *and* $(A_1 - A_2)$ *hold. Then from* $(\lambda_\Omega, 0)$ *bifurcates an unbounded branch* S_Ω *of positive solutions of* (P_Ω) *such that*:

(i) $\Pi(\bar{S}_\Omega) = [\lambda^*, \lambda_0[$, *for some* $\lambda^* \leqq \lambda_\Omega$;

(ii) *On* S_Ω *one has that, for all* $p \geqq 1$, $|u|_p \to +\infty$ *iff* $\lambda \uparrow \lambda_0$.

Proof. It suffices to apply Lemmas 2.1, 2.2, 2.4 and Remark 2.5(i). □

Remarks 2.7 (i) The preceding Theorem improves some results of [1].

(ii) If h is a decreasing function, then (P_Ω) has a unique positive solution for all $\lambda\in]\lambda_\Omega, \lambda_0[$. In such a case the branch of positive solutions is a graph and could have been found by using sub- and super-solutions.

(iii) Completing Remark 2.5(ii), when Ω_0 has zero Lebesgue measure or it is possibly empty, there is an unbounded branch S_Ω of positive solutions of (P_Ω) such that $\Pi(\bar{S}_\Omega)=[\lambda^*, +\infty[$.

(iv) According to the classical Theorem of Bifurcation from the simple eigenvalue, the behavior of the branch S_Ω near λ_Ω depends on the sign of h near 0. In particular, if $h(s) > 0$ in a right neighbourhood of $s=0$, then $\lambda^* < \lambda_\Omega$.

3 A general global bifurcation result

We turn to the Schrödinger equation on all of $\mathbb{R}^N$

$$-\Delta u + q(x)u = \lambda u + g(x)h(u)u, \quad x\in\mathbb{R}^N\ , \qquad (P)$$

where, hereafter, $N \geqq 1$. In the sequel q, g are supposed to satisfy (Q) and (A_1) with $\Omega = \mathbb{R}^N$, as well as h is assumed to verify (A_2). In any case, it is worth recalling that Ω_0 is still assumed to be a bounded domain. We also set

$$a = \liminf_{|x|\to\infty} q(x)(> -\infty)\ .$$

We shall work in the Sobolev space $H = W^{1,2}(\mathbb{R}^N)$ equipped with the usual norm

$$\|u\|^2 = \int_{\mathbb{R}^N} [|\nabla u|^2 + u^2]\, dx .$$

A positive solution of (P) will be an $u \in H$, $u > 0$, which satisfies (P) weakly.

Let us recall (see, for example, [3] or [12]) that if (Q) holds then the spectrum of the linear eigenvalue problem

$$-\Delta u + q(x)u = \lambda u, \quad x \in \mathbb{R}^N \tag{3.1}$$

contains eigenvalues provided

$$\Lambda = \Lambda[q] := \inf_{u \in H,\ |u|_2 = 1} \int_{\mathbb{R}^N} [|\nabla u|^2 + qu^2] < a . \tag{3.2}$$

Moreover, if (3.2) holds then Λ is the principal eigenvalue of (3.1).

The existence of a branch of positive solutions for (P) will be proved by an approximating procedure, carried out by means of the following topological lemma (see [14], Theorem 9.1).

Lemma 3.1 *Let X_n be a sequence of connected subsets of a complete metric space X. If*

(i) $\liminf(X_n) \neq \emptyset$;

(ii) $\bigcup X_n$ *is precompact*;

Then $\limsup(X_n)$ *is not empty, compact and connected.*

Above, $\liminf(X_n)$ and $\limsup(X_n)$ denote the set of all $x \in X$ such that any neighbourhood of x intersects all but finitely many of X_n, infinitely many of X_n respectively. In order to use the preceding Lemma, let B_R be the ball in $\mathbb{R}^N$ centered at the origin, of radius $R > 0$ and let (P_R) denote problem (P_Ω) with $\Omega = B_R$. We will always take R sufficiently large in such a way that $\Omega_0 \subset B_R$. We also set $\lambda_R := \lambda_{B_R}$. According to Theorem 2.6, there exists a branch S_R, of positive solutions of (P_R), that bifurcates from $(\lambda_R, 0)$, and blows up as $\lambda \uparrow \lambda_0$.

It is well known that λ_R (is decreasing with respect to $R > 0$ and) converges to Λ as $R \to +\infty$. For future reference we add that, in particular, since $\underline{\lambda} < \lambda_R < \lambda_0$ for R large, one has that $\underline{\lambda} \leqq \Lambda < \lambda_0$.

To carry over the limiting procedure an uniform a-priori bound is in order. We suppose

(B) *There exist $b \in]\Lambda, \lambda_0[$ and $\Psi \in L^\infty \cap L^2$, $\Psi > 0$, such that $u < \Psi$, for all $(\lambda, u) \in S_R$, uniformly in $R > 0$ and $\lambda \in [\underline{\lambda}, b]$.*

We set $T = [\underline{\lambda}, b]$ and $X = T \times H$. Let $R_n \to +\infty$ and denote by X_n the connected component of the set $\{(\lambda, u) \in S_{R_n}\colon \lambda \in T\}$, such that $(\lambda_R, 0) \in \overline{X}_n$. In view of the properties of S_{R_n} discussed in the preceding section, $X_n \neq \emptyset$ and b belongs to $\Pi(X_n)$, for all n large.

Lemma 3.2 *$\bigcup X_n$ is precompact in X.*

Proof. Let $(\lambda_k, u_k) \in \bigcup X_n$ (we assume that the diameters of $\operatorname{supp}(u_k) \to \infty$, otherwise, the result is trivial). By (B), it follows that

$$|u_k|_2 \leqq c_1 \ . \tag{3.3}$$

Moreover, one has

$$\int_{\mathbb{R}^N} |\nabla u_k|^2 + \int_{\mathbb{R}^N} q(x) u_k^2 = \lambda_k \int_{\mathbb{R}^N} u_k^2 + \int_{\mathbb{R}^N} g(x) h(u_k) u_k^2 \ . \tag{3.4}$$

Since $h(u_k) \leqq C_0$ then (3.3) and (3.4) imply $|\nabla u_k|_2 \leqq c_2$ and hence

$$\|u_k\| \leqq c_3 \ .$$

Therefore, up to a subsequence, $u_k \rightharpoonup u$ in H and in L^2. It is easy to see that u satisfies

$$\int_{\mathbb{R}^N} \nabla u \nabla \phi + \int_{\mathbb{R}^N} q(x) u \phi = \lambda \int_{\mathbb{R}^N} u \phi + \int_{\mathbb{R}^N} g(x) h(u) u \phi, \quad \forall \phi \in C_0^\infty \ . \tag{3.5}$$

By density, (3.5) holds for all $\phi \in H$ and, in particular, for $\phi = u_k$. To prove that u_k converges strongly to u we consider

$$\|u_k - u\|^2 = \|u_k\|^2 + \|u\|^2 - 2 \int_{\mathbb{R}^N} \nabla u_k \nabla u - 2 \int_{\mathbb{R}^N} u_k u \ .$$

From (3.4) we find

$$\|u_k\|^2 = \int_{\mathbb{R}^N} |\nabla u_k|^2 + \int_{\mathbb{R}^N} u_k^2 = \int_{\mathbb{R}^N} F_{\lambda_k}(x, u_k) u_k + \int_{\mathbb{R}^N} u_k^2 \tag{3.6}$$

where

$$F_\lambda(x, u) = \lambda u - q(x) u + g(x) h(u) u \ . \tag{3.7}$$

From (3.5) with $\phi = u_k$ and $\phi = u$ respectively, we infer

$$\int_{\mathbb{R}^N} \nabla u_k \nabla u = \int_{\mathbb{R}^N} F_\lambda(x, u) u_k \ , \tag{3.8}$$

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \int_{\mathbb{R}^N} F_\lambda(x, u) u \ . \tag{3.9}$$

Putting together (3.6), (3.8) and (3.9) we find

$$\begin{aligned} \|u_k - u\|^2 &= \int_{\mathbb{R}^N} F_{\lambda_k}(x, u_k) u_k + \int_{\mathbb{R}^N} u_k^2 + \int_{\mathbb{R}^N} F_\lambda(x, u) u + \int_{\mathbb{R}^N} u^2 - 2 \int_{\mathbb{R}^N} F_\lambda(x, u) u_k - 2 \int_{\mathbb{R}^N} u_k u \\ &= \int_{\mathbb{R}^N} [F_{\lambda_k}(x, u_k) - F_\lambda(x, u)] u_k + \int_{\mathbb{R}^N} F_\lambda(x, u)[u - u_k] + \int_{\mathbb{R}^N} u_k [u_k - u] + \int_{\mathbb{R}^N} u [u - u_k] \ . \end{aligned}$$

Since $u_k < \Psi$ we deduce

$$\begin{aligned} \|u_k - u\|^2 \leqq & \int_{\mathbb{R}^N} |F_{\lambda_k}(x, u_k) - F_\lambda(x, u)| \Psi + \int_{\mathbb{R}^N} |F_\lambda(x, u)| \, |u - u_k| \\ & + \int_{\mathbb{R}^N} |u_k - u| \Psi + \int_{\mathbb{R}^N} |u| \, |u - u_k| \ . \end{aligned}$$

The last three integrals converge to zero. As for the first one, since F_λ is locally Lipschitzian, one has

$$\int_{\mathbb{R}^N} |F_{\lambda_k}(x,u_k) - F_\lambda(x,u)|\Psi \leqq |\lambda_k - \lambda| \int_{\mathbb{R}^N} \Psi^2 + c_4 \int_{\mathbb{R}^N} |u_k - u|\Psi \to 0 \, .$$

In conclusion, it follows that $u_k \to u$ strongly in H. □

Let $S := \limsup(X_n)\backslash\{(\Lambda,0)\}$.

Lemma 3.3 *If* $(\lambda,u)\in S$ *then* u *is a* (*nontrivial*) *positive solution of* (P).

Proof. It is easy to check that any $(\lambda,u)\in S$ is a non-negative solution of (P). We need to prove that $u \not\equiv 0$. We shall consider separately the cases $\lambda > \Lambda$ and $\lambda < \Lambda$ (obviously, when $\lambda = \Lambda$ the result is trivial, according to the definition of S).

Step 1. Let $\lambda > \Lambda$ and suppose there exists a sequence $(\lambda_k,u_k)\in X_{n_k}$, $n_k \to \infty$, such that $(\lambda_k,u_k) \to (\lambda,0)$. We can assume that all $\lambda_k > \Lambda+\delta$ for some $\delta > 0$. Since $\lambda_R \downarrow \Lambda$, there exists $R > 0$ such that $\lambda_R < \Lambda + \delta < \lambda$. Let m be such that $R_{n_k} > R$ for all $k > m$. It is easy to check that, for such k, (λ_k,u_k) are super-solutions of the problem

$$\begin{cases} -\Delta u + q(x)u = (\Lambda+\delta)u + g(x)h(u)u, & |x| < R \, , \\ u(x) = 0 & |x| = R \, . \end{cases} \tag{3.10}$$

Small subsolutions of type $\varepsilon_k\phi_R$ can also be obtained, and thus we can find a sequence of positive solutions of (3.10) that converges to 0. In other words $(\Lambda+\delta,0)$ is bifurcation of positive solutions for (P_R). Since $(\lambda_R,0)$ is the only possible bifurcation for positive solutions for (P_R), this is a contradiction.

Step 2. Suppose that $\lambda < \Lambda$ and let $M > 0$ be such that $q(x)+M \geqq 1$ in $\mathbb{R}^N$. It is convenient to endow the space H with the norm

$$|||u|||^2 := \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} [q+M]u^2 \, ,$$

which is equivalent to $\|\cdot\|$. By the variational expression of Λ, we deduce that

$$0 < (\Lambda+M) \int_{\mathbb{R}^N} u^2 \leqq |||u|||^2, \quad \forall u \in H \, .$$

Let $0 < \sigma < \kappa$ be such that $h(s) < c_1 s^\sigma$ and $H \subset L^{\sigma+2}(\mathbb{R}^N)$.

Let (λ,w) be a positive solution of some (P_R). Then

$$\int_{\mathbb{R}^N} (|\nabla w|^2 + q(x)w^2) = \lambda \int_{\mathbb{R}^N} w^2 + \int_{\mathbb{R}^N} g(x)h(w)w^2$$

and hence

$$\begin{aligned} |||w|||^2 &= (\lambda+M) \int_{\mathbb{R}^N} w^2 + \int_{\mathbb{R}^N} g(x)h(w)w^2 \leqq \frac{\lambda+M}{\Lambda+M} |||w|||^2 + c_2 \int_{\mathbb{R}^N} w^{\sigma+2} \\ &\leqq \frac{\lambda+M}{\Lambda+M} |||w|||^2 + c_3 |||w|||^{\sigma+2} \, . \end{aligned}$$

In particular, for $\lambda < \Lambda$,

$$c_3|||w|||^\sigma \geqq 1 - \frac{\lambda + M}{\Lambda + M} > 0\,.$$

Then the approximating branchs are uniformly bounded away from zero. □

Theorem 3.4 *Suppose that* $(Q), (A_1 - A_2)$ *and* (B) *hold. Then there exists a branch*, S, *of positive solutions of* (P) *bifurcating from* $(\Lambda, 0)$, *such that* $\Pi(\overline{S}) = [\Lambda^*, b]$, *for some* $\Lambda^* \leqq \Lambda$.

Proof. We apply Lemma 3.1 with X and X_n defined before. Since $\lambda_{R_n} \downarrow \Lambda$, it immediately follows that $(\Lambda, 0) \in \liminf X_n$. Moreover, Lemma 3.2 shows that the compactness property (ii) of Lemma 3.1 is satisfied. As a consequence $S = \limsup(X_n)\backslash\{\Lambda, 0)\}$ is (not empty), connected and $(\Lambda, 0) \in \overline{S}$. By Lemma 3.3 $(\lambda, u) \in S$ is a positive solution of (P). Moreover, since $\Lambda < b < \lambda_0$ then $b \in \Pi(X_n)$ for all n large, and the preceding compactness arguments imply that $b \in \Pi(S)$. □

Remarks 3.5 (i) Similarly as in Remark 2.5(ii), if $g(x) > 0$ a.e. in $\mathbb{R}^N$ then it is understood that b could be any number greater than Λ.

(ii) We have found solutions in $W^{1,2}$. In fact in the specific applications discussed in the sequel the a-priori bound Ψ will have an exponential decay at zero as $|x| \to \infty$ and thus the solutions on S will have the same asymptotic behaviour.

(iii) Assumption (B) can be slightly weakened. Actually it suffices to assume

$(\widetilde{B})$ *There exist* $b \in]\Lambda, \lambda_0[$ *and* $\Psi \in L^\infty \cap L^2$, $\Psi > 0$, *such that* $u < \Psi$, *for all* $(\lambda, u) \in X_n$, *uniformly in* n *and* $\lambda \in [\underline{\lambda}, b]$.

If for each $\lambda \in [\underline{\lambda}, b]$ problem (P_R) has a unique positive solution, (B) and $(\widetilde{B})$ obviously coincide. However, in the applications discussed later on, where multiple solutions could arise, we will be able to prove $(\widetilde{B})$, only.

(iv) According to Remark 2.3 one obviously has that $\Lambda^* \geqq \underline{\lambda}$. □

Theorem 3.4 deals with the specific problem (P). But a similar result holds true for a more general equation like

$$-\Delta u = F_\lambda(x, u), \quad x \in \mathbb{R}^N\,,$$

where F is locally Lipschitzian. It suffices that each approximated problem on B_R possesses a branch of positive solutions bifurcating from some $(\lambda_R, 0)$, that λ_R converge to some Λ and that assumption (B), or $(\widetilde{B})$, holds for some $\underline{\lambda}$, b such that $\underline{\lambda} < \Lambda < b$. The preceding arguments can be still carried out and yield the existence of a global bifurcating branch.

This is, for example, the case of problem

$$-\Delta u = \lambda \rho(x) u^\alpha, \quad x \in \mathbb{R}^N\,, \tag{3.11}$$

where $\rho \in L^\infty$ and $0 < \alpha < 1$, studied in [5]. Here we can show:

Theorem 3.6 *Assume that there exists* $U \in L^\infty \cap L^2$ *satisfying*

$$-\Delta U = \rho(x), \quad x \in \mathbb{R}^N . \tag{3.12}$$

Then there exists a branch S of positive, $W^{1,2}$ *solutions of* (3.11) *bifurcating from* (0,0) *and such that* $\Pi(\overline{S}) = [0, \infty)$.

Proof. We only give an outline of the proof and leave the details to the reader. For every $R > 0$ the boundary value problem

$$-\Delta u = \lambda \rho(x) u^\alpha, \quad x \in B_R, \ u = 0, \ x \in \partial B_R ,$$

has a bifurcating branch S_R emanating from (0,0) such that $\Pi(\overline{S}_R) = [0, \infty)$. Moreover, one easily shows that assumption (B) holds true with $\Lambda = \underline{\lambda} = 0$, any $b > 0$ and $\Psi(x) = CU(x)$, where U satisfies (3.12) and $C > 0$ is sufficiently large. Then the result follows by the arguments discussed before. □

Remark 3.7 In [5] it is only assumed that $U \in L^\infty$. On the other hand, we find here (a branch of) solutions in $W^{1,2}$, not merely bounded solutions. □

4 Existence of a global branch of positive solutions of (*P*)

A global branch of positive solutions of problem (P) will be found through an application of Theorem 3.4. Here we assume in addition to $(Q), (A_1 - A_2)$, that

(A_3) for all compact $\mathscr{K} \subset \mathbb{R}^N \setminus \overline{\Omega}_0$ there exists $g_0 = g_0(\mathscr{K})$, such that $\inf\{g(x) \colon x \in \mathscr{K}\} \geqq g_0 > 0$;

(A_4) $\Lambda < a - \overline{g} C_0$,

where according to the notation introduced in Sect. 2, $\overline{g} = \sup_{\mathbb{R}^N} g(x)$. We set

$$l = \min\{\lambda_0, a - \overline{g} C_0\} ,$$

Let us point out that (A_4) implies $\Lambda < l$, because $\Lambda < \lambda_0$.

Our goal is to show that assumption ($\widetilde{B}$) holds for any $b < l$.

a) Construction of Ψ. We assume that Ω_0 is not emptry: if not, the first part of this subsection can be avoided.

Let $b < l \leqq a - \overline{g} C_0$, and let Ω_δ denote the δ-neighborhood of Ω_0. Taking $\delta > 0$ sufficiently small, there results

$$b < \lambda_{\Omega_\delta} < \lambda_0 . \tag{4.1}$$

Fix a', with $b + \overline{g} C_0 < a' < a$, and let $\beta(x)$ denote the function defined by setting

$$\beta(x) := (q(x) - a')^- .$$

Since $a' < a$, it follows that the support $K = \mathrm{supp}(\beta)$ of β is compact. Letting $\rho > 0$ be such that

$$K \cup \Omega_\delta \subset B_\rho$$

we set $\gamma_\alpha(x) = \gamma_\alpha(|x|)$, $\alpha > 0$, where

$$\gamma_\alpha(t) = \begin{cases} -\alpha & \text{if } \ t \leqq \rho, \\ \alpha(t-\rho-1) & \text{if } \ \rho < t < \rho+1, \\ 0 & \text{if } \ t \geqq \rho+1. \end{cases}$$

Consider the linear Schrödinger equation

$$-\Delta u + \gamma_\alpha(x)u = \mu u, \quad x \in \mathbb{R}^N , \tag{4.2}$$

and let

$$\mu_\alpha = \inf_{u\in H,\, |u|_2=1} \int_{\mathbb{R}^N} [|\nabla u|^2 + \gamma_\alpha(x)u^2] .$$

The properties of μ_α are collected in the following Lemma.

Lemma 4.1 (i) $\mu_\alpha \leqq 0$ *and there exists* $\alpha^* \geqq 0$ *such that* $\mu_\alpha < \mu_{\alpha^*} = 0$ *provided* $\alpha > \alpha^*$;

(ii) *if* $\alpha > \alpha^*$ *then* μ_α *is the principal eigenvalue of* (4.2) *with corresponding positive eigenfunction* $\varphi_\alpha \in W^{2,p}(\mathbb{R}^N)$ *for all* $p > 1$.

(iii) μ_α *depends continuously on* α;

Proof. Since $\gamma_\alpha(x) \leqq 0$ it follows that

$$\mu_\alpha \leqq \inf_{|u|_2=1} |\nabla u|_2^2 = 0 .$$

Moreover, $\mu_\alpha < 0$ provided α is large enough (for example, $\alpha > \lambda_{B_\rho}[0]$ suffices). Property (ii) is well known (see, for example, [3] or [12]) and (iii) is standard. □

From Lemma 4.1 it follows that there exists $\alpha_0 > 0$ such that for $\mu_0 := \mu_{\alpha_0}$ one has

$$b - a' + \overline{g}C_0 < \mu_0 < 0 . \tag{4.3}$$

Let $\psi = \psi_b : \mathbb{R}^N \to \mathbb{R}$ be any strictly positive C^2-function such that

$$\psi(x) = \begin{cases} \phi_\delta(x) & x \in \Omega_{\delta/2} , \\ \varphi_0(x) & x \in \mathbb{R}^N \backslash B_\rho , \end{cases}$$

where $\varphi_0 = \varphi_{\alpha_0}$ and $\phi_\delta \in W_0^{1,2}(\Omega_\delta)$ satisfies

$$-\Delta\phi_\delta + q(x)\phi_\delta = \lambda_{\Omega_\delta}\phi_\delta, \quad x \in \Omega_\delta .$$

Finally we define $\Psi(x) = \Psi_b(x) := C\psi(x)$.

b) A priori bounds. We first show

Lemma 4.2 *There exists* $C > 0$ *such that* $\Psi = C\psi$ *is a supersolution for* (P_R) *for all* $R > 0$ *and all* $\lambda \leqq b$.

Proof. Taking C large enough, one has that $h(\Psi(x)) < 0$ for all $x \in \Omega_{\delta/2}$. Then, the definition of ψ, (4.1) and $\lambda \leqq b$ imply

$$-\Delta\Psi + q(x)\Psi = \lambda_{\Omega_\delta}\Psi > b\Psi \geqq \lambda\Psi + g(x)h(\Psi)\Psi, \quad x \in \Omega_{\delta/2} .$$

Let

$$M = \sup\{\Delta\psi - q(x)\psi + b\psi \colon x \in B_{\rho+1}\backslash\Omega_{\delta/2}\} .$$

Since $\psi(x) \geqq c_1 > 0$ on $B_{\rho+1}\backslash\Omega_{\delta/2}, (A_2 - A_3)$ allow us to take C in such a way that

$$c_1 g(x)h(C\psi(x)) \leqq -M, \quad \forall x \in B_{\rho+1}\backslash\Omega_{\delta/2} ,$$

whence

$$-\Delta\Psi + q(x)\Psi \geqq b\Psi - MC \geqq \lambda\Psi + g(x)h(\Psi)\Psi, \quad \forall x \in B_{\rho+1}\backslash\Omega_{\delta/2} .$$

Finally, for $|x| \geqq \rho + 1$, $\gamma_\alpha \equiv 0$ and $\psi = \varphi_0$ satisfies $-\Delta\psi = \mu_0\psi$. Using (4.3) and since $\mathrm{supp}(\beta) \subset B_\rho$, it follows that

$$\begin{aligned} -\Delta\Psi + q(x)\Psi &= (\mu_0 + q(x))\Psi \geqq (\mu_0 + a')\Psi \\ &\geqq (b + \overline{g}C_0)\Psi \geqq \lambda\Psi + g(x)h(\Psi)\Psi . \end{aligned}$$

Let us remark that if $\Omega_0 = \emptyset$ one can simply define $\Psi(x) = C\varphi_0(x)$ and the Lemma follows from the preceding inequality. □

Lemma 4.3 *For all $b < l$ condition $(\widetilde{B})$ holds with the function Ψ defined above.*

Proof. We use the notation introduced in Sect. 3. For a fixed n, we set $B_n = B_{R_n}$. Let $K > 0$ be such that $F_\lambda(x, s) + Ks$ (see (3.7) for the definition) is strictly increasing with respect to $s \in [0, \max_{B_n} \Psi]$, for all $\lambda \in [\underline{\lambda}, b]$. Let v_n denote the solution of the b.v.p.

$$\begin{cases} -\Delta v_n + Kv_n = F_b(x, \Psi) + K\Psi, & |x| < R_n , \\ v_n(x) = 0, & |x| = R_n . \end{cases}$$

Since $F_b(x, \Psi) + K\Psi \geqq 0$, the maximum principle implies that v_n is strictly positive in B_n, and has negative outer normal derivative at every point on ∂B_n. This means that v_n lies in the interior of the cone, $\mathscr{P}$, of positive functions of $C_0^1(B_n)$. Moreover, by Lemma 4.2 it follows that $-\Delta\Psi \geqq F_b(x, \Psi)$, whence

$$-\Delta(\Psi - v_n) + K(\Psi - v_n) \geqq F_b(x, \Psi) + K\Psi - F_b(x, \Psi) - K\Psi = 0, \quad |x| < R_n.$$

Since, for $|x| = R_n$ one has $\Psi(x) - v_n(x) = \Psi(x) > 0$, the maximum principle again implies

$$0 < v_n(x) < \Psi(x), \quad \forall |x| < R .$$

Furthermore, $F_b(x, \cdot) + K$ increasing and $F_b(x, s) \geqq F_\lambda(x, s)$ for all $b \geqq \lambda$, yield

$$-\Delta v_n > F_b(x, v_n) \geqq F_\lambda(x, v_n), \quad \forall |x| < R .$$

In particular, $v_n \in C_0^1(B_n)$ is a super-solution (but not a solution) of (P_{R_n}) for every $\lambda \leqq b$. To conclude the proof it remains to show that $v_n > u$ whenever $(\lambda, u) \in X_n$. Consider the set $\Sigma_n = \{(\lambda, v_n - u)\colon (\lambda, u) \in X_n\}$. Let us explicitely point out that Σ_n is *connected* because X_n is. Moreover observe that $(\lambda_{R_n}, v_n) \in \Sigma_n$ and therefore $\Sigma_n \cap (T \times \mathring{\mathscr{P}}) \neq \emptyset$. We claim that $\Sigma_n \subset T \times \mathring{\mathscr{P}}$. If not, the connection of Σ_n allows us to find $(\lambda, u) \in X_n$ such that $v_n - u \in \partial\mathscr{P}$. In particular, since v_n is not a solution of (P_{R_n}), one infers that $v_n \geqq (\neq) u$ in B_n. Then

$$-\Delta(v_n - u) + K(v_n - u) \geqq F_\lambda(x, v_n) + Kv_n - F_\lambda(x, u) - Ku \geqq 0, \quad \forall |x| < R_n \,.$$

By the strong maximum principle, we deduce that $v_n - u \in \mathring{\mathscr{P}}$, which is a contradiction. Then $\forall (\lambda, u) \in X_n$ one has

$$0 < u(x) < v_n(x) < \Psi(x), \quad \forall |x| < R_n \,.$$

This completes the proof. □

c) Global bifurcation. We are now in position to prove:

Theorem 4.4 *Suppose that* (Q), $(A_1 - A_2 - A_3 - A_4)$ *hold. Then there exist a branch*, S, *of positive solutions of* (P) *bifurcating from* $(\Lambda, 0)$, *such that* $\Pi(\overline{S}) = [\Lambda^*, l[$, *for some* $\Lambda^* \leqq \Lambda$.

Moreover, *if* $\Omega_0 \neq \emptyset$ *and*

(A_5) $\quad \lambda_0 < a - \overline{g}C_0,$

then $\Pi(\overline{S} = [\Lambda^*, \lambda_0[$.

Proof. By Lemma 4.3 $(\overline{B})$ holds for all $b < l$. Then an application of Theorem 3.4 yields the existence of a bifurcation branch S_b of positive solutions of (P) such that $\Pi(\overline{S}_b) = [\Lambda^*, b]$. Taking $S = \bigcup_b S_b$ it follows that $\Pi(\overline{S}) = [\Lambda^*, l[$.

Finally, to prove the last statement, it suffices to remark that (P) has no positive solution for $\lambda \geqq \lambda_0$. Otherwise, if u_0 is a positive solution of (P) for some $\lambda \geqq \lambda_0$, such a u_0 is a supersolution of (P_Ω), for such λ, for any bounded domain Ω with $\Omega_0 \subset \Omega$. Since $\varepsilon\phi_0$ (see notation introduced in Sect. 2) is a subsolution for any $\varepsilon > 0$ small, it follows that (P_Ω) has a positive solution for that $\lambda \geqq \lambda_0$, in contradiction with Lemma 2.2. □

Remarks 4.5 (i) Since $\Lambda < \lambda_0$ (see Sect. 3), condition (A_5) implies (A_4).

(ii) When $\Omega_0 = \emptyset$ Theorem 4.4 improves the existence results of [7], where, in addition to $(A_1 - A_2 - A_4)$ and to some regularity and growth restriction on q and g, it is assumed that $g(x) > 0$ on $\mathbb{R}^N$ and that $h(s)$ is decreasing. Of course, in this case (A_4) is nothing but $\Lambda < a$.

(iii) Let us point out that (A_4) is, in general, a necessary condition for the existence of a positive solution of (P). For example, see [7], if $g(x) > 0$, $h(s)$ is decreasing and $q(x) \to a$ as $|x| \to \infty$, then (P) has no positive solution if $\lambda \geqq a$.

A case in which (A_4) holds is, for example, when there exists $R > 0$ such that $q(x) \leqq -\lambda_{B_R}[0] + a - \overline{g}C_0$ in B_R.

(iv) Theorem 4.4 applies to the specific problem (1.1) discussed in the Introduction. In such a case one immediately finds that (A_5) holds provided $-\alpha_1 + \pi^2/4d^2 < -\alpha_2$. □

5 Blow up

In this section we will study the behaviour of S when $\lambda \uparrow \lambda_0$. Here we deal with the case $\Omega_0 \neq \emptyset$, and assume that (A_5) holds.

Consider $\widehat{h}: \mathbb{R}^+ \to \mathbb{R}$, a strictly decreasing function satisfying (A_2) and such that

$$\widehat{h}(s) < h(s), \quad \forall s > 0 .$$

Let $k: \mathbb{R}^+ \to \mathbb{R}$ be a strictly decreasing function such that

$$k(s) > h(s), \quad \forall s \geqq 0 .$$

Setting $\widetilde{h}(s) = k(s) - k(0)$, suppose that k has been chosen in such a way that $\widetilde{h}$ satisfies (A_2).

Denote by $(\widehat{P}_R)$, resp. (P), the problem (P_R), resp. (P), with $\widehat{h}$ instead of h, and moreover, setting

$$\widetilde{q}(x) = q(x) - g(x)k(0), \quad \forall x \in \mathbb{R}^N ,$$

we denote by $(\widetilde{P}_R)$, resp. $(\widetilde{P})$, the problem (P_R), resp. (P), with $\widetilde{h}$ and $\widetilde{q}$ instead of h and q. We also set $\widetilde{a} = \liminf \widetilde{q}(x)$ and $\widetilde{\Lambda} = \Lambda[\widetilde{q}]$.

Remark that $\lambda_{\Omega_0}[\widetilde{q}] = \lambda_0$, because $\widetilde{q}(x) = q(x)$, for all $x \in \Omega_0$. Moreover, since $k(0)$ can be taken sufficiently close to C_0, it follows that

$$\widetilde{\Lambda} < \lambda_0 < \widetilde{a} . \tag{5.1}$$

We denote by $\widehat{S}_R, \widetilde{S}_R$, the branches of positive solutions of $(\widehat{P}_R)$, and $(\widetilde{P}_R)$, found in Theorem 2.6. By Remark 2.7(ii), for all $\lambda \in \Pi(\widehat{S}_R) \cap \Pi(\widetilde{S}_R)$, there exists a unique $\widehat{u}_{\lambda,R}$ such that $(\lambda, \widehat{u}_{\lambda,R}) \in \widehat{S}_R$, and a unique $\widetilde{u}_{\lambda,R}$ such that $(\lambda, \widetilde{u}_{\lambda,R}) \in \widetilde{S}_R$.

According to (5.1) Theorem 4.4 applies to problem $(\widetilde{P})$ and yields a branch of positive solutions $\widetilde{S}$.

Roughly, we will show that S 'lies between' the branches $\widehat{S}_{R_0}$ and $\widetilde{S}$ in such a way that the blow up of the latters will imply that of S.

Lemma 5.1 *Fixed $R_0 > 0$, with $B_{R_0} \supset \Omega_0$, then $\forall R \geqq R_0$, there results*

$$\widehat{u}_{\lambda,R_0} \leqq u_R, \quad \forall (\lambda, u_R) \in S_R, \quad \lambda > \lambda_{R_0} , \tag{5.2}$$

$$u_R \leqq \widetilde{u}_{\lambda,R} \quad \forall (\lambda, u_R) \in S_R . \tag{5.3}$$

Proof. Since $\widehat{h} < h$ and $R_0 < R$, it follows that u_R is a super-solution for problem $(\widehat{P}_{R_0})$. As usual, a small sub-solution of $(\widehat{P}_{R_0})$ can be found as before, and thus the unique solution $\widehat{u}_{\lambda,R_0}$ of $(\widehat{P}_{R_0})$ satisfies $\widehat{u}_{\lambda,R_0} \leqq u_R$.

Similarly, it is easy to check that u_R is a sub-solution of $(\widetilde{P}_R)$, and $\Psi = C\psi$ is a super-solution of $(\widetilde{P}_R)$ (provided that C is sufficiently large). Hence $u_R \leqq \widetilde{u}_{\lambda,R}$. □

Theorem 5.2 *Suppose* (Q) *and* $(A_1 - A_2 - A_3 - A_5)$ *hold. Then, on* S *one has that for all* $p \geqq 1$, $|u|_p \to \infty$, *iff* $\lambda \uparrow \lambda_0$.

Proof. (a) Let $\lambda \uparrow \lambda_0$. By (h) of Theorem 2.6, $|\widehat{u}_{\lambda,R_0}|_p \to \infty$. Using (5.2) it follows that $|u|_p \to \infty$ as $\lambda \uparrow \lambda_0$.

(b) Since $(\widetilde{P}_R)$ has a unique solution for all $\lambda \in \Pi(\widetilde{S}_R)$, the Ψ found in Subsect. 4b, satisfies not merely $(\widetilde{B})$ but also (B) and therefore it becomes an a-priori bound for *all* the solutions of $(\widetilde{P})$, with $\lambda \leqq b$. This implies that $\widetilde{S}$ can only blow up in L^p as $\lambda \to \lambda_0$. Taking limits in (5.3) one finds that $u \leqq \widetilde{u}$ for all $(\lambda, u) \in S$ and $(\lambda, \widetilde{u}) \in \widetilde{S}$. Then $|u|_p \to \infty$ on S implies that $\lambda \to \lambda_0$. □

Remark 5.3 When $\Omega_0 = \emptyset$ the blow up of S has been studied in Sects. 6 and 7 of [7]. It is worth pointing out that Theorem 5.2 is quite different from the results of [7]. Actually, in the latter the branch blows up at $\lambda = a$ in L^p, for a certain range of p, related to the dimension N and to the asymptotic properties of q, g and h. We also note that such a behaviour of S remains valid also if $\Omega_0 \neq \emptyset$, $\Lambda < a < \lambda_0$ and q, h satisfy the same assumptions of [7]: it suffices to use, with minor changes, the arguments in [7]. □

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