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Topological AdS/CFT

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Abstract: We define a holographic dual to the Donaldson-Witten topological twist of $\mathcal{N} = 2$ gauge theories on a Riemannian four-manifold. This is described by a class of asymptotically locally hyperbolic solutions to $\mathcal{N} = 4$ gauged supergravity in five dimensions, with the four-manifold as conformal boundary. Under AdS/CFT, minus the logarithm of the partition function of the gauge theory is identified with the holographically renormalized supergravity action. We show that the latter is independent of the metric on the boundary four-manifold, as required for a topological theory. Supersymmetric solutions in the bulk satisfy first order differential equations for a twisted Sp(1) structure, which extends the quaternionic Kähler structure that exists on any Riemannian four-manifold boundary. We comment on applications and extensions, including generalizations to other topological twists.

Keywords: AdS-CFT Correspondence, Conformal Field Theory, Extended Supersymmetry, Supersymmetric Gauge Theory

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1 Introduction and outline

The AdS/CFT correspondence is a conjectured duality relating certain quantum field theories (QFTs) to quantum gravity [1]. This typically relates a strong coupling limit in field theory to semi-classical gravity, and quantitative comparisons between the two sides usually rely on additional symmetries, such as supersymmetry or integrability. Starting with the work of [2], recently localization techniques in supersymmetric gauge theories defined on rigid supersymmetric backgrounds have led to new exact computations. Moreover, the appropriate strong coupling limits have been successfully matched to semi-classical gravity calculations, in a variety of different set-ups.[1] On the other hand, localization in QFT originated in [4], where the topological twist was introduced to define a topological quantum field theory (TQFT). It is natural to then ask whether one can define and study holography in this topological setting. Indeed, what does gravity tells us about TQFT, and *vice versa*? In this paper, we take some first steps in this direction.

1.1 Background

In [4], Witten gave a physical construction of Donaldson invariants of four-manifolds [5–7] as certain correlation functions in a TQFT. This theory is constructed by taking pure $\mathcal{N} = 2$ Yang-Mills gauge theory and applying a topological twist: identifying a background SU(2) R-symmetry gauge field with the right-handed spin connection results in a conserved scalar supercharge $\mathcal{Q}$, on any oriented Riemannian four-manifold (M_4, g). The path integral localizes onto Yang-Mills instantons, and correlation functions of $\mathcal{Q}$-invariant operators localize to integrals of certain forms over the instanton moduli space $\mathcal{M}$. These are precisely Donaldson's invariants of M_4. They are, under certain general conditions, independent of the choice of metric g on M_4, but in general depend on the diffeomorphism type of M_4. In particular, Donaldson invariants can sometimes distinguish manifolds which are homeomorphic but not diffeomorphic. That this is possible is because the instanton equations are PDEs, which depend on the differentiable structure. From the TQFT point of view, independence of the choice of metric follows by showing that metric deformations lead to $\mathcal{Q}$-exact changes in the integrand of the path integral. For example, the stress-energy tensor is $\mathcal{Q}$-exact, implying that the partition function is invariant under arbitrary metric deformations, and hence (formally at least) is a diffeomorphism invariant.

Donaldson-Witten theory is typically studied for pure $\mathcal{N} = 2$ Yang-Mills, with gauge group $\mathscr{G} = \mathrm{SU}(2)$ or $\mathscr{G} = \mathrm{SO}(3)$. However, the topological twist may be applied to any $\mathcal{N} = 2$ theory with matter, and also for any gauge group $\mathscr{G}$. For example, $\mathscr{G} = \mathrm{SU}(N)$ Donaldson invariants were first studied in [8], with further mathematical work in [9]. In particular the latter reference contains some explicit large N results for the partition function on certain four-manifolds. The procedure of topological twisting may also be applied to theories with different amounts of supersymmetry, and in various dimensions. For example, the larger SU(4) R-symmetry of four-dimensional $\mathcal{N} = 4$ Yang-Mills leads to three inequivalent twists [10]. Viewing the $\mathcal{N} = 4$ theory as an $\mathcal{N} = 2$ theory coupled to an adjoint matter multiplet, applying the Donaldson-Witten twist leads to a TQFT that is referred to as the "half-twisted" $\mathcal{N} = 4$ theory. This theory is relevant for the construction in the present paper. The other two twists are the Vafa-Witten twist [11], and the twist studied by Kapustin-Witten in [12], relevant for the Geometric Langlands programme. Historically the development of Donaldson-like invariants took a rather different direction after the introduction of Seiberg-Witten invariants in [13]. The former may be expressed (conjecturally) in terms of the latter, but Seiberg-Witten theory is simpler and easier to compute with.

[1] A review of some of these results appears in [3], although many more results have appeared since.

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The Donaldson-Witten twist of $\mathcal{N} = 2$ gauge theories can be understood as a special case of rigid supersymmetry. Soon after Witten's paper, Karlhede-Roček interpreted the construction as coupling the gauge theory to a background (*i.e.* non-dynamical) $\mathcal{N} = 2$ conformal gravity [14]. The background SU(2) R-symmetry gauge field is part of this gravity multiplet, and is embedded into the spin connection in such a way that the Killing spinor equations of the theory admit a constant solution, leading to the conserved scalar supercharge $\mathcal{Q}$. There is also an auxiliary scalar field turned on in this background gravity multiplet, proportional to the Ricci scalar curvature of (M_4, g). Motivated by the work of Pestun in [2], the last few years have seen considerable interest in defining rigid supersymmetry more generally on Riemannian manifolds. Unlike the topological twist, this generally requires the background d-manifold (M_d, g) to possess some additional geometric structure, and correlation functions of $\mathcal{Q}$-invariant observables then usually depend on this structure. For example, one can couple four-dimensional $\mathcal{N} = 1$ theories with a U(1) R-symmetry to a background new minimal supergravity. Geometrically this construction requires (M_4, g) to be a Hermitian four-manifold, with an integrable complex structure [15, 16]. Generalizing [14], similarly $\mathcal{N} = 2$ theories may be coupled to a background $\mathcal{N} = 2$ conformal supergravity [17]. Generically this requires the existence of a conformal Killing vector on (M_4, g), but the topological twist arises as a degenerate special case, in which (M_4, g) is arbitrary.

An interesting application of these constructions is to the AdS/CFT correspondence. Here strong coupling (typically large rank N) gauge theory computations are related to semi-classical gravity. The general idea is as follows. Rigid supersymmetry generically equips the background manifold (M_d, g), on which the gauge theory is defined, with certain additional geometric structure, such as the integrable complex structure mentioned for four-dimensional $\mathcal{N} = 1$ theories above. In the gravitational dual description one seeks solutions to an appropriate supergravity theory in $d + 1$ dimensions, where (M_d, g) arises as a conformal boundary. That is, the $(d + 1)$-dimensional metric is asymptotically locally hyperbolic, approximated by $\frac{\mathrm{d}z^2}{z^2} + \frac{1}{z^2} g$ to leading order in z near the conformal boundary at $z = 0$. A saddle point approximation to quantum gravity in this bulk then identifies

$$Z[M_d] = \sum \mathrm{e}^{-S[Y_{d+1}]} \ . \tag{1.1}$$

Here $Z[M_d]$ denotes the partition function of the gauge theory defined on M_d, while $S[Y_{d+1}]$ is the holographically renormalized supergravity action, evaluated on an asymptotically locally hyperbolic solution to the equations of motion of the $(d + 1)$-dimensional theory. The manifold $M_d = \partial Y_{d+1}$ is the conformal boundary, with the boundary conditions for supergravity fields on Y_{d+1} fixed by the rigid background structure of M_d.

The general AdS/CFT relation (1.1) is somewhat schematic, and both sides must be interpreted appropriately. For example, in order to make sense of the left hand side for topologically twisted four-dimensional $\mathcal{N} = 2$ SCFTs it can be refined, as discussed in section 6.1. On the other hand, the sum on the right hand side of (1.1) is not well understood. One should certainly include all saddle point solutions on smooth manifolds Y_{d+1}. However, the existence of such a filling immediately implies that M_d has trivial class in the oriented bordism group Ω_d^{SO}, in general constraining the choice of M_d.[2] That said, various explicit examples (see, for example, [18–20]) suggest that requiring Y_{d+1} to be smooth is in any case too strong: one should allow for certain types of singular fillings of (M_d, g), and indeed these may even be the dominant contribution in (1.1) (especially for non-trivial topologies of M_d). There are some clear constraints, although no general prescription.[3] The supergravity action S typically scales with a positive power of N, and in the $N \to \infty$ limit only the solution of least action contributes to (1.1) at leading order, with contributions from other solutions being exponentially suppressed.

1.2 Outline

In this paper we construct a holographic dual to the Donaldson-Witten twist of four-dimensional $\mathcal{N}=2$ gauge theories. As already mentioned, this twist may be interpreted as coupling the theory to a particular background $\mathcal{N}=2$ conformal gravity multiplet. On the other hand, four-dimensional $\mathcal{N}=2$ conformal gravity arises on the conformal boundary of asymptotically locally hyperbolic solutions to the Romans [22] $\mathcal{N}=4^+$ gauged supergravity in five dimensions [23]. The real Euclidean signature version of this theory described in section 2 has, in addition to the bulk metric $G_{\mu\nu}$, an SU(2) R-symmetry gauge field $\mathcal{A}^I_\mu$ $(I=1,2,3)$, a one-form $\mathcal{C}$, and a scalar field X. (In general there is also a doublet of B-fields, but this is zero for the topological twist boundary condition, and moreover may be consistently set to zero in the Romans theory.)

The main property of a topological field theory is that appropriate correlation functions, including the partition function, are independent of any choice of metric. Assuming one is given an appropriate solution to the Romans theory with (M_4, g) as conformal boundary, we therefore expect the holographically renormalized action to be independent of g. Here one can mimic the field theory argument in [4], and attempt to show that arbitrary deformations $g_{ij} \to g_{ij} + \delta g_{ij}$ leave this action invariant. We have the general holographic Ward identity formula

$$\delta S = \int_{M_4} \mathrm{d}^4 x \sqrt{\det g} \left(\frac{1}{2} T_{ij}\, \delta g^{ij} + \mathscr{J}^i_I\, \delta A^I_i + \Xi\, \delta X_1 \right) . \tag{1.2}$$

Here S is the renormalized supergravity action of the Euclidean Romans theory, defined in section 2, while (g_{ij}, A^I_i, X_1) are the non-zero background fields in the $\mathcal{N}=2$ conformal gravity multiplet for the topological twist. Equivalently, these arise as boundary values of the Romans fields: in particular A^I_i is simply the restriction of the bulk SU(2) R-symmetry gauge field to the boundary at $z=0$, while $X_1 = \lim_{z\to 0}(X-1)/z^2 \log z$. For the topological twist these quantities are all fixed by the choice of metric g_{ij}: A^I_i is fixed to be the right-handed spin connection, while $X_1 = -R/12$, where $R = R(g)$ is the Ricci scalar for g. Thus the variations of these fields appearing in (1.2) are all determined by the

[2] For example, in the case of interest in this paper $d=4$, and $\Omega_4^{SO} \cong \mathbb{Z}$ with the map to the integers being given by the signature $\sigma(M_4) = b_2^+(M_4) - b_2^-(M_4) = \frac{1}{3}\int_{M_4} p_1(M_4)$, where p_1 denotes the first Pontryagin class. A generator of $\Omega_4^{SO} \cong \mathbb{Z}$ is the complex projective plane.

[3] One might also speculate that the dominant contribution may come from complex saddle points; that is, from complex-valued metrics — see, for example, [21]. In this paper we focus on real solutions.

metric variation δg_{ij}. On the other hand, T_{ij}, $\mathscr{J}_I^i$ and Ξ are respectively the holographic vacuum expectation values (VEVs) of the operators for which these boundary fields are the sources. In particular T_{ij} is the holographic stress-energy tensor. As is well-known, the expansion of the equations of motion near $z = 0$ does not fix these VEVs in terms of boundary data on M_4, but rather they are only determined by regularity of the solution in the interior. Determining these quantities for fixed boundary data is thus an extremely non-linear problem. What allows progress in this case is supersymmetry: the partition function should be described by a supersymmetric solution to the Romans theory.[4] By similarly solving the Killing spinor equations in a Fefferman-Graham-like expansion, we are able to compute these VEVs for a general supersymmetric solution. This still leaves certain unknown data, ultimately determined by regularity in the interior, but remarkably these constraints are sufficient to prove that (1.2) is indeed zero, for arbitrary δg_{ij}! More precisely, we show that the integrand on the right hand side is a total derivative, and its integral is then zero provided M_4 is closed, without boundary. The computation, although in principle straightforward, is not entirely trivial, and along the way we require some interesting identities that are specific to Riemannian four-manifolds (notably the quadratic curvature identity of Berger [24]). This is the main result of the paper, but it immediately raises a number of interesting questions. We postpone our discussion of these until later in the paper, notably at the end of section 4, and in sections 5 and 6.

The outline of the paper is as follows. In section 2 we define the relevant five-dimensional Euclidean $\mathcal{N} = 4^+$ gauged supergravity theory, and holographically renormalize its action S. In section 3 we show that on the conformal boundary of an asymptotically locally hyperbolic solution to this theory one obtains the supersymmetry equations [17] of Euclidean $\mathcal{N} = 2$ conformal supergravity, which admits [14] the topological twist as a solution. We then expand the bulk supersymmetry equations in a Fefferman-Graham-like expansion. Section 4 contains the main proof that $\delta S/\delta g_{ij} = 0$, while in section 5 we reformulate the supersymmetry equations in terms of a first order differential system for a twisted Sp(1) structure. On the conformal boundary this induces the canonical quaternionic Kähler structrure that exists on any oriented Riemannian four-manifold. This paper raises a number of interesting questions, prompting further computations, and the results may potentially be extended and generalized in a number of different directions. We comment on some of these issues in section 6.

2 Holographic supergravity theory

We begin in section 2.1 by defining a real Euclidean section of $\mathcal{N} = 4^+$ gauged supergravity in five dimensions. A Fefferman-Graham expansion of asymptotically locally hyperbolic solutions to this theory is constructed in section 2.2, for arbitrary conformal boundary four-manifold (M_4, g). Using this, in section 2.3 we holographically renormalize the action.

[4] If the dominant saddle point in (1.1) were non-supersymmetric, this would presumably be interpreted as spontaneous breaking of supersymmetry in the dual TQFT. This is certainly not expected in the case at hand, but would be interesting to investigate further.

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2.1 Euclidean Romans $\mathcal{N}=4^+$ theory

The Lorentzian signature Romans $\mathcal{N}=4^+$ theory [22] is a five-dimensional $SU(2)\times U(1)$ gauged supergravity which admits a supersymmetric AdS_5 vacuum. It is a consistent truncation of both Type IIB supergravity on S^5 [25], and also eleven-dimensional supergravity on an appropriate class of six-manifolds N_6 [26]. The bosonic sector comprises the metric $G_{\mu\nu}$, a dilaton ϕ, an $SU(2)_R$ Yang-Mills gauge field $\mathcal{A}^I_\mu$ $(I=1,2,3)$, a $U(1)_R$ gauge field $\mathcal{A}_\mu$, and two real anti-symmetric tensors $B^\alpha_{\mu\nu}$, $\alpha=4,5$, which transform as a charged doublet under $U(1)_R\cong SO(2)_R$. It is convenient to introduce the scalar field $X\equiv e^{-\frac{1}{\sqrt{6}}\phi}$ and the complex combinations $\mathcal{B}^\pm\equiv B^4\pm iB^5$. The associated field strengths are $\mathcal{F}=d\mathcal{A}$, $\mathcal{F}^I=d\mathcal{A}^I-\frac{1}{2}\epsilon^I{}_{JK}\mathcal{A}^J\wedge\mathcal{A}^K$, and $H^\pm=d\mathcal{B}^\pm\mp i\mathcal{A}\wedge\mathcal{B}^\pm$. We have set the gauged supergravity gauge coupling to 1.[5]

The bosonic action and equations of motion in Lorentzian signature appear in [25]. However, as we are interested in holographic duals to TQFTs defined on Riemannian four-manifolds, we require the Euclidean signature version of this theory. The Wick rotation in particular introduces a factor of i into the Chern-Simons couplings, leading to the Euclidean action

$$\begin{aligned} I=-\frac{1}{2\kappa_5^2}\int\Big[&R*1-3X^{-2}dX\wedge *dX+4(X^2+2X^{-1})*1-\frac{1}{2}X^4\,\mathcal{F}\wedge*\mathcal{F} \\ &-\frac{1}{4}X^{-2}\left(\mathcal{F}^I\wedge*\mathcal{F}^I+\mathcal{B}^-\wedge*\mathcal{B}^+\right)+\frac{1}{8}\mathcal{B}^-\wedge H^+-\frac{1}{8}\mathcal{B}^+\wedge H^--\frac{i}{4}\mathcal{F}^I\wedge\mathcal{F}^I\wedge\mathcal{A}\Big]. \end{aligned} \tag{2.1}$$

Here $R=R(G)$ denotes the Ricci scalar of the metric $G_{\mu\nu}$, and $*$ is the Hodge duality operator acting on forms. The associated equations of motion are:[6]

$$\begin{aligned} d(X^{-1}*dX)=&\frac{1}{3}X^4\,\mathcal{F}\wedge*\mathcal{F}-\frac{1}{12}X^{-2}\left(\mathcal{F}^I\wedge*\mathcal{F}^I+\mathcal{B}^-\wedge*\mathcal{B}^+\right) \\ &-\frac{4}{3}(X^2-X^{-1})*1\,, \end{aligned} \tag{2.2}$$

$$d(X^{-2}*\mathcal{F}^I)=\epsilon^I{}_{JK}X^{-2}*\mathcal{F}^J\wedge\mathcal{A}^K-i\mathcal{F}^I\wedge\mathcal{F}\,, \tag{2.3}$$

$$d(X^4*\mathcal{F})=-\frac{i}{4}\mathcal{F}^I\wedge\mathcal{F}^I-\frac{i}{4}\mathcal{B}^-\wedge\mathcal{B}^+\,, \tag{2.4}$$

$$H^\pm=\pm X^{-2}*\mathcal{B}^\pm\,, \tag{2.5}$$

$$\begin{aligned} R_{\mu\nu}=&\,3X^{-2}\partial_\mu X\partial_\nu X-\frac{4}{3}(X^2+2X^{-1})G_{\mu\nu}+\frac{1}{2}X^4\left(\mathcal{F}_\mu{}^\rho\mathcal{F}_{\nu\rho}-\frac{1}{6}G_{\mu\nu}\mathcal{F}^2\right) \\ &+\frac{1}{4}X^{-2}\left(\mathcal{F}^{I\rho}_\mu\mathcal{F}^I_{\nu\rho}-\frac{1}{6}G_{\mu\nu}(\mathcal{F}^I)^2+\mathcal{B}^-{}_{(\mu}{}^\rho\mathcal{B}^+_{\nu)\rho}-\frac{1}{6}G_{\mu\nu}\mathcal{B}^-{}_{\rho\sigma}\mathcal{B}^{+\rho\sigma}\right). \end{aligned} \tag{2.6}$$

Here $\mathcal{F}^2\equiv\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$, $(\mathcal{F}^I)^2\equiv\sum_{I=1}^3\mathcal{F}^I_{\mu\nu}\mathcal{F}^{I\mu\nu}$. In general equations (2.2)–(2.6) are complex, and solutions will likewise be complex. However, note that setting $i\mathcal{A}\equiv\mathcal{C}$ effectively

[5] In addition we have rescaled the $SU(2)_R$ gauge field and the anti-symmetric tensors by a factor of $1/\sqrt{2}$, compared to [25].

[6] Equation (2.3) incorporates a correction to the Lorentzian equation, in line with [26].

removes all factors of i. We may then consistently define a real section of this Euclidean theory in which all fields, and in particular $\mathcal{C}$ and $\mathcal{B}^{\pm} = B^4 \pm \mathrm{i}B^5$, are real. We henceforth impose these reality conditions. Although globally $\mathcal{A}$ is a $\mathrm{U}(1)_R$ gauge field in the original Lorentzian theory, after the above Wick rotation the real field $\mathcal{C} = \mathrm{i}\mathcal{A}$ effectively becomes an $\mathrm{SO}(1,1)_R$ gauge field. We may then think of $\mathcal{C}$ as a global one-form, but for which the theory has a symmetry $\mathcal{C} \to \mathcal{C} - \mathrm{d}\lambda$, for any global function λ. We denote the corresponding field strength as $\mathcal{G} \equiv \mathrm{d}\mathcal{C} = \mathrm{i}\mathcal{F}$.

In the Lorentzian theory the fermionic sector contains four gravitini and four dilatini, which together with the spinor parameters ϵ all transform in the fundamental **4** representation of the $\mathrm{Sp}(2)_R$ global R-symmetry group. The $\mathrm{SU}(2) \times \mathrm{U}(1) \subset \mathrm{Sp}(2)$ gauge symmetry arises as a gauged subgroup. Since $\mathrm{Sp}(2) \cong Spin(5)$ it is natural to introduce the associated Clifford algebra $\mathrm{Cliff}(5,0)$, with generators Γ_A, $A = 1,\ldots,5$, satisfying $\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}$. We then decompose $I,J,K = 1,2,3$, transforming in the **3** of SU(2), and $\alpha,\beta = 4,5$ in the **2** of U(1). In Euclidean signature the conditions for preserving supersymmetry are then the vanishing of the following supersymmetry variations of the gravitini and dilatini, respectively:

$$\begin{aligned}
0 &= D_\mu\epsilon + \frac{\mathrm{i}}{3}\gamma_\mu\Big(X + \frac{1}{2}X^{-2}\Big)\Gamma_{45}\epsilon \\
&\quad + \frac{\mathrm{i}}{24}(\gamma_\mu{}^{\nu\rho} - 4\delta_\mu^\nu\gamma^\rho)\left(X^{-1}(\mathcal{F}^I_{\nu\rho}\Gamma_I + B^\alpha_{\nu\rho}\Gamma_\alpha) + X^2\mathcal{F}_{\nu\rho}\right)\epsilon\,, \qquad (2.7)\\
0 &= \frac{\sqrt{3}}{2}\mathrm{i}\gamma^\mu X^{-1}\partial_\mu X\epsilon + \frac{1}{\sqrt{3}}\Big(X - X^{-2}\Big)\Gamma_{45}\epsilon \\
&\quad + \frac{1}{8\sqrt{3}}\gamma^{\mu\nu}\Big(X^{-1}(\mathcal{F}^I_{\mu\nu}\Gamma_I + B^\alpha_{\mu\nu}\Gamma_\alpha) - 2X^2\mathcal{F}_{\mu\nu}\Big)\epsilon\,, \qquad (2.8)
\end{aligned}$$

where the covariant derivative is

$$D_\mu\epsilon \equiv \nabla_\mu\epsilon + \frac{1}{2}\mathcal{A}_\mu\Gamma_{45}\epsilon + \frac{1}{2}\mathcal{A}^I_\mu\Gamma_{I45}\epsilon\ . \qquad (2.9)$$

Here γ_μ, $\mu = 1,\ldots,5$, are generators of the Euclidean spacetime Clifford algebra, satisfying $\{\gamma_\mu, \gamma_\nu\} = 2G_{\mu\nu}$, where recall $G_{\mu\nu}$ is the metric. Given the gauging it is natural to introduce the following choice of generators:

$$\Gamma_I = \sigma_3 \otimes \sigma_I\,, \quad I = 1,2,3\,, \qquad \Gamma_4 = \sigma_1 \otimes 1_2\,, \qquad \Gamma_5 = \sigma_2 \otimes 1_2\,, \qquad (2.10)$$

where σ_I are the Pauli matrices, and 1_2 denotes the 2×2 identity matrix. In particular notice that $\Gamma_{45} = \mathrm{i}\sigma_3 \otimes 1_2$ squares to -1_4, and we may write

$$\epsilon = \begin{pmatrix}\epsilon^+ \\ \epsilon^-\end{pmatrix}, \qquad (2.11)$$

where the spinor doublets $\epsilon^\pm$ denote projections onto the $\pm\mathrm{i}$ eigenspaces of Γ_{45}, respectively. One then has

$$\Gamma_I\epsilon = \begin{pmatrix}\sigma_I\epsilon^+ \\ -\sigma_I\epsilon^-\end{pmatrix}, \qquad B^\alpha_{\mu\nu}\Gamma_\alpha\epsilon = \begin{pmatrix}\mathcal{B}^-_{\mu\nu}\epsilon^- \\ \mathcal{B}^+_{\mu\nu}\epsilon^+\end{pmatrix}. \qquad (2.12)$$

We next introduce the charge conjuguation matrix $\mathscr{C}$ for the Euclidean spacetime Clifford algebra. By definition $\gamma_\mu^* = \mathscr{C}^{-1}\gamma_\mu\mathscr{C}$, and one may choose Hermitian generators $\gamma_\mu^\dagger = \gamma_\mu$ together with the conditions $\mathscr{C} = \mathscr{C}^* = -\mathscr{C}^T$, $\mathscr{C}^2 = -1$. We may then define the following charge conjugate spinor in Euclidean signature

$$\epsilon^c \equiv (\sigma_3 \otimes \mathrm{i}\sigma_2)\,\mathscr{C}\epsilon^* \ . \tag{2.13}$$

It is straightforward to check that $(\epsilon^c)^c = \epsilon$. Moreover, provided $\mathcal{C} = \mathrm{i}\mathcal{A}$ and $\mathcal{B}^\pm$ (and all other bosonic fields) are real, then one can show that ϵ satisfies the gravitini and dilatini equations (2.7), (2.8) if and only if its charge conjugate ϵ^c satisfies the same equations. Given this property, we may consistently impose the symplectic Majorana condition $\epsilon^c = \epsilon$. We will be interested in solutions that satisfy these reality conditions.

2.2 Fefferman-Graham expansion

In this section we determine the Fefferman-Graham expansion [27] of asymptotically locally hyperbolic solutions to this Euclidean Romans theory. This is the general solution to the bosonic equations of motions (2.2)–(2.6), expressed as a perturbative expansion in a radial coordinate near the conformal boundary.

We take the form of the metric to be [27]

$$G_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \frac{1}{z^2}\mathrm{d}z^2 + \frac{1}{z^2}\mathrm{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j = \frac{1}{z^2}\mathrm{d}z^2 + h_{ij}\mathrm{d}x^i\mathrm{d}x^j \ . \tag{2.14}$$

where the AdS radius $\ell = 1$, and in turn we have the expansion

$$\mathrm{g}_{ij} = \mathrm{g}^0_{ij} + z^2\mathrm{g}^2_{ij} + z^4\big(\mathrm{g}^4_{ij} + h^0_{ij}(\log z)^2 + h^1_{ij}\log z\big) + o(z^4) \ . \tag{2.15}$$

Here $\mathrm{g}^0_{ij} = g_{ij}$ is the boundary metric induced on the conformal boundary M_4 at $z = 0$.

It is convenient to introduce the inner product $\langle\alpha, \beta\rangle$ between two p-forms α, β via

$$\alpha \wedge *\beta = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p}\beta^{\mu_1\cdots\mu_p}\,\mathrm{vol} = \frac{1}{p!}\langle\alpha, \beta\rangle\,\mathrm{vol}\,, \tag{2.16}$$

where vol denotes the volume form, with associated Hodge duality operator $*$. The volume form for the five-dimensional bulk metric (2.14) is

$$\mathrm{vol}_5 = \frac{1}{z^5}\mathrm{d}z \wedge \mathrm{vol}_{\mathrm{g}} = \frac{1}{z^5}\mathrm{d}z \wedge \sqrt{\det \mathrm{g}}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^4 \ . \tag{2.17}$$

The determinant may then be expanded in a series in z, around that for g^0, as follows

$$\begin{aligned}\sqrt{\det \mathrm{g}} = \sqrt{\det \mathrm{g}^0}\,\bigg[1 + \frac{z^2}{2}t^{(2)} + \frac{z^4}{2}\bigg(&t^{(4)} - \frac{1}{2}t^{(2,2)} + \frac{1}{4}(t^{(2)})^2 \\ &+ u^{(0)}(\log z)^2 + u^{(1)}\log z\bigg)\bigg] + o(z^4)\,.\end{aligned} \tag{2.18}$$

Here we have denoted $t^{(n)} \equiv \mathrm{Tr}\big[(\mathrm{g}^0)^{-1}\mathrm{g}^n\big]$, $u^{(n)} \equiv \mathrm{Tr}\big[(\mathrm{g}^0)^{-1}h^n\big]$ and $t^{(2,2)} \equiv \mathrm{Tr}\big[(\mathrm{g}^0)^{-1}\mathrm{g}^2\big]^2$.

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The remaining bosonic fields are likewise expanded as follows:

$$X = 1 + z^2\,(X_1 \log z + X_2) + z^4(X_3 \log z + X_4) + o(z^4)\,, \tag{2.19}$$

$$\mathcal{A}^I = A^I + z^2(a_1^I \log z + a_2^I) + o(z^2)\,, \tag{2.20}$$

$$\mathcal{A} = \mathrm{a} + z^2(\mathrm{a}_1 \log z + \mathrm{a}_2) + o(z^2)\,, \tag{2.21}$$

$$\mathcal{B}^\pm = \frac{1}{z} b^\pm + \mathrm{d}z \wedge b_1^\pm + z(b_2^\pm \log z + b_3^\pm) + o(z)\,, \tag{2.22}$$

A priori there are additional terms that appear in these expansions. However, these may either be gauged away, or turn out to be set to zero by the equations of motion, and we have thus removed them in order to streamline the presentation.

We now substitute the above expansions into the equations of motion (2.2)–(2.6) and solve them order by order in the radial coordinate z in terms of the boundary data $\mathrm{g}^0 = g, X_1, A^I, \mathrm{a}$ and $b^\pm$. This will leave a number of terms undetermined. For the Einstein equation (2.6) we will need the Ricci tensor of the metric (2.14):

$$R_{zz} = -\,\frac{4}{z^2} - \frac{1}{2}\Big(\mathrm{Tr}\left[\mathrm{g}^{-1}\partial_z^2\mathrm{g}\right] - \frac{1}{z}\mathrm{Tr}\left[\mathrm{g}^{-1}\partial_z\mathrm{g}\right] - \frac{1}{2}\mathrm{Tr}\left[\mathrm{g}^{-1}\partial_z\mathrm{g}\right]^2\Big)\,, \tag{2.23}$$

$$\begin{aligned} R_{ij} = -\,\frac{4}{z^2}\mathrm{g}_{ij} - \Big(&\frac{1}{2}\partial_z^2\mathrm{g} - \frac{3}{2z}\partial_z\mathrm{g} - \frac{1}{2}(\partial_z\mathrm{g})\mathrm{g}^{-1}(\partial_z\mathrm{g}) + \frac{1}{4}(\partial_z\mathrm{g})\mathrm{Tr}\left[\mathrm{g}^{-1}\partial_z\mathrm{g}\right] \\ &- R(\mathrm{g}) - \frac{1}{2z}\mathrm{g}\mathrm{Tr}\left[\mathrm{g}^{-1}\partial_z\mathrm{g}\right]\Big)_{ij}\,, \end{aligned} \tag{2.24}$$

$$R_{zi} = -\,\frac{1}{2}(\mathrm{g}^{-1})^{jk}\Big(\nabla_i\mathrm{g}_{jk,z} - \nabla_k\mathrm{g}_{ij,z}\Big)\,. \tag{2.25}$$

Here ∇ is the covariant derivative for g, and we have corrected the sign of $R(\mathrm{g})_{ij}$ and the right hand side of (2.25) compared to [28].

Examining first the equation (2.5) gives at leading order

$$*_{\mathrm{g}^0}\, b^\pm = \mp b^\pm\,, \tag{2.26}$$

so that the boundary B-fields b^+, b^- are required to be anti-self-dual and self-dual, respectively. At subleading orders one finds

$$b_1^\pm = \mp *_{\mathrm{g}^0}\left(\mathrm{d}b^\pm \mp \mathrm{i}\mathrm{a}\wedge b^\pm\right)\,, \qquad *_{\mathrm{g}^0}\, b_2^\pm = \pm(b_2^\pm - 2X_1\, b^\pm)\,. \tag{2.27}$$

In particular notice that the first equation fixes $b_1^\pm$ in terms of boundary data, while the second equation determines only the anti-self-dual/self-dual parts of $b_2^\pm$, respectively. An equation may also be derived for $b_3^\pm$, although we will not need this in what follows.

Next the gauge field equations (2.3), (2.4) determine

$$\begin{aligned} \mathrm{a}_1 &= -\,\frac{1}{2} *_{\mathrm{g}^0} \mathrm{d} *_{\mathrm{g}^0} \mathrm{f} + \frac{\mathrm{i}}{8} *_{\mathrm{g}^0}\left(b^-\wedge b_1^+ + b^+\wedge b_1^-\right)\,, \\ a_1^I &= -\,\frac{1}{2} *_{\mathrm{g}^0} \mathcal{D} *_{\mathrm{g}^0} F^I\,, \end{aligned} \tag{2.28}$$

in terms of boundary data, where the curvatures are $\mathrm{f} \equiv \mathrm{d}\mathrm{a}$, $F^I \equiv \mathrm{d}A^I - \frac{1}{2}\epsilon^I{}_{JK}A^J \wedge A^K$, and we have introduced a gauge covariant derivative with respect to the boundary SU(2) field: $\mathcal{D}\alpha^I \equiv \mathrm{d}\alpha^I - \epsilon^I{}_{JK}A^J \wedge \alpha^K$. In addition we have the constraints

$$\mathrm{d} *_{\mathrm{g}^0} \mathrm{a}_2 = -\frac{\mathrm{i}}{8} F^I \wedge F^I\,, \qquad \mathcal{D} *_{\mathrm{g}^0} a_2^I = 0\,, \tag{2.29}$$

which leave a_2 and a_2^I partially undetermined.

Turning next to the scalar equation of motion (2.2) we find

$$4X_3 = -\nabla^2 X_1 - 2\left(t^{(2)}X_1 - 2X_1^2\right) - \frac{1}{24}\left(\langle b^+, b_2^-\rangle_{\mathrm{g}^0} + \langle b^-, b_2^+\rangle_{\mathrm{g}^0}\right)\,, \tag{2.30}$$

$$\begin{aligned} 4X_4 = &-\nabla^2 X_2 - \left(t^{(2)}X_1 + 2t^{(2)}X_2 - X_1^2 - 4X_1X_2 + 4X_3\right) - \frac{1}{24}\langle F^I, F^I\rangle_{\mathrm{g}^0} + \frac{1}{6}\langle \mathrm{f}, \mathrm{f}\rangle_{\mathrm{g}^0} \\ &- \frac{1}{12}\langle b_1^+, b_1^-\rangle_{\mathrm{g}^0} + \frac{1}{12}\langle b^-, \mathrm{g}^2 \circ b^+\rangle_{\mathrm{g}^0} - \frac{1}{24}\left(\langle b^+, b_3^-\rangle_{\mathrm{g}^0} + \langle b^-, b_3^+\rangle_{\mathrm{g}^0}\right)\,. \end{aligned} \tag{2.31}$$

We regard these as determining X_3, X_4 in terms of X_1 (a boundary field), and X_2 (which is undetermined by the equations of motion), together with the other fields in the expansion. In the second equation we have used the definition

$$(\mathrm{g}^2 \circ \alpha)_{i_1\cdots i_p} \equiv (\mathrm{g}^2)_{[i_1}{}^j \alpha_{|j|i_2\cdots i_p]}\,, \tag{2.32}$$

where α is a p-form on M_4. Here indices are always raised with g^0, so $(\mathrm{g}^2)_i{}^j \equiv (\mathrm{g}^2)_{ik}(\mathrm{g}^0)^{kj}$.

Finally, we introduce the matter modified boundary Ricci tensor

$$\mathscr{R}_{ij} = \mathscr{R}_{ij}(\mathrm{g}^0) \equiv R_{ij}(\mathrm{g}^0) - \frac{1}{4}(b^+)_{(i}{}^k(b^-)_{j)k}\,. \tag{2.33}$$

Notice the scalar curvature is $\mathscr{R}(\mathrm{g}^0) = R(\mathrm{g}^0)$, due to the opposite duality properties (2.26) of $b^\pm$. From the ij component of the Einstein equation (2.6), using (2.24) gives

$$\mathrm{g}_{ij}^2 = -\frac{1}{2}\left(\mathscr{R}_{ij} - \frac{1}{6}\mathrm{g}_{ij}^0\mathscr{R}\right). \tag{2.34}$$

The right hand side is a matter modified form of the Schouten tensor. From this expression we immediately deduce the traces

$$t^{(2)} = -\frac{1}{6}\mathscr{R}\,, \qquad t^{(2,2)} = \frac{1}{4}\left(\mathscr{R}_{ij}\mathscr{R}^{ij} - \frac{2}{9}\mathscr{R}^2\right)\,. \tag{2.35}$$

The zz component of the Einstein equation in (2.6), together with (2.23), determines the traces of higher order components in the expansion of the bulk metric:

$$u^{(0)} = -2X_1^2\,, \tag{2.36}$$

$$u^{(1)} = -4X_1X_2 + \frac{1}{96}\left(\langle b^+, b_2^-\rangle_{\mathrm{g}^0} + \langle b^-, b_2^+\rangle_{\mathrm{g}^0}\right)\,, \tag{2.37}$$

$$\begin{aligned} 4t^{(4)} = &\, t^{(2,2)} - u^{(0)} - 3u^{(1)} - 3X_1^2 - 8X_2^2 - 12X_1X_2 + \frac{1}{12}\left(\langle \mathrm{f}, \mathrm{f}\rangle_{\mathrm{g}^0} + \frac{1}{2}\langle F^I, F^I\rangle_{\mathrm{g}^0}\right) \\ &- \frac{1}{6}\langle b_1^+, b_1^-\rangle_{\mathrm{g}^0} - \frac{1}{12}\langle b^-, (\mathrm{g}^2 \circ b^+)\rangle_{\mathrm{g}^0} + \frac{1}{24}\left(\langle b^+, b_3^-\rangle_{\mathrm{g}^0} + \langle b^-, b_3^+\rangle_{\mathrm{g}^0}\right)\,. \end{aligned} \tag{2.38}$$

Returning to the ij component we may determine the logarithmic terms in (2.15):

$$\begin{aligned}
h^0_{ij} &= \frac{1}{4}\mathrm{g}^0_{ij}(u^{(0)}+2u^{(1)}+8X_1X_2) \\
&\quad -\frac{1}{16}\left[(b^+)_{(i}{}^k(b^-_2)_{j)k}+(b^-)_{(i}{}^k(b^+_2)_{j)k}-\frac{1}{6}\mathrm{g}^0_{ij}\big(\langle b^+,b^-_2\rangle_{\mathrm{g}^0}+\langle b^-,b^+_2\rangle_{\mathrm{g}^0}\big)\right], \qquad (2.39) \\
h^1_{ij} &= -\frac{1}{2}h^0_{ij}+\mathrm{g}^2_{ik}(\mathrm{g}^0)^{kl}\mathrm{g}^2_{lj}+\frac{1}{4}\mathrm{g}^0_{ij}\big(4t^{(4)}-2t^{(2,2)}+u^{(1)}+8X_2^2\big) \\
&\quad +\frac{1}{4}\big(\nabla^k\nabla_i\mathrm{g}^2_{jk}+\nabla^k\nabla_j\mathrm{g}^2_{ik}-\nabla^2\mathrm{g}^2_{ij}-\nabla_i\nabla_jt^{(2)}\big)-\frac{1}{8}\left((b^+_1)_{(i}(b^-_1)_{j)}-\frac{1}{3}\mathrm{g}^0_{ij}\langle b^+_1,b^-_1\rangle_{\mathrm{g}^0}\right) \\
&\quad +\frac{1}{8}\left[(b^-)_{(i|k|}(\mathrm{g}^2)^{kl}(b^+)_{j)l}-\frac{1}{3}\mathrm{g}^0_{ij}(b^-)_k{}^m(\mathrm{g}^2)^{kl}(b^+)_{lm}\right] \\
&\quad -\frac{1}{8}\left[(b^+)_{(i}{}^k(b^-_3)_{j)k}+(b^-)_{(i}{}^k(b^+_3)_{j)k}-\frac{1}{6}\mathrm{g}^0_{ij}\big(\langle b^+,b^-_3\rangle_{\mathrm{g}^0}+\langle b^-,b^+_3\rangle_{\mathrm{g}^0}\big)\right] \\
&\quad -\frac{1}{4}\left[\mathrm{f}_{ik}\mathrm{f}_j{}^k+\frac{1}{2}F^I_{ik}F^{Ik}_j-\frac{1}{6}\mathrm{g}^0_{ij}\left(\langle \mathrm{f},\mathrm{f}\rangle_{\mathrm{g}^0}+\frac{1}{2}\langle F^I,F^I\rangle_{\mathrm{g}^0}\right)\right]. \qquad (2.40)
\end{aligned}$$

The structure of the ij component of the Einstein equation in four dimensions is such that g^4 always appears with zero coefficient, and so is left undetermined. In the original literature [29] the iz component has been used to determine g^4 up to an arbitrary symmetric divergence-free tensor. However, in the supergravity we are considering the presence of a $(\log z)^2$ contribution to the bulk scalar field expansion means that X_2 appears without a derivative, which hence spoils this approach. In section 3.4 we will see that by imposing supersymmetry we obtain further constraints on the fields, and in particular this leads to an expression for g^4 in terms of other data.

2.3 Holographic renormalization

Having solved the bulk equations of motion to the relevant order, we are now in a position to holographically renormalize the Euclidean Romans theory. The bulk action (2.1) is divergent for an asymptotically locally hyperbolic solution, but can be rendered finite by the addition of appropriate local counterterms. The corresponding computations in Lorentzian signature have been carried out in [23].

We begin by taking the trace of the Einstein equation (2.6). Substituting the result together with (2.5) into the Euclidean action (2.1), we arrive at the bulk on-shell action

$$\begin{aligned}
I_{\text{on-shell}} = \frac{1}{2\kappa_5^2}\int_{Y_5}\Big[&\frac{8}{3}(X^2+2X^{-1})*1+\frac{1}{3}X^4\mathcal{F}\wedge*\mathcal{F}+\frac{1}{6}X^{-2}\mathcal{F}^I\wedge*\mathcal{F}^I \\
&-\frac{1}{12}X^{-2}\mathcal{B}^-\wedge*\mathcal{B}^++\frac{\mathrm{i}}{4}\mathcal{F}^I\wedge\mathcal{F}^I\wedge\mathcal{A}\Big]. \qquad (2.41)
\end{aligned}$$

Here Y_5 is the bulk five-manifold, with boundary $\partial Y_5 = M_4$. In order to obtain the equations of motion (2.2)–(2.6) from the original bulk action (2.1) on a manifold with boundary, one has to add the Gibbons-Hawking term

$$I_{\mathrm{GH}} = -\frac{1}{\kappa_5^2}\int_{\partial Y_5}\mathrm{d}^4x\,\sqrt{\det h}\,K = \frac{1}{\kappa_5^2}\int_{\partial Y_5}\mathrm{d}^4x\,z\partial_z\sqrt{\det h}\,. \qquad (2.42)$$

Here, more precisely, one cuts Y_5 off at some finite radial distance, or equivalently non-zero $z > 0$, and (M_4, h) is the resulting four-manifold boundary, with trace of the second fundamental form being K. Recall from (2.14) that $h_{ij} = \frac{1}{z^2}\mathrm{g}_{ij}$.

The combined action $I_{\text{on-shell}} + I_{\text{GH}}$ suffers from divergences as the conformal boundary is approached. To remove these divergences we use the standard method of holographic renormalization [28–30]. Namely, we introduce a small cut-off $z = \delta > 0$, and expand all fields via the Fefferman-Graham expansion of section 2.2 to identify the divergences. These may be cancelled by adding local boundary counterterms. We find

$$\begin{aligned} I_{\text{counterterm}} = \frac{1}{\kappa_5^2}\int_{\partial Y_5} \mathrm{d}^4x\sqrt{\det h}\,\Big\{ & 3 + \frac{1}{4}R(h) + 3(X-1)^2 - \frac{1}{32}\langle \mathcal{B}^-, \mathcal{B}^+\rangle_h \\ & + \log\delta\left[-\frac{1}{8}\left(\mathscr{R}_{ij}(h)\mathscr{R}^{ij}(h) - \frac{1}{3}\mathscr{R}(h)^2\right) + \frac{3}{2}(\log\delta)^{-2}(X-1)^2 \right. \\ & \left. + \frac{1}{48}\langle H^-, H^+\rangle_h + \frac{1}{8}\langle \mathcal{F}, \mathcal{F}\rangle_h + \frac{1}{16}\langle \mathcal{F}^I, \mathcal{F}^I\rangle_h \right]\Big\} . \end{aligned} \tag{2.43}$$

Notice the somewhat unusual form of the logarithmic term for the scalar field X, but cf. the expansion (2.19). As is standard, we have written the counterterm action (2.43) covariantly in terms of the induced metric h_{ij} on $M_4 = \partial Y_5$. The total renormalized action is then

$$S = \lim_{\delta\to 0}\left(I_{\text{on-shell}} + I_{\text{GH}} + I_{\text{counterterm}}\right) , \tag{2.44}$$

which by construction is finite.

The choice of local counterterms (2.43) defines a particular renormalization scheme, that is in some sense a "minimal scheme" in the case at hand. However, we are free to consider a non-minimal scheme where we add local counterterms to the action which remain finite as $\delta \to 0$. For the supergravity theory we are considering, the following are an independent set of finite counterterms that are both diffeomorphism and gauge invariant:[7]

$$\begin{aligned} I_{\text{ct, finite}} = -\frac{1}{\kappa_5^2}\int_{\partial Y_5} \mathrm{d}^4x\sqrt{\det h}\,\Big[& \zeta_1 R^2 + \zeta_2 C_{ijkl}C^{ijkl} + \zeta_3\mathcal{F}_{ij}\mathcal{F}^{ij} + \zeta_4\mathcal{F}^I_{ij}\mathcal{F}^{Iij} \\ & + \zeta_5\mathcal{E} + \zeta_6\mathcal{P} + \zeta_7\epsilon^{ijkl}\mathcal{F}_{ij}\mathcal{F}_{kl} + \zeta_8\epsilon^{ijkl}\mathcal{F}^I_{ij}\mathcal{F}^I_{kl}\Big] . \end{aligned} \tag{2.45}$$

Here $\zeta_1, \ldots, \zeta_8$ are arbitrary constant coefficients, C_{ijkl} denotes the Weyl tensor of the metric h_{ij}, while the Euler scalar $\mathcal{E}$ and Pontryagin scalar $\mathcal{P}$ are respectively

$$\mathcal{E} = R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2 , \qquad \mathcal{P} = \frac{1}{2}\epsilon^{ijkl}R_{ijmn}R_{kl}{}^{mn} . \tag{2.46}$$

In particular, notice that for compact $M_4 = \partial Y_5$ without boundary, the second line of (2.45) are all topological invariants: they are proportional to the Euler number $\chi(M_4)$, the signature $\sigma(M_4)$, and the Chern numbers $\int_{M_4} c_1(\mathcal{L})^2$, $\int_{M_4} c_2(\mathcal{V})$ respectively, where $\mathcal{L}$ and $\mathcal{V}$ denote the rank 1 and rank 2 complex vector bundles associated to the $\mathrm{U}(1)_R$ and $\mathrm{SU}(2)_R$

[7] We may also add finite local counterterms constructed from the B-field. For example, terms proportional to $\int_{\partial Y_5}\mathrm{d}^4x\sqrt{\det h}\,\langle H^-, H^+\rangle_h$, or $\int_{\partial Y_5}\mathrm{d}^4x\sqrt{\det h}\,R(h)\langle \mathcal{B}^-, \mathcal{B}^+\rangle_h$. However, for the topological twist we will later set the B-field to zero, and these terms will not be relevant to our discussion.

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gauge bundles, respectively. In the real Euclidean theory in which we are working, recall that $\mathcal{F} = \mathrm{d}\mathcal{A}$ is globally exact (and purely imaginary), and in any case for the topological twist studied later in the paper we will have $\mathcal{A}\mid_{M_4} = 0$. Being topological invariants, the variation of the action we shall compute in section 4 will be insensitive to the choice of constants $\zeta_5, \ldots, \zeta_8$.

As emphasized in [31], in order to make quantitative comparisons in AdS/CFT it is important to match choices of renormalization schemes on the two sides. In particular, localization calculations in QFT make a (somewhat implicit) choice of scheme. In the case at hand, we note that in [32] a supersymmetric Rényi entropy, computed in field theory using localization, was successfully matched to a gravity calculation involving a supersymmetric black hole in the $\mathcal{N} = 4^+$ Romans theory. Here the supergravity action was computed using the minimal scheme. Our computation in section 4 will imply that this minimal scheme is indeed the correct one to compare to the topological twist of [4]. We shall make further comments on this, and the relation to recent papers [31, 33–35], in section 4.2.

Given the renormalized action we may compute the following VEVs:

$$\begin{aligned} \langle T_{ij} \rangle &= \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{ij}} \,, \qquad & \langle \Xi \rangle &= \frac{1}{\sqrt{g}} \frac{\delta S}{\delta X_1} \,, \\ \langle \mathscr{J}_I^i \rangle &= \frac{1}{\sqrt{g}} \frac{\delta S}{\delta A_i^I} \,, \qquad & \langle \mathbb{J}^i \rangle &= \frac{1}{\sqrt{g}} \frac{\delta S}{\delta \mathrm{a}_i} \,. \end{aligned} \tag{2.47}$$

Here, as usual in AdS/CFT, the boundary fields $\mathrm{g}^0_{ij} = g_{ij}$, X_1, A_i^I and a_i act as sources for operators, and the expressions in (2.47) compute the vacuum expectation values of these operators. Similar expressions may also be written for the boundary fields $b^{\pm}$ for $\mathcal{B}^{\pm}$, but these will be zero for the topological twist of interest and play no role in the present paper. Using the above holographic renormalization we may write (2.47) as the following limits:

$$\begin{aligned} \langle T_{ij} \rangle = \frac{1}{\kappa_5^2} \lim_{\delta \to 0} \frac{1}{\delta^2} \Bigg[& - K_{ij} + K h_{ij} - \left(3 + 3(X-1)^2\right) h_{ij} + \frac{1}{2} \left(\mathscr{R}_{ij}(h) - \frac{1}{2} \mathscr{R}(h)\, h_{ij} \right) \\ & + \log\delta \left(\frac{1}{4} \mathscr{B}_{ij}(h) + \frac{1}{2} \mathcal{F}_{ik} \mathcal{F}_j{}^k - \frac{1}{8} h_{ij} \langle \mathcal{F}, \mathcal{F} \rangle_h + \frac{1}{4} \mathcal{F}^I_{ik} \mathcal{F}^{Ik}_j - \frac{1}{16} h_{ij} \langle \mathcal{F}^I, \mathcal{F}^I \rangle_h \right. \\ & \left. + \frac{1}{8} H^-_{ikl} H^+{}_j{}^{kl} - \frac{1}{48} h_{ij} \langle H^-, H^+ \rangle_h - \frac{3}{2} (\log\delta)^{-2} (X-1)^2 h_{ij} \right) \Bigg] \,, \end{aligned} \tag{2.48}$$

where K_{ij} is the second fundamental form of the cut-off hypersurface (M_4, h_{ij}) and the B-field modified Bach tensor is (cf. (2.33))

$$\begin{aligned} \mathscr{B}_{ij} = & -\frac{2}{3} \nabla_i \nabla_j \mathscr{R} - \nabla^2 \left(\mathscr{R}_{ij} - \frac{1}{6} h_{ij} \mathscr{R} \right) + 2 \nabla_k \nabla_{(i} \mathscr{R}^k{}_{j)} - 2 \mathscr{R}_{ik} \mathscr{R}^k{}_j + \frac{2}{3} \mathscr{R} \mathscr{R}_{ij} \\ & + \frac{1}{2} h_{ij} \left(\mathscr{R}_{kl} \mathscr{R}^{kl} - \frac{1}{3} \mathscr{R}^2 \right) , \end{aligned} \tag{2.49}$$

together with

$$
\begin{aligned}
\langle \Xi \rangle &= \frac{1}{\kappa_5^2} \lim_{\delta\to 0} \frac{\log\delta}{\delta^2}\Big[-3X^{-2}\delta\partial_\delta X + 6(X-1) + 3(\log\delta)^{-1}(X-1)\Big]\,, \\
\langle \mathscr{J}^{Ii} \rangle &= \frac{1}{4\kappa_5^2} \lim_{\delta\to 0} \frac{1}{\delta^4}\Big\{ -*_h\Big[\mathrm{d}x^i \wedge (X^{-2} *_5 \mathcal{F}^I + \mathrm{i}\mathcal{F}^I \wedge \mathcal{A})\Big] + \log\delta\, \mathcal{D}_j \mathcal{F}^{Iij} \Big\}\,, \\
\langle \mathbb{J}^i \rangle &= \frac{1}{2\kappa_5^2} \lim_{\delta\to 0} \frac{1}{\delta^4}\Big[-*_h\Big(\mathrm{d}x^i \wedge X^4 *_5 \mathcal{F}\Big) + \log\delta\, \nabla_j \mathcal{F}^{ij}\Big]\,.
\end{aligned} \tag{2.50}
$$

Here $*_h$ denotes the Hodge duality operator for the metric h_{ij}. A computation then gives the finite expressions

$$
\begin{aligned}
\langle T_{ij} \rangle = \frac{1}{\kappa_5^2}\Big[& 2\mathrm{g}^4_{ij} + \frac{1}{2}h^1_{ij} - \frac{1}{2}(4t^{(4)} - 2t^{(2,2)} - \frac{1}{2}u^{(1)})\mathrm{g}^0_{ij} - 3\mathrm{g}^0_{ij}X_2^2 - \mathrm{g}^2_{ij}t^{(2)} \\
& + \frac{1}{4}\Big(\nabla^k\nabla_i \mathrm{g}^2_{jk} + \nabla^k\nabla_j \mathrm{g}^2_{ik} - \nabla^2 \mathrm{g}^2_{ij} - \nabla_i\nabla_j t^{(2)}\Big) + \frac{1}{4}\mathrm{g}^0_{ij}(\mathrm{g}^2_{kl}R^{kl}) - \frac{1}{4}\mathrm{g}^2_{ij}R \\
& - \frac{1}{8}\Big[(b^+)_{(i}{}^k(b_3^-)_{j)k} + (b^-)_{(i}{}^k(b_3^+)_{j)k} - \frac{1}{2}\mathrm{g}^0_{ij}(\langle b^+, b_3^-\rangle_{\mathrm{g}^0} + \langle b^-, b_3^+\rangle_{\mathrm{g}^0})\Big] \\
& + \frac{1}{8}\Big[(b^+)_{(i|k|}(\mathrm{g}^2)^{kl}(b^-)_{j)l} - \frac{1}{2}\mathrm{g}^0_{ij}\langle b^-, (\mathrm{g}^2 \circ b^+)\rangle_{\mathrm{g}^0}\Big]\Big]\,,
\end{aligned} \tag{2.51}
$$

$$
\langle \Xi \rangle = \frac{3}{\kappa_5^2} X_2\,, \tag{2.52}
$$

$$
\langle \mathscr{J}^I_i \rangle = -\frac{1}{4\kappa_5^2}\left[(a^I_1)_i + 2(a^I_2)_i - \mathrm{i}\big(*_4 (\mathrm{a}\wedge F^I)\big)_i\right]\,, \tag{2.53}
$$

$$
\langle \mathbb{J}_i \rangle = -\frac{1}{2\kappa_5^2}\left[(\mathrm{a}_1)_i + 2(\mathrm{a}_2)_i\right]\,. \tag{2.54}
$$

Notice that these expressions contain a number of terms that are not determined, in terms of boundary data, by the Fefferman-Graham expansion of the bosonic equations of motion. In particular the g^4_{ij} term in the stress-energy tensor T_{ij}, the scalar X_2 that determines Ξ, and a^I_2, a_2 appearing in the $SU(2)_R$ and $U(1)_R$ current, respectively. The general holographic Ward identity corresponding to the first three variations of the action is given by equation (1.2). We will need the expressions (2.51)–(2.53) in section 4.

3 Supersymmetric solutions

In this section we study supersymmetric solutions to the Euclidean $\mathcal{N} = 4^+$ theory. We begin in section 3.1 by deriving the Killing spinor equations on the conformal boundary, starting from the bulk equations (2.7), (2.8). We precisely recover the Euclidean $\mathcal{N} = 2$ conformal supergravity equations of [17]. In section 3.2 we then recall from [14] how the topological twist arises as a special solution to these Killing spinor equations, that exists on any Riemannian four-manifold (M_4, g). We rephrase this in terms of the quaternionic Kähler structure that exists on any such manifold, involving (locally) a triplet of self-dual two-forms J^I. Finally, in section 3.4 we expand solutions to the bulk spinor equations in a Fefferman-Graham-like expansion.

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3.1 Boundary spinor equations

We begin by expanding the bulk Killing spinor equations (2.7), (2.8) to leading order near the conformal boundary at $z = 0$. We will consequently need the Fefferman-Graham expansion of an orthonormal frame for the metric (2.14), (2.15), together with the associated spin connection. The following is a choice of frame $\mathrm{E}_{\mu}^{\overline{\mu}}$ for the metric (2.14):

$$\mathrm{E}_{z}^{\overline{z}} = \frac{1}{z}\,, \qquad \mathrm{E}_{i}^{\overline{z}} = \mathrm{E}_{z}^{\overline{i}} = 0\,, \qquad \mathrm{E}_{i}^{\overline{i}} = \frac{1}{z}\mathsf{e}_{i}^{\overline{i}}\,, \tag{3.1}$$

where $\mathsf{e}_{i}^{\overline{i}}$ is a frame for the z-dependent metric g. The latter then has the expansion (2.15), but for the present subsection we shall only need that

$$\mathsf{e}_{i}^{\overline{i}} = \mathrm{e}_{i}^{\overline{i}} + O(z^2)\,, \tag{3.2}$$

where $\mathrm{e}_{i}^{\overline{i}}$ is a frame for the boundary metric $\mathsf{g}^0 = g$. The non-zero components of the spin connection $\Omega_{\mu}{}^{\overline{\nu}\overline{\rho}}$ at this order are correspondingly

$$\Omega_{i}{}^{\overline{z}\overline{j}} = \frac{1}{z}\mathrm{e}_{i}^{\overline{j}} + O(z)\,, \qquad \Omega_{i}{}^{\overline{j}\overline{k}} = (\omega^{(0)})_{i}{}^{\overline{j}\overline{k}} + O(z^2)\,, \tag{3.3}$$

where $(\omega^{(0)})_{i}{}^{\overline{j}\overline{k}}$ denotes the boundary spin connection.

The generators $\gamma_{\overline{\mu}}$ of the Clifford algebra Cliff$(5,0)$ in this frame are chosen to obey

$$\gamma_{\overline{z}} = \gamma_{\overline{1}\overline{2}\overline{3}\overline{4}}\,. \tag{3.4}$$

It follows that $\gamma_{\overline{z}}^2 = 1$, and we may identify $-\gamma_{\overline{z}}$ with the boundary chirality operator. The bulk Killing spinor is then expanded as

$$\epsilon = z^{-1/2}\varepsilon + z^{1/2}\eta + o(z^{1/2})\,. \tag{3.5}$$

As in (2.11), we may further decompose the spinors ε, η into their projections $\varepsilon^{\pm}$, $\eta^{\pm}$ onto the $\pm$i eigenspaces of Γ_{45}. At leading order in the z-component of the gravitino equation (2.7) one then finds

$$-\gamma_{\overline{z}}\varepsilon^{\pm} = \pm\varepsilon^{\pm}\,, \tag{3.6}$$

so that the Γ_{45} eigenvalue of the leading order spinor ε is correlated with its boundary chirality. Similarly, at the next order in the gravitino equation one finds the opposite correlation for the spinor η:

$$-\gamma_{\overline{z}}\eta^{\pm} = \mp\eta^{\pm}\,. \tag{3.7}$$

Recall that the boundary B-fields satisfy $*_4 b^{\pm} = \mp b^{\pm}$ (see (2.26)). This together with the chirality conditions (3.6) implies that

$$b^{\pm}\cdot\varepsilon^{\pm} = 0\,, \tag{3.8}$$

where $\cdot$ denotes the Clifford product (using the boundary frame). Using this, the leading order term in the i-component of the gravitino equation is then seen to be identically satisfied. The next order gives the pair of boundary Killing spinor equations:

$$\mathcal{D}_i^{(0)}\varepsilon^{\pm} - \frac{\mathrm{i}}{4}b_{ij}^{\mp}\gamma^j\varepsilon^{\mp} \mp \gamma_i\eta^{\pm} = 0\,, \tag{3.9}$$

where we have defined the covariant derivative

$$\mathcal{D}_i^{(0)} \equiv \nabla_i^{(0)} \pm \frac{\mathrm{i}}{2}\mathrm{a}_i + \frac{\mathrm{i}}{2}A_i^I \sigma_I \ . \tag{3.10}$$

Here $\nabla_i^{(0)}$ denotes the Levi-Civita spin connection of the boundary metric $\mathrm{g}_{ij}^0 = g_{ij}$, and $\gamma_i = \gamma_{\bar{i}}\, \mathrm{e}_i^{\bar{i}}$, so that $\{\gamma_i, \gamma_j\} = 2g_{ij}$.

Turning to the bulk dilatino equation (2.8), the leading order term is in fact equivalent to the duality properties of $b^\pm$, given the chiralities of $\varepsilon^\pm$. At the next order we obtain the boundary dilatino equation

$$-\,\mathrm{f}\cdot\varepsilon^\pm \pm \frac{1}{2}F^I\sigma_I\cdot\varepsilon^\pm \mp 3\mathrm{i}X_1\,\varepsilon^\pm + \frac{1}{2}b^\mp\cdot\eta^\mp \mp \frac{1}{2}b_1^\mp\cdot\varepsilon^\mp = 0\ . \tag{3.11}$$

The supersymmetry equations for four-dimensional Euclidean off-shell $\mathcal{N} = 2$ conformal supergravity have been studied[8] in [17], and our equations (3.9), (3.11) precisely reproduce the equations in this reference.[9] Notice in particular that one can solve for the (conformal) spinor η by taking the trace of (3.9) with γ^i, to obtain

$$\eta^\pm = \pm\frac{1}{4}\not{\mathcal{D}}^{(0)}\varepsilon^\pm\,, \tag{3.12}$$

where $\not{\mathcal{D}}^{(0)} \equiv \gamma^i\mathcal{D}_i^{(0)}$ is the Dirac operator. Taking the covariant derivative of (3.9) and using the integrability condition for $[\mathcal{D}_i^{(0)}, \mathcal{D}_j^{(0)}]$ then leads to the following form of the dilatino equation

$$\not{\mathcal{D}}^{(0)}\not{\mathcal{D}}^{(0)}\varepsilon^\pm - \mathrm{i}\mathcal{D}_i(b^\pm)^i{}_j\gamma^j\varepsilon^\mp + \left(4X_1 + \frac{1}{3}R\right)\varepsilon^\pm \mp 2\mathrm{i}\,\mathrm{f}\cdot\varepsilon^\pm = 0\,, \tag{3.13}$$

where $R = R(g)$ is the Ricci scalar of the boundary metric. Requiring the boundary fields $g_{ij}, X_1, \mathrm{a}, A^I, b^\pm$ to solve the spinor equations (3.9), (3.11) for $\varepsilon^\pm$ in general imposes geometric constraints. Remarkably, in [17] it is shown that generically these conditions are equivalent to the boundary manifold (M_4, g) admitting a conformal Killing vector. However, the topological twist background of [14] arises as a very degenerate case, where in fact (M_4, g) may be an arbitrary Riemannian four-manifold. We turn to this case in the next subsection.

3.2 Topological twist

The topological twist background of [14] is obtained by setting

$$\varepsilon^- = 0\,, \qquad \mathrm{a} = 0\,, \qquad b^\pm = 0\,, \qquad \eta^\pm = 0\ . \tag{3.14}$$

The boundary Killing spinor equation (3.9) immediately implies that ε^+ is covariantly constant

$$\mathcal{D}_i^{(0)}\varepsilon^+ = 0\ . \tag{3.15}$$

[8] See [36] for related earlier work and [37] for a recent construction of Euclidean $\mathcal{N} = 2$ conformal supergravity from a timelike reduction of a five-dimensional theory.

[9] The explicit notation change is $A_4^{\mathrm{KZ}} = -\mathrm{i}\mathrm{a}$, $A_{\mathrm{KZ}}^I = A^I$, $T_{\mathrm{KZ}}^\pm = -b^\pm$, $\epsilon_\pm^{\mathrm{KZ}} = \varepsilon^\mp$, $\tilde{d}_{\mathrm{KZ}} = 2X_1$.

The dilatino equation, in the form (3.13), then fixes

$$X_1 = -\frac{1}{12}R\,. \tag{3.16}$$

Recall that ε^+ is a doublet of positive chirality spinors: the Pauli matrices σ_I act on these doublet indices, while the Clifford matrices $\gamma_{\bar{i}}$ act on the spinor indices. We may write out the covariant derivative in (3.15) more explicitly by first introducing the following explicit Hermitian representation

$$\gamma_{\bar{a}} = \begin{pmatrix} 0 & \mathrm{i}\sigma_{\bar{a}} \\ -\mathrm{i}\sigma_{\bar{a}} & 0 \end{pmatrix}\,, \quad \gamma_{\bar{4}} = \begin{pmatrix} 0 & -1_2 \\ -1_2 & 0 \end{pmatrix}\,, \quad \gamma_{\bar{z}} = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}\,. \tag{3.17}$$

Here $\bar{a} = 1,2,3$. Since $\gamma_{\bar{z}}\varepsilon^+ = -\varepsilon^+$, we may identify each of the two spinors in the doublet ε^+ with a two-component spinor, acted on by the second 2×2 block. With these choices (3.15) reads

$$\mathcal{D}_i^{(0)}\varepsilon^+ = \partial_i\varepsilon^+ + \frac{\mathrm{i}}{4}\eta^{\bar{a}}_{\bar{j}\bar{k}}(\omega^{(0)})_i{}^{\bar{j}\bar{k}}\sigma_{\bar{a}}\varepsilon^+ + \frac{\mathrm{i}}{2}A_i^I\sigma_I\varepsilon^+ = 0\,, \tag{3.18}$$

where $\eta^{\bar{a}}_{\bar{i}\bar{j}}$ are the self-dual 't Hooft symbols, and recall that $(\omega^{(0)})_i{}^{\bar{j}\bar{k}}$ is the spin connection for the boundary metric g_{ij}. One may then solve (3.18) by taking

$$A_i^I = \frac{1}{2}\eta^I_{\bar{j}\bar{k}}(\omega^{(0)})_i{}^{\bar{j}\bar{k}}\,, \qquad (\varepsilon^+)^i{}_\alpha = (\mathrm{i}\sigma_2)^i{}_\alpha\, c\,. \tag{3.19}$$

Here $i = 1,2$ labels the doublet indices, while $\alpha = 1,2$ labels the positive chirality spinor indices, and notice that the frame index $\bar{a} = 1,2,3$ is identified with the gauge indices $I = 1,2,3$. It is straightforward to check that (3.19) solves (3.18), for any constant c. The $\mathrm{SU}(2)_R$ gauge field A^I given by (3.19) is precisely the right-handed part of the spin connection, where recall that $\mathrm{Spin}(4) = \mathrm{SU}(2)_- \times \mathrm{SU}(2)_+$. Thus the $\mathrm{SU}(2)_R$ gauge bundle is identified with $\mathrm{SU}(2)_+$.

More invariantly, ε^+ is a section of $\mathcal{S}^+ \otimes \mathcal{V}$, where $\mathcal{S}^+$ denotes the positive chirality spinor bundle over M_4, while $\mathcal{V}$ is the rank 2 complex vector bundle for which A^I is an associated $\mathrm{SU}(2)$ connection. *A priori* this makes sense globally only when M_4 is a spin manifold, when $\mathcal{S}^+$ and $\mathcal{V}$ both exist as genuine vector bundles. However, the topological twist (3.19) identifies $\mathcal{V}$ with $\mathcal{S}^+$, and their tensor product then always exists globally, even when M_4 is not spin.[10] This topological construction of a spin-type bundle on a manifold which is not necessarily spin was first suggested in [38], and is sometimes referred to as a $Spin_{\mathscr{G}}$ structure, where here the group $\mathscr{G} = \mathrm{SU}(2)$. Perhaps more familiar are $Spin^c$ structures, where instead $\mathscr{G} = \mathrm{U}(1)$. (For example, this arises in Seiberg-Witten theory.)

It will be convenient later to introduce the triplet of self-dual two-forms

$$\mathrm{J}^I_{ij} \equiv \eta^I_{\bar{i}\bar{j}}\,\mathrm{e}^{\bar{i}}_i\,\mathrm{e}^{\bar{j}}_j\,, \tag{3.20}$$

[10]There are various ways to see this. For example, the lack of a spin structure on M_4 is detected by a non-zero second Stiefel-Whitney class $w_2(M_4) \in H^2(M_4, \mathbb{Z}_2)$. Concretely this means the cocycle condition for the spin lift of the frame bundle fails up to some minus signs. However, if two copies are tensored together all such signs square to $+1$, and the tensor product is a well-defined bundle.

where recall that $\mathrm{e}_i^{\bar{i}}$ is the boundary frame for g_{ij}. More explicitly, these read

$$\mathrm{J}^1 = \mathrm{e}^2 \wedge \mathrm{e}^3 + \mathrm{e}^1 \wedge \mathrm{e}^4 , \quad \mathrm{J}^2 = \mathrm{e}^3 \wedge \mathrm{e}^1 + \mathrm{e}^2 \wedge \mathrm{e}^4 , \quad \mathrm{J}^3 = \mathrm{e}^1 \wedge \mathrm{e}^2 + \mathrm{e}^3 \wedge \mathrm{e}^4 . \tag{3.21}$$

Of course, in general a frame $\mathrm{e}_i^{\bar{i}}$ is only defined locally on M_4, in an appropriate open set, and likewise the J^I in (3.21) are then well-defined forms only locally. More globally, local frames are patched together with SO(4). The spin cover is $\mathrm{Spin}(4) \cong \mathrm{SU}(2)_- \times \mathrm{SU}(2)_+$, and the self-dual/anti-self-dual two-forms are precisely the representations associated to $\mathrm{SO}(3)_\pm = \mathrm{SU}(2)_\pm/\mathbb{Z}_2$. In particular, the $\{\mathrm{J}^I\}$ rotate as a 3-vector under $\mathrm{SO}(3)_+ \subset \mathrm{SO}(4)$. In this sense the J^I in general don't exist individually as global two-forms on M_4, but instead as a triplet of forms that rotate appropriately. We comment further on this below.

One can also write the J^I in terms of spinor bilinears. Recall from the end of section 2.1 that the bulk spinors satisfy a symplectic Majorana reality condition. In particular the boundary spinor ε^+ satisfies

$$(\varepsilon^+)^c \equiv \mathrm{i}\sigma_2 \mathscr{C} (\varepsilon^+)^* = \varepsilon^+ , \tag{3.22}$$

where recall that $\mathscr{C}$ is the charge conjugation matrix for the spacetime Clifford algebra. In the explicit basis (3.17) we may take

$$\mathscr{C} = \begin{pmatrix} \mathrm{i}\sigma_2 & 0 \\ 0 & \mathrm{i}\sigma_2 \end{pmatrix} . \tag{3.23}$$

Given the solution (3.19) one finds that the reality condition (3.22) is satisfied provided the constant $c \in \mathbb{R}$. Explicitly, the components of the doublet ε^+ are

$$(\varepsilon^+)^1 = (0,0,0,c)^{\mathrm{T}} , \qquad (\varepsilon^+)^2 = (0,0,-c,0)^{\mathrm{T}} . \tag{3.24}$$

We then define the boundary spinor

$$\chi \equiv (\varepsilon^+)^1 . \tag{3.25}$$

This has square norm $\bar{\chi}\chi = c^2$, where the bar denotes Hermitian conjugate, and χ of course has positive chirality, $-\gamma_{\bar{5}}\chi = \chi$. One easily checks that

$$\mathrm{J}^2 + \mathrm{i}\mathrm{J}^1 = \frac{1}{\bar{\chi}\chi} \bar{\chi}^c \gamma_{(2)} \chi , \qquad \mathrm{J}^3 = \frac{\mathrm{i}}{\bar{\chi}\chi} \bar{\chi} \gamma_{(2)} \chi , \tag{3.26}$$

where $\chi^c \equiv \mathscr{C}\chi^*$.

From the original definition (3.20), the J^I inherit a number of algebraic identities from those for the 't Hooft symbols. For example,

$$\mathrm{J}^I_{ij}\mathrm{J}^I_{kl} = g_{ik}g_{jl} - g_{il}g_{jk} + \epsilon_{ijkl} . \tag{3.27}$$

Using the metric to raise an index, one obtains a triplet $(\mathrm{I}^I)^i{}_j \equiv g^{ik}(\mathrm{J}^I)_{kj}$ of endomorphisms of the tangent bundle of M_4. These satisfy the quaternionic algebra

$$\mathrm{I}^I \circ \mathrm{I}^J = -\delta^{IJ} - \epsilon^{IJ}{}_K \mathrm{I}^K . \tag{3.28}$$

One also finds that

$$\nabla_i \mathrm{J}^I_{jk} = \epsilon^I{}_{JK} A^J_i \mathrm{J}^K_{jk} \,, \tag{3.29}$$

where the R-symmetry gauge field A^I here is precisely the right-handed spin connection given by the topological twist (3.19). Notice that we may correspondingly write the curvature as

$$F^I_{ij} = \frac{1}{2} \mathrm{J}^I_{kl} R_{ij}{}^{kl} \,, \tag{3.30}$$

where R_{ijkl} is the boundary Riemann tensor.

In general a *quaternionic Kähler manifold* is a Riemannian manifold of dimension $4n$ with holonomy $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(4n)$.[11] Such manifolds admit, locally, a triplet of skew endomorphisms I^I of the tangent bundle satisfying (3.28), for which the corresponding triplet of two-forms J^I satisfy (3.29). Here A^I is the Riemannian connection corresponding to the Sp(1) part of this holonomy group. For $n = 1$ notice that $\mathrm{Sp}(1) \cdot \mathrm{Sp}(1) = \mathrm{SO}(4)$, and such a structure exists on any Riemannian four-manifold (M_4, g) (as we have just seen). Crucially, the two-forms (3.21) are not in general defined globally, but are (in our language) twisted by the R-symmetry gauge field, transforming as a vector under $\mathrm{SO}(3)_R = \mathrm{SU}(2)_R/\mathbb{Z}_2$. As such, they don't define a reduction of the structure group to $\mathrm{SU}(2)_-$, as a global set of such forms would do. Indeed, the globally defined tensor on a quaternionic Kähler manifold is the four-form $\Psi \equiv \mathrm{J}^I \wedge \mathrm{J}^I$ (summed over I), and in four dimensions ($n = 1$) this is proportional to the volume form. The stabiliser of Ψ is $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$, which is SO(4) when $n = 1$.

In dimensions $n \geq 2$ irreducible quaternionic Kähler manifolds are automatically Einstein. Some authors choose to *define* a quaternionic Kähler four-manifold to be an Einstein manifold with self-dual Weyl tensor, but we shall not use this terminology.

3.3 $\mathrm{U}(1)_R$ current

Before continuing to expand the spinor equations into the bulk, in this subsection we pause briefly to consider the VEV of the $\mathrm{U}(1)_R$ current given by (2.54). In the topological twist background equation, (2.28) gives $\mathrm{a}_1 = 0$, so that $\langle \mathbb{J} \rangle = -\mathrm{a}_2/\kappa_5^2$. On the other hand, from (2.29) we obtain the $\mathrm{U}(1)_R$ anomaly equation

$$\mathrm{d} *_4 \langle \mathbb{J} \rangle = \frac{\mathrm{i}}{8\kappa_5^2} F^I \wedge F^I \,, \tag{3.31}$$

where $*_4$ denotes the Hodge duality operator on (M_4, g). Using equations (3.30) and (3.27) this may be rewritten as

$$\mathrm{d} *_4 \langle \mathbb{J} \rangle = \frac{\mathrm{i}}{32\kappa_5^2} (\mathcal{E} + \mathcal{P}) \, \mathrm{vol}_4 \,, \tag{3.32}$$

where $\mathcal{E}$ and $\mathcal{P}$ are the Euler and Pontryagin densities, (2.46). On a compact M_4 without boundary these integrate to $\int_{M_4} \mathcal{E} \, \mathrm{vol}_4 = 32\pi^2 \chi(M_4)$, $\int_{M_4} \mathcal{P} \, \mathrm{vol}_4 = 48\pi^2 \sigma(M_4)$, so that

[11] See, for example, [39].

integrating (3.32) over M_4 gives[12]

$$\int_{M_4} \mathrm{d} *_4 \langle \mathbb{J} \rangle = \frac{\mathrm{i}\pi^2}{2\kappa_5^2} \left[2\chi(M_4) + 3\sigma(M_4) \right] \ . \tag{3.33}$$

It follows that if a_2, or equivalently $\langle \mathbb{J} \rangle$, is a *global* one-form on M_4, then by Stokes' theorem the left hand side of (3.33) is zero, implying the topological constraint

$$2\chi(M_4) + 3\sigma(M_4) = 0 \ . \tag{3.34}$$

Indeed, in section 2.1 we noted that we are studying gravitational saddle points in the real Euclidean Romans theory, where the $\mathrm{U}(1)_R$ gauge field $\mathcal{A}$ is a (purely imaginary) global one-form. Related to this, the $\mathrm{U}(1)_R$ symmetry effectively becomes an $\mathrm{SO}(1,1)_R$ symmetry after Wick rotation, as also emphasized in [17] (see also [2]). A number of gravity expressions that we shall obtain below only make sense if a_2 is interpreted as a global one-form on M_4, at least in the set-up we have defined. Thus (3.34) already restricts the topology of M_4. Interestingly, in section 6.1 we shall see that (3.34) also plays an important role in the dual TQFT. Specifically, if (3.34) does not hold, the partition function is zero![13]

3.4 Supersymmetric expansion

In this section we continue to expand the bulk spinor equations to higher order in z. From this we extract further information about some of the fields which are not fixed, in terms of boundary data, by the bosonic equations of motion. We will continue to use the boundary conditions appropriate to the topological twist. In particular we note that the boundary B-fields $b^{\pm} = 0$ in this case, and that setting the bulk $\mathcal{B}^{\pm} = 0$ is a consistent truncation of the Euclidean $\mathcal{N} = 4^+$ theory. Moreover, in this case the bulk spinors $\epsilon^{\pm}$ satisfy decoupled equations, and since the leading order term $\varepsilon^- = 0$ it is then also consistent to set the bulk $\epsilon^- = 0$. We henceforth work in this truncated theory. This subsection is somewhat technical. All of the relevant formulas that we need in section 4 are in any case summarized in that section, and a reader uninterested in the details may safely skip the present subsection.

The frame, spin connection and spinor expansions beyond the leading order given in section 3.1 will be needed, so we first give details of these. The frame expansion is

$$\mathsf{e}_i^{\bar{i}} = \mathrm{e}_i^{\bar{i}} + z^2 (\mathrm{e}^{(2)})_i^{\bar{i}} + z^4 \left[(\log z)^2 (\mathring{\mathrm{e}}^{(4)})_i^{\bar{i}} + \log z (\tilde{\mathrm{e}}^{(4)})_i^{\bar{i}} + (\mathrm{e}^{(4)})_i^{\bar{i}} \right] + o(z^4) \,, \tag{3.35}$$

[12] A little less laboriously we can instead note that F^I is the curvature of the bundle of self-dual two-forms $\Lambda_2^+ M_4$, and the integral of the right hand side of (3.31) is proportional to the first Pontryagin class $p_1(\Lambda_2^+ M_4) = 2\chi(M_4) + 3\sigma(M_4)$.

[13] In passing we note that (3.34) corresponds (with an appropriate choice of orientation) to equality in the Hitchin-Thorpe inequality. In particular the only Einstein manifolds satisfying this condition are the flat torus, a K3 surface, or a quotient thereof [40]. A non-example is S^4, for which $2\chi(S^4) + 3\sigma(S^4) = 4$. On the other hand, for a complex surface (3.34) is equivalent to $\int_{M_4} c_1 \wedge c_1 = 0$, where $c_1 = c_1(M_4)$ is the first Chern class of the holomorphic tangent bundle (the anti-canonical class).

where in particular $\mathrm{e}_i^{\bar{i}}$ is a frame for the boundary metric. The additional spin connection components we will need are

$$\Omega_i{}^{\overline{zi}} = \frac{1}{z}\mathsf{e}_i^{\bar{i}} - \frac{1}{2}\mathsf{g}^{jk}\mathsf{e}_j^{\bar{i}}\partial_z\mathsf{g}_{ik} \qquad \Omega_z{}^{\overline{ij}} = \mathsf{g}^{ij}\mathsf{e}_i^{[\bar{i}}\partial_z\mathsf{e}_j^{\bar{j}]}\,. \tag{3.36}$$

The bulk spinor has $\epsilon^- = 0$ in our truncated theory, and we thus henceforth drop the superscript on $\epsilon^+ \to \epsilon$, $\varepsilon^+ \to \varepsilon$ (we hope this abuse of notation won't lead to any confusion). The bulk spinor then has the following expansion

$$\epsilon = z^{-1/2}\varepsilon + z^{3/2}\varepsilon^3 + z^{5/2}(\log z\,\tilde{\varepsilon}^5 + \varepsilon^5) + z^{7/2}\big((\log z)^2\,\tilde{\tilde{\varepsilon}}^7 + \log z\,\tilde{\varepsilon}^7 + \varepsilon^7\big) + o(z^{7/2})\,, \tag{3.37}$$

where ε is constant with positive chirality under $-\gamma_{\bar{z}}$. As in equation (3.22) the bulk spinor ϵ satisfies the reality condition

$$\epsilon^c \equiv \mathrm{i}\sigma_2\mathscr{C}\epsilon^* = \epsilon\ . \tag{3.38}$$

We start by analysing the bulk dilatino equation. At lowest order we find

$$0 = X_1\,\varepsilon + \frac{\mathrm{i}}{6}F^I\cdot(\sigma^I\varepsilon) = \left(X_1 + \frac{1}{12}R\right)\varepsilon\,, \tag{3.39}$$

which is satisfied identically, where we have used (3.16) and (3.30). At the next order we find

$$\mathrm{i}a_1^I\cdot(\sigma_I\varepsilon) = -\frac{1}{4}(\mathrm{d}R)\cdot\varepsilon\,. \tag{3.40}$$

This is effectively a matrix equation, of which we shall see many more. Components of such equations may be extracted by first noting that

$$\varepsilon = \begin{pmatrix}\chi\\ -\mathscr{C}\chi^*\end{pmatrix}\,, \tag{3.41}$$

in the notation of section 3.2. For example, one can then take the first component of (3.40), and apply $\bar{\chi}\gamma_j$ on the left. Taking the real part, and using the definitions (3.26) of J^I in terms of spinor bilinears, one obtains

$$(a_1^I)^i\,\mathrm{J}^I_{ij} = \frac{1}{4}\nabla_j R\,. \tag{3.42}$$

We shall make use of similar manipulations throughout this subsection. Focusing on (3.42), recall that a_1^I is already fixed in terms of the SU(2) covariant divergence of F^I, via equation (2.28). The latter reads $(a_1^I)_i = \frac{1}{2}\mathcal{D}^j F^I_{ij}$. Starting from this and (3.30), and using the identity $\alpha_{pq}\mathrm{J}^I_m{}^p\mathrm{J}^I_n{}^q = \alpha_{mn} - 2(*\alpha)_{mn}$, where α_{pq} is any two-form, one can show that (3.42) is an identity. We may then differentiate (3.42) and, upon using the quaternionic Kähler equation (3.29), we obtain

$$(\mathcal{D}a_1^I)^{ij}\mathrm{J}^I_{ij} = -\frac{1}{4}\nabla^2 R\,. \tag{3.43}$$

This relation appears frequently hereafter.

At the next order in the dilatino equation we find an equation involving several undetermined fields:

$$\mathrm{i}a_2^I\cdot(\sigma_I\varepsilon) = \left(2\mathrm{i}\mathrm{a}_2 + 3\mathrm{d}X_2 + \frac{1}{8}\mathrm{d}R\right)\cdot\varepsilon\,, \tag{3.44}$$

from which we similarly extract

$$(a_2^I)^i \mathrm{J}_{ij}^I = -2\mathrm{i}(\mathrm{a}_2)_j - 3\nabla_j X_2 - \frac{1}{8}\nabla_j R\,. \tag{3.45}$$

From this expression, taking a covariant derivative and symmetrizing indices gives

$$3\nabla_i\nabla_j X_2 = \mathcal{D}_{(i}(a_2^I)^k \mathrm{J}_{j)k}^I - 2\mathrm{i}\nabla_{(i}(\mathrm{a}_2)_{j)} - \frac{1}{8}\nabla_i\nabla_j R\,. \tag{3.46}$$

At higher order still we have

$$X_3\,\varepsilon = X_1(1+\gamma_{\bar z})\varepsilon^3 - \frac{\mathrm{i}}{12}\mathcal{D}a_1^I\cdot(\sigma_I\varepsilon)\,. \tag{3.47}$$

As ε has positive chirality we can act with $P_- = \frac{1}{2}(1+\gamma_{\bar z})$ to deduce that ε^3 also has positive chirality. It then follows that

$$X_3 = -\frac{1}{12}(\mathcal{D}a_1^I)^{ij}\mathrm{J}_{ij}^I = \frac{1}{48}\nabla^2 R\,. \tag{3.48}$$

where we have used (3.43). This expression for X_3 is equivalent to that in (2.30), for the topological twist. Finally, at order $\mathcal{O}(z^{7/2})$ we have

$$\begin{aligned} X_4\,\varepsilon = &-\frac{1}{2}X_3\,\varepsilon - \frac{1}{2}X_1\,\varepsilon^3 - \frac{\mathrm{i}}{12}\Big[(\mathcal{D}a_2^I)\cdot(\sigma_I\varepsilon) - 2\mathrm{f}_2\cdot\varepsilon + F^I\cdot(\sigma_I\varepsilon^3)\Big] \\ &-\frac{\mathrm{i}}{12}\,\mathrm{e}_{\bar i}^{i}\,(\mathrm{e}^{(2)})_{\bar j}^{j}F_{ij}^I\gamma^{\overline{ij}}(\sigma_I\varepsilon)\,. \end{aligned} \tag{3.49}$$

Here $\mathsf{e}_{\bar i}^{i}$ is the inverse frame to $\mathsf{e}_i^{\bar i}$, with $\mathrm{e}_{\bar i}^{i}$ and $(\mathrm{e}^{(2)})_{\bar i}^{i}$ being coefficients in its expansion, precisely as in (3.35). We have also defined $\mathrm{f}_2 = \mathrm{d}\mathrm{a}_2$. Since ε^3 is so far undetermined, we cannot yet extract an expression for X_4. This concludes the expansion of the bulk dilatino equation.

Turning next to the bulk gravitino equation, at lowest order in the z direction we find, after using the fact that ε^3 has positive chirality, that

$$\varepsilon^3 = \frac{1}{48}R\,\varepsilon - \frac{1}{4}g^{ij}\,\mathrm{e}_i^{\bar i}\,(\mathrm{e}^{(2)})_j^{\bar j}\gamma_{\overline{ij}}\,\varepsilon\,. \tag{3.50}$$

As a metric defines the frame only up to an arbitrary local SO(4) rotation, it is convenient to gauge fix this arbitrariness. A consistent gauge choice is $(\mathrm{e}^{(2)})_i^{\bar i} = \frac{1}{2}(\mathsf{g}^2)^{\bar i}{}_{\bar j}\,\mathrm{e}_i^{\bar j}$ and $(\mathrm{e}^{(2)})_{\bar i}^{i} = -\frac{1}{2}\mathrm{e}_{\bar j}^{i}\,(\mathsf{g}^2)^{\bar j}{}_{\bar i}$, where recall that g^2 is fixed in terms of the boundary Schouten tensor via (2.34). This then implies that

$$g_{ij}\,\mathrm{e}_{\bar i}^{i}\,(\mathrm{e}^{(2)})_{\bar j}^{j} = -\frac{1}{2}\mathsf{g}_{\overline{ij}}^2\,, \qquad g^{ij}\,\mathrm{e}_i^{\bar i}\,(\mathrm{e}^{(2)})_j^{\bar j} = \frac{1}{2}(\mathsf{g}^2)^{\overline{ij}}\,, \tag{3.51}$$

and, being symmetric, their contraction with any anti-symmetric tensor automatically vanishes. Consequently, this gauge choice reduces the relation between the spinors ε and ε^3 to simply

$$\varepsilon^3 = \frac{1}{48}R\,\varepsilon\,. \tag{3.52}$$

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Having found this relation we may substitute for ε^3 into the right hand side of (3.49), extract X_4 and then substitute for g^2, X_1, X_3 and F^I to obtain

$$X_4 = \frac{1}{288}R^2 - \frac{1}{48}R_{kl}R^{kl} - \frac{1}{96}\nabla^2 R - \frac{1}{24}\left(\mathcal{D}a_2^I\right)^{ij} \mathrm{J}^I_{ij}\,. \tag{3.53}$$

Here strictly speaking we have taken the real part of this equation, where the term involving f_2 is purely imaginary, and thus doesn't appear. Using the trace of (3.46), together with several other equations derived so far, one can check that the expression (3.53) for X_4 agrees with the expression (2.31), obtained from the equations of motion.

At the next orders we find

$$(5-\gamma_{\bar z})\,\varepsilon^5 = -\,2\,\tilde\varepsilon^5 + 2(\mathrm{i}\mathrm{a}_2 + \mathrm{d}X_2)\cdot\varepsilon\,, \tag{3.54}$$

$$(5-\gamma_{\bar z})\,\tilde\varepsilon^5 = \frac{2\mathrm{i}}{3}a_1^I\cdot(\sigma_I\varepsilon) = -\frac{1}{6}\mathrm{d}R\cdot\varepsilon\,. \tag{3.55}$$

We could continue and analyse higher order terms in this z component of the gravitino equation, but the subsequent expressions are not required, nor particularly enlightening, and so we stop here.

The remaining equation to study is the i direction of the gravitino equation. Crucially this involves the spin connection components $\Omega_i{}^{\overline{zi}}$, which introduce the metric expansion fields from (2.15). Of course, the leading order equation is satisfied by construction. Remarkably, at the next order we find a non-trivial equation which is also identically satisfied given the chirality of ε^3 and the algebraic properties of the Riemann tensor. At the following order we find another condition on $\tilde\varepsilon^5$:

$$\gamma_{\bar i}\left[3\mathrm{i}(1+\gamma_{\bar z})\tilde\varepsilon^5 + a_1^I\cdot(\sigma_I\varepsilon)\right] = 0\,, \tag{3.56}$$

which, used in conjunction with (3.55), allows us to determine

$$\gamma_{\bar z}\tilde\varepsilon^5 = \tilde\varepsilon^5\,, \qquad \tilde\varepsilon^5 = -\frac{1}{24}\mathrm{d}R\cdot\varepsilon\,. \tag{3.57}$$

We now substitute $\tilde\varepsilon^5$ into equation (3.54):

$$(5-\gamma_{\bar z})\varepsilon^5 = \left(2\mathrm{i}\mathrm{a}_2 + 2\mathrm{d}X_2 + \frac{1}{12}\mathrm{d}R\right)\cdot\varepsilon\,. \tag{3.58}$$

Acting on this last equation with $\gamma_{\bar z}$, and taking the difference, implies that ε^5 is a negative chirality spinor: $\gamma_{\bar z}\varepsilon^5 = \varepsilon^5$. We thus find

$$\varepsilon^5 = \left(\frac{\mathrm{i}}{2}\mathrm{a}_2 + \frac{1}{2}\mathrm{d}X_2 + \frac{1}{48}\mathrm{d}R\right)\cdot\varepsilon\,. \tag{3.59}$$

At the next order we begin to see the metric fields appearing:

$$h^0_{\bar i\bar j}\gamma^{\bar j}\varepsilon = -\frac{1}{288}R^2\gamma_{\bar i}\varepsilon - \frac{1}{2}\gamma_{\bar i}(1+\gamma_{\bar z})\mathring{\varepsilon}^7\,. \tag{3.60}$$

Using the chiral projector P_- again we see that $\mathring{\varepsilon}^7$ has positive chirality, and we may extract h^0:

$$h^0_{ij} = -\frac{1}{288}R^2 g_{ij}\,. \tag{3.61}$$

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This agrees with the expression $h^0_{ij} = -\frac{1}{2} g_{ij} X_1^2$, given by equation (2.39), derived from the expansion of the bosonic field equations. The next order gives

$$
\begin{aligned}
h^1_{\bar{i}\bar{j}} \gamma^{\bar{j}} \varepsilon = & -\frac{1}{2}\gamma_{\bar{i}}(1+\gamma_{\bar{z}})\, \tilde{\varepsilon}^7 - \frac{1}{2} h^0_{\bar{i}\bar{j}} \gamma^{\bar{j}} \varepsilon - X_1 X_2 \gamma_{\bar{i}}\, \varepsilon + \nabla_{\bar{i}} \tilde{\varepsilon}^5 + \frac{\mathrm{i}}{2} A^I_{\bar{i}} (\sigma_I \tilde{\varepsilon}^5) \\
& - \frac{\mathrm{i}}{24} X_1 (\gamma_{\bar{i}}{}^{\overline{jk}} - 4\delta^{\bar{j}}_{\bar{i}} \gamma^{\bar{k}}) F^I_{\overline{jk}} (\sigma_I \varepsilon) + \frac{\mathrm{i}}{24} (\gamma_{\bar{i}}{}^{\overline{jk}} - 4\delta^{\bar{j}}_{\bar{i}} \gamma^{\bar{k}}) (\mathcal{D} a^I_1)_{\overline{jk}} (\sigma_I \varepsilon) \,. \qquad (3.62)
\end{aligned}
$$

As before, we can show that $\tilde{\varepsilon}^7$ has positive chirality and hence drops out of (3.62). Now using the definition of $\tilde{\varepsilon}^5$ in (3.57) allows us to write everything acting on the spinor ε. After using the intermediate result

$$
-\frac{1}{4} \mathrm{J}^I{}_{(i}{}^k (\mathcal{D} a^I_1)_{j)k} = -\frac{1}{8} \left(R_i{}^k R_{jk} + R_{iklj} R^{kl} - \nabla^2 R_{ij} + \frac{1}{2} \epsilon_{(j|kmn|} R^{kl} R^{mn}{}_{i)l} \right) , \qquad (3.63)
$$

and substituting for the known expressions, we can then read off h^1_{ij}:

$$
\begin{aligned}
h^1_{ij} = & \frac{1}{192} g_{ij} R^2 + \frac{1}{12} g_{ij} R X_2 - \frac{1}{48} R R_{ij} - \frac{1}{24} \nabla_i \nabla_j R - \frac{1}{48} g_{ij} \nabla^2 R \\
& - \frac{1}{8} \left(R_i{}^k R_{jk} + R_{iklj} R^{kl} - \nabla^2 R_{ij} + \frac{1}{2} \epsilon_{(j|kmn|} R^{kl} R^{mn}{}_{i)l} \right) . \qquad (3.64)
\end{aligned}
$$

Once again, we have found another expression for something we have already derived: h^1_{ij} is also given by equation (2.40). However, in this instance the equality of the two expressions (3.64) and (2.40) is non-trivial. It is equivalent to the equation

$$
\begin{aligned}
0 \,=\, & \left(R R_{ij} - 2 R_i{}^k R_{jk} + 2 R_{iklj} R^{kl} + R_{mnik} R^{mn}{}_j{}^k \right) - \frac{1}{4} g_{ij} \left(R^2 - 4 R_{kl} R^{kl} + R_{mnkl} R^{mnkl} \right) \\
& + \frac{1}{2} \left[\epsilon_{mnpq} \left(-\frac{1}{4} g_{ij} R^{mn}{}_{kl} R^{pqkl} + g_{jk} R^{mn}{}_{il} R^{pqkl} \right) - 2 \epsilon_{(j|kmn|} R^{kl} R^{mn}{}_{i)l} \right] . \qquad (3.65)
\end{aligned}
$$

The first line quite remarkably is known to be zero for any Riemannian four-manifold, and is called Berger's identity [24]. One can also show that the second line is equal to zero, which amounts to an algebraic identity that holds for any tensor sharing the algebraic symmetries of the Riemann tensor.

Finally, at the last order we find[14]

$$
\begin{aligned}
(4\mathrm{g}^4_{\bar{i}\bar{j}} + h^1_{\bar{i}\bar{j}}) \gamma^{\bar{j}} \varepsilon = & -2\gamma_{\bar{i}}(1+\gamma_{\bar{z}}) \varepsilon^7 + 4 \left(\nabla_{\bar{i}}\, \varepsilon^5 + \frac{\mathrm{i}}{2} A^I_{\bar{i}} (\sigma_I \varepsilon^5) \right) - 2 X_2^2 \gamma_{\bar{i}}\, \varepsilon - 2\mathrm{g}^2_{\bar{i}\bar{j}} \gamma^{\bar{j}} \varepsilon^3 \\
& + \frac{\mathrm{i}}{6} (\gamma_{\bar{i}}{}^{\overline{jk}} - 4\delta^{[\bar{j}}_{\bar{i}} \gamma^{\bar{k}]}) \Big[(\mathcal{D} a^I_2)_{\overline{jk}} (\sigma_I \varepsilon) + (\mathrm{f}_2)_{\overline{jk}} \varepsilon + F^I_{\overline{jk}} (\sigma_I \varepsilon^3) - X_2 F^I_{\overline{jk}} (\sigma_I \varepsilon) \\
& + 2 \mathrm{e}^j_{\bar{j}} (\mathrm{e}^{(2)})^k_{\bar{k}} F^I_{jk} (\sigma_I \varepsilon) \Big] - 2 \Big[\mathrm{e}^i_{\bar{i}} (\mathrm{e}^{(2)})^j_{\bar{j}} + (\mathrm{e}^{(2)})^i_{\bar{i}} \mathrm{e}^j_{\bar{j}} \Big] \mathrm{g}^2_{ij} \gamma^{\bar{j}} \varepsilon \,. \qquad (3.66)
\end{aligned}
$$

[14]Of course, knowing $h^1_{\bar{i}\bar{j}}$ we could write an expression for $\mathrm{g}^4_{\bar{i}\bar{j}}$ alone, but it is only the combination $4\mathrm{g}^4_{\bar{i}\bar{j}} + h^1_{\bar{i}\bar{j}}$ which we shall need in the next section.

Again there is a positive chirality condition on ε^7 which removes it from the above equation. Using the many intermediate results we have derived, we then find

$$\begin{aligned}4\mathrm{g}^4_{ij} + h^1_{ij} =& 2\nabla_i\nabla_j\left(X_2 + \frac{1}{24}R\right) + 2\mathrm{i}\nabla_{(i}(\mathrm{a}_2)_{j)} + \left(X_2 - \frac{1}{12}R\right)R_{ij}\\ &+ g_{ij}\left(-\frac{1}{6}RX_2 - 2X_2^2 + \frac{1}{12}R_{kl}R^{kl}\right) + \frac{1}{4}R_{ik}R^k{}_j\\ &-\frac{1}{8}\epsilon^{mnk}{}_j R_{mnli}R_k{}^l + \frac{1}{4}R_{iklj}R^{kl} + \frac{1}{3}[2\mathcal{D}a_2^I - *(\mathcal{D}a_2^I)]_{(i|k|}\mathrm{J}^{Ik}{}_{|j)}\,. \end{aligned} \tag{3.67}$$

4 Metric independence

Our aim in this section is to show that, for any supersymmetric asymptotically locally hyperbolic solution to the Euclidean $\mathcal{N} = 4^+$ supergravity theory, with the topologically twisted boundary conditions on an arbitrary Riemannian four-manifold (M_4, g), the variation (1.2) of the holographically renormalized action is identically zero. As explained in the introduction, this implies that the right hand side of (1.1) is independent of the choice of metric g, precisely as expected for the holographic dual of a topological QFT. We find that this is indeed the case, using the minimal holographic renormalization scheme described in section 2.3. We comment further on this at the end of section 4.2.

4.1 Variation of the action

As discussed in section 3.2, the Donaldson-Witten topological twist corresponds to the following boundary conditions on the supergravity fields on M_4:

$$0 = b^{\pm} = \mathrm{a} = \varepsilon^- \,, \qquad X_1 = -\frac{1}{12}R\,, \qquad A^I = \frac{1}{2}\omega_i{}^{\overline{jk}}\mathrm{J}^I_{\overline{jk}}\,\mathrm{d}x^i\,. \tag{4.1}$$

Here the boundary Riemannian metric g_{ij} on M_4 is arbitrary, with $\omega_i{}^{\overline{jk}}$ being the spin connection, R being the Ricci scalar curvature, and the triplet of self-dual two-forms J^I being given by (3.21). The holographic Ward identity for the variation of the renormalized action (2.44) with respect to general variations of the non-zero boundary fields is

$$\delta S \;=\; \delta_g S + \delta_{A^I} S + \delta_{X_1} S \;=\; \int_{\partial Y_5 = M_4} \mathrm{d}^4 x\,\sqrt{\det g}\left[\frac{1}{2}T_{ij}\delta g^{ij} + \mathscr{J}^i_I\delta A^I_i + \Xi\,\delta X_1\right]. \tag{4.2}$$

It is worth pausing to consider carefully why this equation holds. A variation of the boundary data on M_4 will induce a corresponding variation of the bulk solution that fills it. However, we are evaluating the action on a solution to the equations of motion, and by definition these are stationary points of the bulk action. Thus the resulting variation of the on-shell action is necessarily a boundary term, and this is the expression on the right hand side of (4.2). This argument requires that the equations of motion are solved everywhere in the interior of Y_5: if the latter has internal boundaries, or singularities, the above in general breaks down, and one will encounter additional terms around these boundaries/singularities on the right hand side of (4.2).

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For the topological twist all boundary fields are determined by the metric g_{ij}. Since $X_1 = -\frac{1}{12}R$, to compute δX_1 we need the variation of the Ricci scalar:

$$\delta R = R_{ij}\delta g^{ij} + \nabla_i\left(g^{jk}\delta\Gamma^i_{jk} - g^{ij}\delta\Gamma^k_{jk}\right), \tag{4.3}$$

with the variation of the Christoffel symbols being

$$\delta\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\nabla_k\delta g_{lj} + \nabla_j\delta g_{lk} - \nabla_l\delta g_{jk}\right). \tag{4.4}$$

After integrating by parts twice we obtain

$$\delta_{X_1}S = -\frac{1}{12}\int_{\partial Y_5}\left[\left(\Xi R_{ij} + g_{ij}\nabla^2\Xi - \nabla_i\nabla_j\Xi\right)\delta g^{ij}\,\mathrm{vol}_4 + \frac{1}{\kappa_5^2}\mathscr{D}_{X_1}\mathrm{vol}_4\right], \tag{4.5}$$

where $\mathrm{vol}_4 \equiv \sqrt{\det g}\,\mathrm{d}^4x$ is the Riemannian volume form on (M_4, g), and all geometric quantities appearing are computed using the boundary metric g_{ij}. Substituting the value of Ξ from (2.52) leads to

$$\delta_{X_1}S = -\frac{1}{4\kappa_5^2}\int_{\partial Y_5}\left[\left(X_2R_{ij} + g_{ij}\nabla^2X_2 - \nabla_i\nabla_jX_2\right)\delta g^{ij}\,\mathrm{vol}_4 + \frac{1}{3}\mathscr{D}_{X_1}\mathrm{vol}_4\right], \tag{4.6}$$

where the total derivative term is

$$\mathscr{D}_{X_1} \equiv -3\nabla_i\left[\nabla^kX_2g^{ij}\delta g_{jk} - \nabla^iX_2g^{jk}\delta g_{jk} - X_2g^{jk}g^{il}(\nabla_k\delta g_{lj} - \nabla_l\delta g_{jk})\right]. \tag{4.7}$$

For δA^I_i we first need the variation of the spin connection. After a short calculation we have

$$\delta\omega_i{}^{\overline{jk}} = \frac{1}{2}\,\mathrm{e}^{l\bar{j}}\,\mathrm{e}^{m\bar{k}}\left(\nabla_m\delta g_{il} - \nabla_l\delta g_{im}\right). \tag{4.8}$$

Thus

$$\delta A^I_i = \frac{1}{2}\delta\omega_i{}^{\overline{jk}}\mathrm{J}^I_{\overline{jk}} = \frac{1}{2}(\nabla_k\delta \mathrm{g}_{ij})\mathrm{J}^{Ijk}. \tag{4.9}$$

After integrating by parts, the $\mathrm{SU}(2)_R$ current contribution is hence

$$\delta_{A^I}S = -\frac{1}{8\kappa_5^2}\int_{\partial Y_5}\left\{\left[\mathcal{D}^k(a^I_1 + 2a^I_2)_i\,\mathrm{J}^I_{jk}\right]\delta g^{ij}\,\mathrm{vol}_4 + \mathscr{D}_{A^I}\mathrm{vol}_4\right\}, \tag{4.10}$$

where we have substituted for the $\mathrm{SU}(2)_R$ current using (2.53), and used the quaternionic Kähler identity (3.29). The object in square brackets is a tensor with indices ij: only the symmetric part contributes. The total derivative term is

$$\mathscr{D}_{A^I} \equiv \nabla_i\left[(a^I_1 + 2a^I_2)^k\mathrm{J}^{Iij}\delta g_{jk}\right]. \tag{4.11}$$

It remains to evaluate the stress-energy tensor contribution (2.51) and combine it with (4.6) and (4.10). Doing so leads to

$$\delta S = \frac{1}{4\kappa_5^2}\int_{\partial Y_5}\left(\mathcal{T}_{ij}\,\delta g^{ij}\,\mathrm{vol}_4 + \mathscr{D}_S\,\mathrm{vol}_4\right), \tag{4.12}$$

where the total derivative term is

$$\mathscr{D}_S \equiv -\frac{1}{3}\mathscr{D}_{X_1} - \frac{1}{2}\mathscr{D}_{A^I}\,, \tag{4.13}$$

and

$$\begin{aligned}\mathcal{T}_{ij} = \Bigg[&4\mathrm{g}^4_{ij} + h^1_{ij} - 4g_{ij}\left(t^{(4)} - \frac{1}{2}t^{(2,2)} - \frac{1}{8}u^{(1)}\right) - 2\mathrm{g}^2_{ij}t^{(2)} - 6g_{ij}X_2^2 \\ &+ \frac{1}{2}(\nabla^k\nabla_i\mathrm{g}^2_{jk} + \nabla^k\nabla_j\mathrm{g}^2_{ik} - \nabla^2\mathrm{g}^2_{ij} - \nabla_i\nabla_j t^{(2)}) - \frac{1}{2}\mathrm{g}^2_{ij}R + \frac{1}{2}g_{ij}(\mathrm{g}^2_{kl}R^{kl})\Bigg] \\ &- (X_2R_{ij} + g_{ij}\nabla^2X_2 - \nabla_i\nabla_jX_2) - \frac{1}{2}\left[\mathcal{D}^k(a_1^I + 2a_2^I)_{(i}\,\mathrm{J}^I{}_{j)k}\right]\,. \end{aligned} \tag{4.14}$$

Here the first two lines come from the stress-energy tensor (2.51), while the last line combines (4.6) and (4.10). Provided M_4 is a closed manifold, without boundary, the integral of the total derivative term is zero, and we have simply

$$\delta S = \frac{1}{4\kappa_5^2}\int_{\partial Y_5 = M_4}\mathcal{T}_{ij}\,\delta g^{ij}\,\mathrm{vol}_4\,. \tag{4.15}$$

The tensor $\mathcal{T}_{ij}$ is thus an *effective* stress-energy tensor, for variations of the renormalized on-shell action with respect to the boundary metric, all boundary data being determined by this choice of metric. Our claim that the on-shell action is invariant under an arbitrary metric deformation δg_{ij} is thus equivalent to the statement that $\mathcal{T}_{ij} \equiv 0$, for every Riemannian four-manifold. Remarkably, despite there being several undetermined quantities in (4.14), using the results of sections 2.3 and 3.4 we will show that indeed $\mathcal{T}_{ij} \equiv 0$ in the next subsection.

4.2 Proof that $\delta S/\delta g_{ij} = 0$

We begin by substituting expressions from section 2.2 into (4.14), which recall follow from the Fefferman-Graham expansion of the bosonic equations of motion. In particular we substitute for $\nabla^2 X_2$ using equation (2.31), as well as various metric quantities, except for the combination $4\mathrm{g}^4_{ij} + h^1_{ij}$. With the topological twist boundary conditions (4.1) this leads to the expression

$$\begin{aligned}\mathcal{T}_{ij} = &\left(\frac{1}{12}R - X_2\right)R_{ij} - \frac{1}{2}R_{ik}R^k{}_j - \frac{1}{2}R_{iklj}R^{kl} - \frac{1}{4}\nabla_i\nabla_jR + \nabla_i\nabla_j\left(X_2 + \frac{1}{6}R\right) \\ &+ \frac{1}{4}\nabla^2R_{ij} + g_{ij}\left(2X_2^2 - \frac{1}{72}R^2 + \frac{1}{6}RX_2 - \frac{1}{24}\nabla^2R + 4X_3 + 4X_4\right) \\ &+ 4\mathrm{g}^4_{ij} + h^1_{ij} - \frac{1}{2}\left[\mathcal{D}^k(a_1^I + 2a_2^I)_{(i}\,\mathrm{J}^I{}_{j)k}\right]\,. \end{aligned} \tag{4.16}$$

In particular we have used the identity

$$-\frac{1}{2}\nabla_k\nabla_{(i}R^k{}_{j)} = -\frac{1}{2}R_{ik}R^k{}_j - \frac{1}{2}R_{iklj}R^{kl} - \frac{1}{4}\nabla_i\nabla_jR\,, \tag{4.17}$$

in deriving (4.16).

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The equations of motion, or equivalently supersymmetry conditions, determine

$$X_3 = \frac{1}{48}\nabla^2 R\,, \quad X_4 = \frac{1}{288}R^2 - \frac{1}{48}R_{kl}R^{kl} - \frac{1}{96}\nabla^2 R - \frac{1}{24}\left(\mathcal{D}a_2^I\right)^{ij}\mathrm{J}^I_{ij}\,. \tag{4.18}$$

On the other hand, in section 3.4 the expansion of the supersymmetry conditions led to the expression (3.67), which we repeat here:

$$\begin{aligned}4\mathrm{g}^4_{ij} + h^1_{ij} = {} & 2\nabla_i\nabla_j\left(X_2 + \frac{1}{24}R\right) + 2\mathrm{i}\nabla_{(i}(\mathrm{a}_2)_{j)} + \left(X_2 - \frac{1}{12}R\right)R_{ij} \\ & + g_{ij}\left(-\frac{1}{6}RX_2 - 2X_2^2 + \frac{1}{12}R_{kl}R^{kl}\right) + \frac{1}{4}R_{ik}R^k{}_j \\ & - \frac{1}{8}\epsilon^{mnk}{}_j R_{mnli}R_k{}^l + \frac{1}{4}R_{iklj}R^{kl} + \frac{1}{3}[2\mathcal{D}a_2^I - *(\mathcal{D}a_2^I)]_{(i|k|}\mathrm{J}^{Ik}{}_{|j)}\,.\end{aligned} \tag{4.19}$$

Substituting into (4.16), after several immediate cancellations we are left with

$$\begin{aligned}\mathcal{T}_{ij} = {} & \frac{1}{4}\nabla^2 R_{ij} - \frac{1}{8}\epsilon^{mnk}{}_j R_{mnpi}R_k{}^p - \frac{1}{4}R_{ik}R^k{}_j - \frac{1}{4}R_{iklj}R^{kl} + 3\nabla_i\nabla_j X_2 - \frac{1}{2}\mathcal{D}^k(a_1^I)_{(i}\mathrm{J}^I{}_{j)k} \\ & + 2\mathrm{i}\nabla_{(i}(\mathrm{a}_2)_{j)} - \frac{1}{6}g_{ij}\left(\mathcal{D}a_2^I\right)^{kl}J^I_{kl} + \frac{1}{3}(2\mathcal{D}a_2^I - *\mathcal{D}a_2^I)_{(i|k|}\mathrm{J}^{Ik}{}_{j)} - \mathcal{D}^k(a_2^I)_{(i}\mathrm{J}^I{}_{j)k}\,.\end{aligned} \tag{4.20}$$

Using the expression

$$(a_1^I)_i = -\frac{1}{4}\mathrm{J}^I_{mn}\nabla_j R^{mnj}{}_i\,, \tag{4.21}$$

together with the contracted second Bianchi identity, we find that

$$\mathcal{D}^k(a_1^I)_i\mathrm{J}^I_{jk} = -\frac{1}{2}\epsilon_j{}^{kmn}\nabla_k\nabla_m R_{ni} - \frac{1}{2}\nabla^k\nabla^l R_{jkli}\,. \tag{4.22}$$

Substituting this expression, together with equation (3.46), into $\mathcal{T}_{ij}$ in (4.20), we arrive at

$$\begin{aligned}\mathcal{T}_{ij} = {} & \frac{1}{4}\nabla^2 R_{ij} - \frac{1}{8}\nabla_i\nabla_j R + \frac{1}{4}\nabla^k\nabla^l R_{jkli} - \frac{1}{4}R_{ik}R^k{}_j - \frac{1}{4}R_{iklj}R^{kl} \\ & - \frac{1}{6}g_{ij}\left(\mathcal{D}a_2^I\right)^{kl}J^I_{kl} + \frac{1}{3}\left[2\mathcal{D}a_2^I - *(\mathcal{D}a_2^I)\right]_{(i|k|}\mathrm{J}^{Ik}{}_{j)} - (\mathcal{D}a_2^I)_{(i|k|}\mathrm{J}^{Ik}{}_{j)} \\ & + \frac{1}{8}\epsilon_j{}^{kmn}(2\nabla_k\nabla_m R_{ni} - R_{mni}{}^l R_{kl}) \\ = {} & 0\,.\end{aligned} \tag{4.23}$$

Here, remarkably, each of the three lines vanishes separately. The first line is zero using again (4.17) and the contracted second Bianchi identity, whilst the terms in the second line combine to give zero after using the self-duality property of the J^I tensors to remove the Hodge dual acting on the field strength $\mathcal{D}a_2^I$. The final line is zero after applying the Ricci identity for a rank two covariant tensor, followed by the first Bianchi identity and using the symmetry of the summed indices.

We emphasize again that this proof that $\delta S/\delta g_{ij} = 0$ uses the minimal holographic renormalization scheme defined in section 2.3. Up to finite counterterms in (2.45) that are topological invariants, which have identically zero variations, another choice of scheme would spoil the above result. Another important comment is that the original path integral arguments in [4] are essentially classical (see footnote 10 of [4]). In particular there might have been an anomaly, implying that the partition function (and other correlation functions) are not invariant under arbitrary metric deformations. In this case, the topological twist would not have led to a TQFT. This might seem like a strange comment, given that the topologically twisted $\mathcal{N}=2$ Yang-Mills theory of [4] at least formally reproduces Donaldson theory, which of course certainly does rigorously define diffeomorphism invariants of M_4. However, it has recently been argued that precisely such an anomaly exists for four-dimensional rigid $\mathcal{N}=1$ supersymmetry [34, 35]. The computations in these papers are in fact holographic, and rely on the fact that in AdS/CFT the semi-classical gravity computation is a fully quantum computation on the QFT side, including any potential anomalies. Specifically, it is argued that there is an anomalous transformation of the supercurrent under rigid supersymmetry on the conformal boundary, implying that the partition function is not invariant under certain metric deformations that are classically $\mathcal{Q}$-exact. These particular anomalous transformations were first discovered in [31, 33], via essentially the same computation we have followed in this paper, although this was not interpreted as an anomaly in [31, 33]. It remains an open problem to directly derive this anomalous transformation from the QFT in a new minimal supergravity background. Returning to our present problem, the QFT is in any case coupled to an $\mathcal{N}=2$ conformal supergravity background, and for the $\mathcal{N}=2$ topological twist we find no anomaly. In particular our topologically twisted supergravity theory, *formally* at least, defines a topological theory. We discuss this further in section 5.3 and section 6.

5 Geometric reformulation

In this section we present a geometric reformulation of the bulk supersymmetry equations. In section 5.1 we describe how (twisted) differential forms built out of bilinears in the bulk spinor define a twisted Sp(1) structure on Y_5, and in section 5.2 we then derive a set of first order differential constraints on this structure. On the conformal boundary this restricts to the quaternionic Kähler structure that exists on any oriented Riemannian four-manifold (M_4, g), described in section 3.2. We also discuss some general aspects of the filling problem in section 5.3.

5.1 Twisted Sp(1) structure

Recall from section 2.1 that the bulk spinor ϵ of the Romans $\mathcal{N}=4^+$ theory is originally a quadruplet of spinors. These split into two doublets $\epsilon^{\pm}$, with eigenvalues $\pm$i under Γ_{45} (see equation (2.11)). Beginning in section 3.2, we worked in a truncated theory in which $\mathcal{B}^{\pm}=0$ and $\epsilon^-=0$. We may then define

$$\epsilon^+ = \begin{pmatrix} \zeta \\ -\zeta^c \end{pmatrix} , \tag{5.1}$$

where ζ is a spinor on Y_5, and recall that $\zeta^c \equiv \mathscr{C}\zeta^*$. Equation (5.1) is the solution to the symplectic Majorana condition $(\epsilon^+)^c = \epsilon^+$. More globally, and as on the conformal boundary M_4, the spinor ϵ^+ in (5.1) is a $Spin_{\mathscr{G}}$ spinor, where $\mathscr{G} = \mathrm{SU}(2)_R$ — see section 3.2.

With this notation we may define the following (local) differential forms

$$\begin{aligned} S &\equiv \bar{\zeta}\zeta\,, & \mathcal{K} &\equiv \frac{1}{S}\bar{\zeta}\gamma_{(1)}\zeta\,, \\ \mathcal{J}^3 &\equiv \frac{\mathrm{i}}{S}\bar{\zeta}\gamma_{(2)}\zeta, & \mathcal{J}^2+\mathrm{i}\mathcal{J}^1 &\equiv \frac{1}{S}\bar{\zeta}^c\gamma_{(2)}\zeta\,, \end{aligned} \tag{5.2}$$

where in our Hermitian basis of Clifford matrices recall that a bar denotes Hermitian conjugate. There are a number of global comments to make. First, as in the discussion in section 3.2, the fact that ζ is globally a twisted spinor, rather than a spinor, means that (5.2) in general only locally defines an $\mathrm{SU}(2)\cong\mathrm{Sp}(1)$ structure.[15] More globally, the $\mathcal{J}^I$ are twisted via the $\mathrm{SU}(2)_R$ symmetry, transforming as a triplet. We shall call this a *twisted* Sp(1) structure. Another comment is that in any case the structure is well-defined only where $\zeta\neq 0$. In general there may be solutions to the spinor equations where $\zeta=0$ on some locus. We should hence more precisely define $Y_5^{(0)}\equiv Y_5\setminus\{\zeta=0\}$, so that (5.2) is well-defined on $Y_5^{(0)}$. One will then need to impose certain boundary conditions on this structure, near $\{\zeta=0\}$, in order that the solution on Y_5 is appropriately regular. The bilinears (5.2) define a twisted Sp(1) structure on $Y_5^{(0)}$.

The expansion of the spinor (3.37) implies that near the conformal boundary

$$\zeta = z^{-1/2}\chi + z^{3/2}\left(\frac{1}{48}R\right)\chi + z^{5/2}\left(\log z\,\mathrm{d}R+\frac{\mathrm{i}}{2}\mathrm{a}_2+\frac{1}{2}\mathrm{d}X_2+\frac{1}{48}\mathrm{d}R\right)\cdot\chi + o(z^3)\,, \tag{5.3}$$

where χ is the boundary spinor defined in section 3.2. In particular for the topological twist this is constant, with constant square norm $\bar{\chi}\chi=c^2$ (see equations (3.24), (3.25)). Without loss of generality we henceforth set $c=1$, so that

$$S = \frac{1}{z}+\frac{z}{24}R+o(z^{5/2})\ . \tag{5.4}$$

In particular notice that $\zeta\neq 0$ near to the conformal boundary at $z=0$.

5.2 Differential system

Starting from the bulk Killing spinor equations (2.7), (2.8) one can derive a system of differential equations for the twisted Sp(1) structure (5.2). In the notation (5.1) the spinor equations read

$$\begin{aligned} \nabla_\mu\zeta =& -\frac{\mathrm{i}}{2}\mathcal{A}_\mu\zeta+\frac{\mathrm{i}}{2}\left(\mathcal{A}^1_\mu-\mathrm{i}\mathcal{A}^2_\mu\right)\zeta^c-\frac{\mathrm{i}}{2}\mathcal{A}^3_\mu\zeta+\frac{1}{3}\left(X+\frac{1}{2}X^{-2}\right)\gamma_\mu\zeta \\ &+\frac{\mathrm{i}}{24}X^{-1}(\mathcal{F}^1_{\nu\rho}-\mathrm{i}\mathcal{F}^2_{\nu\rho})(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho)\zeta^c-\frac{\mathrm{i}}{24}\big(X^{-1}\mathcal{F}^3_{\nu\rho}+X^2\mathcal{F}_{\nu\rho}\big)(\gamma_\mu{}^{\nu\rho}-4\delta^\nu_\mu\gamma^\rho)\zeta\,, \\ 0 =& \frac{3}{2}\mathrm{i}\,X^{-1}\partial_\mu X\gamma^\mu\zeta+\mathrm{i}\big(X-X^{-2}\big)\zeta-\frac{1}{8}X^{-1}(\mathcal{F}^1_{\mu\nu}-\mathrm{i}\mathcal{F}^2_{\mu\nu})\gamma^{\mu\nu}\zeta^c \\ &+\frac{1}{8}(X^{-1}\mathcal{F}^3_{\mu\nu}-2X^2\mathcal{F}_{\mu\nu})\gamma^{\mu\nu}\zeta\,. \end{aligned} \tag{5.5}$$

[15] A general discussion of global Sp(1) structures on five-manifolds may be found in [41].

As in section 2.1, it will be convenient to introduce the real one-form

$$\mathcal{C} \equiv \mathrm{i}\mathcal{A} \ . \tag{5.6}$$

Using these equations, a standard calculation[16] leads to

$$X^{-2}\mathcal{K} = \mathrm{d}\log(XS) + \mathcal{C} \,, \tag{5.7}$$

together with the triplet of equations

$$\begin{aligned} \mathrm{d}(S\mathcal{J}^I) = &- \mathcal{C} \wedge S\mathcal{J}^I + (2X + X^{-2})\mathcal{K} \wedge S\mathcal{J}^I + \epsilon^I{}_{JK}\mathcal{A}^J \wedge S\mathcal{J}^K \\ &+ \frac{1}{4}X^{-1}S\,(*\mathcal{F}^I + \mathcal{K} \wedge \mathcal{F}^I) \,. \end{aligned} \tag{5.8}$$

Here the Hodge dual is constructed from the volume form $\mathrm{vol}_5 = -\mathcal{K} \wedge \mathrm{vol}_4$, where $\mathrm{vol}_4 \equiv \frac{1}{2}\mathcal{J}^I \wedge \mathcal{J}^I$ (no sum over I). The sign here is chosen to match our earlier choice of orientation, via (2.17), as we shall see shortly.

We may read the first equation (5.7) as determining the one-form $\mathcal{C}$ in terms of geometric data and the function X:

$$\mathcal{C} = X^{-2}\mathcal{K} - \mathrm{d}\log(XS) \ . \tag{5.9}$$

In particular, the associated flux is then

$$\mathcal{G} \equiv \mathrm{d}\mathcal{C} = \mathrm{i}\mathcal{F} = \mathrm{d}(X^{-2}\mathcal{K}) \ . \tag{5.10}$$

Substituting (5.9) into (5.8), the latter simplifies to

$$\mathrm{d}\mathcal{J}^I = \epsilon^I{}_{JK}\mathcal{A}^J \wedge \mathcal{J}^K + (\mathrm{d}\log X + 2X\mathcal{K}) \wedge \mathcal{J}^I + \frac{1}{4}X^{-1}(*\mathcal{F}^I + \mathcal{K} \wedge \mathcal{F}^I) \ . \tag{5.11}$$

Recall that in the original Lorentzian theory $\mathcal{A}$ is a $\mathrm{U}(1)_R$ gauge field. In the real Euclidean section we have defined $\mathcal{C} = \mathrm{i}\mathcal{A}$, which is a real one-form, but there is then a residual part of the (complexified) gauge symmetry $\mathcal{C} \to \mathcal{C} - \mathrm{d}\lambda$, where λ is a global real function. The fields transform as follows:

$$\zeta \ \to \ \mathrm{e}^{\lambda/2}\zeta \,, \qquad S \ \to \ \mathrm{e}^{\lambda}S \,, \qquad \mathcal{C} \ \to \ \mathcal{C} - \mathrm{d}\lambda \,, \tag{5.12}$$

with everything else invariant. In particular it is immediate to see that (5.9), (5.11) are invariant under these gauge transformations. In our boundary value problem recall that we fixed $\mathcal{C} \mid_{M_4} = 0$, and in order to preserve this gauge condition on the conformal boundary one should restrict to gauge transformations that vanish there, so that $\lambda \mid_{M_4} = 0$. With this caveat, one might use this gauge freedom to effectively remove one of the functional degrees of freedom.

[16] For example, see [19].

Let us look at the asymptotic form of the differential conditions near the conformal boundary at $z=0$. Recalling the Fefferman-Graham expansion of the fields (2.19)–(2.21), together with the topological twist boundary conditions (4.1), we have

$$\begin{aligned} X &= 1 - \frac{1}{12} z^2 \log z\, R + z^2 X_2 + o(z^2)\,, \\ \mathcal{A}^I &= A^I - \frac{1}{4} z^2 \log z\, \mathrm{J}^I_{mn} \nabla_j R^{mnj}{}_i\, \mathrm{d}x^i + z^2 a_2^I + o(z^2)\,, \\ \mathcal{C} &= z^2\, \mathrm{ia}_2 + o(z^2)\ . \end{aligned} \tag{5.13}$$

Here recall that R is the boundary Ricci scalar, the boundary gauge field is

$$A^I = \frac{1}{2} \omega_i{}^{\overline{jk}} \mathrm{J}^I_{\overline{jk}}\, \mathrm{d}x^i\,, \tag{5.14}$$

where $\omega_i{}^{\overline{jk}}$ is the boundary spin connection, R_{mnij} is the boundary Riemann tensor, and J^I are the boundary triplet of self-dual two-forms. The one-form ia_2 is real. Using also (5.4), equation (5.7) then implies that

$$\mathcal{K} = -\frac{\mathrm{d}z}{z} + z^2 \left(2 \log z\, \mathrm{d}R + \mathrm{ia}_2 + \mathrm{d}X_2 + \frac{1}{24} \mathrm{d}R \right) + o(z^{5/2})\,. \tag{5.15}$$

Recall that in section 3.2 we defined the triplet of boundary almost complex structures $(\mathrm{I}^I)^i{}_j \equiv g^{ik} (\mathrm{J}^I)_{kj}$. If we define the boundary (almost) Ricci two-forms

$$\rho^I_{ij} \equiv R_{k[i} (\mathrm{I}^I)^k{}_{j]}\,, \tag{5.16}$$

where R_{ij} is the boundary Ricci tensor, then similarly from the definition (3.26) we have

$$\begin{aligned} \mathcal{J}^I &= \frac{1}{z^2} \mathrm{J}^I + \frac{1}{12} R\, \mathrm{J}^I - \frac{1}{2} \rho^I \\ &\quad + z \mathrm{d}z \wedge \mathrm{I}^I \left(2 \log z\, \mathrm{d}R + \mathrm{ia}_2 + \mathrm{d}X_2 + \frac{1}{24} \mathrm{d}R \right) + o(z^{3/2})\ . \end{aligned} \tag{5.17}$$

Here $\mathrm{I}^I(\eta)_i = (\mathrm{I}^I)^j{}_i \eta_j$ for a one-form η tangent to the boundary. It is interesting to note that the $O(1)$ terms in $\mathcal{J}^I$ above may also be written as $\frac{1}{12} R\, \mathrm{J}^I - \frac{1}{2}\rho^I = (\mathsf{g}^2 \circ J^I)$, where recall from equation (2.34) that g^2 is (minus) the Schouten tensor of the conformal boundary. From (5.11) we hence read off the leading order the boundary equation

$$\mathrm{dJ}^I = \epsilon^I{}_{JK} A^J \wedge \mathrm{J}^K\ . \tag{5.18}$$

Equation (5.18) follows from taking the skew symmetric part of (3.29). In fact since the exterior derivatives of the boundary SU(2) structure J^I completely determine the intrinsic torsion (this is true for an SU(n) structure in real dimension $2n$ [42]), it follows that (5.18) also implies (3.29).

We may always choose a frame $\mathscr{E}^{\overline{\mu}}_\mu$ for the bulk metric on Y_5 such that

$$\begin{aligned} \mathcal{K} &= -\mathscr{E}^5\,, & \mathcal{J}^1 &= \mathscr{E}^2 \wedge \mathscr{E}^3 + \mathscr{E}^1 \wedge \mathscr{E}^4\,, \\ \mathcal{J}^2 &= \mathscr{E}^3 \wedge \mathscr{E}^1 + \mathscr{E}^2 \wedge \mathscr{E}^4\,, & \mathcal{J}^3 &= \mathscr{E}^1 \wedge \mathscr{E}^2 + \mathscr{E}^3 \wedge \mathscr{E}^4\ . \end{aligned} \tag{5.19}$$

In particular (5.15) identifies $\mathscr{E}^5 \sim \mathrm{d}z/z$ to leading order, and the sign for $\mathcal{K}$ in (5.19) follows since $-\gamma_{\bar{z}}\chi = \chi$, where $\mathrm{E}^{\bar{z}} = \mathrm{d}z/z$. The volume form is $\mathrm{vol}_5 = \mathscr{E}^{12345}$. Notice that the expansions (5.15), (5.17) imply that in general we may not identify $\mathscr{E}^{\overline{\mu}}_{\mu}$ near the conformal boundary with the Fefferman-Graham frame $\mathrm{E}^{\overline{\mu}}_{\mu}$ in (3.1), except to leading order.

5.3 Filling problem

As explained in the introduction, given a Riemannian-four manifold (M_4, g) as a fixed conformal boundary, at least to a zeroth order approximation in AdS/CFT one wants to find the least action supersymmetric solution to the five-dimensional $\mathcal{N} = 4^+$ supergravity theory, with this boundary data. Such a solution will be the dominant saddle point on the right hand side of (1.1). In this subsection we make some comments on this problem, with further comments in section 6.1.

As we have seen in the previous subsection, supersymmetric solutions on Y_5 are characterized geometrically in terms of a set of first order differential equations (5.9), (5.11) for a certain twisted Sp(1) structure. In particular there is a triplet of twisted two-forms $\mathcal{J}^I$, $I = 1, 2, 3$, which locally at the conformal boundary restrict to an orthonormal set of self-dual two-forms on (M_4, g). The differential equations become tautological on the boundary, and are equivalent to the fact that every oriented Riemannian four-manifold has a quaternionic Kähler structure, i.e. has holonomy group $\mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \cong \mathrm{SO}(4)$. This differential system on Y_5, regarded as extending that on (M_4, g), clearly deserves closer study. In particular, these are necessary conditions for a solution, but one would also like to know whether they are sufficient. It should also be possible to rewrite the renormalized supergravity action (2.44) in terms of this geometric data. The computation in section 4 implies that, given any one-parameter family of metrics on M_4, the action of any family of fillings of the boundary is independent of the parameter. What type of invariant is this? *A priori* it depends on the choice of Y_5 filling M_4, and on the twisted Sp(1) structure on Y_5.

An important question is what are the global constraints on Y_5? As mentioned in the introduction, topologically a smooth filling Y_5 of M_4 exists if and only if the signature $\sigma(M_4) = 0$. Moreover, as explained in section 6.1, for solutions embedded in string theory one also needs these manifolds to be spin.[17] This restriction would seem to rule out many interesting four-manifolds.[18] However, as also mentioned in the introduction, requiring Y_5 to be smooth is almost certainly too strong. Already from AdS/CFT in other contexts, it is clear that the dominant saddle point contribution can be singular, and one might anticipate that this is somewhat generic, at least for general M_4. Perhaps the appropriate question is then: what are the relevant singularities of Y_5, for a given M_4?[19] Mathematically one would need control over existence and uniqueness of the differential equations for the twisted Sp(1) structure, for appropriate Y_5 (with singularities/appropriate internal boundary conditions) filling M_4. However, one might also anticipate that the supergravity action (2.44) could

[17] The relevant spin bordism group is $\Omega_4^{Spin} \cong \mathbb{Z}$, generated by a K3 surface, where the map to the integers is $\sigma(M_4)/16$.

[18] Although it leaves, for example, $M_4 = S^1 \times M_3$, for any oriented three-manifold M_3, and products of Riemann surfaces.

[19] We thank S. Gukov for discussions on this, and indeed for posing this precise question!

be evaluated without knowing the detailed form of the solution, but instead in terms of appropriate global data, and perhaps local data associated to singularities. Notice that one constraint on such singularities/internal boundaries is that they do not contribute to the variation of the action (4.2) — see the discussion after this equation.[20]

Less ambitiously, one might also try to find explicit solutions; for example, via symmetry reduction so that the equations reduce to coupled ODEs. An obvious case is solutions with $Y_5 = S^1 \times B_4$, where B_4 is a four-ball so that $\partial Y_5 = M_4 = S^1 \times S^3$, and seek solutions invariant under $\mathrm{U}(1) \times \mathrm{SU}(2)$ (the latter acting on the left on $S^3 \cong \mathrm{SU}(2)$).

Finally, the present problem may be contrasted to the general hyperbolic filling problem described in [43]. Here one also begins with an arbitrary Riemannian (M_4, g), which is a conformal boundary, but one instead asks for the filling to be an Einstein metric of negative curvature. This problem is still quite poorly understood: there are in general obstructions and non-uniqueness, and one should at least impose that g has a conformal representative with positive scalar curvature [44] (physically, so that the CFT is stable). The geometric problem in the present paper is likely to be much better behaved: the equations are first order, not second order, and the solutions should be dual to a TQFT.

6 Discussion

We conclude with a discussion of "topological AdS/CFT" in section 6.1, followed by various extensions and generalizations in section 6.2.

6.1 Topological AdS/CFT

An application of the ideas developed in this paper would be to a topologically twisted version of the AdS/CFT correspondence. To make quantitative comparisons between calculations on the two sides, as in (1.1) (appropriately interpreted), the construction needs embedding in string theory. This is straightforward: the Romans theory is a consistent truncation of both Type IIB supergravity on S^5 [25], and also of eleven-dimensional supergravity on N_6 [26], where N_6 are the geometries classified by Lin-Lunin-Maldacena [45]. This means that any solution to the five-dimensional Romans theory uplifts (at least locally — see below) to a string/M-theory solution.

In order to be concrete, let us focus on the case of $\mathcal{N} = 4$ Yang-Mills theory. Applying the Donaldson-Witten twist leads to the half-twisted theory referred to in the introduction. For general gauge group $\mathscr{G}$ the path integral localizes [46, 47] onto solutions to a non-Abelian [48] version of the Seiberg-Witten equations, in which the spinor field is in the adjoint representation of $\mathscr{G}$. For $\mathscr{G} = \mathrm{SU}(N)$, AdS/CFT should relate the large N limit of this theory to an appropriate class of solutions to the Romans $\mathcal{N} = 4^+$ theory in five dimensions, uplifted on S^5 to give full solutions of Type IIB string theory. This is where the restriction that M_4 is spin enters: if M_4 is not spin then the background SU(2) R-symmetry gauge field we turn on is not globally a connection on an SU(2) bundle over

[20]For example, the singularities in the gravity fillings in [18, 19] are isolated conical singularities. Provided the radial dependence of fields near to the singular point are no worse than for smooth fields in flat space, such singularities will not spoil the result (4.2).

M_4. On the other hand, the Type IIB solution is an S^5 fibration over the filling Y_5, where $S^5 \subset \mathbb{C}^2 \oplus \mathbb{C}$, and SU(2) acts on $\mathbb{C}^2$ in the fundamental representation. Thus if M_4 is not spin, this associated bundle is not well-defined. This is also directly visible in the TQFT: for the half-twist of $\mathcal{N} = 4$ Yang-Mills there are still spinors in the twisted theory, which only make sense if M_4 is spin.

There is some discussion of the half-twisted $\mathcal{N} = 4$ theory for general gauge group $\mathscr{G}$ in [49]. In particular the (virtual) dimension of the the relevant non-Abelian monopole moduli space $\mathcal{M}$ may be computed using index theory, leading to

$$\dim \mathcal{M} = -\frac{1}{4} \dim \mathscr{G} \cdot [2\chi(M_4) + 3\sigma(M_4)] \ . \tag{6.1}$$

Because of the associated fermion zero modes, the partition function of the theory vanishes unless the right hand side of (6.1) is also zero. We have already seen precisely this condition in the holographic dual set-up, namely equation (3.34). In the gravity context this followed from $\mathcal{A}$ being a global one-form, and then integrating the divergence of the VEV of the $\mathrm{U}(1)_R$ current (the $\mathrm{U}(1)_R$ anomaly) over a compact M_4 without boundary, as in (3.33). In fact the two are directly related, since the virtual dimension (6.1) of $\mathcal{M}$ computed in field theory is proportional to this integrated $\mathrm{U}(1)_R$ anomaly. In the current holographic set-up, we can see this explicitly by first noting that for the large N limit of the $\mathscr{G} = \mathrm{SU}(N)$ half-twisted $\mathcal{N} = 4$ Yang-Mills theory, a standard AdS/CFT formula fixes the dual effective five-dimensional Newton constant as

$$\frac{1}{\kappa_5^2} = \frac{N^2}{4\pi^2} \ . \tag{6.2}$$

This fixes the overall normalization of the supergravity action. In the large N limit, using (3.33) we may then write

$$\dim \mathcal{M} = 2\mathrm{i} \int_{M_4} \mathrm{d} *_4 \langle \mathbb{J} \rangle \,, \tag{6.3}$$

in terms of the integrated (holographic) $\mathrm{U}(1)_R$ anomaly.

Another important observation is that (6.1) is independent of the topology of the gauge bundle over M_4, unlike the corresponding case for Donaldson theory (pure $\mathcal{N} = 2$ Yang-Mills with gauge group $\mathscr{G}$). Because of this, all choices of gauge bundle contribute to the partition function at the same time. The left hand side of (1.1) then needs appropriately interpreting for such twists of four-dimensional $\mathcal{N} = 2$ SCFTs, as taken at face value it may be divergent. There is a standard way to deal with this,[21] namely to refine the partition function via the $\mathrm{U}(1)_R$ charge. For example, this is discussed at the end of section 2 of [50], and in [51]. This should play an important role in making sense also of the right hand side of (1.1), in addition to the comments on this in section 5.3. For example, a very concrete case mentioned in the latter subsection is $M_4 = S^1 \times S^3$. Here the refined partition function is closely related to the Coulomb branch index, as explained in [52]. One might then try to reproduce this from a dual supergravity solution for which $Y_5 = S^1 \times B_4$, with

[21] We are again grateful to S. Gukov for pointing this out.

$\partial Y_5 = S^1 \times S^3$. More generally, for a four-manifold $S^1 \times M_3$ with product metric both $\mathcal{E}$ and $\mathcal{P}$ vanish, and the holographic $\mathrm{U}(1)_R$ current is conserved, as can be seen from (3.32). The associated conserved holographic R-charge might then provide a natural holographic correspondent to the refinement of the partition function for the twisted four-dimensional SCFT. The AdS/CFT relation (1.1) in particular implies that the logarithm of the TQFT partition function, appropriately refined as above, scales as N^2 as $N \to \infty$, when it is non-zero. On the other hand, when the right hand side of (6.1) is positive, one obtains non-zero invariants in the TQFT by inserting appropriate $\mathcal{Q}$-exact operators into the path integral. We briefly discuss the dual holographic computation in section 6.2. In particular, such insertions will change the boundary conditions on supergravity fields we have imposed in this paper.

As far as we are aware, computations of topological observables in the half-twisted $\mathcal{N} = 4$ theory, for general $\mathscr{G} = \mathrm{SU}(N)$, have not been done explicitly. However, for $\mathscr{G} = \mathrm{SU}(2)$ the partition function and topological correlation functions have been computed explicitly for simply-connected spin four-manifolds of simple type [47]. This is done by giving masses, explicitly breaking $\mathcal{N} = 4$ to $\mathcal{N} = 2$, leading to an $\mathcal{N} = 2$ gauge theory with a massive adjoint hypermultiplet, a twisted version of the $\mathcal{N} = 2^*$ theory. The twisted theory is still topological, and the relevant observables are written in terms of Seiberg-Witten invariants using the methods of [53]. Observables for the original theory are then identified with the massless limit of these formulae (when this makes sense), although the validity of this assertion is not completely clear. In any case, to compare to the holographic construction in this paper one should compute the large N limit for gauge group $\mathscr{G} = \mathrm{SU}(N)$. We note that an analogous large N limit of Donaldson invariants (for pure $\mathcal{N} = 2$ $\mathrm{SU}(N)$ Yang-Mills) has been computed in [9]. Unlike the formula (6.1), here the dimension of the moduli space of instantons depends on the topology of the gauge bundle. One can then choose this bundle in such a way that $\dim \mathcal{M} = 0$. The partition function is a certain signed count of the points that make up $\mathcal{M}$, and the large N limit was computed for a certain class of four-manifolds in [9].[22]

We conclude this subsection by noting that similar remarks apply to twists of $\mathcal{N} = 2$ SCFTs with M-theory duals. Indeed, an important restriction on the class of $\mathcal{N} = 2$ gauge theories to which this holographic description applies is that they are conformal theories.[23] A large number of examples arise as class $\mathcal{S}$ theories [54], obtained by wrapping M5-branes over punctured Riemann surfaces, for which the gravity dual was found in [55] using the construction of [45]. Romans solutions uplift on the corresponding internal spaces N_6 to solutions of M-theory [26]. At the level of the five-dimensional theory, all that changes is the formula (6.2) for the effective Newton constant, which in general reads [56]

$$\frac{1}{\kappa_5^2} = \frac{a}{\pi^2}\,, \tag{6.4}$$

[22]In particular the final section of [9] computes the large N limit of the partition function Z for a four-manifold with boundary, constructed as $S^1 \times M_3$ where M_3 is a knot complement. One finds $Z \sim N \log \alpha$, where α is a certain knot invariant (the Mahler measure).

[23]In particular this is not true of pure $\mathcal{N} = 2$ Yang-Mills, from which the original Donaldson invariants are constructed.

where a is the a central charge. In the supergravity limit recall that $a = c$. For the above-mentioned M5-brane theories the central charge scales with N^3 as $N \to \infty$. Indeed, the partition function will *a priori* depend on both the choice of $\mathcal{N} = 2$ SCFT that is being twisted, and also on the four-manifold M_4 on which it is defined. The choice of theory corresponds to the choice of internal space in the uplifting to ten or eleven dimensions. The structure of the dual supergravity solution as a fibration of the internal space over the spacetime filling of M_4 then implies that the large N limits of the partition functions should also factorize. That is, the dependence on the choice of theory should only be visible via the central charge a, which via (6.4) fixes the overall normalization of the supergravity action. On the other hand, the dependence on the choice of M_4 is then captured by the effective five-dimensional Romans theory we have described.[24]

6.2 Generalizations

We have already discussed a number of open problems and directions for future work. Here we briefly mention some further generalizations:

- Perhaps the most immediate generalization of the computations in this paper would be to the so-called Ω-background of [57]. Here (M_4, g, ξ) is an arbitrary Riemannian four-manifold, equipped with a Killing vector field ξ. As for the pure topological twist, this geometry also arises by coupling an $\mathcal{N} = 2$ gauge theory to a certain background of $\mathcal{N} = 2$ conformal supergravity, and is briefly mentioned at the end of section 3 of [17]. The non-zero Killing vector ξ requires turning on a boundary B-field: specifically one needs to take b^- (or b^+) proportional to the self-dual (or anti-self-dual) part of the two-form $\mathrm{d}\xi^\flat$, where $\xi^\flat$ is the Killing one-form dual to ξ. Correspondingly, both boundary spinor doublets ε^+ and ε^- are now non-zero, and one needs to work with the full Romans theory, rather than the truncated version with $\mathcal{B}^\pm = 0$ we used from section 3.2 onwards. Nevertheless, the computations should not be too much more involved than those in the present paper. One expects the supergravity action now to depend on the choice of Killing vector ξ on M_4, but otherwise not on the metric. One should thus look at metric deformations $g_{ij} \to g_{ij} + \delta g_{ij}$, where $\mathcal{L}_\xi \, \delta g_{ij} = 0$.

- As mentioned in the introduction, there are three inequivalent topological twists of $\mathcal{N} = 4$ Yang-Mills. The half-twist, relevant to this paper, was discussed in the previous subsection. The other two twists are the Vafa-Witten twist [11], and the twist studied by Kapustin-Witten in [12]. In particular in the former theory the only non-trivial observable is the partition function, and this has been studied for gauge group $\mathscr{G} = \mathrm{SU}(N)$ in [58]. These twists require the larger $\mathrm{SU}(4)_R$ R-symmetry of the $\mathcal{N} = 4$ theory, meaning for the holographic dual one needs to start with a

[24] This structure can already be seen in the more general formula for $\dim \mathcal{M}$ given in [50]. For the general class of twisted field theories considered there, equation (2.42) of [50] implies that in the large N limit where $a = c$, one has $\dim \mathcal{M} = -a[2\chi(M_4) + 3\sigma(M_4)]$, generalizing (6.1). The central charge appears as an overall factor, at large N. Of course, this precisely agrees with our holographic formula (6.3), using (3.33) and (6.4).

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Euclidean form of $\mathcal{N} = 8$ gauged supergravity theory. Optimistically, one might hope to embed within the SU(4) $\sim$ SO(6) truncation of the latter theory studied in [59], which is a consistent truncation of Type IIB supergravity on S^5, and contains the five-dimensional Romans $\mathcal{N} = 4^+$ theory (with zero B-field) as a further truncation.

- Topological twists exist in a variety of dimensions. In three dimensions the R-symmetry group is Spin($\mathcal{N}$). The analogous amount of supersymmetry to that studied in the present paper is $\mathcal{N} = 4$, leading to a Spin(4) = SU(2) $\times$ SU(2) R-symmetry group. On the other hand Spin(3) = SU(2), and this leads to two inequivalent three-dimensional $\mathcal{N} = 4$ topological twists — see, for example, the diagram in section 1 of [60]. One of these twists is closely related (by dimensional reduction on a circle) to the Donaldson-Witten twist. The relevant holographic construction should begin with four-dimensional $\mathcal{N} = 4$ gauged supergravity. This contains an Spin$(4)_R$ gauge field, as required, and is a consistent truncation of eleven-dimensional supergravity on S^7 [61]. The uplifted solutions should be holographically dual to twists of the ABJM theory [62] on N M2-branes, in the large N limit. This is currently under investigation [63].

- Finally, in this paper we have focused exclusively on the partition function. However, in general TQFTs have non-trivial topological correlation functions, involving the insertion of $\mathcal{Q}$-invariant operators into the path integral. For example, this is true of Donaldson theory, where such insertions are required to obtain non-zero invariants in field theory whenever $\dim \mathcal{M} = d > 0$, due to fermion zero modes. Geometrically these invariants arise as the integral of a d-form over $\mathcal{M}$, where this top form is itself constructed as a wedge product of certain closed forms. The operators are constructed via a descent procedure [4]. It would be very interesting to understand the holographic dual computation of these correlation functions. Of course, correlation functions are well studied in AdS/CFT. In the present setting one would again hope to be able to work in a truncated supergravity theory, containing the fields whose boundary values act as sources for the operators. Being topological, the correlation functions should be independent of the positions at which the local operators are inserted, and also independent of the metric. These statements might be proven along similar lines to the present paper. We leave this, and other interesting questions, for future work.

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References

[1] J.M. Maldacena, *The Large-N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [hep-th/9711200] [INSPIRE].

[2] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[3] V. Pestun et al., *Localization techniques in quantum field theories*, *J. Phys.* **A 50** (2017) 440301 [arXiv:1608.02952] [INSPIRE].

[4] E. Witten, *Topological Quantum Field Theory*, *Commun. Math. Phys.* **117** (1988) 353 [INSPIRE].

[5] S.K. Donaldson, *An application of gauge theory to four-dimensionaltopology*, *J. Diff. Geom.* **18** (1983) 279 [INSPIRE].

[6] S.K. Donaldson, *Polynomial invariants for smooth manifolds*, *Topology* **29** (1990) 257 [INSPIRE].

[7] S.K. Donaldson and P.B. Kronheimer, *The Geometry of Four-Manifolds*, *Oxford Mathematical Monographs*, Clarendon Press, Oxford U.K. (1997).

[8] M. Mariño and G.W. Moore, *The Donaldson-Witten function for gauge groups of rank larger than one*, *Commun. Math. Phys.* **199** (1998) 25 [hep-th/9802185] [INSPIRE].

[9] P.B. Kronheimer, *Four-manifold invariants from higher-rank bundles*, *J. Diff. Geom.* **70** (2005) 59 [math/0407518].

[10] J.P. Yamron, *Topological Actions From Twisted Supersymmetric Theories*, *Phys. Lett.* **B 213** (1988) 325 [INSPIRE].

[11] C. Vafa and E. Witten, *A Strong coupling test of S duality*, *Nucl. Phys.* **B 431** (1994) 3 [hep-th/9408074] [INSPIRE].

[12] A. Kapustin and E. Witten, *Electric-Magnetic Duality And The Geometric Langlands Program*, *Commun. Num. Theor. Phys.* **1** (2007) 1 [hep-th/0604151] [INSPIRE].

[13] E. Witten, *Monopoles and four manifolds*, *Math. Res. Lett.* **1** (1994) 769 [hep-th/9411102] [INSPIRE].

[14] A. Karlhede and M. Roček, *Topological Quantum Field Theory and $N = 2$ Conformal Supergravity*, *Phys. Lett.* **B 212** (1988) 51 [INSPIRE].

[15] T.T. Dumitrescu, G. Festuccia and N. Seiberg, *Exploring Curved Superspace*, *JHEP* **08** (2012) 141 [arXiv:1205.1115] [INSPIRE].

[16] C. Klare, A. Tomasiello and A. Zaffaroni, *Supersymmetry on Curved Spaces and Holography*, *JHEP* **08** (2012) 061 [arXiv:1205.1062] [INSPIRE].

[17] C. Klare and A. Zaffaroni, *Extended Supersymmetry on Curved Spaces*, *JHEP* **10** (2013) 218 [arXiv:1308.1102] [INSPIRE].

[18] L.F. Alday, M. Fluder and J. Sparks, *The Large-N limit of M2-branes on Lens spaces*, *JHEP* **10** (2012) 057 [arXiv:1204.1280] [INSPIRE].

[19] L.F. Alday, M. Fluder, C.M. Gregory, P. Richmond and J. Sparks, *Supersymmetric solutions to Euclidean Romans supergravity*, *JHEP* **02** (2016) 100 [arXiv:1505.04641] [INSPIRE].

[20] N. Banerjee, S. Banerjee, R.K. Gupta, I. Mandal and A. Sen, *Supersymmetry, Localization and Quantum Entropy Function*, *JHEP* **02** (2010) 091 [arXiv:0905.2686] [INSPIRE].

[21] A. Maloney and E. Witten, *Quantum Gravity Partition Functions in Three Dimensions*, *JHEP* **02** (2010) 029 [arXiv:0712.0155] [INSPIRE].

[22] L.J. Romans, *Gauged $N = 4$ Supergravities in Five-dimensions and Their Magnetovac Backgrounds*, *Nucl. Phys.* **B 267** (1986) 433 [INSPIRE].

[23] T. Ohl and C.F. Uhlemann, *The Boundary Multiplet of $N = 4$ $\mathrm{SU}(2) \times \mathrm{U}(1)$ Gauged Supergravity on Asymptotically-AdS$_5$*, *JHEP* **06** (2011) 086 [arXiv:1011.3533] [INSPIRE].

[24] M. Berger, *Quelques formules de variation pour une structure riemannienne*, *Annales Sci. Ecole Norm. Sup.* **3** (1970) 285.

[25] H. Lü, C.N. Pope and T.A. Tran, *Five-dimensional $N = 4$, $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauged supergravity from type IIB*, *Phys. Lett.* **B 475** (2000) 261 [hep-th/9909203] [INSPIRE].

[26] J.P. Gauntlett and O. Varela, *$D = 5$ $\mathrm{SU}(2) \times \mathrm{U}(1)$ Gauged Supergravity from $D = 11$ Supergravity*, *JHEP* **02** (2008) 083 [arXiv:0712.3560] [INSPIRE].

[27] C. Fefferman and C.R. Graham, *The ambient metric*, arXiv:0710.0919 [INSPIRE].

[28] M. Taylor, *More on counterterms in the gravitational action and anomalies*, hep-th/0002125 [INSPIRE].

[29] S. de Haro, S.N. Solodukhin and K. Skenderis, *Holographic reconstruction of space-time and renormalization in the AdS/CFT correspondence*, *Commun. Math. Phys.* **217** (2001) 595 [hep-th/0002230] [INSPIRE].

[30] R. Emparan, C.V. Johnson and R.C. Myers, *Surface terms as counterterms in the AdS/CFT correspondence*, *Phys. Rev.* **D 60** (1999) 104001 [hep-th/9903238] [INSPIRE].

[31] P. Benetti Genolini, D. Cassani, D. Martelli and J. Sparks, *Holographic renormalization and supersymmetry*, *JHEP* **02** (2017) 132 [arXiv:1612.06761] [INSPIRE].

[32] M. Crossley, E. Dyer and J. Sonner, *Super-Rényi entropy & Wilson loops for $\mathcal{N} = 4$ SYM and their gravity duals*, *JHEP* **12** (2014) 001 [arXiv:1409.0542] [INSPIRE].

[33] P. Benetti Genolini, D. Cassani, D. Martelli and J. Sparks, *The holographic supersymmetric Casimir energy*, *Phys. Rev.* **D 95** (2017) 021902 [arXiv:1606.02724] [INSPIRE].

[34] I. Papadimitriou, *Supercurrent anomalies in 4d SCFTs*, *JHEP* **07** (2017) 038 [arXiv:1703.04299] [INSPIRE].

[35] O.S. An, *Anomaly-corrected supersymmetry algebra and supersymmetric holographic renormalization*, arXiv:1703.09607 [INSPIRE].

[36] R.K. Gupta and S. Murthy, *All solutions of the localization equations for $N = 2$ quantum black hole entropy*, *JHEP* **02** (2013) 141 [arXiv:1208.6221] [INSPIRE].

[37] B. de Wit and V. Reys, *Euclidean Supergravity*, *JHEP* **12** (2017) 011 [arXiv:1706.04973] [INSPIRE].

[38] A. Back, P.G.O. Freund and M. Forger, *New Gravitational Instantons and Universal Spin Structures*, *Phys. Lett.* **B 77** (1978) 181.

[39] S.M. Salamon, *Quaternion-Kähler geometry*, in *Surveys in differential geometry VI: essays on Einstein manifolds*, International Press, Boston U.S.A. (1999), pg. 83.

[40] N.J. Hitchin, *Compact four-dimensional Einstein manifolds*, *J. Diff. Geom.* **9** (1974) 435.

[41] D. Conti and S. Salamon, *Generalized Killing spinors in dimension 5*, *Trans. Am. Math. Soc.* **359** (2007) 5319 [math/0508375].

[42] A. Gray and L.M. Hervella, *The sixteen classes of almost-Hermitian manifolds and their linear invariants*, *Ann. Math. Pura Appl.* **123** (1980) 35.

[43] M.T. Anderson, *Geometric aspects of the AdS/CFT correspondence*, *IRMA Lect. Math. Theor. Phys.* **8** (2005) 1 [hep-th/0403087] [INSPIRE].

[44] E. Witten and S.-T. Yau, *Connectedness of the boundary in the AdS/CFT correspondence*, *Adv. Theor. Math. Phys.* **3** (1999) 1635 [hep-th/9910245] [INSPIRE].

[45] H. Lin, O. Lunin and J.M. Maldacena, *Bubbling AdS space and* 1/2 *BPS geometries*, *JHEP* **10** (2004) 025 [hep-th/0409174] [INSPIRE].

[46] J.M.F. Labastida and C. Lozano, *Mathai-Quillen formulation of twisted* $N = 4$ *supersymmetric gauge theories in four-dimensions*, *Nucl. Phys.* **B 502** (1997) 741 [hep-th/9702106] [INSPIRE].

[47] J.M.F. Labastida and C. Lozano, *Duality in twisted* $N = 4$ *supersymmetric gauge theories in four-dimensions*, *Nucl. Phys.* **B 537** (1999) 203 [hep-th/9806032] [INSPIRE].

[48] J.M.F. Labastida and M. Mariño, *NonAbelian monopoles on four manifolds*, *Nucl. Phys.* **B 448** (1995) 373 [hep-th/9504010] [INSPIRE].

[49] C. Lozano, *Duality in topological quantum field theories*, hep-th/9907123 [INSPIRE].

[50] S. Gukov, *Trisecting non-Lagrangian theories*, *JHEP* **11** (2017) 178 [arXiv:1707.01515] [INSPIRE].

[51] S. Gukov, P. Putrov and C. Vafa, *Fivebranes and 3-manifold homology*, *JHEP* **07** (2017) 071 [arXiv:1602.05302] [INSPIRE].

[52] M. Dedushenko, S. Gukov and P. Putrov, *Vertex algebras and 4-manifold invariants*, arXiv:1705.01645 [INSPIRE].

[53] G.W. Moore and E. Witten, *Integration over the* u *plane in Donaldson theory*, *Adv. Theor. Math. Phys.* **1** (1997) 298 [hep-th/9709193] [INSPIRE].

[54] D. Gaiotto, $N = 2$ *dualities*, *JHEP* **08** (2012) 034 [arXiv:0904.2715] [INSPIRE].

[55] D. Gaiotto and J. Maldacena, *The gravity duals of* $N = 2$ *superconformal field theories*, *JHEP* **10** (2012) 189 [arXiv:0904.4466] [INSPIRE].

[56] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [hep-th/9806087] [INSPIRE].

[57] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, *Prog. Math.* **244** (2006) 525 [hep-th/0306238] [INSPIRE].

[58] J.M.F. Labastida and C. Lozano, *The Vafa-Witten theory for gauge group* SU(N), *Adv. Theor. Math. Phys.* **3** (1999) 1201 [hep-th/9903172] [INSPIRE].

[59] M. Cvetič, H. Lü, C.N. Pope, A. Sadrzadeh and T.A. Tran, *Consistent* SO(6) *reduction of type IIB supergravity on* S^5, *Nucl. Phys.* **B 586** (2000) 275 [hep-th/0003103] [INSPIRE].

[60] B. Geyer and D. Mülsch, $N(T) = 4$ *equivariant extension of the* $3-D$ *topological model of Blau and Thompson*, *Nucl. Phys.* **B 616** (2001) 476 [hep-th/0108042] [INSPIRE].

[61] M. Cvetič, H. Lü and C.N. Pope, *Four-dimensional* $N = 4$, SO(4) *gauged supergravity from* $D = 11$, *Nucl. Phys.* **B 574** (2000) 761 [hep-th/9910252] [INSPIRE].

[62] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, $N = 6$ *superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [arXiv:0806.1218] [INSPIRE].

[63] Work in progress.