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Exact partition functions for deformed $\mathcal{N} = 2$ theories with $N_f = 4$ flavours

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ABSTRACT: We consider the Ω -deformed $\mathcal{N} = 2$ SU(2) gauge theory in four dimensions with $N_f = 4$ massive fundamental hypermultiplets. The low energy effective action depends on the deformation parameters $\varepsilon_1, \varepsilon_2$, the scalar field expectation value a , and the hypermultiplet masses $\mathbf{m} = (m_1, m_2, m_3, m_4)$. Motivated by recent findings in the $\mathcal{N} = 2^*$ theory, we explore the theories that are characterized by special fixed ratios $\varepsilon_2/\varepsilon_1$ and $\mathbf{m}/\varepsilon_1$ and propose a simple condition on the structure of the multi-instanton contributions to the prepotential determining the effective action. This condition determines a finite set Π_N of special points such that the prepotential has N poles at fixed positions independent on the instanton number. In analogy with what happens in the $\mathcal{N} = 2^*$ gauge theory, the full prepotential of the Π_N theories may be given in closed form as an explicit function of a and the modular parameter q appearing in special combinations of Eisenstein series and Jacobi theta functions with well defined modular properties. The resulting finite pole partition functions are related by AGT correspondence to special 4-point spherical conformal blocks of the Virasoro algebra. We examine in full details special cases where the closed expression of the block is known and confirms our Ansatz. We systematically study the special features of Zamolodchikov's recursion for the Π_N conformal blocks. As a result, we provide a novel effective recursion relation that can be exactly solved and allows to prove the conjectured closed expressions analytically in the case of the Π_1 and Π_2 conformal blocks.

KEYWORDS: Extended Supersymmetry, Nonperturbative Effects, Supersymmetric Effective Theories

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1 Introduction and results

In this paper, we consider the four dimensional $\mathcal{N} = 2$ SU(2) theory with N_f flavour hypermultiplets in the fundamental representation of the gauge group. At the special value $N_f = 4$ the one loop β -function vanishes and conformal invariance is broken by the flavour masses $\mathbf{m} = (m_1, \dots, m_4)$. Due to $\mathcal{N} = 2$ supersymmetry, the full effective action may be expressed in terms of the analytic prepotential $\mathcal{F}(a, \mathbf{m})$, where a is the vacuum expectation value of the scalar in the adjoint gauge multiplet [1]. Non-perturbative effects are present due to instantons and they are captured by the exact Seiberg-Witten (SW) solution [2, 3]. The instanton expansion is weighted by the instanton counting parameter

x which is a cross-ratio of the four roots of the original SW curve. In the following, it will be convenient to trade x by the modular parameter q defined by, see appendix A,

$$q = \exp \left(-\pi \frac{\mathbb{K}(1-x)}{\mathbb{K}(x)} \right), \quad (1.1)$$

where $\mathbb{K}$ is the complete elliptic integral of the first kind. Indeed, the quantity τ appearing in the parametrization $q = e^{i\pi\tau}$ receives only corrections proportional to the hypermultiplet masses. It is the analogue of the complexified gauge coupling constant of the related $\mathcal{N} = 2^*$ theory whose matter content is a massive adjoint hypermultiplet (see [4] for a detailed recent discussion).

As it is well known, an important alternative to the SW approach is localization [5–7]. The many-instanton moduli space is regularized by considering the theory on the Ω -background, i.e. a 4d Poincaré breaking deformation depending on two parameters $\varepsilon = (\varepsilon_1, \varepsilon_2)$ [8–15].¹ In this approach, the deformed partition function $Z^{\text{inst}}(\varepsilon, a, \mathbf{m})$ is well defined and its associated non-perturbative ε -deformed prepotential is introduced according to

$$F^{\text{inst}}(\varepsilon, a, \mathbf{m}) = -\varepsilon_1 \varepsilon_2 \log Z^{\text{inst}}(\varepsilon, a, \mathbf{m}). \quad (1.2)$$

Besides, the complete prepotential include a (simpler) perturbative contribution F^{pert} taking into account the 1-loop renormalization of the gauge coupling by terms proportional to the masses $\mathbf{m}$. Due to superconformal invariance, this is the full perturbative effect. The partition function in the ε -background is an interesting object because its expansion around vanishing deformation parameters generates higher genus amplitudes of the $\mathcal{N} = 2$ topological string [20–26] and satisfies a holomorphic anomaly equation [27–30].

At finite values of $\varepsilon_1, \varepsilon_2$, the deformed partition function is a central quantity in the Alday-Gaiotto-Tachikawa (AGT) correspondence [31].² This is a map between deformed $\mathcal{N} = 2$ instanton partition functions and conformal blocks of a suitable CFT with certain worldsheet genus and operator insertions. AGT correspondence may be checked perturbatively in the number of instantons [34] and has been proved in various cases like $\mathcal{N} = 2^*$ theory [35] or linear quivers on sphere [36]. For the $\mathcal{N} = 2^*$ ε -deformed $\text{SU}(2)$ gauge theory the relevant CFT quantity is the one-point conformal block on the torus. In the case of the $\text{SU}(2)$ theory with $N_f = 4$ the generalized prepotential is related to the logarithm of the conformal blocks of four Liouville operators on a sphere with the bare gauge coupling constant being associated with the cross-ratio of the punctures where the four operators are inserted [37–45].

¹For the string interpretation of the Ω -background and its BPS excitations see [16–19]. The approach in these papers provides a geometric interpretation for the localization (in terms of a dilaton potential), and gives a clear understanding of all the possible deformation parameters (ε and mass terms). It also gives a way to write down an effective action that contains exactly the same information as the partition function.

²A simpler setup is the Nekrasov-Shatashvili (NS) limit [32] where one of the two ε parameters vanishes. The supersymmetric vacua of the deformed theory are related to the eigenstates of a quantum integrable system. In the case of the $\mathcal{N} = 2^*$ theory, the relevant integrable system is the elliptic Calogero-Moser system [32] and the associated spectral problem reduces to the study of the celebrated Lamé equation. If the hypermultiplet mass m is taken to be proportional to ε with definite special ratios $\frac{m}{\varepsilon} = n + \frac{1}{2}$, where $n \in \mathbb{N}$, the spectral problem is n -gap and major simplifications occur in the k -instanton prepotential contributions [33].

The AGT perspective suggests to inspect the modular properties that have been studied in the undeformed case [46, 47] as well as for $\varepsilon_{1,2} \neq 0$ [48, 49]. It is possible to sum in closed form the instanton expansion order by order at large vacuum expectation value $a \rightarrow \infty$. In more details, for the $\mathcal{N} = 2^*$ theory, the coefficients of the various $(1/a)^n$ terms are written in terms of quasi-modular Eisenstein series depending on q . Similar results are available for the $N_f = 4$ theory by adding contributions involving the Jacobi theta functions ϑ_2, ϑ_4 . This approach has been successful in the gauge theory [50–56], see also [57–62] for a semiclassical WKB analysis.

In a recent paper [63], the Ω -deformed partition function of the $\mathcal{N} = 2^*$ theory is analyzed in the (α, β) plane where α, β are the ratios $\alpha = m/\varepsilon_1$ (m being the hypermultiplet mass) and $\beta = \varepsilon_2/\varepsilon_1$. At each instanton number, the partition function is a rational function of the ratio $\nu = 2a/\varepsilon_1$. Special points (α, β) do exist with the property that the k -instanton prepotential has poles at a fixed set of positions $\nu \in \{\nu_1, \dots, \nu_N\}$ independent on k . At these special N -poles points, the instanton partition function and the perturbative part of the prepotential take an exact closed form to be reviewed in section 2. These explicit expressions satisfy the modular anomaly equation expressing $\mathcal{S}$ -duality formulated in [50–55]. Besides, by applying the AGT correspondence, it is possible to obtain particular toroidal blocks in closed form at specific values of the central charge c and of the inserted operator conformal dimension. Similarly, the perturbative part of the prepotential can also be given in closed form and provides interesting special cases of the 3-point DOZZ Liouville correlation function [64–66]. In [63], the complete list of all the $N \leq 4$ poles points has been given, i.e. a set of 4, 7, 12, and 11 solutions at $N = 1, 2, 3, 4$ respectively. Recently, these exact partition functions have been analyzed from the perspective of the Zamolodchikov’s recursion relation for toroidal and spherical conformal blocks in [67]. This approach reveals once more the non-triviality of the solutions. In particular, the all order resummation of the instanton expansion in terms of quasi-modular forms remains an open question from the purely CFT point of view.

Building on these results, in this paper we extend the analysis to the N_f theory with four hypermultiplet in the fundamental representation.³ Indeed, as we discuss in section 3, the explicit results obtained in [50] and our previous experience in the $\mathcal{N} = 2^*$ theory suggest the possibility of obtaining again closed partition functions in terms of modular Eisenstein series plus additional contributions involving two Jacobi theta functions. To illustrate our results, we begin the discussion by presenting in great details two special one-pole and one two-poles partition functions. The first example has central charge $c = 1$

³A complementary analysis has been performed in [68] on a class of $\mathcal{N} = 2$ gauge theories on S^4 with gauge group $U(N)$, and N fundamental plus N anti-fundamental hypermultiplets. It has been shown that for a specific values of the masses (being linear combinations of ε_i and a), the partition function is non trivial only at certain values of a so that the full partition function $\int da |Z(a)|^2$ is given in terms of a finite number of residues and can be explicitly evaluated by generalized hypergeometric functions. From the AGT perspective, these configurations are associated with four point correlators involving the insertion of a degenerated field. Our setup is quite different. Our hyper masses are proportional to linear combinations of the $\varepsilon_{1,2}$ parameters only and the instanton partition functions will be a non-trivial function of a . Using AGT, the associated conformal blocks will have a generic intermediate conformal dimension and well defined modular properties.

(in the dual AGT conformal theory) and it is discussed in section 4. The mass and ε parameters are constrained by the relations

$$\frac{m_1}{\varepsilon_1} = \frac{m_3}{\varepsilon_1} = \frac{1}{2}, \quad \frac{m_2}{\varepsilon_1} = \frac{m_4}{\varepsilon_1} = 1, \quad \varepsilon_2 = -\varepsilon_1. \quad (1.3)$$

An educated guess based on explicit high instanton number computation reads⁴

$$Z^{\text{inst}}(\varepsilon, a, \mathbf{m}; x) = \left(\frac{x}{16q} \right)^{-\frac{\nu^2}{4}} (1-x)^{-\frac{9}{4}} \vartheta_3^{-5} \left[1 + \frac{1}{3\nu^2} (-E_2 + \vartheta_4^4 + 2\vartheta_2^4) \right]. \quad (1.4)$$

This expression may be read as an all-order truncation of the results of [50] at the special point (1.3). To prove it, we can exploit AGT correspondence. As explained later, this amounts to the calculation of Virasoro conformal block in closed form with external conformal dimensions $\Delta_i = (\frac{1}{16}, \frac{9}{16}, \frac{9}{16}, \frac{1}{16})$, and internal *generic* dimension $\Delta = \frac{\nu^2}{4}$. This block can be computed in the Ramond sector of a massless scalar field and has been fully worked out by Zamolodchikov and Apikyan in [69], generalizing the celebrated determination of the block of primary fields of dimension $\Delta_i = (\frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16})$ and generic internal dimension Δ in [70]. The result confirms perfectly the guessed expression (1.4).

We remark that the calculation in [69] is standard but somewhat laborious, in particular if one wants to extend it to other cases. It would be quite nice if it were possible to obtain results like (1.4) by directly solving the Zamolodchikov recursion for conformal blocks at special points like (1.3). This happens to be quite non-trivial, according to the analysis of [67] that is plagued by important technical problems. In particular, the special combination of quasi-modular forms in (1.4) is a fact that remains a conjecture within this approach — although it is true according to the monodromy approach of [69]. We solve these difficulties and provide a new *effective* recursion that allows to compute the block in closed form. Again, this may appear as a mere confirmation of the result in [69], but the advantage of the effective recursion is that it may be applied to other situations and prove more complicated expressions extending (1.4) to two-poles partition functions, as explained later.⁵ We also emphasize that arbitrary conformal blocks at $c = 1$ are computed in principle by generic Painlevé VI tau functions [80], see also [81–85]. Special Painlevé solutions (of Riccati, Picard, Chazy and algebraic type) are in correspondence with specific blocks. For instance, the $(\frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16})$ conformal block is obtained from the Picard solution of [86] and it would be interesting to discuss the block associated with (1.4) in this perspective.

In the special deformed point (1.3), we show that it is possible to give also the perturbative part of the prepotential in closed form. Thus, we obtain the exact quantum prepotential as

$$F(\varepsilon, a, \mathbf{m}) = F^{\text{pert}}(\varepsilon, a, \mathbf{m}) + F^{\text{inst}}(\varepsilon, a, \mathbf{m}), \quad (1.5)$$

⁴Here, E_2 is the second Eisenstein series and the dependence on q is understood in all modular functions.

⁵Related remarks may be found in the study of Zamolodchikov recursion for superconformal models, see [38, 71–79] and the dissertation at the following [link](#).

where the perturbative and instanton parts are respectively (Λ is a perturbative ultraviolet scale)

$$F^{\text{pert}}(\varepsilon, a, \mathbf{m}) = 2\varepsilon_1^2 \log \frac{a}{\Lambda} - 4a^2 \log 2, \quad (1.6)$$

$$F(\varepsilon, a, \mathbf{m}) = 2\varepsilon_1^2 \log \frac{a}{\Lambda} + \varepsilon_1^2 \log \left(1 + \frac{\varepsilon_1^2}{12a^2} (-E_2 + \vartheta_4^4 + 2\vartheta_2^4) \right).$$

The second example that we illustrate in details has central charge $c = -2$ and is discussed in section 5. The mass and ε parameters are constrained by the relations

$$\frac{m_1}{\varepsilon_1} = \frac{1}{2}, \quad \frac{m_2}{\varepsilon_1} = -\frac{m_4}{\varepsilon_1} = \frac{1}{4}, \quad \frac{m_3}{\varepsilon_1} = 1, \quad \varepsilon_2 = -\frac{1}{2}\varepsilon_1. \quad (1.7)$$

Our guess is in this case

$$Z^{\text{inst}}(\varepsilon, a, \mathbf{m}; x) = \left(\frac{x}{16q} \right)^{-\frac{\nu^2}{2}} (1-x)^{-2} \vartheta_3^{-5} \left[1 + \frac{-3 + 2E_2 + 2\vartheta_2^4 + \vartheta_4^4}{3(1-4\nu^2)} \right]. \quad (1.8)$$

and it matches several terms of the small instanton expansion. Now, the associated conformal block has external dimensions $\Delta_i = (-\frac{3}{32}, \frac{5}{32}, \frac{5}{32}, \frac{21}{32})$ and *generic* internal dimension $\Delta = \frac{1}{8}(4\nu^2 - 1)$. It may be realized in the $\mathbb{Z}_4$ orbifold sector of the (η, ξ) ghost logarithmic CFT considered in [87]. Many 4-point functions are available for such theories, see for instance [88–94]. However, our case in (1.8), and with generic internal dimension Δ , is new. Nevertheless, we are able to compute it by applying the effective recursion mentioned above. The result is in full agreement with the Ansatz (1.8). As in the previous $c = 1$ case, the perturbative part can be computed exactly, and we are able to give a closed formula for the full prepotential

$$F(\varepsilon, a, \mathbf{m}) = \varepsilon_1^2 \log \frac{a}{\Lambda} + \frac{1}{2}\varepsilon_1^2 \log \left(1 - \frac{\varepsilon_1^2 (2(E_2 + \vartheta_2^4) + \vartheta_4^4)}{48a^2} \right). \quad (1.9)$$

In the same section 5 we also discuss a more complex example, where the scaling relations among the ε and the $\mathbf{m}$ lead to a two-poles Nekrasov partition function. While it is slightly more complex from the technical point of view, we can again provide an exact form for the partition function and the complete prepotential

$$F(\varepsilon, a, \mathbf{m}) = 2\varepsilon_1^2 \log \frac{a}{\Lambda} + \frac{1}{2}\varepsilon_1^2 \log \left(1 + \frac{\varepsilon_1^2 (-4E_2 + 6\vartheta_2^4 + 3\vartheta_4^4)}{16a^2} + \frac{\varepsilon_1^4 (4E_2^2 - 12E_2\vartheta_2^4 + \vartheta_4^4 (11\vartheta_2^4 - 6E_2) + 11\vartheta_2^8 + 2\vartheta_4^8)}{768a^4} \right). \quad (1.10)$$

In section 6.1, the previous examples are generalized and we provide a full list of one-pole Nekrasov partition functions with their associated exact conformal blocks. They are characterized by special values of the 5 ratios $\alpha = \mathbf{m}/\varepsilon_1$ and $\beta = \varepsilon_2/\varepsilon_1$. The general form of the one-pole instanton partition function is

$$Z^{\text{inst}} = \left(\frac{x}{16q} \right)^{\frac{\nu^2}{4\beta}} (1-x)^{\frac{(\beta+1+\sum_{i=1}^4 \alpha_i)^2}{4\beta}} \vartheta_3(q)^{\frac{-(\beta+1)^2+2\sum_{i=1}^4 \alpha_i^2}{\beta}} H_\nu(q), \quad (1.11)$$

where

$$H_\nu^{1\text{-pole}}(q) = \frac{\nu^2 + \frac{1}{\beta} (c_1(\alpha, \beta) E_2 + c_2(\alpha, \beta) \vartheta_2^4 + c_3(\alpha, \beta) \vartheta_4^4)}{\nu^2 - \nu_1^2}, \quad (1.12)$$

with certain explicit coefficients $c_{1,2,3}(\alpha, \beta)$. The relevant 1-pole sets $\Pi_1 = \{(\alpha, \beta)\}$ contain 29 elements, up to additional discrete transformations to be discussed later. In each case, the perturbative prepotential can be given in closed form and the full quantum prepotential turns out to be given by the following compact expression (this is (6.11) copied here for the sake of presentation)

$$F = F^{\text{pert}} + F^{\text{inst}} = -\frac{1}{2}\varepsilon_1^2 \left(\beta(\beta + 1) - 2 \sum_i \alpha_i^2 + 1 \right) \log \left(\frac{a}{\Lambda} \right) \\ - \beta \varepsilon_1^2 \log \left(1 + \frac{\varepsilon_1^2}{4a^2\beta} (c_1(\alpha, \beta) E_2 + c_2(\alpha, \beta) \vartheta_2^4 + c_3(\alpha, \beta) \vartheta_4^4) \right). \quad (1.13)$$

The prepotential (1.13) may be checked to pass a non trivial test, i.e. the $\mathcal{S}$ -duality modular anomaly equation. All these solutions reduce to the massless $N_f = 4$ theory in the undeformed limit $\varepsilon \rightarrow 0$. Nevertheless, in the deformed theory, they are non-trivial exact special partition functions.

Finally, in section 6.2, we present similar results for 2-pole solutions. In this case we find a set Π_2 with 74 elements. The instanton partition function takes again the form (1.11), but with a function $H_\nu^{2\text{-poles}}(q)$ fully discussed in (6.13) and involving higher degree modular forms. Again, the perturbative prepotential can be computed in closed form and nicely combine with the instanton contribution.

To conclude, the expression (1.13) and the analogous ones for the two-pole solutions, see section 6.2, together with the full data characterizing the sets Π_1 and Π_2 are the main result of our paper. These are results for the gauge theory quantum prepotential and, due to AGT correspondence, they may be equivalently read as closed expressions for specific Virasoro conformal blocks. In principle, all of them can be proved by applying the novel tool of the effective recursion relation discussed in section 4.3. Actually, it is very interesting to exploit gauge theory insights about some expected features in order to provide educated guesses. For instance, the special combinations of Eisenstein series and theta functions in (1.13) have well defined modular properties connected with S-duality of the gauge theory, a fact that is not at all obvious from the Virasoro perspective.

2 Exact partition functions in the deformed $\mathcal{N} = 2^*$ theory

In this section we briefly summarize the results of [63] for the Ω -deformed $\mathcal{N} = 2^*$ SU(2) gauge theory. Exact partition functions are found at particular values of the ratios

$$\alpha = \frac{m}{\varepsilon_1}, \quad \beta = \frac{\varepsilon_2}{\varepsilon_1}, \quad (2.1)$$

where m is the mass of the hypermultiplet transforming in the adjoint representation. We can parametrize the scalar field expectation value by setting $a = \frac{\nu}{2} \varepsilon_1$. The partition

function is then

$$Z\left(\varepsilon_1, \beta \varepsilon_1, \frac{\nu}{2} \varepsilon_1, \alpha \varepsilon_1\right) = Z\left(1, \beta, \frac{\nu}{2}, \alpha\right) = \tilde{Z}_{(\alpha, \beta)}(\nu). \quad (2.2)$$

where we exploited the dimensional scaling independence to remove ε_1 and we use a tilde to emphasize the new variables. We also define

$$\tilde{F}_{(\alpha, \beta)}(\nu) = -\beta \log \tilde{Z}_{(\alpha, \beta)}(\nu), \quad (2.3)$$

which is the deformed prepotential that can be decomposed in its classical, perturbative, and instanton contributions

$$\tilde{F}_{(\alpha, \beta)}(\nu) = \tilde{F}_{(\alpha, \beta)}^{\text{class}}(\nu) + \tilde{F}_{(\alpha, \beta)}^{\text{pert}}(\nu) + \tilde{F}_{(\alpha, \beta)}^{\text{inst}}(\nu). \quad (2.4)$$

The main claim of [63] is that there exists a finite set of N -poles points (α, β) such that the k -instanton prepotential is a rational function of ν with poles at a fixed set of positions $\nu \in \{\nu_1, \dots, \nu_N\}$ independent on k . This claim is definitely non-trivial and it has important consequences. At the special N -poles points, it has been shown that the instanton partition function and the perturbative part of the prepotential read, see appendix A

$$\begin{aligned} \tilde{Z}_{(\alpha, \beta)}^{\text{inst}}(\nu) &= \frac{\nu^{2N} + \sum_{n=1}^N \nu^{2(N-n)} \mathcal{M}_{2n}(q)}{(\nu^2 - \nu_1^2) \dots (\nu^2 - \nu_N^2)} [q^{-\frac{1}{12}} \eta(\tau)]^{2(\Delta_m - 1)}, \\ \tilde{F}_{(\alpha, \beta)}^{\text{pert}}(\nu) &= -\beta \Delta_m \log \frac{\nu}{\Lambda} - \beta \log \prod_{n=1}^N \left(1 - \frac{\nu_n^2}{\nu^2}\right), \quad \Delta_m = \frac{(\beta + 1)^2 - 4\alpha^2}{4\beta}, \end{aligned} \quad (2.5)$$

where $\mathcal{M}_{2n}$ is a polynomial in the Eisenstein series E_2, E_4, E_6 with modular degree $2n$ and coefficients depending on α, β , and $\Delta_m \in \mathbb{N}$. The quantity Λ is the Seiberg-Witten UV scale. The total prepotential is therefore remarkably simple and reads

$$\tilde{F}_{(\alpha, \beta)}(\nu) = -\beta \Delta_m \log \frac{\nu}{\Lambda} - \beta \log \left(1 + \sum_{n=1}^N \frac{\mathcal{M}_{2n}(q)}{\nu^{2n}}\right). \quad (2.6)$$

The properties of this result have been discussed in the introduction. Here, we just add that according to AGT duality, the instanton partition function is related to the 1-point torus conformal block $\mathcal{F}_{\Delta_m}^{\Delta}$ by the relation [31]⁶

$$Z^{\text{inst}}(q, a, m) = \left[\prod_{k=1}^{\infty} (1 - q^{2k}) \right]^{2\Delta_m - 1} \mathcal{F}_{\Delta_m}^{\Delta}(q). \quad (2.7)$$

Here, the conformal dimension of the internal operator of the block is

$$\Delta = \frac{(\beta + 1)^2 - \nu^2}{4\beta}, \quad (2.8)$$

⁶We remind that the torus conformal block $\mathcal{F}_{\Delta_m}^{\Delta}(q)$ is the trace of the *external* primary $\mathcal{O}_{\Delta_m}$ over the descendants of $\mathcal{O}_{\Delta}$ with a suitable q -dependent weight.

and gives the dependence on the vacuum expectation value a appearing in ν , see (2.2). Finally, the central charge is

$$c = 13 + 6 \left(\beta + \frac{1}{\beta} \right). \quad (2.9)$$

The four 1-pole solutions found in [63] are associated with the CFT data ($c = 0, \Delta_m = 2$) and ($c = -2, \Delta_m = 3$). The prediction for the conformal block is then

$$\mathcal{F}_{\Delta_m}^{\Delta}(q) = \frac{q^{\frac{1}{12}}}{\eta(\tau)} \left[1 + \frac{c-1}{24\Delta} (E_2(q) - 1) \right], \quad (2.10)$$

with similar results for the 2- and 3-pole solutions. In the rest of the paper, we shall generalize such kind of results to the $N_f = 4$ theory. Here, AGT predicts exact expressions for spherical conformal blocks with four external operators and an exchanged internal operator with generic conformal dimension.⁷

3 Partition function and prepotential in the deformed $N_f = 4$ theory

After the brief review of the $\mathcal{N} = 2^*$ case, we now move to the Ω -deformed $\mathcal{N} = 2$ SYM theory with gauge group $SU(2)$ and four flavour hypermultiplets in the fundamental representation. For non-vanishing masses of the hypermultiplets, the classical prepotential of the theory receives both perturbative (1-loop) and non-perturbative (instanton) quantum corrections

$$F = F^{\text{pert}} + F^{\text{inst}}, \quad (3.1)$$

where the instanton contribution is related to the Nekrasov partition function by the well known relation (1.2) that we repeat here for the reader's convenience

$$F^{\text{inst}}(\varepsilon, a, \mathbf{m}) = -\varepsilon_1 \varepsilon_2 \log Z^{\text{inst}}(\varepsilon, a, \mathbf{m}). \quad (3.2)$$

The perturbative part instead can be expressed in terms of the generalized Barnes Γ functions [5, 6]. The precise definition is the following. Given the function

$$\gamma_{\varepsilon_1, \varepsilon_2}(y) = \frac{d}{ds} \left(\frac{\Lambda^s}{\Gamma(s)} \int_0^\infty \frac{1}{t} \frac{t^s e^{-ty}}{(e^{-\varepsilon_1 t} - 1)(e^{-\varepsilon_2 t} - 1)} dt \right) \Big|_{s=0}, \quad (3.3)$$

the perturbative part of the prepotential is

$$F^{\text{pert}}(\varepsilon, a, \mathbf{m}) = \varepsilon_1 \varepsilon_2 [f(a) + f(-a)], \quad (3.4)$$

with

$$f(a) = \gamma_{\varepsilon_1, \varepsilon_2}(2a) - \sum_{f=1}^4 \gamma_{\varepsilon_1, \varepsilon_2} \left(a + m_f + \frac{\varepsilon_+}{2} \right), \quad (3.5)$$

where we used the shorthand notation $\varepsilon_+ = \varepsilon_1 + \varepsilon_2$.⁸ As anticipated in the introduction, our exact results for the Nekrasov partition functions (and consequently for the prepotential) will involve simple combinations of modular and Jacobi theta functions. Similar structures

⁷Of course, in a definite CFT the internal operator is constrained by the fusion algebra, but the conformal block may be defined *off-shell* and as such it is the object appearing in AGT correspondence.

⁸We warn the reader that there are different conventions in the literature on the parametrization of the hypermultiplets masses, differing by a sign choice for m_f and/or shift of the masses by $(\varepsilon_1 + \varepsilon_2)/2$. Our conventions are the same, for example, of [50].

appear in the large a expansion of the prepotential analysed in [50]. We recall here explicitly some of the results of [50], since they will be useful in the following sections for a direct comparison. The complete quantum prepotential can be arranged in a series expansion in inverse powers of a of the form

$$F^{\text{inst}}(\varepsilon, a, \mathbf{m}) + F^{\text{pert}}(\varepsilon, a, \mathbf{m}) = h_0 \log \frac{a}{\Lambda} - \sum_{k=1}^{\infty} \frac{h_k}{2^{k+1} k} \frac{1}{a^{2k}}. \quad (3.6)$$

The logarithmic term comes from the perturbative part, and both the perturbative and instanton parts contribute to the coefficients h_k . Rewriting the expansion (3.6) in terms of q instead of x , the authors of [50] managed to reconstruct the complete q dependence of the first coefficients h_k , i.e. their results encode the complete all-instanton contribution. The result is that the functions h_k can be expressed as combinations of modular forms E_2, E_4, E_6 and Jacobi theta functions ϑ_2, ϑ_4 , with ε_i and m_f dependent polynomial coefficients. Explicitly, the first cases are

$$\begin{aligned} h_0 &= \frac{1}{2} (4R - \varepsilon_+^2 + \varepsilon_1 \varepsilon_2), \\ h_1 &= \frac{1}{24} E_2 (4R - \varepsilon_+^2 + \varepsilon_1 \varepsilon_2) (4R - \varepsilon_+^2 + 3\varepsilon_1 \varepsilon_2) - 4 (T_1 \vartheta_4^4 - T_2 \vartheta_2^4), \\ h_2 &= \frac{1}{144} E_2^2 (4R - \varepsilon_+^2 + \varepsilon_1 \varepsilon_2) (4R - \varepsilon_+^2 + 3\varepsilon_1 \varepsilon_2) (4R - \varepsilon_+^2 + 4\varepsilon_1 \varepsilon_2) \\ &\quad + \frac{4}{3} E_2 (4R - \varepsilon_+^2 + 4\varepsilon_1 \varepsilon_2) (T_2 \vartheta_2^4 - T_1 \vartheta_4^4) \\ &\quad + \frac{1}{720} E_4 \left[2304N + 2\varepsilon_1 \varepsilon_2 (2(80R^2 - 88R\varepsilon_+^2 + 17\varepsilon_+^4) + \varepsilon_1 \varepsilon_2 (98R - 47\varepsilon_+^2 + 15\varepsilon_1 \varepsilon_2)) \right. \\ &\quad \left. + (4R - 13\varepsilon_+^2) (\varepsilon_+^2 - 4R)^2 \right] \\ &\quad - \frac{8}{3} (R - \varepsilon_+^2 + \varepsilon_1 \varepsilon_2) (T_2 \vartheta_2^8 + 2(T_1 + T_2) \vartheta_4^4 \vartheta_2^4 + T_1 \vartheta_4^8), \end{aligned} \quad (3.7)$$

where R, T_1, T_2 and N are given by the following $\text{SO}(8)$ invariant combinations of the masses $\mathbf{m}$

$$\begin{aligned} R &= \frac{1}{2} \sum_f m_f^2, \\ T_1 &= \frac{1}{12} \sum_{f < f'} m_f^2 m_{f'}^2 - \frac{1}{24} \sum_f m_f^4, \\ T_2 &= -\frac{1}{24} \sum_{f < f'} m_f^2 m_{f'}^2 + \frac{1}{48} \sum_f m_f^4 - \frac{1}{2} m_1 m_2 m_3 m_4, \\ N &= \frac{3}{16} \sum_{f < f' < f''} m_f^2 m_{f'}^2 m_{f''}^2 - \frac{1}{96} \sum_{f \neq f'} m_f^2 m_{f'}^4 + \frac{1}{96} \sum_f m_f^6. \end{aligned} \quad (3.8)$$

Starting from h_3 , the expressions involve also the modular form E_6 . Note that the combination of masses defined in (3.8) transform non trivially under the action of the modular group [3]. Taking into account the modular properties of the Jacobi theta functions, these

transformations ensure that each h_k transforms as a modular form of degree $2k$, up to the anomalous term coming from E_2 , see [50] for details.

Compared to the previous expansion, the meaning of the result (1.11) is that it is possible to choose special ratios m_i/ε_1 and $\varepsilon_2/\varepsilon_1$ in such a way that all the higher degree modular forms and powers of Jacobi theta functions in the prepotential are generated by the expansion of the logarithm of a finite number of terms in the partition function.

3.1 Instanton partition function

The instanton partition function in the Ω -deformed $\mathcal{N} = 2$ SYM theory can be expanded order by order in the number of instantons according to

$$Z^{\text{inst}}(\varepsilon, a, \mathbf{m}) = 1 + \sum_{k=1}^{\infty} Z_k(\varepsilon, a, \mathbf{m}) x^k. \quad (3.9)$$

As we discussed in the Introduction, it is convenient to trade x by the modular parameter q by means of the inverse of the relation (1.1)

$$x = \frac{\vartheta_2^4(q)}{\vartheta_3^4(q)}. \quad (3.10)$$

We will refer to the Z_k in expansion (3.9) as the *Nekrasov functions*. At fixed instanton number k , the Nekrasov function can be calculated as a sum over pairs of $U(k)$ Young diagrams (Y_1, Y_2) , such that the total number of boxes $|Y_1| + |Y_2| = k$ [31]. To give a glimpse of the results and fix the notation, the simplest function Z_1 is given by the sum of the two terms

$$\begin{aligned} Z_{(\square, \bullet)} &= \frac{\prod_{f=1}^4 (2(a - m_f) - \varepsilon_+)}{32 a \varepsilon_1 \varepsilon_2 (\varepsilon_+ - 2a)}, \\ Z_{(\bullet, \square)} &= -\frac{\prod_{f=1}^4 (2(a + m_f) + \varepsilon_+)}{32 a \varepsilon_1 \varepsilon_2 (2a + \varepsilon_+)}, \\ Z_1(\varepsilon, a, \mathbf{m}) &= Z_{(\square, \bullet)} + Z_{(\bullet, \square)}. \end{aligned} \quad (3.11)$$

Increasing k , the expressions for the Nekrasov functions become quickly cumbersome, but the structure of the Z_k is always a rational function of the parameters, i.e.,

$$Z_k(\varepsilon, a, \mathbf{m}) = \frac{P_k(\varepsilon, a, \mathbf{m})}{Q_k(\varepsilon, a, \mathbf{m})}, \quad (3.12)$$

where $P_k(\varepsilon, a, \mathbf{m})$ and $Q_k(\varepsilon, a, \mathbf{m})$ are polynomials in all the variables with increasing degree as k grows.

3.2 Predictions from AGT correspondence

Before discussing AGT in this context, let us briefly summarize our notation for the conformal blocks and recall Zamolodchikov's recursion. Given a four point correlation function of primaries with dimensions Δ_i

$$\langle \mathcal{O}_{\Delta_4}(z_1) \mathcal{O}_{\Delta_3}(z_2) \mathcal{O}_{\Delta_2}(z_3) \mathcal{O}_{\Delta_1}(z_4) \rangle, \quad (3.13)$$

we can use a conformal transformation to set $z_1 = \infty, z_2 = 1, z_3 = x$ and $z_4 = 0$, where x now is the usual cross ratio $x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}$. The four point correlator can be expanded as a sum over intermediate states of dimension Δ

$$\langle \mathcal{O}_{\Delta_4}(\infty) \mathcal{O}_{\Delta_3}(1) \mathcal{O}_{\Delta_2}(x) \mathcal{O}_{\Delta_1}(0) \rangle = \sum_{\Delta} C_{\Delta_2, \Delta_1, \Delta} C_{\Delta, \Delta_3, \Delta_4} \left| x^{\Delta - \Delta_1 - \Delta_2} B_{\Delta} \left[\begin{matrix} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{matrix} \right] (x) \right|^2, \quad (3.14)$$

where $C_{\Delta_i, \Delta_j, \Delta_k}$ are the 3-point structure constants. The function B is the *conformal block* and depends on the external dimensions Δ_i , the internal dimension Δ and the central charge c of the CFT (we omit this obvious dependence).⁹ It may be represented by the diagram

$$B_{\Delta} \left[\begin{matrix} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{matrix} \right] (x) = \begin{array}{c} \Delta_2 \quad \Delta_3 \\ | \quad | \\ \hline \Delta_1 \quad \Delta \quad \Delta_4 \end{array}$$

and admits the following *elliptic* representation [95]

$$B_{\Delta} \left[\begin{matrix} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{matrix} \right] (x) = \left(\frac{x}{16q} \right)^{\frac{c-1}{24} - \Delta} (1-x)^{\frac{c-1}{24} - \Delta_2 - \Delta_3} \vartheta_3(q)^{\frac{1}{2}(c-1-8\sum_i \Delta_i)} H_{\Delta} \left[\begin{matrix} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{matrix} \right] (x), \quad (3.15)$$

where the relation between x and q has been given in (3.10) and the function H obeys the Zamolodchikov's recursion relation [95]. Using the following parametrization for the conformal dimensions in terms of the λ_i variables

$$\Delta_i = \frac{c-1}{24} + \lambda_i^2, \quad (3.16)$$

and defining the quantities ($\Delta_{m,n}$ are the entries in the Kac table)

$$a_{\pm} = \frac{1}{\sqrt{24}} (\sqrt{1-c} \pm \sqrt{25-c}), \quad \lambda_{r,s} = a_+ r + a_- s, \quad (3.17)$$

$$\Delta_{m,n} = \frac{c-1}{24} + \left(\frac{a_+ m + a_- n}{2} \right)^2,$$

the recursion formula for the function H reads

$$H_{\Delta} \left[\begin{matrix} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{matrix} \right] (x) = 1 + \sum_{m,n=1}^{\infty} \frac{16^{mn} q^{mn} R_{m,n}}{\Delta - \Delta_{m,n}} H_{\Delta_{m,n} + m n} \left[\begin{matrix} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{matrix} \right] (x), \quad (3.18)$$

⁹It is convenient to strip off the explicit power of x as in (3.14) so that the small x expansion of the conformal block start at one

$$B_{\Delta} \left[\begin{matrix} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{matrix} \right] (x) = 1 + \frac{(\Delta - \Delta_1 + \Delta_2)(\Delta + \Delta_3 - \Delta_4)}{2\Delta} x + \mathcal{O}(x^2).$$

The coefficients $R_{m,n}$ appearing in (3.18) are defined in terms of the $\lambda_i, \lambda_{r,s}$ introduced in (3.16) and (3.17) as follows

$$\begin{aligned}
 R_{m,n} &= \frac{P_{m,n}}{A_{m,n}}, \\
 P_{m,n} &= -\frac{1}{2} \prod_{r=-m+1, -m+3}^{m-1} \prod_{s=-n+1, -n+3}^{n-1} p_{r,s}, \quad A_{m,n} = \prod_{\substack{k=-m+1 \\ (k,l) \neq (0,0),(m,n)}}^m \prod_{l=-n+1}^n \lambda_{k,l}, \\
 p_{r,s} &= \left(\lambda_1 + \lambda_2 - \frac{\lambda_{r,s}}{2} \right) \left(\lambda_2 - \lambda_1 - \frac{\lambda_{r,s}}{2} \right) \left(\lambda_3 + \lambda_4 - \frac{\lambda_{r,s}}{2} \right) \left(\lambda_3 - \lambda_4 - \frac{\lambda_{r,s}}{2} \right).
 \end{aligned} \tag{3.19}$$

Equation (3.18) is understood as a sum over the residues of the conformal block in the poles $\Delta_{m,n}$. The products appearing in $P_{m,n}$ are exactly those needed to cancel the contribution of null-states at the level $L = nm$ of the Verma module.

3.2.1 The AGT dictionary for the $N_f = 4$ theory

The AGT correspondence states the equivalence between the Nekrasov partition function and a four point conformal block, identified through the following map between the gauge and CFT side parameters [31, 34]

$$\begin{aligned}
 \Delta &= \frac{\varepsilon_+^2}{4\varepsilon_1\varepsilon_2} - \frac{a^2}{\varepsilon_1\varepsilon_2}, & c &= 1 + 6\frac{\varepsilon_+^2}{\varepsilon_1\varepsilon_2}, \\
 \Delta_i &= \frac{\eta_i(\varepsilon_+ - \eta_i)}{\varepsilon_1\varepsilon_2}, & i &= 1 \dots 4, \\
 \eta_1 &= \frac{1}{2}(\varepsilon_+ - m_1 + m_2), & \eta_2 &= \frac{1}{2}(\varepsilon_+ - m_1 - m_2), \\
 \eta_3 &= \frac{1}{2}(\varepsilon_+ - m_3 - m_4), & \eta_4 &= \frac{1}{2}(\varepsilon_+ - m_3 + m_4).
 \end{aligned} \tag{3.20}$$

More precisely, the AGT statement is that the two functions $Z^{\text{inst}}(\varepsilon, a, \mathbf{m}; x)$ and $B(\Delta, \Delta_i, c; x)$ are the same, using the above map, up to an overall $U(1)$ factor

$$Z^{\text{inst}}(\varepsilon, a, \mathbf{m}; x) = (1-x)^\kappa B_\Delta \begin{bmatrix} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{bmatrix} (x) \tag{3.21}$$

with

$$\kappa = \frac{2(\varepsilon_+ - \eta_2)(\varepsilon_+ - \eta_3)}{\varepsilon_1\varepsilon_2} = -\frac{(\sqrt{1-c+24\Delta_2} + \sqrt{1-c})(\sqrt{1-c+24\Delta_3} + \sqrt{1-c})}{12}. \tag{3.22}$$

Following [34], it is convenient to rewrite the Nekrasov partition function in a form that mimics the elliptic form of the conformal block (3.15). For compactness, we introduce the definition of the three exponents

$$X_1 = \frac{a^2}{\varepsilon_1\varepsilon_2}, \quad X_2 = \frac{1}{4\varepsilon_1\varepsilon_2} \left(\varepsilon_+ + \sum_{f=1}^4 m_f \right)^2, \quad X_3 = \frac{1}{\varepsilon_1\varepsilon_2} \left(-\varepsilon_+^2 + 2 \sum_{f=1}^4 m_f^2 \right). \tag{3.23}$$

With these definitions, one can write [34], see (3.15),

$$Z^{\text{inst}}(\epsilon, a, \mathbf{m}; x) = \left(\frac{x}{16q}\right)^{X_1} (1-x)^{X_2} \vartheta_3(q)^{X_3} H_{\Delta} \begin{bmatrix} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{bmatrix} (x). \quad (3.24)$$

In the following sections we are going to analyze particular cases where the Nekrasov functions have a very simple pole structure. In this context, the recursion (3.18) has the clear advantage to make transparent the pole structure from the beginning.

The AGT dictionary is completed with the correspondence between the perturbative part of the prepotential and the DOZZ formula for the 3-point function in the Liouville theory, see section 4.1 of [96].

4 The Ramond sector of scalar CFT and a special point with $c = 1$

In this section we discuss a first example of a special point and we show how the partition function can be obtained in closed form. This is the point specified by the relations (1.3) that we repeat here for convenience

$$\frac{m_1}{\epsilon_1} = \frac{m_3}{\epsilon_1} = \frac{1}{2}, \quad \frac{m_2}{\epsilon_1} = \frac{m_4}{\epsilon_1} = 1, \quad \epsilon_2 = -\epsilon_1. \quad (4.1)$$

Given the AGT map, the special point corresponds to the following set of external dimensions and central charge

$$\Delta_1 = \frac{1}{16}, \quad \Delta_2 = \frac{9}{16}, \quad \Delta_3 = \frac{9}{16}, \quad \Delta_4 = \frac{1}{16}, \quad c = 1. \quad (4.2)$$

Alternative equivalent choices are possible because of the global $\text{SO}(8)$ flavour symmetry of the gauge theory for $N_f = 4$ and this symmetry is non trivial on the CFT side [97]. The choice in (4.2) is particularly convenient for the following analysis.

4.1 Perturbative contribution in closed form

Given the relations (4.1) the calculation of the perturbative part is straightforward. In fact for this special point, the perturbative part reduces to

$$F^{\text{pert}}(\epsilon, a, \mathbf{m}) = 2\epsilon_1^2 \log \frac{a}{\Lambda} - a^2 \log 16. \quad (4.3)$$

In particular, this result means that all the coefficients h_k with $k > 0$ in (3.6) are completely determined by the instanton contribution only.

4.2 Instanton partition function at all orders

A direct, brute force calculation of the Nekrasov partition function leads to the following result, see section 3.1

$$\tilde{Z}^{\text{inst}}(\nu) = 1 + \sum_{k=1}^{\infty} \frac{1}{\nu^2} P_k(\nu) x^k, \quad (4.4)$$

where the polynomials $P_k(\nu)$ have degree $2k + 2$. The first polynomials are

$$\begin{aligned}
 P_1(\nu) &= \frac{1}{8} (\nu^4 + 13\nu^2 + 4), \\
 P_2(\nu) &= \frac{1}{256} (2\nu^6 + 65\nu^4 + 572\nu^2 + 288), \\
 P_3(\nu) &= \frac{1}{6144} (2\nu^8 + 117\nu^6 + 2383\nu^4 + 17484\nu^2 + 11232), \\
 P_4(\nu) &= \frac{1}{393216} (4\nu^{10} + 364\nu^8 + 12771\nu^6 + 207023\nu^4 + 1361924\nu^2 + 1018176), \quad (4.5) \\
 P_5(\nu) &= \frac{1}{15728640} (4\nu^{12} + 520\nu^{10} + 27555\nu^8 + 748070\nu^6 + 10527911\nu^4 \\
 &\quad + 64320240\nu^2 + 53436240), \\
 P_6(\nu) &= \frac{1}{1509949440} (8\nu^{14} + 1404\nu^{12} + 104042\nu^{10} + 4180545\nu^8 + 96592877\nu^6 \\
 &\quad + 1232193096\nu^4 + 7132346928\nu^2 + 6409549440).
 \end{aligned}$$

Going to the function H according to (3.24), we see that the dependence on ν in the polynomials $P_k(\nu)$ is fully removed by the first factor in the r.h.s. of (3.24), i.e. the power of the ratio $x/(16q)$ with a ν -dependent exponent. Including all the other factors appearing in the r.h.s. of (3.24), we obtain

$$H_\nu(q) = 1 + \frac{8q}{\nu^2} + \frac{16q^2}{\nu^2} + \frac{32q^3}{\nu^2} + \frac{32q^4}{\nu^2} + \frac{48q^5}{\nu^2} + \frac{64q^6}{\nu^2} + \mathcal{O}(q^7). \quad (4.6)$$

From this series one may guess the following *exact* q dependence in terms of E_2 , ϑ_2 , and ϑ_4

$$H_\nu(q) = 1 + \frac{1}{3\nu^2} (-E_2 + \vartheta_4^4 + 2\vartheta_2^4). \quad (4.7)$$

Upon the substitution $\Delta = \nu^2/4$, the exact form for H in (4.7) turns into a prediction for the conformal block with conformal data as in (4.2). In other words, our conjecture and AGT imply the remarkable result

$$H_\Delta \left[\begin{matrix} \frac{9}{16} & \frac{9}{16} \\ \frac{1}{16} & \frac{1}{16} \end{matrix} \right] (x) = 1 + \frac{1}{12\Delta} (-E_2 + \vartheta_4^4 + 2\vartheta_2^4). \quad (4.8)$$

In fact, as we mentioned in the Introduction, the relation (4.8) has been obtained by Zamolodchikov and Apikyan in [69] (see also [98]). They considered correlation functions in the Ramond sector of a free scalar field, which contains an infinite set of fields σ_k with dimensions $\Delta_k = (2k + 1)^2/16$. From the analysis of the operator product expansion they derived a set of operatorial relations for the σ_k fields, and using these relations they were able to obtain, algorithmically, exact expressions for the conformal blocks. They also pointed out the simplicity of the pole structure of the conformal blocks. Among the explicit result presented in the paper, the formula (4.23) of [69] is exactly the case under consideration here. Their result reads

$$H_\Delta \left[\begin{matrix} \frac{9}{16} & \frac{9}{16} \\ \frac{1}{16} & \frac{1}{16} \end{matrix} \right] (x) = 1 - \frac{q}{\Delta} \frac{d}{dq} \log \vartheta_4(q) \quad (4.9)$$

It is straightforward to use the properties of Jacobi functions in appendix A to show that (4.8) and (4.9) are equivalent.

4.3 The CFT side: a new method to derive exact conformal blocks

Exact expressions for the conformal blocks are rarely known. In this respect, the example (4.8) considered in this section is a notable exception. In the rest of the paper we will extend the list of exact predictions from AGT, and in all the cases we will find similar structures in terms of the functions E_2 , ϑ_2 , and ϑ_4 . On the other hand, the method adopted in [69] is in principle applicable to other cases, but rather laborious. It is then natural to investigate whether this particular example — where everything is well under control both on the AGT and CFT side — is characterised by some hidden structure that allows for an exact calculation by means of standard CFT techniques.

There are various approaches that allow the perturbative computation of the expansion of the conformal block in powers of the cross-ratio x or of the elliptic parameter q . To give an example, the coefficient B_k of x^k in the expansion of the conformal block can be derived using the so called hypergeometric recursion in the following form [99]¹⁰

$$B_{\Delta} \left[\begin{matrix} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{matrix} \right] (x) = \sum_{k=0} B_k(\Delta) x^k, \quad (4.10)$$

$$B_k(\Delta) = \sum_{\ell=0}^k \chi_{\ell} \frac{(\ell + \Delta + \Delta_2 - \Delta_1)_{(k-\ell)} (\ell + \Delta + \Delta_3 - \Delta_4)_{(k-\ell)}}{(k-\ell)! (2\Delta + 2\ell)_{(k-\ell)}},$$

where $(a)_n$ is the Pochhammer symbol, and the coefficients χ_{ℓ} are functions of the central charge, external and internal dimensions. The χ_{ℓ} coefficients can be computed recursively and quite efficiently as nested sums as explained in [99] (see [100] for a recent application based on this formalism).¹¹ The first non trivial cases, at the CFT data in (4.1), are

$$\begin{aligned} \chi_0 &= 1, & \chi_1 &= 0, \\ \chi_2 &= \frac{1}{128} (2\Delta + 1), & \chi_3 &= \frac{3}{512\Delta (\Delta^2 + 3\Delta + 2)}, \\ \chi_4 &= \frac{(\Delta + 2) (4\Delta^2 + 21\Delta + 23)}{32768(\Delta + 3)}, & \chi_5 &= \frac{3(\Delta + 8)(2\Delta + 9)}{65536\Delta(\Delta + 2)(\Delta + 3)(\Delta + 4)}. \end{aligned} \quad (4.11)$$

Of course, using these χ_{ℓ} , one can reproduce the correct expansion of the complete block,

¹⁰For simplicity of notation, we do not write explicitly the dependence of the coefficients $B_k(\Delta)$ on the external dimensions and the central charge.

¹¹Another option is to evaluate the conformal block using the generalized Dotsenko-Fateev (DF) representation according to the results of [101]. Technically, the hard problem with this representation is the analytical continuation from integer DF exponents to rational values and this is ambiguous. Alternatively, one may exploit $SO(8)$ symmetry of the gauge theory to reduce to integer exponents, but then one needs to evaluate the DF integral with generic number of FD integration variables. The Jack polynomial machinery developed in [102, 103] is useful at low instanton numbers, but becomes increasingly unwieldy at higher orders and is ultimately equivalent to the Nekrasov determinant expressions and basically useless to obtain closed sums over all instantons.

see (3.15)

$$\begin{aligned}
B_{\Delta} \left[\begin{smallmatrix} \frac{9}{16} & \frac{9}{16} \\ \frac{1}{16} & \frac{1}{16} \end{smallmatrix} \right] (x) = & 1 + \frac{(2\Delta+1)^2 x}{8\Delta} + \frac{(\Delta+2)(2\Delta+1)(8\Delta+9)x^2}{128\Delta} \\
& + \frac{(2\Delta+3)(32\Delta^3+204\Delta^2+376\Delta+153)x^3}{3072\Delta} \\
& + \frac{(\Delta+4)(256\Delta^4+2496\Delta^3+8268\Delta^2+10403\Delta+3822)x^4}{98304\Delta} \\
& + \frac{(2\Delta+9)(512\Delta^5+8576\Delta^4+53688\Delta^3+152314\Delta^2+185395\Delta+69990)x^5}{3932160\Delta} \\
& + \frac{1}{188743680\Delta} (\Delta+6)(2\Delta+9)(2048\Delta^5+40704\Delta^4+300704\Delta^3 \\
& + 995460\Delta^2+1381229\Delta+573180)x^6 + \mathcal{O}(x^7).
\end{aligned} \tag{4.12}$$

Computationally the method is highly efficient, and one can push the above expansion to high order. On the other hand, it is hard to see any regularity or structure that may help in the search of the exact solution. From the previous discussion, these difficulties are somehow expected. Using the relations collected in appendix A, one can rewrite the exact expression of the block using only the x variable as follows

$$B_{\Delta} \left[\begin{smallmatrix} \frac{9}{16} & \frac{9}{16} \\ \frac{1}{16} & \frac{1}{16} \end{smallmatrix} \right] (x) = \frac{\pi^{5/2} 16^{\Delta} x^{-\Delta}}{4\sqrt{2}(1-x)^{9/8}\mathbb{K}(x)^{5/2}} \left(1 + \frac{\mathbb{K}(x)(\mathbb{K}(x) - \mathbb{E}(x))}{\pi^2 \Delta} \right) \left(e^{-\frac{\pi \mathbb{K}(1-x)}{\mathbb{K}(x)}} \right)^{\Delta}. \tag{4.13}$$

This form makes clear that the dependence on x of the block is rather intricate.

4.3.1 Effective elliptic recursion

A more promising strategy is provided by the elliptic recursion. Following the results presented in [63], this approach to the problem was investigated in [67]. The solution of the recursion is in principle straightforward. Starting from equation (3.18) and expanding the function H in series of q as

$$H_{\Delta} \left[\begin{smallmatrix} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{smallmatrix} \right] (q) = 1 + \sum_{k=1}^{\infty} H_k(\Delta) q^k, \tag{4.14}$$

one gets a recursive definition for the coefficients $H_k(\Delta)$

$$H_0(\Delta) = 1, \quad H_k(\Delta) = \sum_{\substack{m,n \geq 1 \\ mn \leq k}} \frac{16^{mn} R_{m,n}}{\Delta - \Delta_{m,n}} H_{k-mn}(\Delta_{m,n} + mn). \tag{4.15}$$

The usual strategy, given the external dimensions, is to compute the $R_{m,n}$ leaving c generic, to avoid poles in (3.19), solve the recursion and finally take the limit $c \rightarrow 1$. Following this route one recovers the correct series expansion, but the nested structure of the recursive solution (4.18) is still not very transparent. On the other hand, one knows that at the point under consideration the block has the quite simple dependence on Δ as in

$$H_{\Delta} \left[\begin{smallmatrix} \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_4 \end{smallmatrix} \right] (q) = 1 + \frac{1}{\Delta} \mathcal{H}(\Delta_i; q). \tag{4.16}$$

If one starts at $c = 1$ from the beginning in order to possibly truncate the recursion using (4.16) in (4.15) the problem is that of the reducibility of Verma module. In other words, the equation

$$\Delta_{m,n} + mn = \Delta_{m',n'}, \quad (4.17)$$

has integer solutions. This means that a naive application of (4.15) leads to undetermined ratios $0/0$ preventing truncation, see [67].

Here, we propose another strategy that overcomes these problems and is able to provide the function $\mathcal{H}$ in (4.16) in closed form with little effort. To illustrate the basic idea let us consider the case of the $R_{2,2}$ term. As a function of c , in the special point one gets

$$R_{2,2}(c) = \frac{1}{16^4} \frac{(c-27)^2(c-1)((c-12)c-37)^2}{256(c-28)(c-25)(c+2)}, \quad (4.18)$$

and clearly $\lim_{c \rightarrow 1} R_{2,2}(c) = 0$. Instead, working with $c = 1$, the arguments of the products $P_{m,n}$ and $A_{m,n}$ become $\frac{1}{16}(r-s+1)^2(r-s+2)^2$ and $k-l$ respectively. Taking into account the range of the indices, for $R_{2,2}$ we have a “ 0^2 ” in the numerator from the pair $(r,s) = (-1,1)$, but also two factors vanishing in the denominator for $(k,l) = (-1,-1)$ and $(k,l) = (1,1)$. The final $0^2/0^2$ expression is finite. This order of limits ambiguity has been clearly pointed out in [67].¹² To evaluate practically the products with $c = 1$, we need some regularization. We propose the simplest possible prescription, introducing a shift δ in one of the indices of the products and then taking the $\delta \rightarrow 0$ limit at the end. Our *effective* definition of the $R_{m,n}$ coefficients is then based on the definitions

$$P_{m,n} = -\frac{1}{2} \prod_{r=-m+1, -m+3}^{m-1} \prod_{s=-n+1, -n+3}^{n-1} p_{r,s+\delta}, \quad A_{m,n} = \prod_{\substack{k=-m+1 \\ (k,l) \neq (0,0),(m,n)}}^m \prod_{l=-n+1}^n \lambda_{k,l+\delta}, \quad (4.19)$$

with (in our special $c = 1$ point)

$$p_{r,s+\delta} = \frac{1}{16}(-\delta + r - s + 1)^2(-\delta + r - s + 2)^2, \quad \lambda_{k,l+\delta} = k - l - \delta. \quad (4.20)$$

At the end, we remove the regularization and compute the *effective* coefficients

$$R_{m,n}^{\text{eff}} = \lim_{\delta \rightarrow 0} \frac{P_{m,n}}{A_{m,n}}. \quad (4.21)$$

The first advantage of this approach is that the coefficients $R_{m,n}^{\text{eff}}$ can be easily computed in closed form. They read

$$R_{m,n}^{\text{eff}} = -\frac{2}{16m^2}(-1)^m m^2 \delta_{m,n}, \quad (4.22)$$

and, as expected, they are non vanishing only for $\Delta_{m,n} = 0$. Plugging the Ansatz (4.16) into (4.15) and using $R \rightarrow R^{\text{eff}}$, we get the *effective recursion relation*

$$1 + \frac{1}{\Delta} \mathcal{H} = 1 - 2 \sum_{m=1}^{\infty} \frac{(-1)^m m^2 q^{m^2}}{\Delta} \left(1 + \frac{1}{m^2} \mathcal{H} \right). \quad (4.23)$$

¹²A similar problem is discussed in appendix B of the dissertation at the following [link](#). A theoretical framework for this kind of problems has been recently formulated in [104].

We can solve (4.23) for $\mathcal{H}$ as the ratio of two series

$$\mathcal{H} = -\frac{2 \sum_{m=1} (-1)^m m^2 q^{m^2}}{1 + 2 \sum_{m=1} (-1)^m q^{m^2}}. \quad (4.24)$$

The denominator is nothing but ϑ_4 , while the numerator is its $q \partial_q$. In this way one recovers our result in the form of eq. (4.9). Later, we shall show that this approach is general and powerful providing similar simple computation of the special exact conformal blocks appearing in our analysis.

4.4 Closed form of the full prepotential

Putting together the results for the perturbative and instanton parts we can now write the full prepotential

$$\begin{aligned} F^{\text{pert}}(\varepsilon, a, \mathbf{m}) &= 2 \varepsilon_1^2 \log \frac{a}{\Lambda} - a^2 \log 16, \\ F^{\text{inst}}(\varepsilon, a, \mathbf{m}) &= \varepsilon_1^2 \log \left(\left(\frac{x}{16q} \right)^{-\frac{a^2}{\varepsilon_1^2}} (1-x)^{-\frac{9}{4}} \vartheta_3^{-5} \left(1 + \frac{\varepsilon_1^2}{12a^2} (-E_2 + \vartheta_4^4 + 2\vartheta_2^4) \right) \right), \\ F(\varepsilon, a, \mathbf{m}) &= F^{\text{pert}}(\varepsilon, a, \mathbf{m}) + F^{\text{inst}}(\varepsilon, a, \mathbf{m}). \end{aligned} \quad (4.25)$$

Expanding for $a \rightarrow \infty$, and discarding all the a independent terms, we can compare directly with the large a expansion (3.6). The term $(\frac{x}{16q})^{-a^2/\varepsilon_1^2}$ in the instanton part combines with the corresponding term proportional to a^2 in the perturbative part and gives the renormalization of the gauge coupling constant [50]. The check for the first coefficient h_1 is trivial. Moreover, equation (4.25) implies that *all* the coefficients h_k with $k > 1$ must simply be powers of h_1 , up to the numerical coefficient arising from the expansion of the logarithm. This is indeed what happens. The second coefficient for a^{-4} arising from the expansion of (4.25) is

$$-\frac{1}{288} \varepsilon_1^6 (-E_2 + \vartheta_4^4 + 2\vartheta_2^4)^2, \quad (4.26)$$

which is in agreement with (3.6), evaluating h_2 at the special point and using the identity $E_4 = \vartheta_2^8 + \vartheta_4^4 \vartheta_2^4 + \vartheta_4^8$. The same check can be performed with h_3 , and confirms again the pattern.

5 Logarithmic CFT and examples with $c = -2$

In this section we present other two examples of special points where the Nekrasov partition function has a finite number of poles and can be worked out exactly. They are characterized by the scaling relations, see (1.7)

$$\frac{m_1}{\varepsilon_1} = \frac{1}{2}, \quad \frac{m_2}{\varepsilon_1} = -\frac{m_4}{\varepsilon_1} = \frac{1}{4}, \quad \frac{m_3}{\varepsilon_1} = 1, \quad \varepsilon_2 = -\frac{1}{2} \varepsilon_1. \quad (5.1)$$

and

$$\frac{m_1}{\varepsilon_1} = \frac{1}{2}, \quad \frac{m_2}{\varepsilon_1} = \frac{m_3}{\varepsilon_1} = \frac{3}{4}, \quad \frac{m_4}{\varepsilon_1} = 1, \quad \varepsilon_2 = -\frac{1}{2} \varepsilon_1, \quad (5.2)$$

respectively. For the first one, the Nekrasov functions have again a single pole, for the second choice two poles. Through AGT they are related to a logarithmic CFT with $c = -2$, as discussed in more detail below.

Before moving to the calculation of the instanton partition functions, we give here the result for the perturbative part of the prepotential. In both cases the calculation can be easily done exactly, and the result is

$$F^{\text{pert}}(\epsilon, a, \mathbf{m}) = \kappa \epsilon_1^2 \log \frac{a}{\Lambda} - a^2 \log 16 + \frac{1}{2} \epsilon_1^2 \log \left(1 - \frac{\epsilon_1^2}{16 a^2} \right), \quad (5.3)$$

with $\kappa = 1$ for the first special point, and $\kappa = 2$ for the second one. We remark that the perturbative part of the prepotential may be given in clean form in the undeformed massive limit, see for instance (2.16) of [53]. The simplicity of (5.3) holding for non zero deformation is remarkable and similar to the recent findings in [63].

5.1 A one-pole point

Given the scaling relations in (5.1), the AGT map leads to the corresponding set of external dimensions

$$\Delta_1 = -\frac{3}{32}, \quad \Delta_2 = \frac{5}{32}, \quad \Delta_3 = \frac{5}{32}, \quad \Delta_4 = \frac{21}{32}, \quad (5.4)$$

and the central charge of the CFT is $c = -2$. It is easy to recognize that the values in (5.4) appear in the extended Kac table of the logarithmic minimal model for critical dense polymers, usually denoted by $\mathcal{L}(1, 2)$. This is the first (and best studied) example of a family of integrable logarithmic minimal model $\mathcal{L}(p, p')$, with $1 < p < p'$ and p, p' coprimes, describing non-intersecting loops in a lattice. The CFT description corresponds to the $\mathbb{Z}_4$ sector of the (η, ξ) ghost logarithmic CFT [87], usually referred as *symplectic fermions* [90]. The dimension $-\frac{3}{32}$ corresponds to the ground state of the $\mathbb{Z}_4$ sector. The states with dimensions $\frac{21}{32}$ and $\frac{5}{32}$ are obtained by acting on the ground state with a single creation mode of the η and ξ fields, respectively.

To discuss our results, we follow the same steps of the previous example. The calculation of the Nekrasov partition function gives

$$\tilde{Z}^{\text{inst}}(\nu) = 1 + \sum_{k=1}^{\infty} \frac{1}{4\nu^2 - 1} P_k(\nu) x^k, \quad (5.5)$$

where now the first polynomials $P_k(\nu)$ are given by

$$\begin{aligned} P_1(\nu) &= \frac{1}{8} (8\nu^4 + 42\nu^2 - 15), \\ P_2(\nu) &= \frac{1}{128} (16\nu^6 + 224\nu^4 + 783\nu^2 - 330), \\ P_3(\nu) &= \frac{32\nu^8 + 832\nu^6 + 7330\nu^4 + 20963\nu^2 - 9852}{3072}, \\ P_4(\nu) &= \frac{128\nu^{10} + 5280\nu^8 + 82792\nu^6 + 580950\nu^4 + 1461705\nu^2 - 745290}{196608}, \end{aligned} \quad (5.6)$$

$$\begin{aligned}
 P_5(\nu) &= \frac{1}{7864320} (256 \nu^{12} + 15296 \nu^{10} + 369200 \nu^8 + 4494100 \nu^6 + 27350124 \nu^4 \\
 &\quad + 62752839 \nu^2 - 34153470), \\
 P_6(\nu) &= \frac{1}{377487360} (512 \nu^{14} + 41728 \nu^{12} + 1426880 \nu^{10} + 26202400 \nu^8 + 271958978 \nu^6 \\
 &\quad + 1497332707 \nu^4 + 3199335720 \nu^2 - 1838760300).
 \end{aligned}$$

Again, once the prefactor is removed, the result simplifies drastically and the function H has a very simple expansion

$$H_\nu(q) = 1 - \frac{8q}{4\nu^2 - 1} + \frac{8q^2}{4\nu^2 - 1} - \frac{32q^3}{4\nu^2 - 1} + \frac{40q^4}{4\nu^2 - 1} - \frac{48q^5}{4\nu^2 - 1} + \frac{32q^6}{4\nu^2 - 1} + \mathcal{O}(q^7), \quad (5.7)$$

suggesting the following exact form

$$H_\nu(q) = 1 + \frac{3 - 2E_2 - 2\vartheta_2^4 - \vartheta_4^4}{3(4\nu^2 - 1)}. \quad (5.8)$$

In this second example the relation between ν and the internal conformal dimension Δ is $\nu = \frac{1}{2}\sqrt{8\Delta + 1}$. Thus, equation (5.8) is equivalent to the following prediction for the conformal block

$$H_\Delta \left[\begin{matrix} \frac{5}{32} & \frac{5}{32} \\ -\frac{3}{32} & \frac{21}{32} \end{matrix} \right] (x) = 1 + \frac{1}{24\Delta} (3 - 2E_2 - 2\vartheta_2^4 - \vartheta_4^4). \quad (5.9)$$

To prove (5.9) by a direct CFT calculation we use the effective recursion presented in section 4.3.1. Given the external dimensions and $c = -2$, the arguments of the products $P_{m,n}$ and $A_{m,n}$ are found to be

$$p_{r,s} = \frac{1}{64} (2r - s - 1)(2r - s + 1)(2r - s + 2)(2r - s + 4), \quad \lambda_{k,l} = \sqrt{2}k - \frac{l}{\sqrt{2}}. \quad (5.10)$$

The coefficients $R_{m,n}$ are calculated with the same regularization prescription discussed in section 4.3.1, i.e. we shift one of the indices by δ to compute the products, and take $\delta \rightarrow 0$ at the end. The resulting *effective* coefficients $R_{m,n}^{\text{eff}}$ is again very simple (compare with (4.22))

$$R_{m,n}^{\text{eff}} = \begin{cases} -\frac{1}{16^{m(2m+1)}} m(2m+1), & \text{for } n = 2m+1 \\ -\frac{1}{16^{m(2m-1)}} m(2m-1), & \text{for } n = 2m-1 \\ 0 & \text{otherwise.} \end{cases} \quad (5.11)$$

Inserting in the effective recursion the expression (5.11) together with the Ansatz (4.16), we get the following equation for $\mathcal{H}$

$$\begin{aligned}
 \mathcal{H} &= - \sum_{m=1}^{\infty} m(2m+1) q^{m(2m+1)} \left(1 + \frac{1}{m(2m+1)} \mathcal{H} \right) \\
 &\quad - \sum_{m=1}^{\infty} m(2m-1) q^{m(2m-1)} \left(1 + \frac{1}{m(2m-1)} \mathcal{H} \right),
 \end{aligned} \quad (5.12)$$

which gives

$$\mathcal{H} = -\frac{\sum_{m=1}^{\infty} [m(2m+1)q^{m(2m+1)} + m(2m-1)q^{m(2m-1)}]}{1 + \sum_{m=1}^{\infty} [q^{m(2m+1)} + q^{m(2m-1)}]}. \quad (5.13)$$

Finally, summing the series, one recovers the combination $\frac{1}{24}(3 - 2E_2 - 2\vartheta_2^4 - \vartheta_4^4)$ in equation (5.9), i.e.,

$$\mathcal{H} = \frac{1}{24}(3 - 2E_2 - 2\vartheta_2^4 - \vartheta_4^4). \quad (5.14)$$

To prove this fact, we begin by observing that the denominator of (5.13) may be written as

$$1 + \sum_{m=1}^{\infty} [q^{m(2m+1)} + q^{m(2m-1)}] = \sum_{m=1}^{\infty} q^{m(m-1)/2} = \frac{1}{2} q^{-1/8} \vartheta_2(\sqrt{q}). \quad (5.15)$$

Hence, we have

$$\mathcal{H} = -q \frac{d}{dq} \log \left(\frac{1}{2} q^{-1/8} \vartheta_2(\sqrt{q}) \right). \quad (5.16)$$

Using (A.11), this is

$$\mathcal{H} = -\frac{1}{2} q \frac{d}{dq} \log \left(\frac{1}{2} q^{-1/4} \vartheta_2(q) \vartheta_3(q) \right). \quad (5.17)$$

The derivative can be computed using (A.10). After simplification by (A.9) we prove (5.14).

Complete prepotential. Our results for the perturbative and instanton contribution to the prepotential are

$$\begin{aligned} F^{\text{pert}}(\varepsilon, a, \mathbf{m}) &= \varepsilon_1^2 \log \frac{a}{\Lambda} - 4a^2 \log 2 + \frac{1}{2} \varepsilon_1^2 \log \left(1 - \frac{\varepsilon_1^2}{16a^2} \right), \\ F^{\text{inst}}(\varepsilon, a, \mathbf{m}; x) &= \frac{1}{2} \varepsilon_1^2 \log \left(\left(\frac{x}{16q} \right)^{-\frac{2a^2}{\varepsilon_1^2}} (1-x)^{-2} \vartheta_3^{-5} \left(1 + \frac{-3 + 2E_2 + 2\vartheta_2^4 + \vartheta_4^4}{3(1 - \frac{16a^2}{\varepsilon_1^2})} \right) \right). \end{aligned} \quad (5.18)$$

Summing them, we have again a closed form for $F = F^{\text{pert}}(\varepsilon, a, \mathbf{m}) + F^{\text{inst}}(\varepsilon, a, \mathbf{m}; x)$. In this case the perturbative part contributes to all the h_k coefficient, and combining the logarithms one gets the very compact expression

$$F(\varepsilon, a, \mathbf{m}) = \varepsilon_1^2 \log \frac{a}{\Lambda} + \frac{1}{2} \varepsilon_1^2 \log \left(1 - \frac{\varepsilon_1^2 (2(E_2 + \vartheta_2^4) + \vartheta_4^4)}{48a^2} \right). \quad (5.19)$$

As a further check, we have successfully tested (5.19) by comparing it with the general large a expansions, as we did in the previous example.

5.2 A two poles point

Here we move to the second more complex example, where the Nekrasov partition function and its dual block have two poles. The scaling relations are the ones written in (5.2), and they correspond to the following external dimensions for the conformal block

$$\Delta_1 = -\frac{3}{32}, \quad \Delta_2 = \frac{21}{32}, \quad \Delta_3 = \frac{45}{32}, \quad \Delta_4 = -\frac{3}{32}, \quad (5.20)$$

again with central charge $c = -2$. The complete partition functions is

$$\tilde{Z}^{\text{inst}}(\nu) = 1 + \sum_{k=1}^{\infty} \frac{1}{\nu^2(4\nu^2 - 1)} P_k(\nu) x^k. \quad (5.21)$$

The list of the first polynomials $P_k(\nu)$ in this case is given by

$$\begin{aligned} P_1(\nu) &= \frac{1}{4} \nu^2 (4\nu^4 + 79\nu^2 - 2), \\ P_2(\nu) &= \frac{1}{128} (16\nu^8 + 688\nu^6 + 7687\nu^4 + 1323\nu^2 + 60), \\ P_3(\nu) &= \frac{16\nu^{10} + 1112\nu^8 + 26573\nu^6 + 221431\nu^4 + 75642\nu^2 + 4536}{1536}, \\ P_4(\nu) &= \frac{128\nu^{12} + 12704\nu^{10} + 485160\nu^8 + 8492902\nu^6 + 58818493\nu^4 + 28571244\nu^2 + 2117664}{196608}, \\ P_5(\nu) &= \frac{1}{3932160} (128\nu^{14} + 16928\nu^{12} + 915240\nu^{10} + 25357150\nu^8 + 362520277\nu^6 \\ &\quad + 2207662182\nu^4 + 1342390200\nu^2 + 117034560), \\ P_6(\nu) &= \frac{1}{377487360} (512\nu^{16} + 86272\nu^{14} + 6171584\nu^{12} + 240341920\nu^{10} + 5392008818\nu^8 \\ &\quad + 66661116493\nu^6 + 369193795431\nu^4 + 262864732140\nu^2 + 26081676000). \end{aligned} \quad (5.22)$$

Removing as usual the prefactor, the resulting H has the expansion

$$\begin{aligned} H_\nu(q) &= 1 + \frac{72q}{4\nu^2 - 1} + \frac{24(7\nu^2 + 5)q^2}{\nu^2(4\nu^2 - 1)} + \frac{288(\nu^2 + 2)q^3}{\nu^2(4\nu^2 - 1)} + \frac{72(5\nu^2 + 22)q^4}{\nu^2(4\nu^2 - 1)} \\ &\quad + \frac{432(\nu^2 + 8)q^5}{\nu^2(4\nu^2 - 1)} + \frac{672(\nu^2 + 9)q^6}{\nu^2(4\nu^2 - 1)} + \frac{576(\nu^2 + 18)q^7}{\nu^2(4\nu^2 - 1)} + \frac{24(31\nu^2 + 644)q^8}{\nu^2(4\nu^2 - 1)} \\ &\quad + \frac{72(13\nu^2 + 320)q^9}{\nu^2(4\nu^2 - 1)} + \frac{1008(\nu^2 + 31)q^{10}}{\nu^2(4\nu^2 - 1)} + \frac{864(\nu^2 + 50)q^{11}}{\nu^2(4\nu^2 - 1)} + \mathcal{O}(q^{12}). \end{aligned} \quad (5.23)$$

Despite the structure of the expansion is clearly more involved than in the previous cases, one can guess again a closed form

$$H_\nu(q) = \frac{\nu^4 - 2h_1\nu^2 + 2(h_1^2 - h_2)}{\nu^2(\nu^2 - \frac{1}{4})}, \quad (5.24)$$

where h_1 and h_2 are the coefficients in equation (3.7) evaluated at the special point. Explicitly they are:

$$\begin{aligned} h_1 &= \frac{1}{8} (4E_2 - 6\vartheta_2^4 - 3\vartheta_4^4), \\ h_2 &= \frac{1}{192} (40E_2^2 + 6\vartheta_2^4(7\vartheta_4^4 - 20E_2) - 60E_2\vartheta_4^4 + 44E_4 + 42\vartheta_2^8 - 21\vartheta_4^8). \end{aligned} \quad (5.25)$$

Moving to the CFT side using $\nu = \frac{1}{2}\sqrt{8\Delta + 1}$, it is useful to rewrite the prediction for the conformal block as

$$H_\Delta \left[\begin{matrix} \frac{21}{32} & \frac{45}{32} \\ -\frac{3}{32} & -\frac{3}{32} \end{matrix} \right] (x) = 1 + \frac{H_1}{8\Delta + 1} + \frac{H_2}{8\Delta}. \quad (5.26)$$

where the two functions H_1 and H_2 are given by the following combinations of E_2 and Jacobi theta functions:

$$\begin{aligned} H_1 &= \frac{1}{3} \left(-4E_2^2 + 12E_2\vartheta_2^4 + \vartheta_4^4 (6E_2 - 11\vartheta_2^4) - 11\vartheta_2^8 - 2\vartheta_4^8 \right), \\ H_2 &= \frac{1}{3} \left(\vartheta_2^4 (-12E_2 + 11\vartheta_4^4 + 18) + (9 - 6E_2)\vartheta_4^4 + 4(E_2 - 3)E_2 + 11\vartheta_2^8 + 2\vartheta_4^8 + 3 \right). \end{aligned} \quad (5.27)$$

The strategy to prove our guess is again to use the effective recursion. In this case we have

$$p_{r,s} = \frac{1}{64} (2r - s + 2)(2r - s + 4)(-2r + s - 3)^2, \quad \lambda_{k,l} = \sqrt{2} k - \frac{l}{\sqrt{2}}, \quad (5.28)$$

and the effective $R_{m,n}^{\text{eff}}$ coefficients are given by:

$$R_{m,n}^{\text{eff}} = \begin{cases} \frac{4(-1)^m}{16^{2m^2}} m^2 (16m^2 - 1), & \text{for } n = 2m, \text{ with } \Delta_{m,2m} = -\frac{1}{8}, \\ -\frac{(-1)^m}{16^{m(2m+1)}} m(2m+1)(4m+1)^2, & \text{for } n = 2m+1, \text{ with } \Delta_{m,2m+1} = 0, \\ -\frac{(-1)^m}{16^{m(2m-1)}} m(2m-1)(4m-1)^2, & \text{for } n = 2m-1, \text{ with } \Delta_{m,2m-1} = 0. \\ 0 & \text{otherwise.} \end{cases} \quad (5.29)$$

where we wrote explicitly the corresponding value of the pole $\Delta_{m,n}$. The result for the $R_{m,n}^{\text{eff}}$ implies now an Ansatz for the recursion as

$$H_{\Delta} \left[\begin{matrix} \frac{21}{32} & \frac{45}{32} \\ -\frac{3}{32} & -\frac{3}{32} \end{matrix} \right] (x) = 1 + \frac{\mathcal{H}_1}{8\Delta + 1} + \frac{\mathcal{H}_2}{8\Delta}. \quad (5.30)$$

Inserting this Ansatz in the recursion relation, one gets

$$\frac{\mathcal{H}_1}{8\Delta + 1} + \frac{\mathcal{H}_2}{8\Delta} = \sum_{m,n=1}^{\infty} \frac{16^{mn} q^{mn} R_{m,n}^{\text{eff}}}{\Delta - \Delta_{m,n}} \left(1 + \frac{\mathcal{H}_1}{8(\Delta_{m,n} + mn) + 1} + \frac{\mathcal{H}_2}{8(\Delta_{m,n} + mn)} \right). \quad (5.31)$$

Using the expressions for the $R_{m,n}^{\text{eff}}$ and separating the contributions coming from the two poles, one gets, after some simple algebra, an equation for the two functions $\mathcal{H}_1$ and $\mathcal{H}_2$. As in the previous cases this allows to write the $\mathcal{H}_i$ as a combination of series. The result is slightly more involved than in the one pole cases and reads

$$\begin{aligned} \mathcal{H}_1 &= \frac{32((\mathcal{S}_1 + 1)\mathcal{S}_A - 8\mathcal{S}_2\mathcal{S}_c)}{(\mathcal{S}_1 + 1)(1 - 2\mathcal{S}_B) + 256\mathcal{S}_3\mathcal{S}_c}, \\ \mathcal{H}_2 &= \frac{8(32\mathcal{S}_3\mathcal{S}_A + \mathcal{S}_2(1 - 2\mathcal{S}_B))}{(\mathcal{S}_1 + 1)(2\mathcal{S}_B - 1) - 256\mathcal{S}_3\mathcal{S}_c}, \end{aligned} \quad (5.32)$$

where the $\mathcal{S}_j$ are the following series

$$\begin{aligned} \mathcal{S}_A &= \sum_{m=1}^{\infty} (-1)^m m^2 (16m^2 - 1) q^{2m^2}, \\ \mathcal{S}_B &= \sum_{m=1}^{\infty} (-1)^m (16m^2 - 1) q^{2m^2}, \end{aligned}$$

$$\begin{aligned}
 \mathcal{S}_C &= \sum_{m=1}^{\infty} (-1)^m m^2 q^{2m^2}, \\
 \mathcal{S}_1 &= \sum_{m=1}^{\infty} (-1)^m \left((4m-1)^2 q^{m(2m-1)} + (4m+1)^2 q^{m(2m+1)} \right), \\
 \mathcal{S}_2 &= \sum_{m=1}^{\infty} (-1)^m m \left((4m-1)^2 (2m-1) q^{m(2m-1)} + (2m+1)(4m+1)^2 q^{m(2m+1)} \right), \\
 \mathcal{S}_3 &= \sum_{m=1}^{\infty} (-1)^m m \left((2m-1) q^{m(2m-1)} + (2m+1) q^{m(2m+1)} \right).
 \end{aligned} \tag{5.33}$$

We do not attempt to put (5.30) in the form (5.27), but this equivalence may be checked at very high order in q by exploiting the explicit sums in (5.33). This proves in a very non-trivial way the power of the effective recursion approach.

Complete prepotential. As we did in the previous cases we can collect our results and give the complete prepotential in closed form. Combining (5.3) and the logarithm of the instanton part (5.24) we obtain

$$\begin{aligned}
 F(\varepsilon, a, \mathbf{m}) &= 2\varepsilon_1^2 \log \frac{a}{\Lambda} + \frac{1}{2}\varepsilon_1^2 \log \left(1 + \frac{\varepsilon_1^2 (-4E_2 + 6\vartheta_2^4 + 3\vartheta_4^4)}{16a^2} \right. \\
 &\quad \left. + \frac{\varepsilon_1^4 (4E_2^2 - 12E_2\vartheta_2^4 + \vartheta_4^4 (11\vartheta_2^4 - 6E_2) + 11\vartheta_2^8 + 2\vartheta_4^8)}{768a^4} \right).
 \end{aligned} \tag{5.34}$$

6 Systematic study of one and two poles partition functions

In this section, we present a systematic analysis of the 1- and 2-poles partition functions in the $N_f = 4$ theory. This is an extension of our previous study of 1, 2, 3, and 4-poles partition functions in the $\mathcal{N} = 2^*$ theory [63]. From the conceptual point of view, there is no major difference apart from the appearance of Jacobi theta functions besides Eisenstein series encoding the full instanton dependence. This is expected from the detailed form of S-duality in the gauge theory. In terms of Virasoro conformal blocks, this amounts to very specific Ansatzë resumming the expansion in powers of the q parameter. From a computational point of view, the four mass ratios together with the $\varepsilon_1/\varepsilon_2$ ratio determine a quite large space of parameters where special n -poles partition functions are found. In practice, a brute force study as in [63] is not feasible besides the 1-pole solutions. To overcome these difficulties, we shall adopt a novel approach following from certain structural regularities in the combination of the perturbative and instanton prepotential. These features, to be discussed in details later, were already discussed in [63], but not exploited. This is rather natural in the gauge theory, but rather non-trivial from the Virasoro perspective, i.e. assuming AGT correspondence.

6.1 The set Π_1 of one-pole partition functions

In the previous sections, we provided special one-pole partition functions showing how to extend to the $N_f = 4$ theory the results derived in [63] for the $\mathcal{N} = 2^*$ theory. In

general, one has a set Π_1 of special one-pole points (α, β) where α and β are the following fixed ratios

$$\alpha = \left(\frac{m_1}{\varepsilon_1}, \frac{m_2}{\varepsilon_1}, \frac{m_3}{\varepsilon_1}, \frac{m_4}{\varepsilon_1} \right), \quad \beta = \frac{\varepsilon_2}{\varepsilon_1}. \quad (6.1)$$

In terms of the dual CFT, the ratio β determines the central charge, while α determine the conformal dimensions $\Delta_{1,2,3,4}$ of the relevant 4-point function, according to (3.20). Due to the symmetry of the partition function under a permutation of $\mathbf{m}$, it is obvious that a simple permutation of the α_i will lead to the same result for Z^{inst} (but not to the same set of Δ_i in the dual CFT, see below). Moreover, one can check that if a set α gives a one-pole partition function, then any arbitrary sign choice for the α_i still gives a one-pole Z^{inst} that may be different from the initial one. The position of the pole is uniquely determined by value of the ratio β . For instance, with these definitions, the $c = 1$ case discussed in section 4 belongs to the group of special points

$$\alpha = \left(\frac{1}{2}, \frac{1}{2}, 1, 1 \right), \quad \beta = -1, \quad (6.2)$$

while the one-pole example in section 5 is part of the group

$$\alpha = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 1 \right), \quad \beta = -\frac{1}{2}. \quad (6.3)$$

As stated above, if one is interested in the precise correspondence between a special point and its dual conformal block, the order of the α_i is important, even if Z^{inst} is invariant for permutations. Indeed, the external dimensions for the conformal block in the dual CFT, written as functions of α and β are, see again (3.20),

$$\begin{aligned} \Delta_1 &= \frac{(\alpha_1 - \alpha_2 + \beta + 1)(-\alpha_1 + \alpha_2 + \beta + 1)}{4\beta}, \\ \Delta_2 &= \frac{(-\alpha_1 - \alpha_2 + \beta + 1)(\alpha_1 + \alpha_2 + \beta + 1)}{4\beta}, \\ \Delta_3 &= \frac{(-\alpha_3 - \alpha_4 + \beta + 1)(\alpha_3 + \alpha_4 + \beta + 1)}{4\beta}, \\ \Delta_4 &= \frac{(\alpha_3 - \alpha_4 + \beta + 1)(-\alpha_3 + \alpha_4 + \beta + 1)}{4\beta}, \end{aligned} \quad (6.4)$$

and are *not* invariant for permutations of the α_i . Looking again at the Π_1 point in (6.2), we see that external dimensions associated to the $\alpha = (\frac{1}{2}, \frac{1}{2}, 1, 1)$ are $\Delta = (0, \frac{1}{4}, 1, 0)$. Instead, the exchange $\alpha_2 \leftrightarrow \alpha_3$ gives $\Delta = (\frac{1}{16}, \frac{9}{16}, \frac{9}{16}, \frac{1}{16})$ that are the values we used in (4.2) to exploit the Ramond scalar CFT realization. The symmetry of Z^{inst} ensures that the two conformal blocks have the same analytic form, as one can explicitly check order by order in the instanton expansion.

The examples (6.2) and (6.3) are part of a complete list that may be found in various ways, for instance by a brute force analysis of the Nekrasov functions or, alternatively, by the method suggested in [67]. We exploited these approaches, but we also devised a more interesting search algorithm that is based on the special simple form of F^{pert} at the

points Π_N . As we observed in the illustrative examples and in the analysis of [63], the perturbative prepotential is essentially the sum of $\log(1 - \nu_k^2/\nu^2)$ over all pole positions ν_k . This is a peculiar property of the points Π_N and may be exploited to propose candidate solutions (α, β) . This method is mandatory beyond the one-pole case and is described in appendix B. Each solution in Π_1 has input data (α, β) from which one can write the instanton partition function in the general form

$$Z^{\text{inst}} = \left(\frac{x}{16q} \right)^{\frac{\nu^2}{4\beta}} (1-x)^{X_1} \vartheta_3(q)^{X_2} H_\nu(q), \quad (6.5)$$

where, see (3.23)

$$X_1 = \frac{(\beta + 1 + \sum_{i=1}^4 \alpha_i)^2}{4\beta}, \quad X_2 = \frac{-(\beta + 1)^2 + 2 \sum_{i=1}^4 \alpha_i^2}{\beta}, \quad (6.6)$$

and the function $H_\nu(q)$ can be written (the combination inside the square brackets can be identified with h_1 , in the large a expansion (3.6))

$$H_\nu(q) = \frac{\nu^2 + \frac{1}{\beta} [c_1(\alpha, \beta) E_2 + c_2(\alpha, \beta) \vartheta_2^4 + c_3(\alpha, \beta) \vartheta_4^4]}{(\nu^2 - \nu_1^2)}, \quad (6.7)$$

where ν_1 is the position of the pole, and with

$$\begin{aligned} c_1(\alpha, \beta) &= \frac{1}{24} (2S_2 - (\beta + 1)^2 + \beta) (2S_2 - (\beta + 1)^2 + 3\beta), \\ c_2(\alpha, \beta) &= \frac{1}{12} S_4 - \frac{1}{6} S_{2,2} - 2S_{1,1,1,1}, \\ c_3(\alpha, \beta) &= \frac{1}{6} S_4 - \frac{1}{3} S_{2,2}, \end{aligned} \quad (6.8)$$

and

$$S_2 = \sum_i \alpha_i^2, \quad S_4 = \sum_i \alpha_i^4, \quad S_{2,2} = \sum_{i < j} \alpha_i^2 \alpha_j^2, \quad S_{1,1,1,1} = \alpha_1 \alpha_2 \alpha_3 \alpha_4. \quad (6.9)$$

The quantities in (6.6) and (6.9) are invariant under permutations. Different choices of signs of the four α_i modify only X_1 and $S_{1,1,1,1}$, i.e. the coefficient c_2 .

Before listing all the solutions, we complete the collection of formulas writing also the closed form for the perturbative part, which is given by

$$\begin{aligned} F^{\text{pert}} &= -\frac{1}{2} \varepsilon_1^2 (\beta(\beta + 1) - 2S_2 + 1) \log \left(\frac{a}{\Lambda} \right) - 4a^2 \log 2 \\ &\quad - \beta \varepsilon_1^2 \log \left(1 - \frac{\varepsilon_1^2 \nu_1^2}{4a^2} \right). \end{aligned} \quad (6.10)$$

Putting together the above perturbative part with the logarithm of the instanton partition function (6.5), one gets the complete prepotential. Note that the arguments of the logarithms nicely combine, and the resulting formula (discarding as usual all the a independent

terms) is very compact

$$F = F^{\text{pert}} + F^{\text{inst}} = -\frac{1}{2}\varepsilon_1^2 (\beta(\beta+1) - 2S_2 + 1) \log\left(\frac{a}{\Lambda}\right) - \beta \varepsilon_1^2 \log\left(1 + \frac{\varepsilon_1^2}{4a^2\beta} (c_1(\alpha, \beta) E_2 + c_2(\alpha, \beta) \vartheta_2^4 + c_3(\alpha, \beta) \vartheta_4^4)\right). \quad (6.11)$$

In table 1 and 2, we give the complete list of the groups of one-pole solutions. In particular table 1 collects all the special points such that the single pole is already present in the one-instanton Nekrasov function Z_1 . For the special points in table 2, Z_1 has no poles, and the single pole is present starting from the two-instanton partition function Z_2 . All these solutions have been further cross-checked with a high instanton explicit calculation. According to [67], modular transformation properties imply that the number of poles of a spherical finite pole block is given by the nice expression

$$\# \text{ poles} = \left\lfloor \sum_{i=1}^4 \Delta_i - \frac{c+1}{8} \right\rfloor. \quad (6.12)$$

Consistently with our remarks, this expression depends on α through the combination S_2 and is thus independent on permutations and change of signs of α . One checks that the expression in (6.12) is indeed equal to 1 for the points Π_1 listed in table 1 and 2. We also remark that the analysis of [67] shows that finite pole conformal blocks may only be found at minimal model values of the central charge, $c = 1 - 6\frac{(n-m)^2}{nm}$, with n, m being positive coprime integers. This implies that β must be a negative rational number. Again, this statement is true in all the solutions we found.

As a final comment, we remind that after equation (3.8), we remarked that the h_k combinations are quasi-modular forms, taking into account the transformation properties of the $\text{SO}(8)$ invariant combinations of masses. Once the masses are fixed, as in the special points, this is of course no longer true. Nevertheless, from the explicit form of c_2 , one can note that there are only two possible outcomes for the prepotential within each group of special points. In fact, two distinct outcomes arise only for the special points in table 1. For the special points in table 2 the prepotential is unique for each group, since one of the α_i is zero. It turns out that the $\mathcal{T}$ generator of the modular transformations maps the two outcomes into each other for each group in table 1, while the single prepotentials for any group in table 2 are $\mathcal{T}$ -invariant.

6.2 The set Π_2 of two-poles partition functions

Using the algorithm in section B, we can extend the analysis presented in the previous section and consider the complete list of special points $\Pi_2 = \{(\alpha, \beta)\}$, leading to two poles solutions. Again, we are able to write a closed form for the instanton partition function for all the Π_2 special points. Of course, the complete Z^{inst} is still of the form (6.5), so we write here only the function $H_\nu(q)$, that for the Π_2 cases is given by

$$H_\nu(q) = \frac{\nu^4 + \nu^2 \frac{1}{\beta} \mathcal{H}_1 + \frac{1}{\beta} \left(\frac{1}{2\beta} \mathcal{H}_1^2 + \mathcal{H}_2 \right)}{(\nu^2 - \nu_1^2) (\nu^2 - \nu_2^2)}. \quad (6.13)$$

Special point, β and α	CFT c	ν_1	X_1	X_2	c_1	c_2	c_3
$\beta = -3$, $\{\frac{1}{2}, \frac{3}{2}, 3, 1\}$	-7	2	$-\frac{4}{3}$	-7	9	-3	-3
$\beta = -2$, $\{\frac{1}{2}, \frac{1}{2}, 2, 1\}$	-2	1	$-\frac{9}{8}$	-5	$\frac{4}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$
$\beta = -2$, $\{\frac{1}{2}, \frac{3}{2}, 2, 1\}$	-2	1	-2	-7	4	-4	-4
$\beta = -\frac{3}{2}$, $\{\frac{3}{4}, \frac{1}{4}, 1, \frac{3}{2}\}$	0	$\frac{1}{2}$	$-\frac{3}{2}$	-5	$\frac{3}{4}$	$-\frac{3}{4}$	$-\frac{3}{4}$
$\beta = -\frac{3}{2}$, $\{\frac{3}{4}, \frac{5}{4}, 1, \frac{3}{2}\}$	0	$\frac{1}{2}$	$-\frac{8}{3}$	-7	$\frac{9}{4}$	$-\frac{15}{4}$	$-\frac{15}{4}$
$\beta = -1$, $\{\frac{1}{2}, 1, \frac{1}{2}, 1\}$	1	0	$-\frac{9}{4}$	-5	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$
$\beta = -\frac{2}{3}$, $\{\frac{1}{2}, \frac{1}{6}, 1, \frac{2}{3}\}$	0	$\frac{1}{3}$	$-\frac{8}{3}$	-5	$\frac{4}{27}$	$-\frac{4}{27}$	$-\frac{4}{27}$
$\beta = -\frac{2}{3}$, $\{\frac{1}{2}, \frac{5}{6}, 1, \frac{2}{3}\}$	0	$\frac{1}{3}$	$-\frac{25}{6}$	-7	$\frac{4}{9}$	$-\frac{20}{27}$	$-\frac{20}{27}$
$\beta = -\frac{1}{2}$, $\{\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 1\}$	-2	$\frac{1}{2}$	$-\frac{25}{8}$	-5	$\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{1}{24}$
$\beta = -\frac{1}{2}$, $\{\frac{1}{4}, \frac{3}{4}, \frac{1}{2}, 1\}$	-2	$\frac{1}{2}$	$-\frac{9}{2}$	-7	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$
$\beta = -\frac{1}{3}$, $\{\frac{1}{2}, \frac{1}{6}, \frac{1}{3}, 1\}$	-7	$\frac{2}{3}$	$-\frac{16}{3}$	-7	$\frac{1}{9}$	$-\frac{1}{27}$	$-\frac{1}{27}$

Table 1. First part of the complete list of the one-pole special points $\Pi_1 = \{(\alpha, \beta)\}$. This first group of solutions corresponds to cases where the pole is already present in the first non trivial Nekrasov function Z_1 . The central charge c of the corresponding CFT is obtained from (3.20). In the other columns we report, as example, the values for the two non-trivial exponents X_1 and X_2 and the coefficients c_i for the particular choice (+ + +) for the signs of α . All the other values can be obtained from the formulas in the text.

In the previous formula $\nu_{1,2}$ are the positions of the two poles, $\mathcal{H}_1$ is the same combination $c_1(\alpha, \beta) E_2 + c_2(\alpha, \beta) \vartheta_2^4 + c_3(\alpha, \beta) \vartheta_4^4$ defined in the previous section, see (6.7) and (6.8), and finally $\mathcal{H}_2$ is given by

$$\begin{aligned} \mathcal{H}_2 = & d_1(\alpha, \beta) E_2 \vartheta_2^4 + d_2(\alpha, \beta) \vartheta_2^8 + d_3(\alpha, \beta) E_2 \vartheta_4^4 + d_4(\alpha, \beta) \vartheta_4^8 \\ & + d_5(\alpha, \beta) \vartheta_2^4 \vartheta_4^4 + d_6(\alpha, \beta) E_2^2 + d_7(\alpha, \beta) E_4. \end{aligned} \quad (6.14)$$

The seven coefficients $d_i(\alpha, \beta), i = 1, \dots, 7$, are

$$\begin{aligned} d_1(\alpha, \beta) &= \frac{1}{36} (-2S_{2,2} - 24S_{1,1,1,1} + S_4) (2S_2 - (\beta - 1)^2), \\ d_2(\alpha, \beta) &= -\frac{1}{36} (-2S_{2,2} - 24S_{1,1,1,1} + S_4) (S_2 - 2(\beta(\beta + 1) + 1)), \\ d_3(\alpha, \beta) &= \frac{1}{18} (S_4 - 2S_{2,2}) (2S_2 - (\beta - 1)^2), \\ d_4(\alpha, \beta) &= \frac{1}{18} (S_4 - 2S_{2,2}) (S_2 - 2(\beta(\beta + 1) + 1)), \\ d_5(\alpha, \beta) &= \frac{1}{18} (-2S_{2,2} + 24S_{1,1,1,1} + S_4) (S_2 - 2(\beta(\beta + 1) + 1)), \\ d_6(\alpha, \beta) &= -\frac{1}{144} ((\beta - 1)^2 - 2S_2) ((\beta - 1)\beta - 2S_2 + 1) (\beta(\beta + 1) - 2S_2 + 1), \end{aligned} \quad (6.15)$$

Special point, β and α	CFT c	ν_1	X_1	X_2	c_1	c_2	c_3
$\beta = -4$, $\{0, \frac{3}{2}, 2, \frac{7}{2}\}$	$-\frac{25}{2}$	2	-1	-7	16	0	0
$\beta = -3$, $\{0, 1, \frac{3}{2}, \frac{5}{2}\}$	-7	1	$-\frac{3}{4}$	-5	3	0	0
$\beta = -3$, $\{0, \frac{3}{2}, 2, \frac{5}{2}\}$	-7	1	$-\frac{4}{3}$	-7	9	-3	-3
$\beta = -2$, $\{0, \frac{1}{2}, 1, \frac{5}{2}\}$	-2	2	$-\frac{9}{8}$	-7	4	2	2
$\beta = -2$, $\{0, 1, \frac{3}{2}, \frac{3}{2}\}$	-2	0	$-\frac{9}{8}$	-5	$\frac{4}{3}$	$-\frac{2}{3}$	$-\frac{2}{3}$
$\beta = -\frac{3}{2}$, $\{0, \frac{1}{4}, \frac{1}{2}, \frac{9}{4}\}$	0	2	$-\frac{25}{24}$	-7	$\frac{9}{4}$	$\frac{15}{8}$	$\frac{15}{8}$
$\beta = -\frac{3}{2}$, $\{0, \frac{1}{2}, \frac{3}{4}, \frac{7}{4}\}$	0	1	$-\frac{25}{24}$	-5	$\frac{3}{4}$	$\frac{3}{8}$	$\frac{3}{8}$
$\beta = -\frac{4}{3}$, $\{0, \frac{2}{3}, \frac{5}{6}, \frac{3}{2}\}$	$\frac{1}{2}$	$\frac{2}{3}$	$-\frac{4}{3}$	-5	$\frac{16}{27}$	0	0
$\beta = -1$, $\{0, 0, \frac{1}{2}, \frac{3}{2}\}$	1	1	-1	-5	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$\beta = -1$, $\{0, \frac{1}{2}, 1, \frac{3}{2}\}$	1	1	$-\frac{9}{4}$	-7	1	0	0
$\beta = -\frac{3}{4}$, $\{0, \frac{1}{2}, \frac{5}{8}, \frac{9}{8}\}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{25}{12}$	-5	$\frac{3}{16}$	0	0
$\beta = -\frac{2}{3}$, $\{0, \frac{1}{6}, \frac{1}{3}, \frac{3}{2}\}$	0	$\frac{4}{3}$	$-\frac{49}{24}$	-7	$\frac{4}{9}$	$\frac{10}{27}$	$\frac{10}{27}$
$\beta = -\frac{2}{3}$, $\{0, \frac{1}{3}, \frac{1}{2}, \frac{7}{6}\}$	0	$\frac{2}{3}$	$-\frac{49}{24}$	-5	$\frac{4}{27}$	$\frac{2}{27}$	$\frac{2}{27}$
$\beta = -\frac{1}{2}$, $\{0, \frac{1}{4}, \frac{1}{2}, \frac{5}{4}\}$	-2	1	$-\frac{25}{8}$	-7	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$
$\beta = -\frac{1}{2}$, $\{0, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}\}$	-2	0	$-\frac{25}{8}$	-5	$\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{1}{24}$
$\beta = -\frac{1}{3}$, $\{0, \frac{1}{3}, \frac{1}{2}, \frac{5}{6}\}$	-7	$\frac{1}{3}$	$-\frac{49}{12}$	-5	$\frac{1}{27}$	0	0
$\beta = -\frac{1}{3}$, $\{0, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}\}$	-7	$\frac{1}{3}$	$-\frac{16}{3}$	-7	$\frac{1}{9}$	$-\frac{1}{27}$	$-\frac{1}{27}$
$\beta = -\frac{1}{4}$, $\{0, \frac{3}{8}, \frac{1}{2}, \frac{7}{8}\}$	$-\frac{25}{2}$	$\frac{1}{2}$	$-\frac{25}{4}$	-7	$\frac{1}{16}$	0	0

Table 2. Second part of the complete list of the one-pole special points $\Pi_1 = \{(\alpha, \beta)\}$. The second group of solutions corresponds to the cases where Z_1 has no poles, and the single pole appears starting from Z_2 . The content of the table is as in table 1.

$$\begin{aligned}
 d_7(\alpha, \beta) = & \frac{1}{720} (-24S_{2,4} + 432S_{2,2,2} - 13\beta^6 - 10\beta^5 + \beta^4 (54S_2 - 17) + \beta^3 (40S_2 - 10) \\
 & + \beta^2 (-60S_2^2 + 70S_2 - 17) + \beta (-40S_2^2 + 40S_2 - 10) + 8S_2^3 \\
 & - 60S_2^2 + 54S_2 + 24S_6 - 13),
 \end{aligned}$$

where we introduced the additional quantities (extending the list in (6.9))

$$S_6 = \sum_i \alpha_i^6, \quad S_{2,4} = \sum_{i \neq j} \alpha_i^2 \alpha_j^4, \quad S_{2,2,2} = \sum_{i < j < k} \alpha_i^2 \alpha_j^2 \alpha_k^2. \quad (6.16)$$

In all cases, the perturbative prepotential takes the following natural closed form

$$\begin{aligned}
 F^{\text{pert}} = & -\frac{1}{2} \varepsilon_1^2 (\beta(\beta + 1) - 2S_2 + 1) \log \left(\frac{a}{\Lambda} \right) - 4a^2 \log 2 \\
 & - \beta \varepsilon_1^2 \log \left(\frac{\varepsilon_1^4}{16a^4} \left(\frac{4a^2}{\varepsilon_1^2} - \nu_1^2 \right) \left(\frac{4a^2}{\varepsilon_1^2} - \nu_2^2 \right) \right). \quad (6.17)
 \end{aligned}$$

As in the case of 1-pole points, the two logarithms coming from F^{pert} and $F^{\text{inst}} = -\varepsilon_1 \varepsilon_2 \log Z^{\text{inst}}$ combine nicely in order to give the following simplified form of the full prepotential

$$F = F^{\text{pert}} + F^{\text{inst}} = -\frac{1}{2} \varepsilon_1^2 (\beta(\beta+1) - 2S_2 + 1) \log\left(\frac{a}{\Lambda}\right) - \beta \varepsilon_1^2 \log\left(1 + \frac{8a^2 \varepsilon_1^2 \beta \mathcal{H}_1 + \varepsilon_1^4 (2\beta \mathcal{H}_2 + \mathcal{H}_1^2)}{32a^4 \beta^2}\right). \quad (6.18)$$

The number of solutions Π_2 is 74, significantly larger than for Π_1 , so the result are collected in two tables, table 3 and table 4. All the Π_2 points have been tested successfully by comparing them with a direct calculation of Z^{inst} at high instanton number. One finally checks that the quantity defined in (6.12) is equal to 2 in all cases.

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A Special functions and useful identities

In this paper the relation between the modular parameters τ and q is $q = e^{i\pi\tau}$. Accordingly, the Dedekind η function reads

$$\eta(\tau) = q^{\frac{1}{12}} \prod_{k=1}^{\infty} (1 - q^{2k}). \quad (A.1)$$

The definition of the Jacobi $\vartheta_{2,3,4}(q)$ functions as series expansions around $q = 0$ is

$$\vartheta_2(q) = 2q^{1/4} \sum_{n=0}^{\infty} q^{n(n+1)}, \quad \vartheta_3(q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2}, \quad \vartheta_4(q) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}. \quad (A.2)$$

The first 3 Eisenstein series are

$$E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n}, \quad E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \quad E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}. \quad (A.3)$$

The Eisenstein series can be written in terms of Jacobi ϑ functions (all functions with argument q if not specified) by means of the relations

$$\begin{aligned} E_2 &= 12q \frac{d}{dq} \log \vartheta_3 + \vartheta_4^4 - \vartheta_2^4, \\ E_4 &= \frac{1}{2} (\vartheta_2^8 + \vartheta_3^8 + \vartheta_4^8) = (\vartheta_2^8 + \vartheta_2^4 \vartheta_4^4 + \vartheta_4^8), \\ E_6 &= \frac{1}{2} (\vartheta_2^4 + \vartheta_3^4) (\vartheta_4^4 - \vartheta_2^4) (\vartheta_3^4 + \vartheta_4^4). \end{aligned} \quad (A.4)$$

Special point, β and α	CFT c	ν_1	ν_2	Special point, β and α	CFT c	ν_1	ν_2
$\beta = -6$, $\{0, \frac{5}{2}, 3, \frac{11}{2}\}$	-24	2	4	$\beta = -1$, $\{\frac{1}{2}, \frac{1}{2}, 1, 2\}$	1	0	2
$\beta = -5$, $\{0, 2, \frac{5}{2}, \frac{9}{2}\}$	$-\frac{91}{5}$	1	3	$\beta = -1$, $\{0, 1, \frac{3}{2}, \frac{3}{2}\}$	1	1	1
$\beta = -5$, $\{0, \frac{5}{2}, 3, \frac{9}{2}\}$	$-\frac{91}{5}$	1	3	$\beta = -\frac{4}{5}$, $\{0, \frac{2}{5}, \frac{11}{10}, \frac{3}{2}\}$	$\frac{7}{10}$	$\frac{2}{5}$	$\frac{6}{5}$
$\beta = -5$, $\{\frac{1}{2}, 2, \frac{5}{2}, 5\}$	$-\frac{91}{5}$	2	4	$\beta = -\frac{3}{4}$, $\{\frac{1}{8}, \frac{3}{8}, 1, \frac{3}{2}\}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{4}$
$\beta = -4$, $\{0, 2, \frac{5}{2}, \frac{7}{2}\}$	$-\frac{25}{2}$	0	2	$\beta = -\frac{3}{4}$, $\{\frac{3}{8}, \frac{7}{8}, 1, \frac{3}{2}\}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{5}{4}$
$\beta = -4$, $\{\frac{1}{2}, \frac{3}{2}, 2, 4\}$	$-\frac{25}{2}$	1	3	$\beta = -\frac{3}{4}$, $\{0, \frac{1}{2}, \frac{9}{8}, \frac{13}{8}\}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
$\beta = -4$, $\{\frac{1}{2}, 2, \frac{5}{2}, 4\}$	$-\frac{25}{2}$	1	3	$\beta = -\frac{3}{4}$, $\{0, \frac{1}{2}, \frac{5}{8}, \frac{15}{8}\}$	$\frac{1}{2}$	$\frac{1}{2}$	2
$\beta = -3$, $\{\frac{1}{2}, \frac{3}{2}, 2, 3\}$	-7	0	2	$\beta = -\frac{2}{3}$, $\{0, \frac{1}{3}, \frac{5}{6}, \frac{3}{2}\}$	0	0	$\frac{4}{3}$
$\beta = -3$, $\{0, 1, \frac{3}{2}, \frac{7}{2}\}$	-7	1	3	$\beta = -\frac{2}{3}$, $\{\frac{1}{2}, \frac{2}{3}, 1, \frac{7}{6}\}$	0	$\frac{1}{3}$	$\frac{2}{3}$
$\beta = -3$, $\{1, \frac{3}{2}, \frac{5}{2}, 3\}$	-7	1	2	$\beta = -\frac{2}{3}$, $\{\frac{1}{6}, \frac{1}{2}, 1, \frac{4}{3}\}$	0	$\frac{1}{3}$	1
$\beta = -3$, $\{0, \frac{3}{2}, 2, \frac{7}{2}\}$	-7	1	3	$\beta = -\frac{2}{3}$, $\{0, \frac{1}{6}, \frac{1}{2}, \frac{5}{3}\}$	0	$\frac{1}{3}$	$\frac{5}{3}$
$\beta = -\frac{5}{2}$, $\{\frac{3}{4}, \frac{5}{4}, 2, \frac{5}{2}\}$	$-\frac{22}{5}$	$\frac{1}{2}$	$\frac{3}{2}$	$\beta = -\frac{2}{3}$, $\{\frac{1}{2}, \frac{5}{6}, 1, \frac{4}{3}\}$	0	$\frac{1}{3}$	1
$\beta = -\frac{5}{2}$, $\{\frac{5}{4}, \frac{7}{4}, 2, \frac{5}{2}\}$	$-\frac{22}{5}$	$\frac{1}{2}$	$\frac{3}{2}$	$\beta = -\frac{2}{3}$, $\{\frac{1}{6}, \frac{2}{3}, 1, \frac{3}{2}\}$	0	$\frac{1}{3}$	$\frac{4}{3}$
$\beta = -\frac{5}{2}$, $\{0, \frac{1}{2}, \frac{3}{4}, \frac{15}{4}\}$	$-\frac{22}{5}$	2	4	$\beta = -\frac{2}{3}$, $\{0, \frac{1}{2}, \frac{5}{6}, \frac{5}{3}\}$	0	$\frac{1}{3}$	$\frac{5}{3}$
$\beta = -2$, $\{1, \frac{3}{2}, \frac{3}{2}, 2\}$	-2	0	1	$\beta = -\frac{2}{3}$, $\{0, \frac{1}{3}, \frac{7}{6}, \frac{3}{2}\}$	0	$\frac{2}{3}$	$\frac{4}{3}$
$\beta = -2$, $\{0, 1, \frac{3}{2}, \frac{5}{2}\}$	-2	0	2	$\beta = -\frac{3}{5}$, $\{0, \frac{1}{5}, \frac{7}{10}, \frac{3}{2}\}$	$-\frac{3}{5}$	$\frac{1}{5}$	$\frac{7}{5}$
$\beta = -2$, $\{0, \frac{1}{2}, \frac{1}{2}, 3\}$	-2	1	3	$\beta = -\frac{3}{5}$, $\{\frac{3}{10}, \frac{1}{2}, 1, \frac{6}{5}\}$	$-\frac{3}{5}$	$\frac{2}{5}$	$\frac{4}{5}$
$\beta = -2$, $\{\frac{1}{2}, 1, 2, \frac{5}{2}\}$	-2	1	2	$\beta = -\frac{3}{5}$, $\{0, \frac{7}{10}, \frac{4}{5}, \frac{3}{2}\}$	$-\frac{3}{5}$	$\frac{1}{5}$	$\frac{7}{5}$
$\beta = -2$, $\{0, \frac{1}{2}, \frac{3}{2}, 3\}$	-2	1	3	$\beta = -\frac{1}{2}$, $\{\frac{1}{2}, \frac{3}{4}, \frac{3}{4}, 1\}$	-2	0	$\frac{1}{2}$
$\beta = -\frac{5}{3}$, $\{0, \frac{1}{3}, \frac{7}{6}, \frac{5}{2}\}$	$-\frac{3}{5}$	$\frac{1}{3}$	$\frac{7}{3}$	$\beta = -\frac{1}{2}$, $\{0, \frac{1}{2}, \frac{3}{4}, \frac{5}{4}\}$	-2	0	1

Table 3. First part of the complete list of the two-poles special points. We list the groups $\Pi_2 = \{(\alpha, \beta)\}$, the corresponding value of the CFT central charge and the position of the two poles.

If we define, see (1.1),

$$x = \frac{\vartheta_2^4(q)}{\vartheta_3^4(q)}, \quad (\text{A.5})$$

we can write the following useful alternative forms

$$\begin{aligned}
 E_2(q) &= \frac{6}{\pi} \mathbb{E}(x) \vartheta_3^2(q) - \vartheta_3^4(q) - \vartheta_4^4(q), \\
 E_4(q) &= \left[\frac{2\mathbb{K}(x)}{\pi} \right]^4 [1 - x(1-x)], \\
 E_6(q) &= \left[\frac{2\mathbb{K}(x)}{\pi} \right]^6 (1-2x) \left[1 + \frac{1}{2}x(1-x) \right],
 \end{aligned} \quad (\text{A.6})$$

Special point, β and α	CFT c	ν_1	ν_2	Special point, β and α	CFT c	ν_1	ν_2
$\beta = -\frac{5}{3}, \left\{\frac{1}{2}, \frac{5}{6}, \frac{5}{3}, 2\right\}$	$-\frac{3}{5}$	$\frac{2}{3}$	$\frac{4}{3}$	$\beta = -\frac{1}{2}, \left\{0, \frac{1}{4}, \frac{1}{4}, \frac{3}{2}\right\}$	-2	$\frac{1}{2}$	$\frac{3}{2}$
$\beta = -\frac{5}{3}, \left\{0, \frac{7}{6}, \frac{4}{3}, \frac{5}{2}\right\}$	$-\frac{3}{5}$	$\frac{1}{3}$	$\frac{7}{3}$	$\beta = -\frac{1}{2}, \left\{\frac{1}{4}, \frac{1}{2}, 1, \frac{5}{4}\right\}$	-2	$\frac{1}{2}$	1
$\beta = -\frac{3}{2}, \left\{0, \frac{1}{2}, \frac{5}{4}, \frac{9}{4}\right\}$	0	0	2	$\beta = -\frac{1}{2}, \left\{0, \frac{1}{4}, \frac{3}{4}, \frac{3}{2}\right\}$	-2	$\frac{1}{2}$	$\frac{3}{2}$
$\beta = -\frac{3}{2}, \left\{\frac{3}{4}, 1, \frac{3}{2}, \frac{7}{4}\right\}$	0	$\frac{1}{2}$	1	$\beta = -\frac{2}{5}, \left\{\frac{3}{10}, \frac{1}{2}, \frac{4}{5}, 1\right\}$	$-\frac{22}{5}$	$\frac{1}{5}$	$\frac{3}{5}$
$\beta = -\frac{3}{2}, \left\{\frac{1}{4}, \frac{3}{4}, \frac{3}{2}, 2\right\}$	0	$\frac{1}{2}$	$\frac{3}{2}$	$\beta = -\frac{2}{5}, \left\{\frac{1}{2}, \frac{7}{10}, \frac{4}{5}, 1\right\}$	$-\frac{22}{5}$	$\frac{1}{5}$	$\frac{3}{5}$
$\beta = -\frac{3}{2}, \left\{0, \frac{1}{4}, \frac{3}{4}, \frac{5}{2}\right\}$	0	$\frac{1}{2}$	$\frac{5}{2}$	$\beta = -\frac{2}{5}, \left\{0, \frac{1}{5}, \frac{3}{10}, \frac{3}{2}\right\}$	$-\frac{22}{5}$	$\frac{4}{5}$	$\frac{8}{5}$
$\beta = -\frac{3}{2}, \left\{\frac{3}{4}, \frac{5}{4}, \frac{3}{2}, 2\right\}$	0	$\frac{1}{2}$	$\frac{3}{2}$	$\beta = -\frac{1}{3}, \left\{\frac{1}{6}, \frac{1}{2}, \frac{2}{3}, 1\right\}$	-7	0	$\frac{2}{3}$
$\beta = -\frac{3}{2}, \left\{\frac{1}{4}, 1, \frac{3}{2}, \frac{9}{4}\right\}$	0	$\frac{1}{2}$	2	$\beta = -\frac{1}{3}, \left\{0, \frac{1}{3}, \frac{1}{2}, \frac{7}{6}\right\}$	-7	$\frac{1}{3}$	1
$\beta = -\frac{3}{2}, \left\{0, \frac{3}{4}, \frac{5}{4}, \frac{5}{2}\right\}$	0	$\frac{1}{2}$	$\frac{5}{2}$	$\beta = -\frac{1}{3}, \left\{\frac{1}{3}, \frac{1}{2}, \frac{5}{6}, 1\right\}$	-7	$\frac{1}{3}$	$\frac{2}{3}$
$\beta = -\frac{3}{2}, \left\{0, \frac{1}{2}, \frac{7}{4}, \frac{9}{4}\right\}$	0	1	2	$\beta = -\frac{1}{3}, \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{7}{6}\right\}$	-7	$\frac{1}{3}$	1
$\beta = -\frac{4}{3}, \left\{\frac{1}{6}, \frac{1}{2}, \frac{4}{3}, 2\right\}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{3}$	$\beta = -\frac{1}{4}, \left\{0, \frac{1}{2}, \frac{5}{8}, \frac{7}{8}\right\}$	$-\frac{25}{2}$	0	$\frac{1}{2}$
$\beta = -\frac{4}{3}, \left\{\frac{1}{2}, \frac{7}{6}, \frac{4}{3}, 2\right\}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{5}{3}$	$\beta = -\frac{1}{4}, \left\{\frac{1}{8}, \frac{3}{8}, \frac{1}{2}, 1\right\}$	$-\frac{25}{2}$	$\frac{1}{4}$	$\frac{3}{4}$
$\beta = -\frac{4}{3}, \left\{0, \frac{2}{3}, \frac{3}{2}, \frac{13}{6}\right\}$	$\frac{1}{2}$	$\frac{2}{3}$	2	$\beta = -\frac{1}{4}, \left\{\frac{1}{8}, \frac{1}{2}, \frac{5}{8}, 1\right\}$	$-\frac{25}{2}$	$\frac{1}{4}$	$\frac{3}{4}$
$\beta = -\frac{4}{3}, \left\{0, \frac{2}{3}, \frac{5}{6}, \frac{5}{2}\right\}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{8}{3}$	$\beta = -\frac{1}{5}, \left\{0, \frac{2}{5}, \frac{1}{2}, \frac{9}{10}\right\}$	$-\frac{91}{5}$	$\frac{1}{5}$	$\frac{3}{5}$
$\beta = -\frac{5}{4}, \left\{0, \frac{1}{2}, \frac{11}{8}, \frac{15}{8}\right\}$	$\frac{7}{10}$	$\frac{1}{2}$	$\frac{3}{2}$	$\beta = -\frac{1}{5}, \left\{0, \frac{1}{2}, \frac{3}{5}, \frac{9}{10}\right\}$	$-\frac{91}{5}$	$\frac{1}{5}$	$\frac{3}{5}$
$\beta = -1, \left\{\frac{1}{2}, 1, 1, \frac{3}{2}\right\}$	1	0	1	$\beta = -\frac{1}{5}, \left\{\frac{1}{10}, \frac{2}{5}, \frac{1}{2}, 1\right\}$	$-\frac{91}{5}$	$\frac{2}{5}$	$\frac{4}{5}$
$\beta = -1, \left\{0, 0, \frac{3}{2}, \frac{3}{2}\right\}$	1	1	1	$\beta = -\frac{1}{6}, \left\{0, \frac{5}{12}, \frac{1}{2}, \frac{11}{12}\right\}$	-24	$\frac{1}{3}$	$\frac{2}{3}$

Table 4. Second part of the complete list of the two-poles special points. We list the groups $\Pi_2 = \{(\alpha, \beta)\}$, the corresponding value of the CFT central charge and the position of the two poles.

and also the following identities useful to discuss generic 1-pole partition functions

$$k_1 E_2(q) + k_2 \vartheta_4^4(q) + k_3 \vartheta_2^4(q) = \frac{4}{\pi^2} \mathbb{K}(x) \left\{ 3 k_1 \mathbb{E}(x) + [-2 k_1 + k_2 + (k_1 - k_2 + k_3) x] \mathbb{K}(x) \right\} \quad (\text{A.7})$$

$$\vartheta_2^2(q) = \frac{2}{\pi} \sqrt{x} \mathbb{K}(x), \quad \vartheta_3^2(q) = \frac{2}{\pi} \mathbb{K}(x), \quad \vartheta_4^2(q) = \frac{2}{\pi} \sqrt{1-x} \mathbb{K}(x). \quad (\text{A.8})$$

Additional useful identities are the well known *aequatio identica satis abstrusa*

$$\vartheta_3^4 - \vartheta_2^4 - \vartheta_4^4 = 0, \quad (\text{A.9})$$

and the q -derivatives of theta functions

$$\begin{aligned} q \frac{d}{dq} \log \vartheta_2 &= \frac{1}{12} (E_2 + \vartheta_3^4 + \vartheta_4^4), \\ q \frac{d}{dq} \log \vartheta_3 &= \frac{1}{12} (E_2 + \vartheta_2^4 - \vartheta_4^4), \\ q \frac{d}{dq} \log \vartheta_4 &= \frac{1}{12} (E_2 - \vartheta_2^4 - \vartheta_3^4). \end{aligned} \quad (\text{A.10})$$

We notice also the remarkable duplication identity that has been exploited in the text

$$\vartheta_2(\sqrt{q})^2 = 2 \vartheta_2(q) \vartheta_3(q). \quad (\text{A.11})$$

This can be proved by using the Eulerian product form of the theta functions. To this aim, we define

$$(n) = \prod_{k=1}^{\infty} (1 - q^{kn}), \quad (\text{A.12})$$

and remind that [105]

$$\vartheta_2(q) = 2 q^{1/4} \frac{(4)^2}{(2)}, \quad \vartheta_2(\sqrt{q}) = 2 q^{1/8} \frac{(2)^2}{(1)}, \quad \vartheta_3(q) = \frac{(2)^5}{(1)^2(4)^2}. \quad (\text{A.13})$$

From (A.13), we readily check (A.11).

B An efficient algorithm to find the sets Π_N

Let us illustrate an efficient algorithm that is able to locate special N -pole solutions with moderate computational effort. We examine in details the case of 1-pole solutions. Our starting remark is that the perturbative prepotential takes the special form

$$F^{\text{pert}} = -\beta \varepsilon_1^2 \log \left(1 - \varepsilon_1^2 \frac{\nu_1^2}{\nu^2} \right), \quad (\text{B.1})$$

at all special points. This is an extremely useful Ansatz to locate special points that can be ultimately cross checked by a dedicated high-instanton (≥ 20) computation. As a first step, we match the large a expansion of the perturbative contribution (3.4) with the Ansatz in (B.1). The comparison allows to express the sums $Q_n = \sum_{i=1}^4 \alpha_i^{2n}$ with $n \geq 2$ in terms of the three unknown quantities Q_1 , ν_1^2 , and β . For instance, we find

$$\begin{aligned} Q_2 &= -\frac{\beta^4}{8} - 3\beta\nu_1^2 + \beta^2 \left(\frac{Q_1}{2} - \frac{1}{8} \right) + \frac{Q_1}{2} - \frac{1}{8}, \\ Q_3 &= -\frac{\beta^6}{16} - \frac{15}{4}\beta^3\nu_1^2 + \beta \left(-\frac{15\nu_1^4}{8} - \frac{15\nu_1^2}{4} \right) + \beta^4 \left(\frac{3Q_1}{16} - \frac{11}{64} \right) \\ &\quad + \beta^2 \left(\frac{5Q_1}{8} - \frac{11}{64} \right) + \frac{3Q_1}{16} - \frac{1}{16}, \\ Q_4 &= -\frac{3\beta^8}{128} - \frac{21}{8}\beta^5\nu_1^2 + \beta^3 \left(-\frac{35\nu_1^4}{8} - \frac{35\nu_1^2}{4} \right) + \beta \left(-\frac{7\nu_1^6}{8} - \frac{35\nu_1^4}{8} - \frac{21\nu_1^2}{8} \right) \\ &\quad + \beta^6 \left(\frac{Q_1}{16} - \frac{9}{64} \right) + \beta^4 \left(\frac{7Q_1}{16} - \frac{21}{128} \right) + \beta^2 \left(\frac{7Q_1}{16} - \frac{9}{64} \right) + \frac{Q_1}{16} - \frac{3}{128}, \end{aligned} \quad (\text{B.2})$$

and so on. On the other hand, we can exploit the Viète-Newton theorem [106] to build a polynomial $\mathcal{P}(x)$ with roots $x = \alpha_{1,2,3,4}^2$ compatible with the constraints on $Q_{n \geq 2}$, see for instance (B.2). In particular, the coefficients of the terms $1, x, x^2, x^3, x^4$ are fixed by $Q_{2,3,4,5}$. Instead, the values of $Q_{n \geq 6}$ must be compatible with the claim that $\mathcal{P}(x)$ has

degree 4. These conditions impose algebraic constraints between the three quantities Q_1 , ν_1 , and β . For example, the condition that $\mathcal{P}(x)$ has no x^5 term reads

$$\begin{aligned}
0 = & -\frac{11\beta^{10}}{768} - \frac{125}{64}\beta^7\nu_1^2 + \beta^8\left(\frac{5Q_1}{48} - \frac{55}{512}\right) + \beta^6\left(-\frac{55Q_1^2}{192} + \frac{235Q_1}{384} - \frac{77}{384}\right) \\
& + \beta^5\left(\nu_1^2\left(\frac{185Q_1}{32} - \frac{1415}{128}\right) - \frac{655\nu_1^4}{128}\right) + \beta^4\left(-\frac{75\nu_1^4}{8} + \frac{35Q_1^3}{96} - \frac{455Q_1^2}{384} + \frac{187Q_1}{192} - \frac{77}{384}\right) \\
& + \beta^3\left(-\frac{105\nu_1^6}{32} + \nu_1^4\left(\frac{25Q_1}{4} - \frac{2125}{128}\right) + \nu_1^2\left(-5Q_1^2 + \frac{515Q_1}{32} - \frac{1415}{128}\right)\right) \\
& + \beta^2\left(-\frac{75\nu_1^6}{16} + \nu_1^4\left(\frac{45Q_1}{8} - \frac{75}{8}\right) - \frac{5Q_1^4}{24} + \frac{85Q_1^3}{96} - \frac{455Q_1^2}{384} + \frac{235Q_1}{384} - \frac{55}{512}\right) \\
& + \beta\left(-\frac{45\nu_1^8}{128} + \nu_1^6\left(\frac{35Q_1}{32} - \frac{105}{32}\right) + \nu_1^4\left(-\frac{25Q_1^2}{16} + \frac{25Q_1}{4} - \frac{655}{128}\right)\right. \\
& \left. + \nu_1^2\left(\frac{5Q_1^3}{4} - 5Q_1^2 + \frac{185Q_1}{32} - \frac{125}{64}\right)\right) + \frac{Q_1^5}{24} - \frac{5Q_1^4}{24} + \frac{35Q_1^3}{96} - \frac{55Q_1^2}{192} + \frac{5Q_1}{48} - \frac{11}{768},
\end{aligned} \tag{B.3}$$

with similar conditions from the vanishing of the x^6 and x^7 terms. The solution of this reduced problem is affordable and for each solution (Q_1, ν_1, β) we obtain a specific polynomial $\mathcal{P}(x)$ whose roots can be identified with α_i^2 .

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