

Complex three-form supergravity and membranes

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ABSTRACT: There exist two variants of the old minimal formulation for $\mathcal{N} = 1$ supergravity in four dimensions, in which one or each of the two auxiliary scalars is replaced by the field strength of a gauge three-form. These theories are known as three-form supergravity and complex three-form supergravity, respectively. For each of them, we present a super-Weyl invariant coupling of supergravity to the supermembrane and prove kappa-invariance of the resulting action. In the case of three-form supergravity, we demonstrate that the action constructed reduces to that given by Ovrut and Waldram twenty years ago upon imposing a super-Weyl gauge in which the compensating three-form superfield is set to a constant.

KEYWORDS: p-branes, Supergravity Models, Superspaces

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1 Introduction

The old minimal formulation for $\mathcal{N} = 1$ supergravity in four dimensions, first presented in superspace [1] and soon after developed in the component setting [2, 3], is probably the most famous off-shell supergravity theory.¹ The field content of this theory is known to everyone who studied supersymmetric field theory from the book by Wess and Bagger [4] (part of which is a review and extension of the approach pursued by Wess and Zumino [1]). Its physical fields are the vielbein e_m^a and the Majorana gravitino $(\psi_m^\alpha, \bar{\psi}_{m\dot{\alpha}})$. Its auxiliary fields are the vector b_a , the complex scalar M and its conjugate \bar{M} . The Ferrara-van Nieuwenhuizen formulation [3] made use of the two real auxiliary scalars contained in $M = \text{Re } M + i \text{Im } M$. However, the work by Stelle and West [2] also provided a variant supergravity formulation in which each of the two real auxiliary scalars, $\text{Re } M$ and $\text{Im } M$, was replaced by the field strength of a gauge three-form. Three years later, Gates and Siegel [5] pointed out the existence of one more variant formulation of supergravity in which just one of the two real scalars in $M = \text{Re } M + i \text{Im } M$ was replaced by the field strength of a gauge three-form. The resulting variant formulations of old minimal supergravity are known nowadays as *three-form supergravity* [5] and *complex three-form supergravity* [2].² We will often refer to the off-shell theory presented in [1, 3] as the *standard formulation* or simply old minimal supergravity.

The difference between the standard formulation for supergravity and its variant realisations discussed above can be seen from the corresponding superfield equations of motion.

¹It is quite remarkable that the superspace [1] and the component [2, 3] formulations of old minimal supergravity were published in the same volume of Physics Letters B with an interval of one month.

²There is a plausible explanation as to why Stelle and West did not describe explicitly the three-form supergravity in [2]. The point is that their set of auxiliary fields was inspired by the structure of the gravitational vector superfield and its gauge freedom at the linearised level [6–8].

In terms of the Grimm-Wess-Zumino superspace geometry [9, 10] (see, e.g., [4, 11] for pedagogical reviews), the supergravity equations corresponding to the standard formulation were given in [1]. They are:

$$G_a = 0, \tag{1.1a}$$

$$R = 0. \tag{1.1b}$$

In the case of three-form supergravity, equation (1.1b) is replaced with

$$R + \bar{R} = 0 \implies R - \bar{R} = \text{const}, \tag{1.2}$$

while for complex three-form supergravity it turns into

$$R = \text{const}, \tag{1.3}$$

in accordance with [5]. Equation (1.1a) remains the same for the variant formulations. Even without introducing a supersymmetric cosmological term (which does not exist for complex three-form supergravity, see section 3.3), a negative cosmological constant is generated dynamically in the real and complex three-form supergravity theories for vacuum solutions with $R \neq 0$.³

As is well known, every off-shell formulation for $\mathcal{N} = 1$ supergravity can be realised as $\mathcal{N} = 1$ conformal supergravity coupled to a compensating multiplet (see, e.g., [11, 18, 19] for reviews). Different off-shell formulations correspond to choosing different compensators. This leads to another conceptual way to understand the difference between the standard formulation of old minimal supergravity and its two variants, as was pointed out in [5]. In the standard formulation, the compensator is a *general* chiral scalar superfield [20]. In Minkowski superspace, it obeys the chirality constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$ and can be represented as⁴

$$\Phi = -\frac{1}{4}\bar{D}^2U, \tag{1.4}$$

where the prepotential U is an unconstrained complex superfield. The auxiliary field of Φ , defined by $F(x) := -\frac{1}{4}D^2\Phi(x, \theta)|_{\theta=0}$, is a complex scalar. In the case of three-form supergravity [5], the compensator is a three-form multiplet originally proposed by Gates [21]. It is described by a chiral superfield of the form

$$\Pi = -\frac{1}{4}\bar{D}^2P, \quad \bar{P} = P, \tag{1.5}$$

where P is a real but otherwise unconstrained prepotential. Since the prepotential P is real, Π is no longer a general chiral superfield, for it obeys the condition

$$D^2\Pi - \bar{D}^2\bar{\Pi} = i\partial^a p_a, \quad p_a = (\tilde{\sigma}_a)^{\dot{\alpha}\alpha}[D_{\alpha}, \bar{D}_{\dot{\alpha}}]P, \tag{1.6}$$

which implies that the imaginary part of the auxiliary field F of Π is the field strength of a gauge three-form. In the case of complex three-form supergravity, the compensator is

³The idea that the use of massless gauge three-forms makes it possible to generate a cosmological constant dynamically, has attracted much interest since the early 1980s, see, e.g., [12–17].

⁴As in [4, 11], we make use of the definitions $D^2 = D^{\alpha}D_{\alpha}$ and $\bar{D}^2 = \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}$.

a complex three-form multiplet proposed in [5].⁵ It is described by a chiral superfield of the form

$$\Upsilon = -\frac{1}{4}\bar{D}^2\bar{\Sigma}, \quad D^2\bar{\Sigma} = 0. \quad (1.7)$$

A complex scalar Σ constrained by $\bar{D}^2\Sigma = 0$ is called a complex linear superfield [5].⁶ Since the prepotential $\bar{\Sigma}$ in (1.7) is complex antilinear, Υ is no longer a general chiral superfield, for it obeys the condition

$$D^2\Upsilon = i\partial^a q_a, \quad q_a = (\tilde{\sigma}_a)^{\dot{\alpha}\alpha}[D_\alpha, \bar{D}_{\dot{\alpha}}]\bar{\Sigma}. \quad (1.8)$$

This property tells us that the auxiliary field F of Υ is the field strength of a gauge *complex* three-form.

Unlike the standard formulation of old minimal supergravity, the remarkable feature of three-form supergravity is that it allows a consistent coupling to the four-dimensional supermembrane [23] (the $d = 4$ cousin of the $d = 11$ supermembrane [24, 25]) as demonstrated by Ovrut and Waldram [26]. Since consistency of the supermembrane action requires the presence of a Wess-Zumino term associated with a real gauge three-form in the target superspace [24], it is not surprising that this supergravity formulation plays a special role in this context. It is natural to wonder whether complex three-form supergravity also allows a consistent coupling to the supermembrane. The main goal of this paper is indeed to show that this question has an affirmative answer.

Before we turn to the main body of the paper, a few comments on the literature are in order. The quantum properties of a massless three-form multiplet coupled to supergravity were studied in [27] (see [11] for a review). The superform formulation for the three-form multiplet in supergravity was developed by Binétruy et al. [28] and used in [26] to work out the complete component action for three-form supergravity. The super-Weyl invariant formulation for three-form supergravity was given in [29], as an extension of similar formulations for the non-minimal and new minimal supergravity theories given in section 6.6 of [11]. The formulations described in [11] and [29] were generalised in [30] to construct the super-Weyl invariant formulation for complex three-form supergravity. Various aspects of the dynamics of three-form supergravity coupled to the supermembrane were studied in [31].

This paper is organised as follows. In section 2 we recall the key results concerning the formulation of $\mathcal{N} = 1$ conformal supergravity and its couplings to matter using the geometric framework of [9, 10]. Section 3 elaborates on the super-Weyl invariant formulations for the three versions of old minimal supergravity discussed above. In particular, for both the real and complex three-form multiplets we provide a super-Weyl invariant description of the gauge super 3-forms and gauge-invariant field strengths. Section 4 describes consistent couplings of the real and complex three-form supergravity theories to the supermembrane. Concluding comments are given in section 5.

⁵The name “complex three-form multiplet” was coined in [18].

⁶Such a superfield was first discussed by Zumino [22]. It is primarily used to describe the so-called non-minimal scalar multiplet [5].

2 Conformal supergravity

As reviewed in [11], conformal supergravity can be described using the superspace geometry of [9, 10], which underlies the Wess-Zumino approach to old minimal supergravity [1]. Here we briefly recall the main definitions and conceptual results. The notation and conventions of [11] are used throughout this paper.

Conformal supergravity is formulated in a curved superspace $\mathcal{M}^{4|4}$ parametrised by local bosonic (x^m) and fermionic ($\theta^\mu, \bar{\theta}_{\dot{\mu}}$) coordinates $z^M = (x^m, \theta^\mu, \bar{\theta}_{\dot{\mu}})$, where $m = 0, 1, 2, 3$, $\mu = 1, 2$ and $\dot{\mu} = 1, 2$. The Grassmann variables θ^μ and $\bar{\theta}_{\dot{\mu}}$ are related to each other by complex conjugation: $\overline{\theta^\mu} = \bar{\theta}^{\dot{\mu}}$. We will often make use of a preferred basis of one-forms $E^A = (E^a, E^\alpha, \bar{E}_{\dot{\alpha}})$ with dual basis $E_A = (E_a, E_\alpha, \bar{E}^{\dot{\alpha}})$,

$$E^A = dz^M E_M{}^A, \quad E_A = E_A{}^M \partial_M, \quad (2.1)$$

which will be referred to as the supervielbein and its inverse, respectively. The superspace structure group is $SL(2, \mathbb{C})$. The covariant derivatives have the form

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^{\dot{\alpha}}) = E_A + \Omega_A, \quad (2.2)$$

where Ω_A stands for the Lorentz connection,

$$\Omega_A = \frac{1}{2} \Omega_A{}^{bc} M_{bc} = \Omega_A{}^{\beta\gamma} M_{\beta\gamma} + \Omega_A{}^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}}, \quad (2.3)$$

with $M_{bc} = -M_{cb}$, $M_{\beta\gamma} = \frac{1}{2}(\sigma^{bc})_{\beta\gamma} M_{bc}$ and $\bar{M}_{\dot{\beta}\dot{\gamma}} = -\frac{1}{2}(\bar{\sigma}^{bc})_{\dot{\beta}\dot{\gamma}} M_{bc}$ the Lorentz generators. These act on a covariant vector V_c and two-component spinors Ψ_γ and $\Psi_{\dot{\gamma}}$ as follows:

$$M_{ab} V_c = 2\eta_{c[a} V_{b]}, \quad M_{\alpha\beta} \Psi_\gamma = \varepsilon_{\gamma(\alpha} \Psi_{\beta)}, \quad \bar{M}_{\dot{\alpha}\dot{\beta}} \Psi_{\dot{\gamma}} = \varepsilon_{\dot{\gamma}(\dot{\alpha}} \Psi_{\dot{\beta})}. \quad (2.4)$$

In general, the covariant derivatives enjoy graded commutation relations of the form

$$[\mathcal{D}_A, \mathcal{D}_B] = T_{AB}{}^C \mathcal{D}_C + \frac{1}{2} R_{AB}{}^{cd} M_{cd}, \quad (2.5)$$

where $T_{AB}{}^C$ and $R_{AB}{}^{cd}$ are the torsion and curvature tensors, respectively. To describe supergravity, the covariant derivatives have to obey certain torsion constraints [1, 9, 10] such that their algebra is as follows (the expression for $[\mathcal{D}_a, \mathcal{D}_b]$ is given in [11]):

$$\{\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}\} = -2i\mathcal{D}_{\alpha\dot{\alpha}}, \quad (2.6a)$$

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -4\bar{R}M_{\alpha\beta}, \quad \{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}}\} = 4R\bar{M}_{\dot{\alpha}\dot{\beta}}, \quad (2.6b)$$

$$[\mathcal{D}_\alpha, \mathcal{D}_{\beta\dot{\beta}}] = i\varepsilon_{\alpha\beta} \left(\bar{R} \bar{\mathcal{D}}_{\dot{\beta}} + G^\gamma{}_{\dot{\beta}} \mathcal{D}_\gamma - \mathcal{D}^\gamma G_{\dot{\beta}}{}^\delta M_{\gamma\delta} + 2\bar{W}_{\dot{\beta}}{}^{\dot{\gamma}\delta} \bar{M}_{\dot{\gamma}\delta} \right) + i\bar{\mathcal{D}}_{\dot{\beta}} \bar{R} M_{\alpha\beta}, \quad (2.6c)$$

$$[\bar{\mathcal{D}}_{\dot{\alpha}}, \mathcal{D}_{\beta\dot{\beta}}] = -i\varepsilon_{\dot{\alpha}\dot{\beta}} \left(R \mathcal{D}_\beta + G_{\beta}{}^{\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\gamma}} - \bar{\mathcal{D}}^{\dot{\gamma}} G_{\beta}{}^{\delta} \bar{M}_{\dot{\gamma}\delta} + 2W_{\beta}{}^{\gamma\delta} M_{\gamma\delta} \right) - i\mathcal{D}_\beta R \bar{M}_{\dot{\alpha}\dot{\beta}}. \quad (2.6d)$$

The torsion tensors R , $G_a = \bar{G}_a$ and $W_{\alpha\beta\gamma} = W_{(\alpha\beta\gamma)}$ satisfy the Bianchi identities:

$$\bar{\mathcal{D}}_{\dot{\alpha}} R = 0, \quad \bar{\mathcal{D}}_{\dot{\alpha}} W_{\alpha\beta\gamma} = 0, \quad (2.7a)$$

$$\bar{\mathcal{D}}^{\dot{\gamma}} G_{\alpha\dot{\gamma}} = \mathcal{D}_\alpha R, \quad \mathcal{D}^\gamma W_{\alpha\beta\gamma} = i\mathcal{D}_{(\alpha}{}^{\dot{\gamma}} G_{\beta)\dot{\gamma}}. \quad (2.7b)$$

The definition of the torsion and curvature tensors given by eq. (2.5) can be recast in the language of superforms. Starting from the Lorentz connection Ω_A defined by (2.3), we introduce the connection one-form

$$\Omega = E^C \Omega_C, \quad \Omega V_A = \Omega_A{}^B V_B = E^C \Omega_{CA}{}^B V_B, \quad V_A = (V_a, \Psi_\alpha, \Psi^{\dot{\alpha}}). \quad (2.8)$$

Then the torsion and curvature two-forms are

$$T^C := \frac{1}{2} E^B \wedge E^A T_{AB}{}^C = -dE^C + E^B \wedge \Omega_B{}^C, \quad (2.9a)$$

$$R_C{}^D := \frac{1}{2} E^B \wedge E^A R_{ABC}{}^D = d\Omega_C{}^D - \Omega_C{}^E \wedge \Omega_E{}^D. \quad (2.9b)$$

The gauge group of conformal supergravity includes superspace general coordinate transformations and local Lorentz ones. Such a transformation acts on the covariant derivatives and any tensor superfield U (with its indices suppressed) by the rule

$$\delta_{\mathcal{K}} \mathcal{D}_A = [\mathcal{K}, \mathcal{D}_A], \quad \delta_{\mathcal{K}} U = \mathcal{K} U, \quad (2.10a)$$

where the gauge parameter \mathcal{K} has form

$$\mathcal{K} = \xi^B \mathcal{D}_B + \frac{1}{2} K^{bc} M_{bc} = \xi^B \mathcal{D}_B + K^{\beta\gamma} M_{\beta\gamma} + \bar{K}^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}} = \bar{\mathcal{K}} \quad (2.10b)$$

and describes a coordinate transformation generated by the supervector field $\xi = \xi^B E_B$ and a local Lorentz transformation generated by K^{bc} .

It was first realised by Howe and Tucker [32] that the algebra (2.6) is invariant under super-Weyl transformations of the form

$$\delta_\sigma \mathcal{D}_\alpha = \left(\bar{\sigma} - \frac{1}{2} \sigma \right) \mathcal{D}_\alpha + \mathcal{D}^\beta \sigma M_{\alpha\beta}, \quad (2.11a)$$

$$\delta_\sigma \mathcal{D}_{\alpha\dot{\alpha}} = \frac{1}{2} (\sigma + \bar{\sigma}) \mathcal{D}_{\alpha\dot{\alpha}} + \frac{i}{2} \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\sigma} \mathcal{D}_\alpha + \frac{i}{2} \mathcal{D}_\alpha \sigma \bar{\mathcal{D}}_{\dot{\alpha}} + \mathcal{D}^\beta{}_{\dot{\alpha}} \sigma M_{\alpha\beta} + \mathcal{D}_\alpha{}^{\dot{\beta}} \bar{\sigma} \bar{M}_{\dot{\alpha}\dot{\beta}}, \quad (2.11b)$$

accompanied by the following transformations of the torsion superfields

$$\delta_\sigma R = 2\sigma R + \frac{1}{4} (\bar{\mathcal{D}}^2 - 4R) \bar{\sigma}, \quad (2.12a)$$

$$\delta_\sigma G_{\alpha\dot{\alpha}} = \frac{1}{2} (\sigma + \bar{\sigma}) G_{\alpha\dot{\alpha}} + i \mathcal{D}_{\alpha\dot{\alpha}} (\sigma - \bar{\sigma}), \quad (2.12b)$$

$$\delta_\sigma W_{\alpha\beta\gamma} = \frac{3}{2} \sigma W_{\alpha\beta\gamma}. \quad (2.12c)$$

Here the super-Weyl parameter σ is a covariantly chiral scalar superfield, $\bar{\mathcal{D}}_{\dot{\alpha}} \sigma = 0$.

The gauge group of conformal supergravity is defined to be generated by the local transformations (2.10) and (2.11). It may be shown that this gauge freedom indeed leads to the multiplet of $\mathcal{N} = 1$ conformal supergravity at the component level (see, e.g., [11] for a review).

Of special importance in conformal supergravity are super-Weyl primary multiplets (here we follow the terminology recently used in [33]). A tensor superfield T (with its

indices suppressed) is said to be (super-Weyl) primary of weight (p, q) if its super-Weyl transformation law is

$$\delta_\sigma T = (p\sigma + q\bar{\sigma})T, \tag{2.13}$$

for some parameters p and q . The conformal dimension of T is given by $(p + q)$.

An important class of tensor superfields are covariantly chiral superfields Φ constrained by $\bar{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$. Due to the integrability condition $\{\bar{\mathcal{D}}_{\dot{\alpha}}, \bar{\mathcal{D}}_{\dot{\beta}}\}\Phi = 4R\bar{M}_{\dot{\alpha}\dot{\beta}}\Phi = 0$, such superfields may carry only undotted indices [34]. If Φ is covariantly chiral and super-Weyl primary, its weight is necessarily $(p, 0)$. We will call Φ a chiral primary superfield of weight p .

Covariantly chiral tensor superfields may be constructed using the chiral projection operator [1, 34]

$$\bar{\Delta} := -\frac{1}{4}(\bar{\mathcal{D}}^2 - 4R). \tag{2.14}$$

Given a tensor superfield T with undotted spinor indices only, $\bar{\Delta}T$ is covariantly chiral, $\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\Delta}T = 0$. If T is a super-Weyl primary superfield of weight $(p - 2, 1)$, then $\bar{\Delta}T$ is a chiral primary superfield of weight p . This may be checked by using the super-Weyl transformation of the chiral projection operator

$$\delta_\sigma \bar{\Delta} = (2\sigma - \bar{\sigma})\bar{\Delta} + \frac{1}{2}(\bar{\mathcal{D}}_{\dot{\alpha}}\bar{\sigma})\bar{\mathcal{D}}^{\dot{\alpha}} + \frac{1}{4}(\bar{\mathcal{D}}^2\bar{\sigma}) - \frac{1}{2}(\bar{\mathcal{D}}^{\dot{\alpha}}\bar{\sigma})\bar{\mathcal{D}}^{\dot{\beta}}\bar{M}_{\dot{\alpha}\dot{\beta}}, \tag{2.15}$$

which follows from (2.11).

Given a matter dynamical system coupled to conformal supergravity, its action functional must be invariant under the local transformations (2.10) and (2.11). There are two general action principles. Given a primary real scalar Lagrangian $\mathcal{L} = \bar{\mathcal{L}}$ of weight $(1, 1)$, the action

$$S = \int d^4x d^2\theta d^2\bar{\theta} E \mathcal{L}, \quad E = \text{Ber}(E_M^A), \tag{2.16}$$

is invariant under the supergravity gauge group. Its super-Weyl invariance follows from the transformation law $\delta_\sigma E = -(\sigma + \bar{\sigma})E$. Given a scalar chiral primary Lagrangian \mathcal{L}_c of weight $+3$, the chiral action

$$S_c = \int d^4x d^2\theta \mathcal{E} \mathcal{L}_c \tag{2.17}$$

is invariant under the supergravity gauge group. Its super-Weyl invariance follows from the transformation law $\delta_\sigma \mathcal{E} = -3\mathcal{E}$ of the chiral density \mathcal{E} . The latter may be defined in terms of a chiral prepotential [20]. Alternatively, the chiral density can be read off using the general formalism of integrating out fermionic dimensions, which was developed in [35].⁷ The full superspace action (2.16) can be represented as an integral over the chiral subspace,

$$\int d^4x d^2\theta d^2\bar{\theta} E \mathcal{L} = \int d^4x d^2\theta \mathcal{E} \bar{\Delta}\mathcal{L}. \tag{2.18}$$

⁷This formalism naturally leads to the appearance of the Θ variables postulated in [4].

The chiral action (2.17) can be represented as an integral over the full superspace,

$$S_c = \int d^4x d^2\theta d^2\bar{\theta} E \mathfrak{C} \mathcal{L}_c, \quad (2.19)$$

where \mathfrak{C} is an *improved complex linear* superfield⁸ that is defined by the following properties:

(i) \mathfrak{C} obeys the constraint

$$\bar{\Delta} \mathfrak{C} = 1; \quad (2.20a)$$

(ii) \mathfrak{C} is super-Weyl primary of weight $(-2, 1)$,

$$\delta_\sigma \mathfrak{C} = (\bar{\sigma} - 2\sigma) \mathfrak{C}. \quad (2.20b)$$

A possible choice for \mathfrak{C} is

$$\mathfrak{C} = \frac{\bar{\eta}}{\Delta \bar{\eta}}, \quad \bar{\mathcal{D}}_{\dot{\alpha}} \eta = 0, \quad \delta_\sigma \eta = \sigma \eta, \quad (2.21)$$

for some covariantly chiral superfield η such that $\Delta \eta$ is nowhere vanishing. In case \mathfrak{C} is not required to be super-Weyl primary, it can be identified with R^{-1} ,

$$S_c = \int d^4x d^2\theta d^2\bar{\theta} E \frac{\mathcal{L}_c}{R}, \quad (2.22)$$

provided R is nowhere vanishing. This representation was discovered in [20, 34].

To conclude this section, we point out that there is an alternative way to define the chiral action (2.17) that follows from the superform approach to the construction of supersymmetric invariants [37–41]. It is based on the use of the following super 4-form

$$\begin{aligned} \Xi_4[\mathcal{L}_c] &= 2i \bar{E}_{\dot{\delta}} \wedge \bar{E}_{\dot{\gamma}} \wedge E^b \wedge E^a (\tilde{\sigma}_{ab})^{\dot{\gamma}\dot{\delta}} \mathcal{L}_c + \frac{i}{6} \varepsilon_{abcd} \bar{E}_{\dot{\delta}} \wedge E^c \wedge E^b \wedge E^a (\tilde{\sigma}^d)^{\dot{\delta}\delta} \mathcal{D}_\alpha \mathcal{L}_c \\ &\quad - \frac{1}{96} \varepsilon_{abcd} E^d \wedge E^c \wedge E^b \wedge E^a (\mathcal{D}^2 - 12\bar{R}) \mathcal{L}_c, \end{aligned} \quad (2.23)$$

which was constructed by Binétruy et al. [28] and independently by Gates et al. [41]. This superform⁹ is closed,

$$d \Xi_4[\mathcal{L}_c] = 0. \quad (2.24)$$

The chiral action (2.17) can be recast as an integral of $\Xi_4[\mathcal{L}_c]$ over a spacetime \mathcal{M}^4 ,

$$S_c = \int_{\mathcal{M}^4} \Xi_4[\mathcal{L}_c], \quad (2.25)$$

where \mathcal{M}^4 is the bosonic body of the curved superspace $\mathcal{M}^{4|4}$ obtained by switching off the Grassmann variables. It turns out that the representation (2.25) provides the simplest way to reduce the action from superfields to components.

⁸Such a superfield is the conformal compensator for the non-minimal $\mathcal{N} = 1$ AdS supergravity [36].

⁹In the flat-superspace limit, the superform (2.23) reduces to the one given in [21].

Using the super-Weyl transformation law of the supervielbein

$$\delta_\sigma \bar{E}_{\dot{\alpha}} = \left(\frac{1}{2} \bar{\sigma} - \sigma \right) \bar{E}_{\dot{\alpha}} + \frac{i}{4} (\mathcal{D}^\alpha \sigma) (\sigma_b)_{\alpha\dot{\alpha}} E^b, \quad \delta_\sigma E^a = -\frac{1}{2} (\sigma + \bar{\sigma}) E^a, \quad (2.26)$$

which follows from (2.11), one can check that $\Xi_4[\mathcal{L}_c]$ is super-Weyl invariant,

$$\delta_\sigma \Xi_4[\mathcal{L}_c] = 0. \quad (2.27)$$

This property also follows from the description of this superform given in appendix B of [42] where $\Xi_4[\mathcal{L}_c]$ was formulated in $\mathcal{N} = 1$ conformal superspace [43]. The super-Weyl invariance of $\Xi_4[\mathcal{L}_c]$ will be important for our subsequent analysis.

3 Variant formulations of old minimal supergravity

As described in section 6.6 of [11], any off-shell formulation of $\mathcal{N} = 1$ supergravity may be realised as a super-Weyl invariant coupling of conformal supergravity to a conformal compensator, with conformal supergravity being described as in section 2 above. Here we review the relevant realisations for the three versions of old minimal supergravity discussed in section 1.

3.1 Old minimal supergravity

In the case of the standard formulation, the conformal compensator is a primary chiral scalar superfield Φ of weight +1,

$$\bar{\mathcal{D}}_{\dot{\alpha}} \Phi = 0, \quad \delta_\sigma \Phi = \sigma \Phi, \quad (3.1)$$

which is required to be nowhere vanishing such that Φ^{-1} exists. The locally supersymmetric and super-Weyl invariant action for supergravity is

$$S_{\text{SG,om}} = -\frac{3}{\kappa^2} \int d^4x d^2\theta d^2\bar{\theta} E \bar{\Phi} \Phi + \left\{ \frac{\mu}{\kappa^2} \int d^4x d^2\theta \mathcal{E} \Phi^3 + \text{c.c.} \right\}, \quad (3.2)$$

where κ is the gravitational coupling constant, and μ is a complex parameter related to the cosmological constant. The second term in the action is the supersymmetric cosmological term which was proposed in [44–46] and then recast in the superspace setting in [20].

The super-Weyl gauge freedom allows us to choose the condition $\Phi = 1$. Then (3.2) turns into the supergravity action proposed in [1] for $\mu = 0$ and then generalised to the $\mu \neq 0$ case in [20].

The equation of motion for the chiral compensator is easy to read off from (3.2)

$$\mathbb{R} = \mu, \quad \mathbb{R} := \Phi^{-2} \bar{\Delta} \bar{\Phi}. \quad (3.3)$$

It can be shown [11] that the equation of motion for the gravitational superfield can be written in the form

$$\mathbb{G}_{\alpha\dot{\alpha}} = 0, \quad \mathbb{G}_{\alpha\dot{\alpha}} := \left([\mathcal{D}_\alpha, \bar{\mathcal{D}}_{\dot{\alpha}}] + G_{\alpha\dot{\alpha}} \right) (\Phi \bar{\Phi})^{-1/2}. \quad (3.4)$$

The superfields \mathbb{R} and $\mathbb{G}_{\alpha\dot{\alpha}}$ are super-Weyl invariant. Equation (3.4) is equivalent to eq. (6.6.10) in [11]. The latter states that the supercurrent of the chiral superfield Φ , whose dynamics is described by the action (3.2), vanishes on the mass shell.

In the super-Weyl gauge $\Phi = 1$, the primary superfields \mathbb{R} and $\mathbb{G}_{\alpha\dot{\alpha}}$ turn into the torsion superfields R and $G_{\alpha\dot{\alpha}}$, respectively.

3.2 Three-form supergravity

We now discuss three-form supergravity in some detail. First of all we review the super-Weyl invariant formulation of this theory given in [29]. The corresponding conformal compensator is a three-form multiplet coupled to conformal supergravity. It is described by a covariantly chiral scalar Π and its conjugate $\bar{\Pi}$, with Π defined by

$$\Pi = \bar{\Delta}P \quad \bar{P} = P, \tag{3.5}$$

where the scalar prepotential P is real but otherwise unconstrained. The compensator Π has to be nowhere vanishing so that Π^{-1} exists. We postulate P to be super-Weyl primary of weight (1, 1),

$$\delta_{\sigma}P = (\sigma + \bar{\sigma})P, \tag{3.6a}$$

which implies that Π is also primary,

$$\delta_{\sigma}\Pi = 3\sigma\Pi. \tag{3.6b}$$

As is seen from (3.5), the prepotential P is defined modulo gauge transformations of the form

$$\delta_L P = L, \quad \bar{\Delta}L = 0, \quad \bar{L} = L, \tag{3.7}$$

with the gauge parameter L being a linear multiplet.¹⁰

The action for three-form supergravity is obtained from (3.2) by replacing Φ with $\Pi^{1/3}$. This leads to

$$\begin{aligned} S_{\text{SG,t-f}} &= -\frac{3}{\kappa^2} \int d^4x d^2\theta d^2\bar{\theta} E \left\{ (\bar{\Pi}\Pi)^{\frac{1}{3}} - \frac{1}{2}mP \right\} \\ &= -\frac{3}{\kappa^2} \int d^4x d^2\theta d^2\bar{\theta} E (\bar{\Pi}\Pi)^{\frac{1}{3}} + \left\{ \frac{m}{\kappa^2} \int d^4x d^2\theta \mathcal{E} \Pi + \text{c.c.} \right\}, \end{aligned} \tag{3.8}$$

where m is a real parameter. By construction the action is invariant under gauge transformations (3.7).

Making use of (3.5), the equation of motion for the compensator can be written as

$$\mathbb{R} + \bar{\mathbb{R}} = 2m, \quad \mathbb{R} := \Pi^{-2/3} \bar{\Delta} \bar{\Pi}^{1/3}, \tag{3.9}$$

¹⁰In the case of $\mathcal{N} = 1$ Poincaré supersymmetry, the linear multiplet was first introduced in [47]. It is used to describe the $\mathcal{N} = 1$ tensor multiplet [48].

compare with (3.3). The chirality of \mathbb{R} then implies

$$\mathbb{R} = \mu = \text{const}, \quad \text{Re } \mu = m. \quad (3.10)$$

Unlike (3.2), the action (3.8) for three-form supergravity contains only one real parameter, m , which determines the corresponding supersymmetric cosmological term. However, on the mass shell \mathbb{R} becomes a complex parameter, μ , as in (3.3). The real part of \mathbb{R} is fixed by the equation (3.9), while its imaginary part is generated dynamically.

The three-form multiplet has a geometric realisation in terms of a gauge super 3-form [28] that extends the flat-superspace construction of [21]. Following [28], we consider the real super 3-form

$$\begin{aligned} R_3[P] &= -i\bar{E}_{\dot{\gamma}} \wedge E^{\beta} \wedge E^a (\sigma_a)_{\beta}{}^{\dot{\gamma}} P \\ &\quad - \frac{1}{2} E^{\gamma} \wedge E^b \wedge E^a (\sigma_{ab})_{\gamma\delta} \mathcal{D}^{\delta} P - \frac{1}{2} \bar{E}_{\dot{\gamma}} \wedge E^b \wedge E^a (\tilde{\sigma}_{ab})^{\dot{\gamma}\delta} \bar{\mathcal{D}}_{\delta} P \\ &\quad - \frac{1}{48} \varepsilon_{abcd} E^c \wedge E^b \wedge E^a ((\tilde{\sigma}^d)^{\dot{\gamma}\gamma} [\mathcal{D}_{\gamma}, \bar{\mathcal{D}}_{\dot{\gamma}}] + 12G^d) P, \end{aligned} \quad (3.11)$$

which is constructed in terms of the prepotential P . Its exterior derivative, $R_4 := dR_3$, proves to involve P only via the gauge-invariant field strength $\Pi = \bar{\Delta}P$. For the super 4-form $R_4 \equiv R_4[\Pi]$ we obtain

$$\begin{aligned} R_4[\Pi] &= -\bar{E}_{\dot{\delta}} \wedge \bar{E}_{\dot{\gamma}} \wedge E^b \wedge E^a (\tilde{\sigma}_{ab})^{\dot{\gamma}\delta} \Pi - E^{\delta} \wedge E^{\gamma} \wedge E^b \wedge E^a (\sigma_{ab})_{\gamma\delta} \bar{\Pi} \\ &\quad - \frac{1}{12} \bar{E}_{\dot{\delta}} \wedge E^c \wedge E^b \wedge E^a \varepsilon_{abcd} (\tilde{\sigma}^d)^{\dot{\delta}\alpha} \mathcal{D}_{\alpha} \Pi + \frac{1}{12} E^{\delta} \wedge E^c \wedge E^b \wedge E^a \varepsilon_{abcd} (\sigma^d)_{\delta\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\Pi} \\ &\quad - \frac{1}{192} E^d \wedge E^c \wedge E^b \wedge E^a \varepsilon_{abcd} \left\{ i(\mathcal{D}^2 - 12\bar{R})\Pi - i(\bar{\mathcal{D}}^2 - 12R)\bar{\Pi} \right\}. \end{aligned} \quad (3.12)$$

Note that the real super 4-form R_4 is related to the imaginary part of the complex super 4-form Ξ_4 in eq. (2.23) with the chiral Lagrangian \mathcal{L}_c replaced with Π , that is

$$R_4[\Pi] = \frac{i}{2} \left(\Xi_4[\Pi] - \bar{\Xi}_4[\bar{\Pi}] \right). \quad (3.13)$$

The field strength $R_4[\Pi]$ is invariant under gauge transformations of the potential $R_3[P]$ of the form

$$\delta_L R_3[P] = R_3[L], \quad (3.14)$$

where $R_3[L]$ is obtained from (3.11) by replacing P with a real linear superfield constrained as in (3.7). The super 3-form $R_3[L]$ coincides, modulo an overall numerical factor, with the field strength of the linear multiplet, see, e.g., [42, 49].

The important property of $R_3[P]$, which was not noticed in [28], is that this superform is super-Weyl invariant,

$$\delta_{\sigma} R_3[P] = 0 \quad \implies \quad \delta_{\sigma} R_4[\Pi] = 0. \quad (3.15)$$

The above superform realisation of the three-form multiplet may be given a more geometric setting, in the spirit of [21, 28, 49]. This multiplet can be described by a gauge super 3-form $B_3 = \frac{1}{6}E^C \wedge E^B \wedge E^A B_{ABC}$ defined modulo gauge transformations

$$\delta_\Lambda B_3 = d\Lambda_2, \quad \Lambda_2 = \frac{1}{2}E^B \wedge E^A \Lambda_{AB}, \quad (3.16)$$

with the gauge parameter Λ_2 being an arbitrary super 2-form. In order to obtain an irreducible supermultiplet, the gauge invariant field strength $H_4 = dB_3$ must be subject to certain constraints such that their general solution is given by $H_4 = R_4[\Pi]$, eq. (3.12). Then the gauge freedom (3.16) may be used to choose B_3 in the form $B_3 = R_3[P]$, eq. (3.11). In this gauge, the residual gauge invariance is described by (3.14).

3.3 Complex three-form supergravity

The super-Weyl invariant formulation for complex three-form supergravity was given in [30]. The conformal compensator for this theory is a complex three-form multiplet coupled to conformal supergravity. This multiplet is described in terms of a covariantly chiral scalar Υ and its conjugate $\bar{\Upsilon}$ defined as follows:

$$\Upsilon = \bar{\Delta}\bar{\Sigma}, \quad \bar{\Upsilon} = \Delta\Sigma. \quad (3.17)$$

Here Σ is a covariantly *complex linear* scalar superfield constrained by

$$\bar{\Delta}\Sigma = 0. \quad (3.18)$$

In general, if Σ is chosen to be super-Weyl primary, then its weight has to be $(p-2, 1)$, for some p ,

$$\delta_\sigma \Sigma = [(p-2)\sigma + \bar{\sigma}]\Sigma, \quad (3.19)$$

as a consequence of the condition that the constraint (3.18) be super-Weyl invariant [11]. Requiring the chiral scalar $\Upsilon = \bar{\Delta}\bar{\Sigma}$ to be super-Weyl primary as well, we have to choose $p = 3$, which means

$$\delta_\sigma \Sigma = (\sigma + \bar{\sigma})\Sigma \implies \delta_\sigma \Upsilon = 3\sigma\Upsilon. \quad (3.20)$$

In order for Υ to be used as a conformal compensator, Υ^{-1} must exist.

The general solution to the constraint (3.18) is known [11, 18] to be

$$\Sigma = \bar{\mathcal{D}}_{\dot{\alpha}} \bar{\Psi}^{\dot{\alpha}}, \quad (3.21)$$

where $\bar{\Psi}^{\dot{\alpha}}$ is an unconstrained spinor superfield defined modulo gauge transformations

$$\delta_\Lambda \bar{\Psi}^{\dot{\alpha}} = \bar{\mathcal{D}}_{\dot{\beta}} \bar{\Lambda}^{(\dot{\alpha}\dot{\beta})}, \quad (3.22)$$

which leave Σ invariant. The super-Weyl transformation of the prepotential can be chosen to be

$$\delta_\sigma \bar{\Psi}^{\dot{\alpha}} = \frac{3}{2}\bar{\sigma}\bar{\Psi}^{\dot{\alpha}}, \quad (3.23)$$

and this transformation law implies (3.20).¹¹

¹¹More generally, if the super-Weyl transformation of Σ is given by (3.19), then the prepotential $\bar{\Psi}^{\dot{\alpha}}$ defined by (3.21) transforms as follows: $\delta_\sigma \bar{\Psi}^{\dot{\alpha}} = [(p-3)\sigma + \frac{3}{2}\bar{\sigma}]\bar{\Psi}^{\dot{\alpha}}$, as shown in [11].

The superfields Υ and $\bar{\Upsilon}$ defined by (3.17) are invariant under gauge transformations of the form

$$\delta_L \Sigma = L_1 - iL_2, \quad \bar{\Delta} L_i = 0, \quad \bar{L}_i = L_i. \quad (3.24)$$

This may be recast as a gauge transformation of the prepotential $\bar{\Psi}^{\dot{\alpha}}$ defined by (3.21),

$$\delta_L \bar{\Psi}^{\dot{\alpha}} = \bar{\eta}^{\dot{\alpha}}, \quad \mathcal{D}_\alpha \bar{\eta}^{\dot{\alpha}} = 0. \quad (3.25)$$

The action for complex three-form supergravity is obtained from (3.2) by replacing Φ with $\Upsilon^{1/3}$, which leads to

$$S_{\text{SG,ct-f}} = -\frac{3}{\kappa^2} \int d^4x d^2\theta d^2\bar{\theta} E (\bar{\Upsilon}\Upsilon)^{1/3}. \quad (3.26)$$

No contribution comes from the cosmological term in (3.2) since the replacement $\Phi^3 \rightarrow \Upsilon = \bar{\Delta}\bar{\Sigma}$ and the integration rule (2.18) give a total derivative. In other words, complex three-form supergravity possesses no supersymmetric cosmological term. This is similar to the new minimal formulation for $\mathcal{N} = 1$ supergravity [50–52]. However, unlike new minimal supergravity, a negative cosmological constant is generated dynamically in the case of complex three-form supergravity. Indeed, the equation of motion for the prepotential Ψ_α , which originates in $\bar{\Sigma} = \mathcal{D}^\alpha \Psi_\alpha$, is

$$\mathcal{D}_\alpha \mathbb{R} = 0, \quad \mathbb{R} := \Upsilon^{-2/3} \bar{\Delta} \bar{\Upsilon}^{1/3}. \quad (3.27)$$

Its general solution is $\mathbb{R} = \mu = \text{const}$, where μ is an arbitrary complex constant.

The complex three-form multiplet has a geometric superform realisation that extends the flat-superspace formulation of [18]. Let us consider the following complex super 3-form

$$\begin{aligned} C_3[\bar{\Sigma}] &= -2\bar{E}_{\dot{\gamma}} \wedge E^\beta \wedge E^a (\sigma_a)_\beta^{\dot{\gamma}} \bar{\Sigma} \\ &+ iE^\gamma \wedge E^b \wedge E^a (\sigma_{ab})_{\gamma\delta} \mathcal{D}^\delta \bar{\Sigma} + i\bar{E}_{\dot{\gamma}} \wedge E^b \wedge E^a (\bar{\sigma}_{ab})^{\dot{\gamma}\delta} \bar{\mathcal{D}}_{\dot{\delta}} \bar{\Sigma} \\ &+ \frac{i}{24} \varepsilon_{abcd} E^c \wedge E^b \wedge E^a ((\bar{\sigma}^d)^{\dot{\gamma}\gamma} [\mathcal{D}_\gamma, \bar{\mathcal{D}}_{\dot{\gamma}}] + 12G^d) \bar{\Sigma}. \end{aligned} \quad (3.28)$$

Its exterior derivative, $dC_3[\bar{\Sigma}]$, proves to be constructed entirely in terms of the field strength $\Upsilon = \bar{\Delta}\bar{\Sigma}$. More precisely, it holds that

$$dC_3[\bar{\Sigma}] = \Xi_4[\Upsilon], \quad (3.29)$$

where $\Xi_4[\Upsilon]$ is the complex super 4-form in eq. (2.23) with \mathcal{L}_c replaced by Υ . Similar to the super 3-form (3.11), $C_3[\bar{\Sigma}]$ is super-Weyl invariant,

$$\delta_\sigma C_3[\bar{\Sigma}] = 0 \quad \implies \quad \delta_\sigma \Xi_4[\Upsilon] = 0. \quad (3.30)$$

The field strength $\Xi_4[\Upsilon]$ is invariant under gauge transformations of the potential $C_3[\bar{\Sigma}]$ of the form

$$\delta_L C_3[\bar{\Sigma}] = C_3[L_1 + iL_2], \quad (3.31)$$

where $C_3[L_1 + iL_2]$ is obtained from (3.28) by replacing $\bar{\Sigma} \rightarrow L_1 + iL_2$, with the gauge parameters L_i constrained as in (3.24).

4 Supermembrane coupled to supergravity

We are now in a position to formulate consistent dynamics of a supermembrane propagating in a three-form supergravity background. Our construction will be valid for both the real and complex three-form supergravity theories. We will draw heavily on the results of [24, 26].

The action for a supermembrane propagating in a three-form supergravity background is proposed to be

$$S = T_3 \int d^3\xi \left\{ \frac{1}{2} \sqrt{-\gamma} \gamma^{ij} \Phi \bar{\Phi} E_i^a E_j^b \eta_{ab} + \frac{1}{6} \epsilon^{ijk} E_i^C E_j^B E_k^A B_{ABC} - \frac{1}{2} \sqrt{-\gamma} \right\}. \quad (4.1)$$

Here ξ^i , with $i = 1, 2, 3$, are the coordinates of the world volume with metric γ_{ij} , $\gamma = \det(\gamma_{ij}) = \frac{1}{6} \epsilon^{ijk} \epsilon^{i'j'k'} \gamma_{ii'} \gamma_{jj'} \gamma_{kk'}$, and the Levi-Civita symbol ϵ^{ijk} is normalised as $\epsilon^{123} = 1$. As usual, γ^{ij} denotes the inverse metric such that $\gamma^{ik} \gamma_{kj} = \delta_j^i$. In (4.1) we have used the notation

$$E_i^A = \partial_i z^M(\xi) E_M^A \quad (4.2)$$

for the pull-back supervielbein.

Our action (4.1) involves a composite dilaton $\Phi \bar{\Phi}$, where Φ is a chiral primary superfield of weight +1 such that Φ^{-1} exists. The superfield Φ is assumed to be the compensator of one of the two three-form supergravity theories. In the case of three-form supergravity, we choose Φ to be $\Pi^{1/3}$. The presence of Φ in (4.1) distinguishes our action from that considered in [26]. In the case of complex three-form supergravity, $\Phi = \Upsilon^{1/3}$. The inclusion of the dilaton is necessary since we are working with the super-Weyl invariant formulation for supergravity. The super-Weyl freedom may be fixed by choosing the condition $\Phi = 1$.

The Wess-Zumino term in (4.1) involves the components of a gauge super 3-form $B_3 = \frac{1}{6} E^C \wedge E^B \wedge E^A B_{ABC}$. This superform is chosen as follows: (i) for three-form supergravity, $B_3 = R_3[P]$, with $R_3[P]$ defined by eq. (3.11); and (ii) in the case of complex three-form supergravity, $B_3 = \frac{i}{2} (C_3[\bar{\Sigma}] - \bar{C}_3[\Sigma])$, with $C_3[\bar{\Sigma}]$ given by (3.28). The latter super 3-form, $B_3[\Sigma, \bar{\Sigma}]$, turns out to coincide with the superform $R_3[\Sigma + \bar{\Sigma}]$, which is obtained from (3.11) by replacing P with $(\Sigma + \bar{\Sigma})$. In both cases, the gauge-invariant field strength $H_4 = dB_3$ is such that

$$H_4 = \frac{i}{2} (\Xi_4[\Phi^3] - \bar{\Xi}_4[\bar{\Phi}^3]), \quad (4.3)$$

where Ξ_4 is the superform in eq. (2.23) with \mathcal{L}_c replaced either with $\Phi^3 = \Pi$ or $\Phi^3 = \Upsilon$.

Consistent supermembrane actions must possess a local fermionic κ -symmetry [24].¹² This gauge symmetry ensures that half of the fermionic degrees of freedom can be gauged away and that spacetime and world-volume supersymmetry can be linked to each other. Let us now show that the action (4.1) is consistently κ -symmetric in arbitrary three-form supergravity backgrounds.

¹²The κ -symmetry was first discovered in the cases of massive [53, 54] and massless [55] superparticles. See [56, 57] and references therein for reviews of various aspects of the κ -symmetry.

Defining $\delta_\kappa E^A := \delta_\kappa z^M E_M^A$, we consider the fermionic gauge transformation

$$\delta_\kappa E^a = 0, \quad \delta_\kappa E^\alpha = \Phi^{1/2} \bar{\Phi}^{-1} \left(\kappa^\beta + \bar{\kappa}_{\dot{\alpha}} \bar{\Gamma}^{\dot{\alpha}\alpha} \right), \quad \bar{\kappa}^\alpha \equiv \bar{\kappa}^{\dot{\alpha}}, \quad (4.4a)$$

where the gauge parameter $\kappa^\alpha(\xi)$ is a two-component undotted $\text{SL}(2, \mathbb{C})$ spinor, and a world-volume scalar. The variation $\delta_\kappa \bar{E}_{\dot{\alpha}}$ is the complex conjugate of $\delta_\kappa E^\alpha$, while $\Gamma_{\alpha\dot{\alpha}}$ and $\bar{\Gamma}^{\dot{\alpha}\alpha} = -\varepsilon^{\alpha\beta} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\Gamma}_{\beta\dot{\beta}}$ are given by

$$\Gamma_{\alpha\dot{\alpha}} = -\frac{1}{6\sqrt{-\gamma}} (\Phi \bar{\Phi})^{3/2} \epsilon^{ijk} E_i^a E_j^b E_k^c \varepsilon_{abcd} (\sigma^d)_{\alpha\dot{\alpha}}, \quad (4.4b)$$

$$\bar{\Gamma}^{\dot{\alpha}\alpha} = \frac{1}{6\sqrt{-\gamma}} (\Phi \bar{\Phi})^{3/2} \epsilon^{ijk} E_i^a E_j^b E_k^c \varepsilon_{abcd} (\tilde{\sigma}^d)^{\dot{\alpha}\alpha}. \quad (4.4c)$$

Following [24], we parametrise the variation of the membrane's metric as

$$\delta_\kappa \gamma_{ij} = 2(X_{ij} - \gamma_{ij} X_k^k), \quad (4.5)$$

with X_{ij} to be determined below. We now point out the relation

$$\delta_\kappa E_i^A = \partial_i \delta_\kappa E^A - 2\delta_\kappa E^C E_i^B \Omega_{[BC]}^A + \delta_\kappa E^C E_i^B T_{BC}^A, \quad (4.6)$$

where the Lorentz connection Ω_{BC}^A and the torsion tensor T_{BC}^A are given by eqs. (2.8) and (2.9a), respectively. In conjunction with integration by parts, this relation may be used to bring the variation of the action to the form:

$$\begin{aligned} \delta_\kappa S = T_3 \int d^3\xi \left\{ -\sqrt{-\gamma} X^{ij} (T_{ij} - \gamma_{ij}) + \frac{1}{2} \sqrt{-\gamma} \gamma^{ij} T_{ij} (\bar{\Phi} \delta_\kappa E^\alpha \mathcal{D}_\alpha \Phi + \Phi \delta_\kappa \bar{E}_{\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\alpha}} \bar{\Phi}) \right. \\ \left. - \sqrt{-\gamma} \gamma^{ij} (\Phi \bar{\Phi}) E_i^D \delta_\kappa E^C T_{CD}^a E_j^b \eta_{ab} + \frac{1}{6} \epsilon^{ijk} E_i^D E_j^C E_k^B \delta_\kappa E^A H_{ABCD} \right\}. \quad (4.7) \end{aligned}$$

Here we have denoted $T_{ij} := \Phi \bar{\Phi} E_i^a E_j^b \eta_{ab}$, and H_{ABCD} represents the components of the closed super 4-form

$$H_4 = dB_3 = \frac{1}{4!} E^D \wedge E^C \wedge E^B \wedge E^A \left\{ 4\mathcal{D}_{[A} B_{BCD)} - 6T_{[AB}{}^E B_{|E|CD)} \right\}. \quad (4.8)$$

Since $\delta_\kappa E^a = 0$, in accordance with eq. (4.4a), only dimension-0 and dimension-1/2 components of the torsion tensor appear in the κ -variation (4.7). In the case of the super-space geometry of section 2, no dimension-1/2 torsion is present, and the only dimension-0 torsion is

$$T_\alpha{}^{\dot{\beta}c} = -2i(\sigma^c)_\alpha{}^{\dot{\beta}}. \quad (4.9)$$

The non-trivial components of the superform H_4 defined by (4.3), which appear in the variation of the Wess-Zumino term in eq. (4.1), are

$$H_{ab\gamma\delta} = -4(\sigma_{ab})_{\gamma\delta} \bar{\Phi}^3, \quad H_{abc\delta} = \frac{1}{2} \varepsilon_{abcd} (\sigma^d)_{\delta\dot{\delta}} \bar{\mathcal{D}}^{\dot{\delta}} \bar{\Phi}^3, \quad (4.10)$$

and their complex conjugates. Now, if we make use of the relations (4.4), (4.9) and (4.10), and also choose

$$\begin{aligned}
 X^{ij} = & \left\{ -\frac{2}{\sqrt{-\gamma}} \Phi^{1/2} \bar{\Phi}^2 \epsilon^{kl(i} \gamma^{j)p} E_k^b E_l^c E_p^\alpha (\sigma_{bc})_{\alpha\beta} \right. \\
 & - 6i \Phi^{3/2} (T_{[k}^k + \delta_{[k}^k) \delta_l^{(i} E_{p]\beta} \gamma^{j)l} \gamma^{pq} E_q^a (\sigma_a)_{\beta}{}^{\dot{\beta}} \\
 & + \frac{1}{12\sqrt{-\gamma}} \Phi^{1/2} \bar{\Phi} \epsilon^{klp} \gamma^{ij} E_k^a E_l^b E_p^c \varepsilon_{abcd} (\sigma^d)_{\beta\dot{\beta}} \bar{\mathcal{D}}^{\dot{\beta}} \bar{\Phi} \\
 & \left. + \frac{3}{2} \Phi^{-1/2} \bar{\Phi}^{-1} (T_{[k}^k + \delta_{[k}^k) T_l^l \delta_p^{(i} \gamma^{j)p} \mathcal{D}_\beta \Phi \right\} \kappa^\beta + \text{c.c.}, \tag{4.11}
 \end{aligned}$$

it may be shown, using some algebra, that the action (4.1) is indeed invariant under the fermionic gauge transformation.

We emphasise that the Wess-Zumino term in the supermembrane action (4.1) is constructed in terms of the gauge three-form, for which our results in eqs. (3.11) and (3.28) are essential. On the other hand, the proof of κ -invariance, which was first given in [26] in the gauge $\Phi = 1$, requires only the constraints on the torsion and the field strength four-form, and for this reason is blind to the concrete three-form supergravity we choose, real or complex.

5 Concluding comments

The super-Weyl invariant formulation for the real and complex three-form supergravity theories provides a simple description for conformally flat supergravity backgrounds (compare with section 6.5 of [11]). It is obtained by choosing the supergravity covariant derivatives \mathcal{D}_A to coincide with the flat global ones $D_A = (\partial_a, D_\alpha, \bar{D}^{\dot{\alpha}})$, while keeping the corresponding compensator, Π or Υ , to be arbitrary. In the case of three-form supergravity, our action (4.1) then reduces to that describing the supermembrane coupled to a background three-form multiplet, as constructed by Bandos and Meliveo [58].

Since all supersymmetric actions described in this paper are super-Weyl invariant, our results may be recast in the framework of 4D $\mathcal{N} = 1$ conformal superspace [43].

Recently, nilpotent three-form multiplets have been used to construct theories for spontaneously broken local $\mathcal{N} = 1$ supersymmetry [59, 60]. One can also consider Goldstino models described by a nilpotent complex three-form multiplet. Its off-shell structure is still given by the relations (3.17)–(3.20). But now it is subject to the nilpotency constraint

$$\Upsilon^2 = 0, \tag{5.1}$$

with the additional condition that $\Delta\Upsilon$ is nowhere vanishing such that $(\Delta\Upsilon)^{-1}$ exists. This Goldstino superfield may be coupled to every off-shell formulation for supergravity. In particular, its coupling to old minimal supergravity is described by the action

$$S_{\text{Goldstino}} = \int d^4x d^2\theta d^2\bar{\theta} E \frac{\bar{\Upsilon}\Upsilon}{(\bar{\Phi}\Phi)^2}. \tag{5.2}$$

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