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Super-spectral curve of irregular conformal blocks

Dmitri Polyakov[a,b] **and Chaiho Rim**[c]

[a] *Center for Theoretical Physics, College of Physical Science and Technology Sichuan University, Chengdu 6100064, China*

[b] *Institute of Information Transmission Problems (IITP), Bolshoi Karetny per. 19/1, Moscow 127994, Russia*

[c] *Department of Physics and Center for Quantum Spacetime, Sogang University, Seoul 121-742, Korea*

E-mail: polyakov@sogang.ac.kr, rimpine@sogang.ac.kr

Abstract: We use super-spectral curve to investigate irregular conformal states of integer and half-odd integer rank. The spectral curve is the loop equation of supersymmetrized irregular matrix model. The case of integer rank corresponds to the colliding limit of supersymmetric vertex operators of NS sector and half-odd integer to the Ramond sectors. The spectral curve is simply integrable at Nekrasov-Shatashvili limit and the partition function (inner product of irregular conformal state) is obtained from the superconformal structure manifest in the spectral curve. We present some explicit forms of the partition function of integer (NS sector) and of half-odd ranks (Ramond sector).

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1 Introduction

Irregular conformal state is a conformal state, but is not a primary or descendent state. Rather it is similar to a coherent state since it is a simultaneous eigenstate of some of positive mode of conformal generator. The simplest irregular state is the eigenstate of Virasoro L_{+1} mode which is called Gaiotto state [1] or Whittaker state [2]. More irregular states have been systematically investigated for Virasoro and W-irregular state [3–7].

The irregular state is termed as rank n if it is the simultaneous eigenstate of Virasoro generators L_k with $n \leq k \leq 2n$ or of $W^{(q)}$ generators $W_k^{(q)}$ with $(q-1)n \leq k \leq qn$ with spin q. However, the construction of the irregular state is not easy to find because the eigenvalues are not enough to define the state of rank $n \geq 2$. One needs additional information how the irregular state behaves when all other positive generators applied on the irregular state such as $W_k^{(q)}$ with $0 \leq k < (q-1)n$.

The progress is achieved according to AGT [8] and the idea of colliding limit [9, 10]. AGT connects Nekrasov partition function of N=2 super Yang-Mills theory in 4 dimension with the Liouville conformal block in 2 dimension. Colliding limit of Liouville conformal block describes the irregular state and in turn closely related with Argyres-Douglas theory of N=2 super Yang-Mills theory. Colliding limit of the Liouville conformal block is easily investigated in terms of irregular matrix model. Originally, Penner-type matrix model is suggested from the Liouville conformal block [9, 11, 12] and colliding limit of the Penner-type matrix model results in the irregular matrix model.

The irregular matrix model is successful to describe irregular states and their inner product. The partition function is related with the inner product between primary state and irregular state or between two irregular states depending on the potential of the irregular matrix model. However, the success is limited to the case of irregular states of integer rank. Virasoro irregular state of integer rank n has eigenvalue of the highest Virasoro generator L_{2n}. Question arises. Can we find irregular state with half-odd rank, that is, irregular state of highest Virasoro generator L_{2n-1}? The state of rank 1/2 is easily found from the rank 1 if one limits the eigenvalue of L_2 vanish. However, this trick does not work for rank greater than 1 since this limit does not exist since other eigenvalues diverges unless special limit is achieved so that the state is a simultaneous eigenstate of L_1 and L_{2n-1} only [4].

In this paper, we will present the irregular matrix model of half-odd integer rank using supersymmetrizing the theory. The irregular vertex operator is constructed similar to the regular vertex operator [13–15] and is supersymmetrized in [16]. It is noted that the irregular vertex operator with half-odd rank appears naturally with Ramond sector in the super-symmetrized version. This operator is useful for the free field formalism. If one includes the screening operators, then one can investigate the interacting system of irregular states.

This paper is organized as following. In section 2, we present irregular super-matrix model and its loop equation. The matrix model is related with the N=1 super Liouville conformal block and its colliding limit. The loop equation is simply integrable at Nekrasov-Shatashvili limit (NS limit), which is called super-spectral curve. In section 3, we consider irregular states with integer rank. This state is obtained from the NS sector. Using the super-spectral curve we obtain partition function and present the explicit form of rank 1. In section 4, irregular states with half-odd rank are considered. Partition functions of rank 1/2 and 3/2 are presented. In section 5, we present an idea on RG flow equation corresponding to the operator algebra of the irregular vertices from the string field theory. Section 6 is the conclusion and discussion. Super-spectral curve of the irregular matrix model is presented in the appendix.

2 Irregular super-matrix model and its spectral curve

Super-vertex operator $V_\alpha(\zeta)$ in the NS sector is considered in the super-field formalism

$$V_\alpha(\zeta) = e^{\alpha\Phi}(\zeta) \tag{2.1}$$

where $\zeta = (z, \theta)$ is the holographic super-coordinate, Φ is the super-field and α is the Liouville momentum. Two point correlation of the vertex operator is normalized as in [17]

$$\langle V_{\alpha_1}(\zeta_1) V_{\alpha_2}(\zeta_2)\rangle = (z_{12} - \theta_1\theta_2)^{-\alpha_1\alpha_2} \tag{2.2}$$

where $z_{zb} = z_a - z_b$. To find the multi-point correlation in the super Liouville formalism one may use screening operator $V_b(\zeta)$ in the presence of background charge $Q = b + 1/b$. Primary operator has the conformal dimension $\Delta_\alpha = \alpha(Q - \alpha)/2$ and the superconformal system has central charge $c = 3/2(1 + 2Q^2)$.

Explicitly, $(n+2)$-point holomorphic correlation can be calculated in the presence of N-screening operators $V_b(\zeta)$ and be put into Selberg integrals

$$\left\langle \prod_{A=1}^{n+2} V_{\alpha_A}(\zeta_A) \right\rangle = \int \left[\prod_{I=1}^{N} dz_I d\theta_I\right] \prod_{I<J} (z_{IJ} - \theta_I\theta_J)^{-b^2} \prod_{I,A} (z_{IA} - \theta_I\theta_A)^{-b\alpha_I} \tag{2.3}$$

where neutrality condition $\sum_I \alpha_I + Nb = Q$ is assumed.

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To formulate this integral in terms matrix model, we put $(n+2)$ external operator contribution $(z_{IA}-\theta_I\theta_A)^{-b\alpha_I}$ into an exponential of a super-potential $V(\zeta_I)=\sum_A \hat{\alpha}_I \ln(z_{IA}-\theta_I\theta_A)$ with $\hat{\alpha}=\hbar\alpha$;

$$\mathcal{Z}_n = \int \left[\prod_{I=1}^{N} dz_I d\theta_I\right] \prod_{I<J}(z_{IJ}-\theta_I\theta_J)^{\beta}\, e^{\frac{\sqrt{\beta}}{g}\sum_I V(\zeta_I)}\,. \tag{2.4}$$

This will be called deformed super Penner-type matrix model. Here $\beta=-b^2$ is used instead of b. In addition, $g=i\hbar$ is introduced for later convenience. In terms of the new notations, $b=i\sqrt{\beta}$, $Q=i(\sqrt{\beta}-1/\sqrt{\beta})$ and $\hbar Q=g(\sqrt{\beta}-1/\sqrt{\beta})$.

If one applies the colliding limit by fusing n operators to the one at origin and let the rest to the infinity after accordingly normalizing the partition function at infinity, one obtains a new super potential of the form $V(\zeta_I)=V_B(z_I)+\theta_I V_F(z_I)$: $V_B(z)$ and $V_F(z)$ are bosonic and fermionic part of super-potential.

The loop equation provides the super-spectral curve with the deformed parameter ϵ. (Its derivation is found in appendix A).

$$x_B(z)x_F(z)+\epsilon x_F'(z)=F_F(z) \tag{2.5}$$

$$x_B(z)^2+\epsilon x_B'(z)+x_F(z)V_F'(z)-x_F'(z)V_F(z)=2F_B(z) \tag{2.6}$$

where $x_F(z)$ ($x_B(z)$) is anti-commuting (commuting) one-point resolvent $\omega_F(z)$ ($\omega_B(z)$) shifted by potential term, $x_F(z)=\omega_B(z)-V_F(z)$ ($x_B(z)=\omega_B(z)+V_B'(z)$). F_F (F_B) is also anti-commuting (commuting) holomorphic function and represent spin 3/2 supercurrent (spin 2 Virasoro) symmetry of the partition function.

Explicitly, the potential obtained from the colliding limit of $(n+2)$ number of NS sector of N=1 super Liouville vertex operators is of the form

$$V_B(z_I)=c_0\ln(z_I)-\sum_{k=1}^{n}\frac{c_k}{k z_I^k} \tag{2.7}$$

$$V_F(z_I)=-\sum_{k=0}^{n}\frac{\xi_k}{z_I^{k+1}}. \tag{2.8}$$

where $V_B(z_I)$ is the bosonic part and $V_F(z_I)$ the fermionic part. c_k is a commuting variable defined as $c_k=\sum_A \hat{\alpha}_A z_A^k$ and ξ_k is an anti- commuting variable defined as $\xi_k=\sum_A \hat{\alpha}_A z_A^k\theta_A$. The partition function with the new super potential will be called irregular super-matrix model of integer rank n.

It is noted that the matrix model is closely related with irregular vertex operator was investigated in [16]

$$W_n=e^{\sum_{k=0}^{2n}\gamma_k D_\theta^k \Phi(z,\theta)} \tag{2.9}$$

where $D_\theta=\theta\partial_z+\partial_\theta$. γ_k is commuting (anti-commuting) when k is even (odd). The same potentials $V_B(z_I)$ and $V_F(z_I)$ in (2.7) and (2.8) are obtained if one contracts W_n with N screening operators $V_b(\zeta)$.

There is a slightly different form of the super-matrix model due to Ramond sector. If one uses the vertex operator of Ramond sector

$$W_{n-\frac{1}{2}} = e^{\sum_{k=0}^{2n-1} \gamma_k D_\theta^k \Phi(z,\theta)}, \tag{2.10}$$

and contracts $W_{n-\frac{1}{2}}$ with N screening operators $V_b(\zeta)$, one obtains the irregular matrix model of Ramond sector. The resulting irregular potential is the one similar to (2.7) and (2.8):

$$V_B(z_I) = c_0 \ln(z_I) - \sum_{k=1}^{n} \frac{c_k}{k z_I^k} \tag{2.11}$$

$$V_F(z_I) = -\sum_{k=0}^{n-1} \frac{\xi_k}{z_I^{k+1}}. \tag{2.12}$$

The difference from the model of rank n (NS sector) is that the commuting variable c_k has unusual constraints. c_k contains the product of two anti-commuting variables so that $c_n^2 = 0 = c_n \xi_{n-1}$. This model is called irregular super-matrix model of half-odd rank $(n - 1/2)$.

3 Partition function of integer rank

For the integer rank n, the potential is given in (2.7) and (2.8):

$$V_B(z) = c_0 \ln(z) - \sum_{k=1}^{n} \frac{c_k}{k z^k}, \quad V_F(z) = -\sum_{k=1}^{n} \frac{\xi_k}{z^{k+1}}. \tag{3.1}$$

Here c_k (ξ_k) is a commuting (anti-commuting) variable. We are using the super- spectral curve (2.5) and (2.6) using the explicit form of F_F (F_B) using the potential (3.1).

$$F_F(z) = \sum_{r=1/2}^{2n-1/2} \frac{\Omega_r}{z^{3/2+r}} + \sum_{r=1/2}^{n-1/2} \frac{\eta_r}{z^{3/2+r}} \tag{3.2}$$

$$F_B(z) = \sum_{m=0}^{2n} \frac{\Lambda_m}{z^{2+m}} + \sum_{m=0}^{n} \frac{d_m}{z^{2+m}}. \tag{3.3}$$

Ω_r is anti-commuting number and is defined as $\Omega_r = \sum_k c_k \xi_{r-1/2-k} - \epsilon(\delta_{r,1/2} - (r + 1/2))\xi_{r-1/2}$. On the other hand, Λ_m is commuting number, $\Lambda_m = \sum_{k+l=m} c_k c_l/2 - \epsilon(m + 1)c_m/2$. It is noted in the appendix A that η_r (d_m) is an expectation value $\eta_r = g_r(-\hbar^2 \ln Z)$ with supercurrent g_r ($d_m = \ell_m(-\hbar^2 \ln Z)$ with Virasoro current ℓ_m).

This expectation value is the basic tool to find the partition function from the super-spectral curves (2.5) and (2.6) as noted in bosonic cases [18–20]. The super-flow equation (A.16) and (A.20) is essential to find the moments d_m and η_r. In the following we provide an explicit calculation for the simplest case (rank 1).

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For the rank 1, there are 3 flow equations: 2 bosonic and one fermionic:

$$d_0 = \left(c_1\frac{\partial}{\partial c_1} + \frac{1}{2}\xi_0\frac{\partial}{\partial \xi_0} + \frac{3}{2}\xi_1\frac{\partial}{\partial \xi_1}\right)(-\hbar^2 \ln Z) \tag{3.4}$$

$$\eta_{1/2} = \left(\xi_1\frac{\partial}{\partial c_1} - c_1\frac{\partial}{\partial \xi_0}\right)(-\hbar^2 \ln Z) + \epsilon\xi_0 \tag{3.5}$$

$$d_1 = \xi_1\frac{\partial}{\partial \xi_0}(-\hbar^2 \ln Z). \tag{3.6}$$

To solve the flow equations, we need to find the moments d_0, $\eta_{1/2}$ and d_1 in the left hand side of the flow equations as the functional dependence of variables c_1, ξ_1 and ξ_2 from the super-spectral curve. The moment d_0 is easily identified if one considers the dominant contribution of the bosonic spectral curve (2.6) at large z limit.

$$d_0 = \epsilon N\left(c_0 + \frac{\epsilon(N-1)}{2}\right). \tag{3.7}$$

Therefore, the bosonic flow equation (3.4) requires the partition function to be of the form

$$-\hbar^2 \ln Z = d_0 \log c_1 + A\xi_0\xi_1/c_1^2 + C. \tag{3.8}$$

Here, $\xi_0\xi_1/c_1^2$ is the homogeneous solution and C is a constant independent of c_1, ξ_1 and ξ_2, which can be normalized to be 0.

The fermionic moment $\eta_{1/2}$ is obtained if we use the large z expansion of (2.5):

$$\eta_{1/2} = \epsilon N\xi_0 + \epsilon N_F(\epsilon(N-1) + c_0) \tag{3.9}$$

where $N_F = \langle \sum_I \theta_I \rangle$. To get the information on N_F, we use the fact that d_m and η_r should obey the consistency condition due to commutation relations (A.22) between generators. Note that $[l_m, g_r] = (r - m/2)g_{r+m}$. This requires

$$l_m(\eta_r) - g_r(d_m) = \left(r - \frac{m}{2}\right)\eta_{r+m}, \tag{3.10}$$

Therefore, d_0 in (3.7) and $\eta_{1/2}$ in (3.9) has the relation: $l_0(\eta_{1/2}) = (1/2)\eta_{1/2}$ since $g_{1/2}(d_0) = 0$. This shows that $\eta_{1/2}$ behaves as the primary of conformal dimension 1/2. There are two anti-commuting variables ξ_0 and ξ_1/c_1 of dimension 1/2. This shows that N_F should be proportional to either ξ_0 or ξ_1/c_1. Fermionic filling fraction is anti-commuting and is concentrated at ξ_0 or ξ_1. Putting $N_F = N_1\xi_0 + N_2\xi_1/c_1$ with commuting numbers N_1 and N_2, one has

$$\eta_{1/2} = \xi_0\Big(\epsilon N + \epsilon N_1(c_0 + \epsilon(N-1))\Big) + \left(\frac{\xi_1}{c_1}\right)\Big(c_0 + N_2\epsilon(N-1)\Big). \tag{3.11}$$

The flow equation (3.5) together with (3.8) and (3.11) is rewritten as

$$\xi_0\Big(\epsilon N + \epsilon N_1(c_0 + \epsilon(N-1))\Big) + \left(\frac{\xi_1}{c_1}\right)\Big(c_0 + N_2\epsilon(N-1)\Big) = \xi_0\epsilon + \left(\frac{\xi_1}{c_1}\right)(d_0 - A) \tag{3.12}$$

Therefore, the flow equation reduces to algebraic identities:

$$\begin{aligned}\epsilon N + \epsilon N_1(c_0 + \epsilon(N-1)) &= \epsilon \\ N_2(c_0 + \epsilon(N-1)) &= d_0 - A\end{aligned} \tag{3.13}$$

which fixes N_1 and A as a function of c_0, N and N_2:

$$N_1 = -\frac{N-1}{c_0 + \epsilon(N-1)}, \quad A = d_0 - N_2(c_0 + \epsilon(N-1)). \tag{3.14}$$

As a result, the partition function is given as

$$-\hbar^2 \ln Z = d_0 \ln c_1 + \left(\frac{\xi_0 \xi_1}{c_1^2}\right)(d_0 - N_2(c_0 + \epsilon(N-1)). \tag{3.15}$$

Finally, the bosonic flow equation (3.6) provides additional information on the system. The right hand side of the flow equation vanishes if one uses the partition function of the form (3.15). Therefore, d_1 should vanish. On the other hand one can obtain d_1 using the spectral curve (2.6). It is noted in [21, 22] that the resolvent at NS limit is of the form

$$\omega_B(z) = \epsilon(\ln P(z))' = \epsilon \sum_{\alpha=1}^{N} \frac{1}{z - z_\alpha} \tag{3.16}$$

with a monic polynomial of degree N

$$P(z) = \prod_{\alpha=1}^{N}(z - z_\alpha) = \sum_{k=0}^{N} p_{N-k} z^k \tag{3.17}$$

with $p_0 = 1$. Then, (2.6) results in

$$d_1 = \epsilon N \left(c_1 + p_1(\epsilon(N-1)/2 + c_0)\right). \tag{3.18}$$

Note that p_1 is the sum of all the poles $p_1 = \sum_\alpha z_\alpha$ and is same as the expectation value $\langle \sum_I z_I \rangle$ of the matrix model. Since d_1 vanishes, one concludes that

$$p_1 = -\frac{c_1}{c_0 + \epsilon(N-1)/2}. \tag{3.19}$$

The result (3.19) is consistent with constraint $\ell_0(d_1) = d_1$ since $\ell_1(d_0) = 0$. In general, there are two variables c_1 and $\xi_0\xi_1/c_1$ of conformal dimension 1 for the rank 1 case. However, the term proportional to $\xi_0\xi_1/c_1$ turns out to vanish and the only term proportional to c_1 survives.

The pole structure of the bosonic resolvent also shares with that of the fermionic one. This can be seen from (2.5). First, note that if one uses (3.16), one may put $x_B(z) = \epsilon(\ln \widetilde{P}_N(z))'$ with $\widetilde{P}_N = P_N e^{V_B/\epsilon}$. Then, (2.5) reduces to

$$\epsilon \widetilde{P}_N'(z) x_F(z) + \epsilon \widetilde{P}_N(z) x_F'(z)) = \widetilde{P}_N(z) F_F(z) \tag{3.20}$$

or $(\epsilon \widetilde{P}_N(z) x_F(z))' = \widetilde{P}_N(z) F_F(z)$. Therefore, $x_F(z)$ has the simple expression

$$x_F(z) = \frac{\tau_F(z)}{P_N(z)} \tag{3.21}$$

where $\tau_F(z) = e^{-V_B(z)/\epsilon} \int^z dy F_F(y) \widetilde{P}_N(y)/\epsilon$. Since $\tau_F(z_\alpha)$ is not zero in general (except 0 accidentally), the obvious conclusion is that the pole position z_α is also the pole position of $x_F(z)$.

4 Partition function of half-odd rank

The partition function of integer rank n is interpreted as the inner-product between a primary state and an irregular state of rank n. The spectral curve shows that the irregular state is the simultaneous eigenstate of super-current G_r with $r = n + 1/2, \cdots, 2n - 1/2$ and Virasoro current L_m with $m = n, \cdots, 2n$ if $d_n = 0$ (otherwise, $m = n+1, \cdots, 2n$). One may wonder if the eigenvalue of the highest Virasoro mode L_{2n} vanishes.

Note that the eigenvalue of the highest Virasoro mode is given as $\Lambda_{2n} = c_n^2$. Therefore, unless $c_n = 0$ the case is not achieved in the NS sector. Instead, if one includes the Ramond sector also, one may have the potential of half-odd rank as in (2.11) and (2.12).

$$V_B(z_I) = c_0 \ln(z_I) - \sum_{k=1}^{n} \frac{c_k}{k z_I^k}, \quad V_F(z_I) = -\sum_{k=0}^{n-1} \frac{\xi_k}{z_I^{k+1}}. \tag{4.1}$$

The difference from the NS sector is that the commuting variable vanishes when squared $c_n^2 = 0$. This is because c_n is commuting but is the product of two anti-commuting variables. Therefore, the eigenvalue Λ_{2n} vanishes so that the non-vanishing highest mode becomes L_{2n-1}.

We will consider two simplest cases: rank $1/2$ and $3/2$. The rank $1/2$ has bosonic parameters c_0, c_1 and one fermionic ξ_0 with the constraint $c_1^2 = 0 = c_1\xi_0$. It is clear that $\Lambda_n = 0$ when $n \geq 2$ and $\Lambda_1 = c_1(c_0 - \epsilon)$ so that the irregular state has the eigenvalue of highest Virasoro mode L_1. Note that $G_{3/2}$ annihilates the irregular state since $\Omega_{3/2} = c_1\xi_0 = 0$.

The super-flow equations are simply given as

$$d_0 = \left(c_1 \frac{\partial}{\partial c_1} + \frac{1}{2}\xi_0 \frac{\partial}{\partial \xi_0}\right)(-\hbar^2 \ln Z) \tag{4.2}$$

$$\eta_{1/2} = \left(-c_1 \frac{\partial}{\partial \xi_0}\right)(-\hbar^2 \ln Z) + \epsilon\xi_0. \tag{4.3}$$

The partition function is formally given as

$$-\hbar^2 \ln Z = d_0 \ln c_1 \tag{4.4}$$

where $d_0 = \epsilon N \left(c_0 + \frac{\epsilon(N-1)}{2}\right)$ as given in (3.7).

Non-trivial case starts with rank $3/2$. In this case there are three commuting parameters c_0, c_1, c_2 and two anti-commuting parameters ξ_0, ξ_1. The parameters have the relation with the original γ_k in (2.10) as follows: $c_0 = \hbar\gamma_0$, $c_1 = \hbar(\gamma_1\theta + \gamma_2)$, $c_2 = \hbar\gamma_3\theta$, $\xi_0 = \hbar\gamma_1$, and $\xi_1 = \hbar(\gamma_2\theta + \gamma_3)$. This shows that $c_2^2 = c_2\xi_1 = 0$.

Then we have 4 flow equations

$$d_0 = \left(c_1 \frac{\partial}{\partial c_1} + 2c_2 \frac{\partial}{\partial c_2} + \frac{1}{2}\xi_0 \frac{\partial}{\partial \xi_0} + \frac{3}{2}\xi_1 \frac{\partial}{\partial \xi_1}\right)(-\hbar^2 \ln Z) \tag{4.5}$$

$$\eta_{1/2} = \left(\xi_1 \frac{\partial}{\partial c_1} - c_1 \frac{\partial}{\partial \xi_0} - c_2 \frac{\partial}{\partial \xi_1}\right)(-\hbar^2 \ln Z) + \epsilon\xi_0 \tag{4.6}$$

$$d_1 = \left(c_2 \frac{\partial}{\partial c_1} + \xi_1 \frac{\partial}{\partial \xi_0}\right)(-\hbar^2 \ln Z). \tag{4.7}$$

$$\eta_{3/2} = \left(-c_2 \frac{\partial}{\partial \xi_0}\right)(-\hbar^2 \ln Z) \tag{4.8}$$

The bosonic spectral curve (2.6) shows that d_0 is the same as in (3.7) and the solution of (4.5) is given as

$$-\hbar^2 \ln Z = d_0 \log c_1 + A(t)\xi_0\xi_1/c_1^2 + B(t) \tag{4.9}$$

where we use the fact $t = c_2/c_1^2$ and $\xi_0\xi_1/c_1^2$ are homogeneous solutions.

The fermionic spectral curve (2.5) shows that $\eta_{1/2}$ has the same form (3.11). However, the right hand side of the fermionic flow equation (4.6) has a different result.

$$\begin{aligned}\eta_{1/2} &= \xi_0\Big(\epsilon N + \epsilon N_1(c_0 + \epsilon(N-1))\Big) + \left(\frac{\xi_1}{c_1}\right)\Big(c_0 + N_2\epsilon(N-1)\Big) \\ &= \xi_0(\epsilon + tA(t)) + \left(\frac{\xi_1}{c_1}\right)(d_0 - A(t) - 2tB'(t)).\end{aligned} \tag{4.10}$$

This fermionic flow equation reduces to another algebraic identity whose solves $A(t)$ and $B(t)$:

$$A(t) = \frac{A_1}{t}, \quad B(t) = B_0 \log t + \frac{B_1}{t}. \tag{4.11}$$

so that the partition function is given as

$$-\hbar^2 \ln Z = (d_0 - 2B_0)\log c_1 + B_0 \log c_2 + A_1\xi_0\xi_1/c_2 + B_1 c_1^2/c_2 \tag{4.12}$$

where $A_1 = \epsilon(N - 1 + N_1(c_0 + \epsilon(N-1))$, $B_0 = (d_0 - c_0 + \epsilon N_2(N-1)))/2$ and $B_1 = \epsilon(N - 1 + N_1(c_0 + \epsilon(N-1))/2$. Therefore, the partition function (4.9) is given in terms of potential variables together with N, N_1 and N_2.

Two more flow equations provide additional information on the system. d_1 is given as (3.18) and corresponding flow equation (4.7) shows that

$$\epsilon N\left(c_1 + p_1(c_0 + \epsilon(N-1)/2)\right) = (d_0 - 2B_0)c_2/c_1 + 2B_1 c_1. \tag{4.13}$$

This gives the information on $p_1 = \langle \sum_I z_I \rangle$;

$$p_1 = \frac{(d_0 - 2B_0)c_2/c_1 + (2B_1 - N)c_1}{\epsilon N(c_0 + \epsilon(N-1)/2)}. \tag{4.14}$$

Finally, the fermionic flow equation (4.8) shows that the right side is given as

$$r.h.s. \ = -A_1\xi_1. \tag{4.15}$$

On the other hand, fermionic spectral curve (2.5) shows that the left hand side is

$$l.h.s. \ = \epsilon(c_0 + \epsilon(N-2))q_1 + \epsilon(\xi_0 p_1 + \xi_1 N + (c_1 + \epsilon p_1)N_F) \tag{4.16}$$

where $q_1 = \langle \sum_I z_I\theta_I \rangle$, fermionic partner of p_1. Therefore, the flow equation determines q_1.

$$q_1 = -\frac{A_1\xi_1 + \epsilon(\xi_0 p_1 + \xi_1 N + (c_1 + \epsilon p_1)N_F)}{\epsilon(c_0 + \epsilon(N-2))} \tag{4.17}$$

Note that $\Lambda_n = 0$ when $n \geq 4$ and the positive Virasoro generators L_3 and L_2 have non-vanishing eigenvalues $\Lambda_3 = c_1c_2$ and $\Lambda_2 = c_1^2/2 + (c_0 - 3\epsilon/2)c_2$, respectively. In addition, super-current $G_{n-1/2}$ with $n \geq 4$ annihilates the state and $G_{5/2}$ have non-vanishing eigenvalue $\Omega_{5/2} = c_1\xi_1 + c_2\xi_0$. This eigenvalue is consistent with the commutation algebra $G_{5/2}^2 = -L_5$ since $\Omega_{5/2}^2 = 0$ and $\Lambda_5 = 0$.

5 Irregular vertex operators and RG flow equations

In this section, we provide RG flow equations to the operator algebra of the irregular vertices from the string field theory. The main idea is that, in the formalism of irregular vertex operators, we may have conformal β-function equations on the wavefunctions of these operators, generalized to the off-shell case. For simplicity, we shall limit ourselves to the non-supersymmetric case and to the rank one, however, the discussion is straightforward to generalize to higher ranks and the supersymmetry. The most general form of the rank 1 vertex operator is given by

$$U(\alpha,\beta)=\xi(\alpha,\beta)e^{\alpha\phi+\beta\partial\phi} \tag{5.1}$$

where $\xi(\alpha,\beta)$ is the wavefunction for the irregular state. In case if $U(\alpha,\beta)$ were a regular vertex operator, its leading order contribution to the string sigma-model partition function would be given by $Z_\sigma\sim e^{S(\xi)}$ where $S(\xi)$ is the low-energy effective action, defined by the vanishing β-function condition

$$\frac{\delta S}{\delta\xi}=\Lambda\frac{d\xi}{d\Lambda}\equiv\beta_\xi\sim\Delta\xi+C\xi^2+O(\xi^3)=0 \tag{5.2}$$

where Λ is the worldsheet cutoff and C are the structure constants defined by 3-point worldsheet correlators. The above condition ensures that the conformal invariance is preserved by inserting the on-shell operators on the worldsheet. The irregular vertex operators are, however, the off-shell objects, therefore they do not have any associate β-function in a naive literal sense. Nevertheless, the relation of the type (5.2) still retains some important meaning off-shell, in particular, in the context of background-independent string field theory - and can be related to the flow equations derived above. That is, in the on-shell case, the equations of motion (5.2) define the perturbative background deformations preserving the worldsheet conformal symmetry, ensured by the Weyl invariance combined along with the condition of absence of logarithmic singularities in the partition function due to collisions between vertex operators. It is furthermore important that, in the on-shell case, all the vertex operators have conformal dimension 1, and the only OPE terms contributing to the β-functions as a result of collision of two such vertices, are those involving operators of dimension one. In the off-shell case, such as ours, all these conditions have to be modified. First of all, $U(\alpha,\beta)$ becomes a string field which wavefunction, $\xi(\alpha,\beta)$ now describes a *nonperturbative* background deformation from the original to the one defined by the appropriate analytic solution in string field theory. The "β-function"-like constraint of the type (5.2) is now precisely the condition that the string field U is that analytic solution, producing the nonperturbative background change. Moreover, contrary to the perturbative on-shell-case, the "effective action" $S(\xi)$ is typically nonlocal.

For the irregular vertices, we can no longer require the absence of the OPE singularities for the colliding operators, as this constraint has an essentially on-shell origin in string perturbation theory. However, we still have to retain the Weyl invariance constraints on the operators, since a) these constraints are imposed off-shell even in standard string perturbation theory b) Weyl invariance is essential to fix the (super)conformal gauge which

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we are using here. In order to elucidate the constraints due to the scale invariance, one has to calculate the OPE of the irregular vertex sitting on the disc boundary, with the trace of the stress-energy tensor, integrated over the bulk of the disc, and to extract the logarithmic divergence stemming from the OPE integration. This shall lead to the first set of the constraints, analogous to the flow equations. Straightforward calculation gives:

$$\begin{aligned}
&\lim_{z,\bar z\to\tau} :T_{z\bar z}:(z,\bar z):e^{\alpha\phi+\beta\partial\phi}:(\tau)=\lim_{z,\bar z\to\tau}-\frac{1}{2}:\partial\phi\bar\partial\phi:(z,\bar z)e^{\alpha\phi+\beta\partial\phi}(\tau)\\
&=\left\{\frac{\alpha^2}{2|z-\tau|^2}+\left(\frac{(z-\bar z)^2}{|z-\tau|^4}+\frac{2}{|z-\tau|^2}\right):\beta\partial\phi e^{\alpha\phi+\beta\partial\phi}:(\tau)\right.\\
&\quad+\frac{1}{|z-\tau|^2}\left[\frac{\beta^2}{8}(\alpha\partial\phi)^2-\alpha\partial^2\phi+2(\alpha\partial\phi)(\alpha\partial^2\phi)-\beta\partial^3\phi\right]\\
&\quad\left.-\frac{1}{2}\frac{\alpha\beta}{|z-\tau|^2}(\alpha\partial\phi+\beta\partial^2\phi)\right\}e^{\alpha\phi+\beta\partial\phi}:(\tau)
\end{aligned}\tag{5.3}$$

Integrating over z the contributions proportional to $\sim\int d^2z\frac{1}{|z-\tau|^2}\sim\ln\Lambda$ leads to logarithmic singularities defining the variations of the operators under Weyl transformations. Cancellation condition for these variations defines the flow equations we are looking for. In what follows we shall ignore the OPE terms with higher derivatives of ϕ. That is, the terms proportional to $\partial^2\phi$ and higher derivatives, are only relevant for the RG flows for the higher rank operators, related to variational derivatives with respect momenta, conjugate to higher derivatives in the irregular vertices (e.g. $:\partial^2\phi e^{\alpha\phi+\beta\partial\phi+\gamma\partial^2\phi}:\sim\frac{\partial}{\partial\gamma}e^{\alpha\phi+\beta\partial\phi+\gamma\partial^2\phi}$) Then the flow equation describing the Weyl deformations of the irregular operators is

$$\beta_\xi=\Lambda\frac{d\xi}{d\Lambda}=-\frac{\alpha^2}{2}\xi-\beta\frac{\partial}{\partial\beta}\xi-\frac{\beta^2}{8}\left(\alpha\frac{\partial}{\partial\beta}\right)^2\xi-\frac{1}{2}(\alpha\beta)\alpha\frac{\partial}{\partial\beta}\xi\tag{5.4}$$

This extended β-function relation is related to the Legendre transformed bosonic part of the flow equation (3.4) for the free energy $\ln Z$, expressed in terms of the wavefunction $\xi(\alpha,\beta)$, related to the partition function according to

$$Z_\sigma=\sum_P\frac{1}{P!}\int d\tau_1\dots d\tau_P\xi(\alpha_1,\beta_1)\dots\xi(\alpha_P,\beta_P)<V(\alpha_1,\beta_1,\tau_1)\dots V(\alpha_P,\beta_P,\tau_P)>\tag{5.5}$$

where $V(\alpha,\beta,\tau)=:e^{\alpha\phi+\beta\partial\phi}:(\tau)$.

The relation to the bosonic part of the flow equation (3.4) is not straightforward because the generalized RG flow (5.4) is expressed in terms of very different variables. To obtain this relation, one has to insert the differential operator on the right hand side of (5.4) inside the generating functional $<e^{\int d\tau d\alpha d\beta\xi(\tau,\alpha)V(\alpha,\beta,\beta)}>$. The relation will then follow as the $2d$ Ward identity inside the worldsheet correlators.

It is straightforward to generalize this calculation to the supersymmetric case. In this case, the irregular vertices are not eigenvalues of positive Virasoro generators, but the Jordan blocks. In the simplest rank $\frac{1}{2}$ case such a block has a multiplicity 2 with components:

$$\begin{aligned}
V_1&=\eta_1(\alpha,\beta)(\alpha\psi+\beta\partial\phi)\\
V_2&=\eta_2(\alpha,\beta)e^{\phi}
\end{aligned}\tag{5.6}$$

Applying the Weyl transformation now leads to separate equations on the wavefunctions η_1 and η_2:

$$\begin{aligned}\left(\alpha^2 + \beta\frac{\partial}{\partial\alpha}\right)\eta_1 + \alpha\beta\alpha\frac{\partial}{\partial\alpha}\eta_1 &= 0 \\ (\alpha^2 - 1)\eta_2 &= 0\end{aligned} \tag{5.7}$$

Note that, unlike (4.3) in the pair of the flow equations, one of the equations for the rank $\frac{1}{2}$ is algebraic.

6 Conclusion and discussion

In this work, we analyzed the loop equation in supersymmetric matrix model in the superspace formalism, in order to derive the spectral curve for the Argyres-Douglas limit of $N = 2$ super Yang-Mills theory, related to N=1 super Liouville conformal field theory through generalized AGT conjecture. Noting that the N=1 super Liouville conformal field theory is related with the instanton partition function of $N = 2$ quiver gauge theories on the ALE space $\mathcal{C}^2/\mathcal{Z}_2$ [23–25], we expect that the supersymmetric matrix model at the colliding limit will provide the useful information on the Argyres-Douglas limit.[1] We have been able to derive and to integrate the loop equation in the supersymmetric case and to obtain partition functions associated with irregular blocks of ranks $\frac{1}{2}$, 1 and $\frac{3}{2}$.

The loop equations, as well as the associate flow equations on the free energy, can be reproduced in the irregular vertex operator approach, in terms of the scale invariance constrants for the vertex operators. One particularly promising thing about the vertex operator approach is that it is relatively straightforward to extend to higher ranks, as well as to observe the Jordan cell structure of the flow equations in the supersymmetric case. We hope to be able to extend these results to higher/arbitrary ranks in the future works. It will be also interesting to investigate (super)-spectral curve for the special value of Liouville parameter space as observed in [26, 27].

In general, it is natural to understand the AGT conjecture as an isomorphism between the partition functions of the sigma-models with irregular vertex operators in Toda/superstring theories and those of super Yang-Mills theories. The relation between these theories can be thought of as a generalization of the one between standard string-theoretic sigma-models and their low-energy limit, through the off-shell generalization of the conformal β-functions. The background-independent second-quantized string field theory approach appears to be a promising framework for that. The work in this direction is currently in progress and we hope to be able to elaborate on these issues soon.

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A Super-spectral curve

One may derive the loop equation of the irregular super-matrix model corresponding to the super-conformal symmetry. Spin 3/2 current contribution is obtained if one use the supercoordinate transform [26, 27] $z_I \to z_I + \theta_I \epsilon_F/(z - z_I)$ and $\theta_I \to \theta_I + \epsilon_F/(z - z_I)$ where ϵ_F is the small anti-commuting number. The metric contribution

$$\left[\prod_I dz_I d\theta_I\right] \to \left[\prod_I dz_I d\theta_I\right] \left(1 + \sum_I \frac{\theta_I\, \epsilon_F}{(z - z_I)^2}\right). \tag{A.1}$$

Super-Vandermonde determinant has the contribution

$$\prod_{I<J} (z_{IJ} - \theta_I \theta_J)^\beta \to \prod_{I<J} (z_{IJ} - \theta_I \theta_J)^\beta \left(1 + \beta \left\{\sum_{I,J} \frac{\theta_I}{(z - z_I)(z - z_J)} - \sum \frac{\theta_I}{(z - z_I)^2}\right\} \epsilon_F\right). \tag{A.2}$$

Finally, the potential has the contribution

$$e^{\frac{\sqrt{\beta}}{g} \sum_I V(\zeta_I)} \to e^{\frac{\sqrt{\beta}}{g} \sum_I V(\zeta_I)} \left(1 + \frac{\sqrt{\beta}}{g} \sum_I \left\{\frac{V_B'(z_I)\theta_I - V_F(z_I)}{z - z_I}\right\} \epsilon_F\right). \tag{A.3}$$

Collecting all terms one has

$$\omega_B(z)\omega_F(z) + V_B'(z)\omega_F(z) - V_F(z)\omega_B - \hbar^2 \omega_{BF}(z, z) + \hbar b \omega_F'(z) = f_F(z) \tag{A.4}$$

where prime denotes the derivative with respect to z. $\omega_B(z)$ ($\omega_F(z)$) is one-point commuting (anti-commuting) resolvent

$$\omega_B(z) = g\sqrt{\beta} \left\langle \sum_I \frac{1}{z - z_I} \right\rangle, \quad \omega_F(z) = g\sqrt{\beta} \left\langle \sum_I \frac{\theta_I}{z - z_I} \right\rangle. \tag{A.5}$$

$\omega_{BF}(z, z)$ is the connected two-point resolvent

$$\omega_{BF}(z, w) = \beta \left\langle \sum_I \frac{1}{z - z_I} \sum_J \frac{\theta_I}{w - z_J} \right\rangle_{\text{conn}}. \tag{A.6}$$

f_F is related with the super-potential

$$f_F(z) \equiv g\sqrt{\beta} \left\langle \frac{(V_B'(z) - V_B'(z_I))\theta_I - (V_F(z) - V_F(z_I))}{z - z_I}. \right\rangle \tag{A.7}$$

Virasoro contribution is obtained if one uses the super-coordinate transform $z_I \to z_I + \epsilon/(z - z_I)$ and $\theta_I \to \theta_I(1 + \epsilon/(2(z - z_I)^2)$ where ϵ is an infinitesimal commuting number. The metric contribution is

$$\left[\prod_I dz_I d\theta_I\right] \to \left[\prod_I dz_I d\theta_I\right] \left(1 + \frac{\epsilon}{2} \sum_I \frac{1}{(z - z_I)^2}\right). \tag{A.8}$$

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(Here the anti-commuting measure $[d\theta_I]$ is required to maintain the integral property $\int d\theta_I \theta_I = 1$). Super-Vandermonde determinant has the contribution

$$\prod_{I<J}(z_{IJ} - \theta_I\theta_J)^\beta \to \prod_{I<J}(z_{IJ} - \theta_I\theta_J)^\beta$$
$$\times \left(1 + \epsilon\frac{\beta}{2}\left\{\sum_{I,J}\left[\frac{1}{(z-z_I)(z-z_J)} + \frac{\theta_I}{z-z_I}\frac{\theta_J}{(z-z_J)^2}\right] - \sum_I \frac{1}{(z-z_I)^2}\right\}\right). \quad \text{(A.9)}$$

Finally, the potential has the contribution

$$e^{\frac{\sqrt{\beta}}{g}\sum_I V(\zeta_I)} \to e^{\frac{\sqrt{\beta}}{g}\sum_I V(\zeta_I)}\left(1 + \epsilon\frac{\sqrt{\beta}}{g}\sum_I\left[\frac{(V_B'(z_I) + \theta_I V_F'(z_I))}{z-z_I} + \frac{\theta_I V_F(z_I)}{2(z-z_I)^2}\right]\right). \quad \text{(A.10)}$$

Collecting all terms one has

$$\frac{1}{2}\omega_B(z)^2 + V_B'(z)\omega_B(z) + \frac{1}{2}\Big(\omega_F(z)V_F'(z) - \omega_F'(z)V_F(z)\Big)$$
$$+\frac{\hbar Q}{2}\omega_B'(z) + \frac{1}{2}\hbar^2\left(\omega_{BB}(z,z) + \omega_{FF}^{(1,2)}(z,z)\right) = f_B(z) \quad \text{(A.11)}$$

where $\omega_{BB}(z,z)$ and $\omega_{FF}^{(1,2)}(z,z)$ are the connected two-point resolvents

$$\omega_{BB}(z,w) = \beta\left\langle\sum_I\frac{1}{z-z_I}\sum_J\frac{1}{w-z_J}\right\rangle_{\text{conn}}. \quad \text{(A.12)}$$

and

$$\omega_{FF}^{(1,2)}(z,w) = \beta\left\langle\sum_I\frac{\theta_I}{z-z_I}\sum_J\frac{\theta_J}{(w-z_J)^2}\right\rangle_{\text{conn}}. \quad \text{(A.13)}$$

f_B is related with the super-potential

$$f_B(z) = g\sqrt{\beta}\left\langle\sum_I\frac{(V_B'(z) - V_B'(z_I)) + \theta_I(V_F'(z) - V_F'(z_I))}{z-z_I} + \frac{1}{2}\frac{\theta_I(V_F(z) - V_F(z_I))}{(z-z_I)^2}\right\rangle. \quad \text{(A.14)}$$

It is useful to find the explicit holomorphic structure of $f_F(z)$ and $f_B(z)$ for the given potential (2.7) and (2.8). They are given in terms of the inverse powers of z

$$f_F(z) = \sum_{r=-1/2}^{n-1/2}\frac{\eta_r}{z^{3/2+r}}. \quad \text{(A.15)}$$

The moment η_r is given as an expectation value and $\eta_{-1/2}$ vanishes which is evident from $1/z$ expansion of (A.4). If one uses the explicit form of the potential, one may put the non-vanishing moment into an interesting form as in non-supersymmetric case [18, 20]

$$\eta_r = g_r(-\hbar^2 \log Z) + \delta_{r,1/2}\, g\sqrt{\beta}\xi_0 \quad \text{(A.16)}$$

where g_r is the differential representation of the super current (corresponding to right action)

$$g_r = \sum_k \left(k\xi_{k+r-1/2} \frac{\partial}{\partial c_k} - c_{k+r+1/2} \frac{\partial}{\partial \xi_k} \right). \tag{A.17}$$

This is obtained if one notices that

$$\frac{\sqrt{\beta}}{g} \left\langle \frac{1}{z_I^{k+1}} \right\rangle = k \frac{\partial}{\partial c_k} \ln Z, \quad \frac{\sqrt{\beta}}{g} \left\langle \frac{\theta_I}{z_I^k} \right\rangle = \frac{\partial}{\partial \xi_k} \ln Z \tag{A.18}$$

Likewise, f_B is written in terms of inverse powers of z,

$$f_B(z) = \sum_{m=-1}^{n} \frac{d_m}{z^{2+m}}. \tag{A.19}$$

The moment d_{-1} vanishes from $1/z$ expansion of (A.4). Non-vanishing moment has the form

$$d_m = \ell_m(-\hbar^2 \log Z) \tag{A.20}$$

where ℓ_m is the differential representation of the Virasoro current (corresponding to right action)

$$\ell_m = \sum_k \left(l\, c_{l+m} \frac{\partial}{\partial c_l} + \left(\frac{2\ell + m + 1}{2} \right) \xi_{l+m} \frac{\partial}{\partial \xi_l} \right). \tag{A.21}$$

It can be checked that g_r in (A.17) and l_m in (A.21) satisfy the commutation relation of right action of the super algebra

$$[l_m, g_r] = \left(r - \frac{m}{2} \right) g_{r+m}, \quad \{g_r, g_s\} = -2l_{r+s}, \quad [l_m, l_n] = -(m-n) l_{m+n}. \tag{A.22}$$

At the NS limit ($\hbar \to 0$ and $b \to \infty$ so that $\hbar b = \epsilon$), the loop equations (A.4) and (A.11) can be put in terms of one-point resolvent only, which is called the deformed spectral curve

$$x_B(z) x_F(z) + \epsilon x_F'(z) = F_F(z) \tag{A.23}$$

$$x_B(z)^2 + \epsilon x_B'(z) + x_F(z) V_F'(z) - x_F'(z) V_F(z) = 2F_B(z) \tag{A.24}$$

where we use compact notations: $x_B(z) = \omega_B(z) + V_B'(z)$, $x_F(z) = \omega_B(z) - V_F(z)$, $F_F(z) = f_F(z) - V_B'(z) V_F(z) - \epsilon V_F'(z)$ and $F_B(z) = f_B(z) + \frac{1}{2} V_B'^2 + \epsilon V_B'(z)$.

It is interesting to look into the explicit form of $F_F(z)$ and $F_B(z)$.

$$F_F(z) = \sum_{r=1/2}^{2n+1/2} \frac{\Omega_r + \eta_r}{z^{3/2+r}}, \quad F_B(z) = \sum_{m=0}^{2n} \frac{\Lambda_m + d_m}{z^{2+r}}, \tag{A.25}$$

where Ω_r is an anti-commuting number $\Omega_r = \sum_{k+\ell=r-1/2} c_k \xi_\ell - \epsilon(\delta_{r,1/2} - (r+1/2))\xi_{r-1/2}$ and Λ_m is a commuting number $\Lambda_m = \sum_{k+l=m} c_k c_l / 2 - \epsilon(m+1) c_m / 2$ Non-vanishing η_r ($r = 1/2, \cdots, n-1/2$) and d_m ($m = 0, \cdots, n$) are given in (A.15) and (A.19). The anti-commuting number Ω_r with $r = (n+1/2, n+3/2, \cdots, 2n+1/2)$ corresponds to the eigenvalue of super-current positive mode G_r and the commuting number Λ_m with $m = (n+1, n+2, \cdots, 2n)$ corresponds to the eigenvalue of Virasoro positive mode L_m.

The same analysis can be done for the potential (2.12) and (2.11) of the half-odd rank $(n-1/2)$ similarly if one considers the constraint of the variables, $c_n^2 = 0 = c_n \xi_{n-1}$. This shows that $\Lambda_m = 0$ if $m \geq 2n$ and $\Omega_r = 0$ if $r \geq 2n - 1/2$.

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