Gravitating BPS Skyrmions

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ABSTRACT: The BPS Skyrme model has many exact analytic solutions in flat space. We generalize the model to a curved space or spacetime and find that the solutions can only be BPS for a constant time-time component of the metric tensor. We find exact solutions on the curved spaces: a 3-sphere and a 3-hyperboloid; and we further find an analytic gravitating Skyrme on the 3-sphere. For the case of a nontrivial time-time component of the metric, we suggest a potential for which we find analytic solutions on anti-de Sitter and de Sitter spacetimes in the limit of no gravitational backreaction. We take the gravitational coupling into account in numerical solutions and show that they are well approximated by the analytic solutions for weak gravitational coupling.

KEYWORDS: Integrable Equations in Physics, Solitons Monopoles and Instantons, Classical Theories of Gravity

ARXIV EPRINT: [1510.08735](https://arxiv.org/abs/1510.08735)

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1 Introduction

The Skyrme model, in flat spacetime, was made as an effective field theory in which baryons are solitons in a mesonic field theory [1, 2]. The interest in the Skyrme model increased drastically when it was shown to describe baryons in large- N_c QCD, exactly [3, 4]. Black holes are conjectured to be characterized only by their mass and global charges at spatial infinity, which is called the (weak) no-hair conjecture. Probably the first (stable) counter example to the no-hair conjecture was provided by the black hole with Skyrme hair [5–9] (see also [10, 11]). The Skyrmion black hole solution was also generalized to anti-de Sitter [12, 13] and de Sitter [14] spacetimes. Later some works on the late-time evolution of the radiation coming from a black hole-Skyrmion system was considered [15, 16]. Gravitating sphalerons in the Einstein-Skyrme system were also considered recently [17]. After the marriage of general relativity and the Skyrmion, some studies have put forward potential applications. In particular, one obvious direction of great interest is the application of the system to neutron stars [18–20].

Topological solitons on curved spaces (as opposed to curved spacetimes), have led to important exact solutions in the literature. A few notable examples are: the Skyrmion on the 3-sphere [21] and the vortex on the hyperbolic plane [22].

Topological solitons of other types than Skyrmons have been studied on curved space-time backgrounds (see e.g. [11]) in the literature and in particular on spacetimes with nonzero cosmological constant. It is impossible to make a complete list here, but we would like to mention a few works. The lower-dimensional relative to the Skyrmion, namely the baby-Skyrmion has been studied on the anti-de Sitter background where the curvature

can mimic a mass term [23]. Monopoles were considered in anti-de Sitter space, where monopole condensation in the form of a wall in the bulk can break translational symmetry spontaneously [24, 25]. Gravitating critically coupled vortices give rise to the Einstein-Bogomol'nyi equations [26], and in turn to the conjecture that no coincident vortex solution to latter equation exists (Yang's conjecture), which was proven only recently [27]. Gravitating semilocal strings were studied in ref. [28] and non-Abelian strings were coupled to gravity in ref. [29].

The Skyrme model, although rather phenomenologically successful in describing nuclei of nature, has one short-coming; namely in its minimal formulation it gives rise to too large binding energies. This problem has led to the formulation of a theory with infinitely many mesons [30] and to the BPS Skyrme model¹ [31, 32]. The BPS Skyrme model consists only of the topological baryon current (squared) and a potential (so no Skyrme term). The advantages of this model is that it is integrable and has vanishing binding energies for all baryon numbers. Due to its importance in many applications, we choose to study this model in the context of general relativity in this paper. We would like to mention a third proposal for decreasing the binding energies in the Skyrme model, which is based on energy (mass) bounds [33, 34] and a certain repulsive potential; namely the weakly bound Skyrme model [35].

Recently, the BPS Skyrme model has also been used to describe neutron stars (in analogy with the studies using the normal Skyrme model) [36, 37], where the BPS Skyrme model has a clear advantage. Usually thermodynamic properties are only obtained after averaging over microscopic quantities in a theory. In the BPS Skyrme model, however, averaging is not necessary, as the quantities can be calculated directly from the microscopic fields via target-space integrals [38]. Although many analytic solutions can be found in the BPS Skyrme model on flat (Minkowski) space, the studies of the BPS Skyrmions coupled to gravity, required numerical calculations [36, 37].

In this paper we are considering the BPS Skyrme model on curved spaces and on curved spacetimes and try to find analytic solutions for the gravitating Skyrmions. Our first result (in section 2) is that in order for the BPS equation of the BPS Skyrme model to solve the second-order equation of motion, the time-time component of the metric needs to be a constant ($g_{00} = \text{const.}$). Conversely, for a solution to be BPS, it also requires the same condition to hold true. First in section 3.1, we find analytic solutions on curved (3-)spaces, namely the 3-sphere and the 3-hyperboloid. They are BPS since the time-time component of the metric is constant. Then in section 3.2, we promote the solution on the 3-sphere to be a gravitating Skyrmion, for which the Einstein equation fixes the cosmological constant and the coefficient of the BPS Skyrme term. Since the second-order equation of motion is harder to solve than the first-order BPS equation, we propose in section 4.1, a special potential (for which we have no better name than special potential) which simplifies the Skyrmion equation of motion such that we in section 4.2 find analytic solutions on anti-de Sitter and de-Sitter spacetimes in the limit of vanishing gravitational coupling (i.e. the

¹BPS stands for Bogomol'nyi-Prasad-Sommerfield and the model is named this way because the energy is proportional to the topological charge, namely the baryon charge.

limit of no backreaction onto gravity). These solutions are, however, non-BPS. Finally, in section 4.3 we take into account (a finite) coupling of the Skyrme model to gravity for which we have not been able to find analytic solutions and thus have turned to numerical calculations. The numerical solutions show that the analytic solutions are very good approximations for small values of the gravitational coupling. The results are summarized and discussed in section 5.

2 The model

The model we are interested in in this paper is based on taking the BPS Skyrme model [31, 32] and putting it on a curved background geometry with metric $g_{\mu\nu}$,

$$S = \int d^4x \sqrt{-g} \left[\mathcal{L}_{06} + \frac{1}{4\alpha} (\mathcal{R} - 2\Lambda) \right], \quad (2.1)$$

$$\mathcal{L}_{06} = \mathcal{L}_6 - V, \quad (2.2)$$

where $\mathcal{R}$ is the Ricci scalar, Λ is the cosmological constant, $\alpha \equiv 4\pi G$ is the gravitational coupling, $g = \det g_{\mu\nu}$ and the sextic term, $\mathcal{L}_6$, is given by the square of the baryon current

$$\mathcal{L}_6 = -4\pi^4 c_6 g_{\mu\nu} \mathcal{B}^\mu \mathcal{B}^\nu, \quad (2.3)$$

$$\mathcal{B}^\mu \equiv \frac{1}{24\pi^2 \sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} \text{Tr}[L_\nu L_\rho L_\sigma], \quad (2.4)$$

where the left-current, which is $\mathfrak{su}(2)$ valued, is defined in terms of the derivative of the nonlinear sigma-model field U , as

$$L_\mu \equiv U^\dagger \partial_\mu U, \quad (2.5)$$

and U is given by $U = \sigma \mathbf{1}_2 + i\pi^a \tau^a$, where $a = 1, 2, 3$ is summed over and τ^a are the Pauli matrices. V is an appropriately chosen potential, which for concreteness we will choose to be in the form

$$V_n = \frac{c_{0n}}{n} \left(1 - \frac{1}{2^n} \text{Tr}[U]^n \right), \quad (2.6)$$

with $n = 1, 2$. For $n = 1$ it is simply the standard pion mass term, while for $n = 2$ it is the so-called modified mass term [39–41], see also e.g. refs. [42, 44, 46, 47].

The topological charge, which is called the baryon number or Skyrme number, is defined as

$$B = \int d^3x \sqrt{-g} \mathcal{B}^0. \quad (2.7)$$

The charge is independent of the background geometry by construction, unless a black hole horizon has formed. In that case, the integral is evaluated only from the horizon to spatial infinity, yielding a total charge smaller than unity. If furthermore a cosmological horizon is present, the integral is then only evaluated up to said horizon, which again may give a total baryon charge less than one.

As we want to couple the Skyrme model to gravitational backgrounds, we need the energy-momentum tensor, which can be written as

$$T_{\mu\nu} = 8\pi^4 c_6 \mathcal{B}_\mu \mathcal{B}_\nu - g_{\mu\nu} (4\pi^4 c_6 g_{\rho\sigma} \mathcal{B}^\rho \mathcal{B}^\sigma - V), \quad (2.8)$$

which is the energy-momentum tensor of a perfect fluid [32, 36, 38]. The Einstein equation reads

$$G_{\mu\nu} = 2\alpha T_{\mu\nu} - \Lambda g_{\mu\nu}, \quad (2.9)$$

where the Einstein tensor depends on the background metric.

Let us now consider the Bogomol'nyi completion — keeping track of the metric factors — for the BPS Skyrme model

$$\mathcal{L}_{06} = -g_{\mu\nu} W^\mu W^\nu \mp 4\pi^2 \sqrt{c_6} (g_{00})^{-\frac{1}{2}} \sqrt{V} g_{\mu\nu} \mathcal{B}^\mu \delta^{\nu 0}, \quad (2.10)$$

where we have defined

$$W^\mu \equiv 2\pi^2 \sqrt{c_6} \mathcal{B}^\mu \mp \frac{\delta^{\mu 0} \sqrt{V}}{\sqrt{g_{00}}}. \quad (2.11)$$

This Bogomol'nyi completion is written in the rest frame of the Skyrmion (hence the $\delta^{\mu 0}$). The BPS equation is then simply $W^\mu = 0$, which can be fleshed out as

$$\frac{\sqrt{c_6}}{12} \epsilon^{\mu\nu\rho\sigma} \text{Tr}[L_\nu L_\rho L_\sigma] = \pm \frac{\delta^{\mu 0} \sqrt{-g} \sqrt{V}}{\sqrt{g_{00}}}, \quad (2.12)$$

where the upper (lower) signs are for the Skyrmion (anti-Skyrmion) solution. The energy for configurations that saturate the BPS equation is then the 3-dimensional integral of the cross term coming from the Bogomol'nyi completion

$$M_{\text{BPS}} = \pm \frac{\sqrt{c_6}}{6} \int d^3x \sqrt{g_{00}} \sqrt{V} \epsilon^{ijk} \text{Tr}[L_i L_j L_k]. \quad (2.13)$$

If the time-time component of the metric is constant (i.e. not depending on any spatial coordinate), then the energy is identical to flat space version of the BPS Skyrmion [31, 32]. Otherwise the static energy experiences a warp factor from the time-time component of the metric which thus weighs the integral of the topological charge. Hence the static energy is no longer linearly proportional to the topological charge, i.e. the baryon charge. This often happens for gravitational versions of solitons, see e.g. [23–25, 29]. Note that the mass is positive (semi-)definite independent of the sign.

Using now the BPS equation (2.12), the stress-energy tensor simplifies for BPS saturated configurations to

$$T_{\mu\nu} = \frac{2g_{\mu 0} g_{\nu 0}}{g_{00}} V. \quad (2.14)$$

From this tensor we can first of all observe that the BPS equation yields vanishing pressure ($T_{ii} = 0$), which was already pointed out in the flat-space version of this type of BPS models [48], see also [32, 36, 38]. The second observation is that the energy density for BPS saturated configurations take the form,

$$\mathcal{E}_{\text{BPS}} = 2V, \quad (2.15)$$

and hence depend only on the shape of the potential. This will be important later.

In this paper we will consider only the $B = 1$ sector, for which — with the present type of potential that only depends on $\text{Tr}[U]$ with the vacuum at $U = \mathbf{1}_2$ — we can employ a spherically symmetric Ansatz for the field U as

$$U = \mathbf{1}_2 \cos f + i \hat{x}^a \tau^a \sin f, \quad (2.16)$$

where $\hat{x}^a$ is the Cartesian unit vector and $f = f(r)$ is a radial function, which we will denote the Skyrmion profile function.

Throughout the paper we will assume a diagonal metric of the type

$$ds^2 = N^2 C dt^2 - C^{-1} dr^2 - \Omega (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.17)$$

with $N = N(r)$, $C = C(r)$ and $\Omega = \Omega(r)$ being radial functions; for which the Lagrangian density reads

$$\mathcal{L}_6 = -\frac{c_6 C \sin^4(f) f_r^2}{\Omega^2}, \quad (2.18)$$

where $f_r \equiv \partial_r f$ and we have used the Ansatz (2.16). The BPS equation (2.12), with these Ansätze reads

$$\sqrt{c_6} \sin^2(f) f_r = \mp \frac{\Omega \sqrt{V}}{\sqrt{C}}, \quad (2.19)$$

which we formally can solve as

$$\int df \frac{\sin^2 f}{\sqrt{V}} = \mp \frac{1}{\sqrt{c_6}} \int dr \frac{\Omega}{\sqrt{C}} + \kappa, \quad (2.20)$$

where κ is an appropriately chosen constant. This implicit solution depends both on the choice of the potential and the gravitational background.

The baryon-charge density now reads

$$\sqrt{-g} \mathcal{B}^0 = -\frac{1}{2\pi^2} \sin \theta \sin^2(f) f_r. \quad (2.21)$$

For convenience, we will define the following radial baryon-charge density

$$\mathcal{B}^r \equiv \frac{1}{\Omega} \int d\theta d\phi \sqrt{-g} \mathcal{B}^0 = -\frac{2 \sin^2(f) f_r}{\pi \Omega} = \pm \frac{2}{\pi} \sqrt{\frac{V[f]}{c_6 C}}, \quad (2.22)$$

where we have integrated the two angular variables and in the last equality we have used the BPS equation (2.19). The baryon number is thus simply $B = \int dr \Omega \mathcal{B}^r$. In the following, we will use the terms radial baryon-charge density and baryon-charge density interchangeably.

The integrated baryon charge can be evaluated as

$$B = \int_{r_0}^L dr \Omega \mathcal{B}^r = \frac{2f_0 - \sin 2f_0}{2\pi}, \quad (2.23)$$

where $f_0 \equiv f(r_0)$ is the value of the Skyrmion profile function at r_0 , which is the radius from where the integral begins and L is the compacton size (size of the compact Skyrmion

solution). If a black hole horizon has formed on the background under consideration, $r_0 = r_h$ is the horizon radius, otherwise $r_0 = 0$. In the above expression, we have assumed that $f(L) = 0$, which is also the definition of the compacton size.

The nonzero components of the stress-energy tensor with the Ansatz (2.16) and the metric (2.17) are

$$T_t^t = \frac{c_6 C \sin^4(f) f_r^2}{\Omega^2} + V, \quad (2.24)$$

$$T_r^r = T_\theta^\theta = T_\phi^\phi = -\frac{c_6 C \sin^4(f) f_r^2}{\Omega^2} + V. \quad (2.25)$$

It is straightforward to check that when the BPS equation (2.19) is satisfied then the system possesses a vanishing pressure, as shown generally in eq. (2.14). Using again the BPS equation (2.19) and the definition of the radial baryon-charge density (2.22), we can write the stress-energy tensor for BPS configurations as

$$T_t^t = \frac{c_6 \pi^2 C}{2} (\mathcal{B}^r)^2, \quad T_r^r = T_\theta^\theta = T_\phi^\phi = 0. \quad (2.26)$$

Let us first consider the two potentials of eq. (2.6), for which we can integrate the left-hand side of the BPS solution

$$f = 2 \arccos \sqrt[3]{\pm \frac{3}{4} \sqrt{\frac{c_{01}}{2c_6}} \int_{r_0}^r dr' \frac{\Omega}{\sqrt{C}} + \cos^3 \left(\frac{f_0}{2} \right)}, \quad (2.27)$$

$$f = \arccos \left[\pm \sqrt{\frac{c_{02}}{2c_6}} \int_{r_0}^r dr' \frac{\Omega}{\sqrt{C}} + \cos f_0 \right]. \quad (2.28)$$

The (radial) baryon-charge densities for these two solutions read

$$\mathcal{B}^r = \pm \frac{4}{\pi} \sqrt{\frac{c_{01}}{2c_6}} \sqrt{1 - \left(\pm \frac{3}{4} \sqrt{\frac{c_{01}}{2c_6}} \int_{r_0}^r dr' \frac{\Omega}{\sqrt{C}} + \cos^3 \left(\frac{f_0}{2} \right) \right)^{\frac{2}{3}}} \frac{1}{\sqrt{C}}, \quad (2.29)$$

$$\mathcal{B}^r = \pm \frac{2}{\pi} \sqrt{\frac{c_{02}}{2c_6}} \sqrt{1 - \left(\pm \sqrt{\frac{c_{02}}{2c_6}} \int_{r_0}^r dr' \frac{\Omega}{\sqrt{C}} + \cos f_0 \right)^2} \frac{1}{\sqrt{C}}. \quad (2.30)$$

For completeness, we will review the known analytic solutions for the BPS Skymion in flat space. In this case, the metric simply reads

$$ds^2 = dt^2 - dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.31)$$

which means that $N = C = 1$ and $\Omega = r^2$ and therefore the BPS solutions for the two potentials are given by eqs. (2.27) and (2.28) with, [31, 32]

$$f = 2 \arccos \sqrt[6]{\frac{c_{01}}{2^5 c_6}} r, \quad (2.32)$$

$$f = \arccos \left[\frac{1}{3} \sqrt{\frac{c_{02}}{2c_6}} r^3 - 1 \right], \quad (2.33)$$

where we have fixed the sign and f_0 to match the boundary conditions of the Skymion, as opposed to the anti-Skymion. Since we are in flat space and thus there is no black hole horizon, we have fixed the boundary conditions of the solution in the usual way, such that $f(0) = \pi$ and the size of the Skymion — which is a compacton — is given by $f(L) = 0$ and it reads, respectively, for the two solutions

$$L = \sqrt[6]{\frac{2^5 c_6}{c_{01}}}, \quad L = \sqrt[6]{\frac{2^3 3^2 c_6}{c_{02}}}. \quad (2.34)$$

The baryon-charge densities for the two solutions read

$$\mathcal{B}^r = \frac{4}{\pi} \sqrt{\frac{c_{01}}{2c_6}} \sqrt{1 - \sqrt[3]{\frac{c_{01}}{2^5 c_6}} r^2}, \quad (2.35)$$

$$\mathcal{B}^r = \frac{2}{\pi} \left(\frac{c_{02}}{2c_6} \right)^{\frac{3}{4}} \frac{r^{\frac{3}{2}}}{\sqrt{3}} \sqrt{2 - \frac{r^3}{3\sqrt{2}}}. \quad (2.36)$$

Finally, we should check under what circumstances the BPS equation solves the full equation of motion, which reads

$$2c_6 \sin^2 f \partial_r \left(\frac{NC}{\Omega} \sin^2(f) f_r \right) - N\Omega \frac{\partial V}{\partial f} = 0. \quad (2.37)$$

By inserting the BPS equation (2.19) two times into the above equation of motion, it reduces to

$$\sqrt{c_6} \sin^2 f \partial_r (\sqrt{N^2 C}) \sqrt{V} = 0, \quad (2.38)$$

which means that the BPS equation *only* solves the full equations of motion when the time-time component of the metric is constant. This implies that the Skymion in the BPS Skyrme model can be BPS on a curved space, but not on a curved spacetime. This is also related to the fact that the static energy of the Skymion is only topological when the time-time component of the metric is constant.

As already mentioned, the BPS condition is equivalent to requiring that the pressure vanishes ($T_i^i = 0$). One may ask whether considering the non-BPS extension with constant pressure [37, 38] may ameliorate the problem of the BPS equation not solving the second-order equation of motion on curved spacetime backgrounds. It turns out that one simply gets the same condition as eq. (2.38), with $V \rightarrow V - P$, where P is a constant pressure and so even the corresponding “non-BPS” equation (i.e. the BPS equation with $V \rightarrow V - P$) still requires $\partial_r(\sqrt{N^2 C}) = 0$. This is in fact expected, since the constant pressure is only the conservation of the stress-energy tensor of a static Skymion on flat space (Minkowski space), whereas on a curved spacetime background, the conservation of the stress-energy tensor is $\nabla_\mu T^{\mu\nu} = 0$, with ∇_μ being the covariant derivative, e.g. in the radial direction, we have

$$\nabla_\mu T_r^\mu = \partial_\mu T_r^\mu - T_t^t \left(\frac{C'}{2C} + \frac{N'}{N} \right) + T_r^r \left(\frac{C'}{2C} + \frac{N'}{N} + \frac{\Omega'}{\Omega} \right) - (T_\theta^\theta + T_\phi^\phi) \frac{\Omega'}{2\Omega} = 0,$$

where we have assumed a diagonal stress-energy tensor. The extra terms involving Christoffel symbols will in general not give rise to a constant (isotropic) pressure (but $C = N = \text{const}$ will).

3 BPS solutions

3.1 The BPS Skyrmion on curved space

In this section we will put the BPS Skyrme model on a curved 3-space, which is not a consistent gravitational background (in the absence of matter). The spatial part of the manifold is conformally flat and thus the spaces we consider here correspond to $\mathbb{R} \times S^3$ and $\mathbb{R} \times \mathbb{H}^3$. These spaces were considered for the normal Skyrme model in refs. [21, 49–51]. The metric can be written as

$$ds^2 = dt^2 - \frac{dr^2}{1 - \Lambda r^2} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (3.1)$$

Comparing with the metric (2.17), we have $N^2 C = 1$, $C = 1 - \Lambda r^2$ and $\Omega = r^2$. This means that the solution to the BPS equation is also a solution to the second-order equation of motion. Plugging these functions for C and Ω into the solutions (2.27) and (2.28), we get

$$f = 2 \arccos \sqrt[3]{\frac{3}{8} \sqrt{\frac{c_{01}}{2c_6}} \left(\frac{\arcsin \sqrt{\Lambda} r}{\Lambda^{\frac{3}{2}}} - \frac{r\sqrt{1 - \Lambda r^2}}{\Lambda} \right)}, \quad (3.2)$$

$$f = \arccos \left[\frac{1}{2} \sqrt{\frac{c_{02}}{2c_6}} \left(\frac{\arcsin \sqrt{\Lambda} r}{\Lambda^{\frac{3}{2}}} - \frac{r\sqrt{1 - \Lambda r^2}}{\Lambda} \right) - 1 \right], \quad (3.3)$$

for the two potentials under consideration, respectively. The above solutions are valid for both $\Lambda > 0$ and $\Lambda < 0$; in the limit $\Lambda \rightarrow 0$, they are equal to the flat-space ones of eqs. (2.32) and (2.33).

The solutions are only well defined for $r \leq 1/\sqrt{|\Lambda|}$ (for S^3 this is also the allowed range of the coordinate itself). In order for the compacton to contain a full unit of baryon charge, we get the following constraints

$$\sqrt[6]{\frac{c_{01}}{2c_6}} \sqrt[3]{\frac{3\pi}{16}} \frac{1}{\sqrt{|\Lambda|}} \geq 1, \quad \frac{\pi}{8} \sqrt{\frac{c_{02}}{2c_6}} \frac{1}{|\Lambda|^{\frac{3}{2}}} \geq 1, \quad (3.4)$$

for the two potentials, respectively.

The baryon-charge densities for the above solutions read

$$\mathcal{B}^r = \frac{4}{\pi} \sqrt{\frac{c_{01}}{2c_6}} \sqrt{1 - \left[\frac{3}{8} \sqrt{\frac{c_{01}}{2c_6}} \left(\frac{\arcsin \sqrt{\Lambda} r}{\Lambda^{\frac{3}{2}}} - \frac{r\sqrt{1 - \Lambda r^2}}{\Lambda} \right) \right]^{\frac{2}{3}}} \frac{1}{\sqrt{1 - \Lambda r^2}}, \quad (3.5)$$

$$\mathcal{B}^r = \frac{2}{\pi} \sqrt{\frac{c_{02}}{2c_6}} \sqrt{1 - \left[\frac{1}{2} \sqrt{\frac{c_{02}}{2c_6}} \left(\frac{\arcsin \sqrt{\Lambda} r}{\Lambda^{\frac{3}{2}}} - \frac{r\sqrt{1 - \Lambda r^2}}{\Lambda} \right) - 1 \right]^2} \frac{1}{\sqrt{1 - \Lambda r^2}}. \quad (3.6)$$

In figure 1 are shown plots of the profile functions, the baryon-charge densities and the energy densities for the two solutions (corresponding to the two different potentials, $V_{1,2}$) for $\Lambda = \pm 1/4$. For reference, we have shown the flat-space solutions (2.32) and (2.33). As can readily be seen from figure 1b, the baryon-charge density vanishes at $r = 0$ for $V = V_2$.

Figure 1. (a) Profile functions, f (b) baryon-charge densities and (c) energy densities for analytic Skyrmion solutions on curved spatial backgrounds: $S^3, \mathbb{H}^3$ with $\Lambda = \pm 1/4$ for two choices of potentials: $V_{1,2}$. For comparison, the flat-space solution ($\mathbb{R}^3$) is shown as well. For this figure $c_{01} = c_{02} = c_6 = 1$. In the figure the solutions are rescaled by their respective compacton size: $L_{S^3, V_1} = 1.62202$, $L_{S^3, V_2} = 1.78883$, $L_{\mathbb{H}^3, V_1} = 1.91009$, $L_{\mathbb{H}^3, V_2} = 2.22702$, $L_{\mathbb{R}^3, V_1} = 2^{5/6}$ and $L_{\mathbb{R}^3, V_2} = 3^{1/3}\sqrt{2}$.

This follows from eq. (2.22) because the right-hand side is proportional to $\sqrt{V}$ and hence for BPS solutions, when the potential vanishes, so does the baryon-charge density.

The solutions (3.2) and (3.3) solve the BPS equation and the full second-order equation of motion for the Skyrmion fields as well as the spatial components of the Einstein equations. This is because the spatial part of the metric is conformally flat and the BPS equation ensures that there is no pressure in the system and hence the spatial components of the stress-energy tensor vanish. The time-time component of the Einstein equation is not satisfied in this case, both because the background itself is not a consistent gravitational background (without matter) and the gravitational backreaction has not been taken into account.

The time-time component of the Einstein equation can be satisfied by neglecting the gravitational backreaction, $\alpha = 0$, and setting the cosmological constant to zero: $\Lambda = 0$. This, however, corresponds exactly to the flat-space solutions in eqs. (2.32) and (2.33).

The time-time component of the Einstein tensor is equal to the scalar curvature in this case, which is 2Λ . A solution to the Einstein equation thus requires a constant energy density (constant T_0^0). Although this is possible in the standard Skyrme model without a potential term [52], with for instance the identity map from S^3 to S^3 , the BPS Skyrme model requires a special potential. In fact, from eq. (2.15) we can clearly read off which potential is required for the BPS model to have a constant energy density; namely a constant nonzero potential. We will consider this case in the next section.

3.2 The gravitating BPS Skymion on curved space

In order for the Skymion to be BPS and to solve the second-order equations of motion, we need g_{00} to be a constant and thus for space to be isotropic and homogeneous, it must be conformally flat and take the form of eq. (3.1) (although many other coordinates may be used for the same space). In the previous section we have put BPS Skymions on curved spaces with the latter metric. These BPS Skymions solve the second-order equations of motion as well as the spatial part of the Einstein equations. In order for a BPS Skymion to solve all the Einstein equations, we need to match the scalar curvature — resulting from the Einstein tensor — with the energy density of the Skymion. Since the spaces of eq. (3.1) have constant scalar curvature, we need to choose the following potential

$$V = c_0, \quad (3.7)$$

where $c_0 > 0$ is a positive real constant. On an infinite space, this would imply a solution with infinite energy, but on S^3 , which is compact, this gives rise to a finite-energy solution.

The BPS solution is thus implicit and reads

$$\frac{1}{4}(2f - \sin 2f) = \frac{\pi}{2} - \frac{1}{2}\sqrt{\frac{c_0}{c_6}} \left(\frac{\arcsin \sqrt{\Lambda} r}{\Lambda^{\frac{3}{2}}} - \frac{r\sqrt{1 - \Lambda r^2}}{\Lambda} \right), \quad (3.8)$$

with $\Lambda > 0$. Since BPS-ness implies vanishing pressure ($T_{ii} = 0$) and the background solves the spatial parts of the Einstein equations, only the time-time component of the Einstein equation remains, which is solved by

$$2\Lambda = 2\alpha\mathcal{E}, \quad (3.9)$$

where the energy density of the compacton is

$$\mathcal{E} = \begin{cases} 2c_0, & 0 \leq r \leq L \\ c_0, & r > L \end{cases}, \quad (3.10)$$

which means that we can only solve the time-time component of the Einstein equation by setting the size of the compacton as $L = \frac{1}{\sqrt{\Lambda}}$. In other words, we need to cover the whole 3-sphere for the solution to solve the Einstein equations. This is also natural.² The

²Actually a more contrived solution can be constructed by making a step potential

$$V = c_0 [1 + \Theta(r - L)],$$

where Θ is the Heaviside step function. This potential compensates the missing energy density from the baryon-charge density (squared) so that the energy density is constantly equal to $2c_0$ over all the 3-sphere but the size of the compacton is $L \leq 1/\sqrt{\Lambda}$.

Figure 2. (a) Profile function, f , and (b) baryon-charge density for the analytic gravitating BPS Skyrmion on the curved spatial background, $\mathbb{R} \times S^3$ for $\Lambda = 1/4$ with a constant potential. For this figure $c_0 = c_6 = 1$. In the figure the solution is rescaled by its respective compacton size: $L_{S^3, V=c_0} = 1/\sqrt{\Lambda} = 2$.

solution hence determines the cosmological constant as

$$\Lambda = 2\alpha c_0. \quad (3.11)$$

This solution is consistent on $\mathbb{R} \times S^3$ and indeed a full gravitational BPS Skyrmion.

The mentioned constraint on the compacton size, $L = 1/\sqrt{\Lambda}$, which means that $f(L) = 0$, translates into the following constraint

$$\frac{c_0}{c_6} = 4\Lambda^3. \quad (3.12)$$

Combining this with eq. (3.11) determines c_6 as

$$c_6 = \frac{1}{8\alpha\Lambda^2} = \frac{1}{32\alpha^3 c_0^2}. \quad (3.13)$$

The baryon-charge density for this fully gravitating BPS Skyrmion reads

$$B^r = \frac{4\Lambda^{\frac{3}{2}}}{\pi} \frac{1}{\sqrt{1 - \Lambda r^2}}. \quad (3.14)$$

As can be seen from the above expression, the baryon-charge density blows up at $r = L = 1/\sqrt{\Lambda}$, but the total baryon charge

$$B = \frac{4\Lambda^{\frac{3}{2}}}{\pi} \int_0^{1/\Lambda^2} dr \frac{r^2}{\sqrt{1 - \Lambda r^2}} = 1, \quad (3.15)$$

is finite and indeed integrates to unity as it should.

In figure 2 are shown plots of the profile function and baryon-charge density for the fully gravitating BPS solution on $\mathbb{R} \times S^3$ for $\Lambda = 1/4$. We do not show the energy density as it is simply given by the constant $T_t^t = 2c_0$.

Although the time-time component of the Einstein equations allows for a hyperbolic space by considering a negative $c_0 < 0$, this does not give a real-valued solution for f from eq. (3.8).

The solution (3.8) is similar to that of ref. [52] for the normal Skyrme model. The latter solution fixes the Skyrme-term coefficient and the radius of the 3-sphere in terms of the gravitational coupling, the coefficient of the kinetic term and the cosmological constant. In our solution, on the other hand, we fix the BPS-Skyrme term coefficient and the cosmological constant in terms of the gravitational coupling and the potential.

A comment in store is about the baryon-charge density. Normally, the baryon-charge density vanishes at the compacton radius since it is given by eq. (2.22); from the middle equality it is a product of $\sin^2 f$ and f_r (over $\Omega = r^2$) and hence when $f = 0$, it is natural that the baryon-charge density also vanishes. In the above solution, the derivative of f diverges at $r = 1/\sqrt{\Lambda}$. The compacton fills the 3-sphere and the choice of coordinates on the 3-sphere yields a baryon-charge density that diverges at $r = L = 1/\sqrt{\Lambda}$. The integral of the baryon-charge density — the total baryon charge — is however finite and equals one. All physical observables are thus continuous and convergent in the solution. The mentioned divergence in the derivative is due to the choice of coordinates on the 3-sphere. By switching to, perhaps more natural coordinates

$$ds^2 = dt^2 - r_0^2 (d\psi^2 + \sin^2 \psi d\theta^2 + \sin^2 \psi \sin^2 \theta d\phi^2), \quad (3.16)$$

we can show that the solution is regular. These coordinates are called hyperspherical coordinates and the angles take the values $\psi \in [0, \pi]$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. The radius of the 3-sphere is now given by r_0 . With these coordinates, the appropriate Ansatz for the Skyrme field is ($\psi \rightarrow -\psi$ gives the corresponding anti-Skymion)

$$U = \mathbf{1}_2 \cos \psi - i\tau^1 \sin \psi \cos \theta - i\tau^2 \sin \psi \sin \theta \cos \phi - i\tau^3 \sin \psi \sin \theta \sin \phi. \quad (3.17)$$

This is simply the identity map from the 3-sphere to the 3-sphere and it solves the equations of motion for the matter fields (but not necessarily the BPS equation). The BPS equation now reads

$$\frac{\sqrt{c_6}}{r_0^3} = \sqrt{c_0}, \quad (3.18)$$

which we can solve by choosing the radius of the 3-sphere as

$$r_0 = \sqrt[6]{\frac{c_6}{c_0}}. \quad (3.19)$$

The baryon-charge density is now

$$\mathcal{B}^0 = \frac{1}{2\pi^2 r_0^3}, \quad (3.20)$$

which is the inverse of the volume of the 3-sphere (3.16), giving rise to a unit baryon charge on the 3-sphere

$$B = \int d^3x \sqrt{-g} \mathcal{B}^0 = \frac{1}{2\pi^2 r_0^3} \int_0^\pi d\psi \int_0^\pi d\theta \int_0^{2\pi} d\phi r_0^3 \sin^2 \psi \sin \theta = 1. \quad (3.21)$$

The spatial components of the Einstein equations relate the cosmological constant to the radius of the 3-sphere as $\Lambda = 1/r_0^2$, and thus

$$\Lambda = \sqrt[3]{\frac{c_0}{c_6}}. \quad (3.22)$$

Finally, the time-time component of the Einstein equations determine the coefficient of the BPS Skyrme term as

$$c_6 = \frac{1}{64\alpha^3 c_0^2}. \quad (3.23)$$

What we have done now is merely changing coordinates. In these coordinates on the 3-sphere, however, the Skyrme is an everywhere regular function and the baryon-charge density has no divergences. Up to factors of two (due to different normalization of the new coordinates), the solution is of course physically the same as before.

One may notice that our solution, although similar to that of ref. [52], has one less free parameter than theirs. This is due to the BPS condition implying vanishing pressure, which in turn directly relates the radius of the 3-sphere to the cosmological constant. In that sense, the BPS-ness imposes an additional constraint on the solution and thus we have one less free parameter, compared to the solution of ref. [52].

A final comment about the constant potential is that the Skyrme field f does not have a particular vacuum. The charge-one Skyrme, however, is topological and wraps the 3-sphere once. It solves the equations of motion and the BPS equation. The Bogomol'nyi bound further guarantees that the solution minimizes the static energy. Since the field cannot unwrap the 3-sphere it is topologically protected.

4 Non-BPS solutions

4.1 Special potential

Although eq. (2.38) throws a monkey wrench in using the BPS equation — which is easier to solve than the second-order equation of motion — for gravitational backgrounds with curved spacetime (i.e. having also a non-constant g_{00}), there exists a special potential which makes it easier to solve the second-order matter equation (2.37), namely

$$V_s = \frac{c_s}{4} |2f - \sin 2f|, \quad (4.1)$$

or in terms of $\text{Tr}[U]$,

$$V_s = \frac{c_s}{4} \left| 2 \arccos \left[\frac{1}{2} \text{Tr}[U] \right] - \text{Tr}[U] \left(1 - \frac{1}{4} \text{Tr}[U]^2 \right) \right|. \quad (4.2)$$

The potential is plotted in figure 3 along with the potential of eq. (2.6) with $n = 1, 2$.³

³Actually this potential was considered already in the BPS Skyrme model [38, 53]. For BPS systems, however, it is not as special as in our case, because the BPS equation can handle a very large class of potentials.

Figure 3. The special potential V_s of eq. (4.1) alongside with the potentials V_1, V_2 of eq. (2.6).

The reason for this potential being special, is that

$$\partial_f V_s = c_s \text{sign}(f) \sin^2 f, \quad (4.3)$$

which simplifies the second-order equation of motion for f to

$$2c_6 \partial_r \left(\frac{NC}{\Omega} \sin^2(f) f_r \right) - c_s \text{sign}(f) N\Omega = 0, \quad (4.4)$$

which we can integrate right away

$$\frac{NC}{\Omega} \sin^2(f) f_r = \frac{c_s}{2c_6} \text{sign}(f) \left(\int_{r_0}^r dr' N\Omega - \kappa_1 \right), \quad (4.5)$$

where r_0 is the radius from where the integral begins. If a black hole horizon has formed on the background under consideration, $r_0 = r_h$ is the horizon radius, otherwise $r_0 = 0$. From now on, we will consider only Skymion solutions (as opposed to anti-Skymion solutions) without loss of generality (as they are related by $f \rightarrow 2\pi - f$). Therefore f takes on values in the range $f \in [0, \pi]$ and $\text{sign}(f) = 1$ in the following.

The implicit solution in terms of f reads

$$\frac{1}{4} (2f - \sin 2f) = \frac{1}{4} (2f_0 - \sin 2f_0) + \frac{c_s}{2c_6} \int_{r_0}^r dr' \frac{\Omega}{NC} \left(\int_{r_0}^{r'} dr'' N\Omega - \kappa_1 \right), \quad (4.6)$$

where $f_0 = f(r_0)$ is the value of the profile function at r_0 .

An important observation is that unless $\kappa_1 > 0$, the double integral is positive semi-definite. This means that if $f_0 = \pi$, there is no way for f to flow to its vacuum value $f = 0$. The same conclusion holds for the anti-Skymion.

The next observation is that the outer integral contains a factor of C in the denominator. Since a black hole horizon has a multiplicity-one pole in $1/C$, $N = 1$ at the horizon and $\Omega = r^2$ in the type of spacetime we are considering here, then unless the parenthesis in the outer integral vanishes at the horizon, the outer integral will pick up a logarithmic divergence. Now since $\int_{r_h}^{r_h} dr''$ vanishes, κ_1 must vanish when a black hole horizon is present.

This means, however, that if a black hole horizon is present, no regular Skyrmion (or anti-Skyrmion) solutions with finite energy exists.⁴ Nevertheless, if no black hole horizon is present, we can still have a regular Skyrmion solution.

As it is nontrivial to get the explicit solution for f , it may be useful to notice that in terms of the baryon-charge density, we have

$$\mathcal{B}^r = -\frac{2\sin^2(f)f_r}{\pi\Omega} = -\frac{c_s}{\pi c_6 N C} \left(\int_{r_0}^r dr' N\Omega - \kappa_1 \right). \quad (4.7)$$

It is interesting to note that the baryon-charge density of this solution is dependent on f only through the backreaction of the Skyrmion onto gravity. If the gravitational coupling, α , is sent to zero, then the baryon-charge density is entirely determined by the background geometry of spacetime.

The physical meaning of the integration constant κ_1 is related the baryon-charge density at r_0 , as⁵

$$\kappa_1 = \frac{\pi c_6}{c_s} N C \mathcal{B}^r \Big|_{r=r_0}. \quad (4.8)$$

The two components of the stress-energy tensor corresponding to the energy density and the pressure are

$$T_t{}^t = T + V, \quad T_r{}^r = T_\theta{}^\theta = T_\phi{}^\phi = -T + V, \quad (4.9)$$

$$T \equiv \frac{c_s^2}{4c_6 N^2 C} \left(\int_{r_0}^r dr' N\Omega - \kappa_1 \right)^2, \quad (4.10)$$

$$V \equiv \frac{c_s}{4} (2f_0 - \sin 2f_0) + \frac{c_s^2}{2c_6} \int_{r_0}^r dr' \frac{\Omega}{N C} \left(\int_{r_0}^{r'} dr'' N\Omega - \kappa_1 \right). \quad (4.11)$$

For the solution to be BPS we need the following condition to be true

$$\frac{c_6}{4c_s} (2f_0 - \sin 2f_0) + \frac{1}{2} \int_{r_0}^r dr' \frac{\Omega}{N C} \left(\int_{r_0}^{r'} dr'' N\Omega - \kappa_1 \right) = \frac{1}{4N^2 C} \left(\int_{r_0}^r dr' N\Omega - \kappa_1 \right)^2, \quad (4.12)$$

which generically is not satisfied (and hence not BPS). This condition is tantamount to requiring the pressure to vanish, see eq. (4.9). Taking the derivative with respect to r on both sides of the above equation gives us the following condition

$$\partial_r \left(\frac{1}{N^2 C} \right) \left(\int_{r_0}^r dr' N\Omega - \kappa_1 \right) = 0, \quad (4.13)$$

which is consistent with the condition (2.38), coming from the equation of motion.

⁴We can easily make a regular infinite-energy solution, by letting f flow from 0 to π and having $\kappa_1 = 0$.

⁵This is also consistent with the above reasoning claiming that κ_1 should vanish when a black hole horizon is present.

4.2 The Skyrmion with special potential on curved spacetime

In this section we start by putting the Skyrmion with the special potential (4.1) on a curved spacetime background. Since a curved spacetime implies that $N^2 C$ is not a constant, the BPS condition (2.38) is not satisfied and thus the solutions in this section are not BPS, see also eq. (4.13).

In this section we neglect the backreaction of the Skyrmion to gravity, i.e. we set the gravitational coupling $\alpha = 0$. This limit is a good approximation when the energy/mass scale of the Skyrmion is very small compared to that of gravity, i.e. $1/\sqrt{G}$.

The spacetimes we consider in this section are pure anti-de Sitter and de Sitter spaces, for which we will choose global coordinates or in the case of de Sitter, static coordinates

$$ds^2 = N^2 C dt^2 - C^{-1} dr^2 - \Omega (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.14)$$

where

$$N = 1, \quad C = 1 \pm \frac{r^2}{R^2}, \quad \Omega = r^2, \quad (4.15)$$

which correspond to an AdS (dS) spacetime with $\Lambda = \mp 3/R^2$ for the upper (lower) sign. The implicit solution is thus

$$\frac{1}{4} (2f - \sin 2f) = \frac{\pi}{2} + \mathcal{F}_{\pm}(\rho, \tilde{\kappa}_1), \quad (4.16)$$

and

$$\mathcal{F}_{\pm}(\rho, \tilde{\kappa}_1) \equiv \frac{c_s R^6}{6c_6} \left[\pm \frac{\rho^4}{4} - \frac{\rho^2}{2} \mp 3\tilde{\kappa}_1 \rho \pm \frac{1}{2} \log(1 \pm \rho^2) \pm 3\tilde{\kappa}_1 A_{\pm}(\rho) \right], \quad (4.17)$$

where we have defined the dimensionless coordinate $\rho \equiv r/R$, $\tilde{\kappa}_1 \equiv \kappa_1/R^3$, as well as the function

$$A_{\pm}(\rho) \equiv \begin{cases} \arctan \rho, & + \\ \operatorname{arctanh} \rho, & - \end{cases}. \quad (4.18)$$

R is often called the AdS (or dS) radius and as mentioned above, it is related to the intrinsic curvature of the space time. For convenience, we will use only the dimensionless coordinate from now on.

The baryon-charge density for this solution is

$$\mathcal{B}^r = -\frac{c_s R^3}{3\pi c_6} \frac{\rho^3 - 3\tilde{\kappa}_1}{1 \pm \rho^2}, \quad (4.19)$$

and the energy- and pressure densities are given by eq. (4.9) with

$$T = \frac{c_s^2 R^6}{36c_6} \frac{(\rho^3 - 3\tilde{\kappa}_1)^2}{1 \pm \rho^2}, \quad V = \frac{c_s \pi}{2} + c_s \mathcal{F}_{\pm}(\rho, \tilde{\kappa}_1). \quad (4.20)$$

In order to understand the solution better, let us expand the right-hand side of eq. (4.16),

$$\frac{1}{4} (2f - \sin 2f) = \frac{\pi}{2} - \frac{c_s \tilde{\kappa}_1}{6c_6} \rho^3 + \mathcal{O}(\rho^5). \quad (4.21)$$

Figure 4. Constraint eqs. (4.22) and (4.23) for having solutions on (a) AdS and (b) dS. The line represents solutions which have continuous baryon-charge and energy densities.

This means that for both AdS and dS, $\tilde{\kappa}_1 > 0$ is necessary. Physically, this means that Skyrmion solutions must have a nonvanishing positive baryon-charge density at the origin, which is exactly what one would expect.

Let us first consider the case of AdS and hence the upper signs in the above equations. The function $\mathcal{F}_+$ of eq. (4.17), goes to infinity for $\rho \rightarrow \infty$. Therefore, there is a lower bound on $\tilde{\kappa}_1$ in order for the Skyrmion to satisfy the boundary condition at the compacton radius $f(L) = 0$ and thus contain one unit of baryon charge. It is easy to show that there is a local minimum of $\mathcal{F}_+$ for $\tilde{\kappa}_1 > 0$ at $\rho = \sqrt[3]{3\tilde{\kappa}_1}$ and so by plugging this value of ρ into the solution (4.16), we obtain the following implicit lower bound for $\tilde{\kappa}_1$

$$-\frac{\pi c_6}{2c_s R^6} + \frac{(3\tilde{\kappa}_1)^{\frac{2}{3}}}{12} + \frac{(3\tilde{\kappa}_1)^{\frac{4}{3}}}{8} - \frac{1}{2}\tilde{\kappa}_1 A_+[(3\tilde{\kappa}_1)^{\frac{1}{3}}] - \frac{1}{12} \log \left[1 + (3\tilde{\kappa}_1)^{\frac{2}{3}} \right] \geq 0. \quad (4.22)$$

When the above inequality is satisfied, then $\tilde{\kappa}_1$ is big enough so that the Skyrmion profile function can flow to zero. If, however, the inequality is saturated, i.e. the expression is zero, then the point where $f = 0$ is coincident with where $\mathcal{B}^r = 0$, which is $f(L) = 0$ with $L = \sqrt[3]{3\tilde{\kappa}_1}$ and $\tilde{\kappa}_1$ is the solution to eq. (4.22) equaling zero. The saturated inequality is thus the condition for the baryon-charge density and hence the energy density to be continuous at the compacton radius. Figure 4a shows the constraint given in eq. (4.22) and the line represents solutions which have continuous (but not differentiable) baryon-charge and energy densities.

Let us now consider dS, for which the radial coordinate is only defined in the range $\rho \in [0, 1)$; i.e. there is a cosmological horizon at $\rho = 1$ (and in turn an upper limit on the size of the black hole [54]). Proceeding as in the case of AdS, we find that there is again a local minimum of $\mathcal{F}_-$ at $\rho = \sqrt[3]{3\tilde{\kappa}_1}$. But since there is now a cosmological horizon at $\rho = 1$, we need to require that $\tilde{\kappa}_1 < 1/3$ in the following expression

$$-\frac{\pi c_6}{2c_s R^3} + \frac{(3\tilde{\kappa}_1)^{\frac{2}{3}}}{12} - \frac{(3\tilde{\kappa}_1)^{\frac{4}{3}}}{8} + \frac{1}{2}\tilde{\kappa}_1 A_-[(3\tilde{\kappa}_1)^{\frac{1}{3}}] + \frac{1}{12} \log \left[1 - (3\tilde{\kappa}_1)^{\frac{2}{3}} \right] \geq 0, \quad (4.23)$$

where we have plugged $\rho = \sqrt[3]{3\tilde{\kappa}_1}$ into the right-hand side of eq. (4.16) and picked the lower signs.

If $\tilde{\kappa}_1 > 1/3$, $\mathcal{F}_-$ has no local minimum in the range $\rho \in [0, 1)$. Whereas $\mathcal{F}_- \rightarrow +\infty$ for $\rho \rightarrow 1$ when $\tilde{\kappa}_1 < 1/3$, the divergence changes sign for $\tilde{\kappa}_1 > 1/3$; i.e. $\mathcal{F}_- \rightarrow -\infty$ for $\rho \rightarrow 1$. We will now consider the case of the compacton size being close to unity: $f(L) = 0$ with $L = 1 - \epsilon$. Expanding the right-hand side of eq. (4.16) to first order in ϵ , we get

$$\frac{\pi c_6}{2c_s R^6} - \frac{1}{8} - \frac{1}{12} \log 2 - \frac{1}{4}(\log 2 - 2)\tilde{\kappa}_1 + \frac{1}{12}(3\tilde{\kappa}_1 - 1) \log \epsilon - \frac{3}{8}\epsilon(\tilde{\kappa}_1 - 1) + \mathcal{O}(\epsilon^2) = 0. \quad (4.24)$$

Solving this equation — which is the expansion in ϵ to linear order — gives us

$$\epsilon = -\frac{2(1 - 3\tilde{\kappa}_1)}{9(1 - \tilde{\kappa}_1)} W \left[-\frac{9(1 - \tilde{\kappa}_1)}{1 - 3\tilde{\kappa}_1} 2^{-1 - \frac{1}{1 - 3\tilde{\kappa}_1}} \left(2^{-6\tilde{\kappa}_1} \exp \left(-3 + \frac{12\pi c_6}{c_s R^6} + 12\tilde{\kappa}_1 \right) \right)^{\frac{1}{2(1 - 3\tilde{\kappa}_1)}} \right], \quad (4.25)$$

where W is the Lambert W -function, which is the inverse function of

$$F(W) = We^W. \quad (4.26)$$

Although the profile function f goes to zero at $\rho = 1 - \epsilon$ for $\tilde{\kappa}_1 > 1/3$, the baryon-charge density will be finite at that radius and thus be discontinuous. In turn the energy density will be discontinuous.

Solutions to eq. (4.16), saturating the condition (4.22) [(4.23)], which are (non-BPS) Skyrmons with the special potential (4.1) in the pure AdS [dS] background without taking into account the backreaction onto gravity; i.e. in the limit of $\alpha = 0$, are shown in figure 5.

It is interesting to note that the baryon-charge density has a peak in the case of dS, while it does not in the case of AdS. More concretely, a peak in the baryon-charge density occurs if a real zero of the second derivative of the baryon-charge density is smaller than the compacton size, i.e. if the following condition is satisfied

$$\pm \Xi \pm \frac{9\tilde{\kappa}_1^2 \pm 1}{\Xi} \mp 3\tilde{\kappa}_1 < L = (3\tilde{\kappa}_1)^{\frac{1}{3}}, \quad \Xi \equiv \sqrt[3]{-27\tilde{\kappa}_1^3 \mp 3\tilde{\kappa}_1 + \sqrt{-18\tilde{\kappa}_1^2 \mp 1 \mp 81\tilde{\kappa}_1^4}}. \quad (4.27)$$

Notice also that the pressure density of the solution in AdS [dS] is negative [positive] up to the compacton radius where it vanishes; this compensates the intrinsic curvature of the spacetime.

4.3 The gravitating Skyrmion with special potential

In this section we turn on the gravitational coupling $\alpha > 0$ and consider the system fully coupled to the gravitational background. In order to simplify the problem, we will take the metric to be in the form

$$ds^2 = e^{-2\lambda} C dt^2 - C^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.28)$$

which means that $N = e^{-\lambda}$, and $\Omega = r^2$, where $\lambda = \lambda(r)$ and $C = C(r)$ are radial functions. Since we are still considering the special potential (4.1), eq. (4.6) is still a solution to the matter equation of motion. However, now we also have to solve the inhomogeneous Einstein equations, which boil down to

$$r\lambda' + \frac{2\alpha c_6 \sin^4(f) f_r^2}{r^2} = 0, \quad (4.29)$$

$$-1 + C + rC' + r^2\Lambda + C \frac{2\alpha c_6 \sin^4(f) f_r^2}{r^2} + \frac{1}{2} \alpha c_s r^2 (2f - \sin 2f) = 0. \quad (4.30)$$

Figure 5. (a) Profile functions, f , (b) baryon-charge densities, (c) energy densities and (d) pressure densities, for the analytic (non-BPS) Skyrmons, with the special potential in the pure AdS and dS spacetime backgrounds without backreaction ($\alpha = 0$). We have chosen the value of $\tilde{\kappa}_1 = \frac{1}{3 \times 2^3} = 1/24$ and in turn set $c_6 = [11 - 8 \operatorname{arccot} 2 - 32 \log(5/4)]/(192\pi) \simeq 2.49053 \times 10^{-4}$ for the AdS case and $c_6 = [5 - 64 \log 2 + 36 \log 3]/(192\pi) \simeq 3.12711 \times 10^{-4}$ for the dS case, so that the baryon-charge and energy densities are continuous functions at the compacton radius. The constants are set as $c_s = R = 1$.

By inserting eqs. (4.6) and (4.7) into the above system, we can eliminate the Skyrmion profile function and we have the following integro-differential equations

$$\lambda' + \frac{\alpha c_s^2 r}{2c_6 e^{-2\lambda} C^2} \left(\int_{r_0}^r dr' r'^2 e^{-\lambda} - \kappa_1 \right)^2 = 0, \quad (4.31)$$

$$\begin{aligned} & -1 + C + rC' + r^2 \Lambda + \frac{\alpha c_s^2 r}{2c_6 e^{-2\lambda} C} \left(\int_{r_0}^r dr' r'^2 e^{-\lambda} - \kappa_1 \right)^2 \\ & + \frac{\alpha r^2 c_s^2}{c_6} \int_{r_0}^r dr' \frac{r'^2}{e^{-\lambda} C} \left(\int_{r_0}^{r'} dr'' r''^2 e^{-\lambda} - \kappa_1 \right) + \frac{1}{2} \alpha c_s r^2 (2f_0 - \sin 2f_0) = 0. \end{aligned} \quad (4.32)$$

The first observation, which fits nicely with the results of section 4.2, is that when $\alpha > 0$ is turned on, then $N = e^{-\lambda}$ becomes a nontrivial function (as opposed to the case of $\alpha = 0$ where N is simply unity).

	α	$\tilde{\kappa}_1$	c_6
AdS	0.00	$1/24 \simeq 0.0416667$	$[11 - 8 \operatorname{arccot} 2 - 32 \log \frac{5}{4}]/(192\pi) \simeq 0.000249053$
	0.01	0.0415637	0.000248301
	0.10	0.0419864	0.000257302
	α	$\tilde{\kappa}_1$	c_6
dS	0.00	$1/24 \simeq 0.0416667$	$[5 - 64 \log 2 + 36 \log 3]/(192\pi) \simeq 0.000312711$
	0.01	0.0419284	0.000317556
	0.10	0.0421486	0.000327807

Table 1. Parameters $\tilde{\kappa}_1$ and c_6 for the numerical Skyrmon solutions in AdS and dS with gravitational couplings $\alpha = 0, 0.01, 0.1$.

In principle, one can pick a trial function for λ and determine C from eq. (4.31), which then has to satisfy the remaining Einstein equation, (4.32). Unfortunately, we have not been able to find an analytic solution to this coupled system of integro-differential equations and we therefore turn to numerical methods for finding gravitating Skyrmon solutions (with $\alpha > 0$).

We use a fourth-order Runge-Kutta method and implement a simple trapezoidal integration method in the loop that integrates the two integrals needed for eqs. (4.31) and (4.32). We choose the step size of the numerical integration to be $\Delta\rho = 5 \times 10^{-4}$. Since the gravitational backreaction induces a non-constant λ , we need to adjust $\tilde{\kappa}_1$ in order to make the baryon-charge density vanish at the radius of the compacton. Similarly, we adjust the BPS Skyrme term coefficient, c_6 , such that the Skyrmon profile function, f , vanishes at the same radius as that without gravitational backreaction (the corresponding $\alpha = 0$ solution). The method we use to adjust the coefficients $\tilde{\kappa}_1$ and c_6 is to begin the above-described Runge-Kutta integration with the initial guess of the solution without gravitational backreaction and then calculate the following error

$$e = |f(\ell)| + |\mathcal{B}^r(\ell)|, \quad (4.33)$$

where ℓ is either the zero of f or the minimum value (the turning point). If $\tilde{\kappa}_1$ is too big, f will have a zero, but if it is too small, it will simply turn around with a cusp and increase monotonically from that radius on. Finally, we employ a steepest descent algorithm on the e function (4.33) with the variables $(\tilde{\kappa}_1, c_6)$.

Figure 6 shows numerical solutions for the Skyrmons in anti-de Sitter and de Sitter backgrounds. For comparison, we show the analytic solutions alongside the numerical ones (the $\alpha = 0$ ones), except for λ which vanishes for $\alpha = 0$. As one would expect, the numerical solutions with small α are very well approximated by the analytic solutions; in particular the $\alpha = 0.01$ ones. The values of the parameters, $\tilde{\kappa}_1$ and c_6 for the solutions of figure 6 are shown in table 1.

Figure 6. (a) Profile functions, f , (b) baryon-charge densities, (c) energy densities, (d) pressure densities and metric functions (e) λ and (f) C , for numerical (non-BPS) Skyrmons, with the special potential in AdS and dS backgrounds with backreaction $\alpha = 0.01, 0.1$ and without $\alpha = 0$. The constants are set as $c_s = R = 1$. The numerically integrated baryon charges are $B_{\text{AdS}, V_s, \alpha=0.01} = 0.999999990$, $B_{\text{AdS}, V_s, \alpha=0.1} = 0.999999981$, $B_{\text{dS}, V_s, \alpha=0.01} = 0.999995222$ and $B_{\text{dS}, V_s, \alpha=0.1} = 1.00000109$.

5 Discussion and conclusion

The BPS-ness of Skyrmions in the BPS Skyrme model requires a constant time-time component of the metric or trivial warp factor. It does however allow for a curved 3-space and we have thus found analytic BPS Skyrme solutions on the 3-sphere and on the 3-hyperboloid. The BPS solutions can be found for a large class of potentials. These solutions are pressureless and hence they solve the spatial part of the Einstein equations; they are solutions on a curved space (but not curved spacetime). For the 3-sphere, however, we found a solution to all the Einstein equations, for a constant potential, by fixing the curvature of the 3-sphere (the cosmological constant) in terms of the gravitational coupling multiplied by the potential parameter. In order for the Skyrmon to be BPS and solve the time-time component of the Einstein equations, only a constant potential is allowed. This solution is thus unique.

The pursuit of gravitating Skyrmions on a curved spacetime background (as opposed to a curved spatial background) in the BPS Skyrme model, however, turned out to be easier if a special potential is chosen. For this particular potential, we have been able to find analytic solutions — which however are non-BPS — on both the anti-de Sitter and the de Sitter spacetime backgrounds by neglecting backreaction to gravity. These solutions are analytic solutions in the limit of vanishing gravitational coupling. Once we turn back on the gravitational coupling, $\alpha > 0$, the governing system of equations becomes a coupled set of two integro-differential equations for which we have not been able to find analytic solutions. We have, however, found numerical solutions to this system and shown that for weak gravitational coupling, the analytic solution is a good approximation.

One curiosity has arisen from this study, namely that with the chosen special potential (4.1), we have not been able to find a regular Skyrmon solution with a black hole horizon. This is in contrast to the black holes with Skyrme hair [5, 6, 9, 55] that were the inspiration for this work. In the Einstein-Skyrme theory, regularity of the Skyrmon solution at the horizon dictates the first derivative of the Skyrmon profile function. Logically, there are two possibilities; when a horizon is present, either no regular solution exists for the special potential or no regular solution exists at all in the BPS Skyrme model. One way to approach this question is to turn on a kinetic term (and possibly the Skyrme term) and take the limit where their coefficients vanish. That is beyond the scope of this paper though.

As we mentioned in the end of section 2, the modification of the BPS equation to include a constant pressure, as for instance considered in refs. [37, 38], is not enough for obtaining a first-order equation (BPS-like equation) that solves the full second-order equation of motion on a curved spacetime background. It would be very interesting to find some way or limit in which one may obtain a first-order equation for Skyrmions on a curved spacetime background (other than the special potential that we have considered in this paper). That is, however, beyond the scope of this paper.

In this paper, we have considered numerical solutions only for small values of α . It is expected that there is critical value, α_{cr} , beyond which no Skyrmon exists [9, 56]. We leave this study for the future.

The quadratic potential in eq. (2.6) possesses two discrete vacua and hence admits a domain wall interpolating between them [42–47]. Lumps inside the domain wall are Skyrmions from the bulk point of view that we called domain wall Skyrmions [42, 43, 45–47]. Gravitational domain-wall Skyrmions in the BPS Skyrme model could also be an interesting future problem.

In this paper we have considered only Skyrmions of charge one. In the literature, axially symmetric Skyrmions are well known [57–59] and their gravitational counterparts, i.e. the axially symmetric Skyrmion-black hole systems were considered for the normal Skyrme model coupled to gravity in refs. [60, 61] (see also ref. [17]). This can also be considered in the model considered in this paper. In the case of the BPS solutions with neglected backreaction to gravity, a very simple extension to higher B is possible by increasing the axial winding $\pi^1 + i\pi^2 = e^{i\phi} \rightarrow \pi^1 + i\pi^2 = e^{iB\phi}$, but keeping spherical symmetry for the Skyrmion profile function as in ref. [31]. This is possible because of the volume-preserving diffeomorphism of the BPS system [31]. When the system is not BPS or finite coupling to gravity is considered, then the situation is more complicated. This may be considered in future studies.

Acknowledgments

S.B.G. would like to thank Andrzej Wereszczynski and Christoph Adam for comments on the draft and Roberto Auzzi for discussions. S.B.G. also thanks the Recruitment Program of High-end Foreign Experts for support. The work of M.N. is supported in part by a Grant-in-Aid for Scientific Research on Innovative Areas “Topological Materials Science” (KAKENHI Grant No. 15H05855) and “Nuclear Matter in Neutron Stars Investigated by Experiments and Astronomical Observations” (KAKENHI Grant No. 15H00841) from the the Ministry of Education, Culture, Sports, Science (MEXT) of Japan. The work of M.N. is also supported in part by the Japan Society for the Promotion of Science (JSPS) Grant-in-Aid for Scientific Research (KAKENHI Grant No. 25400268) and by the MEXT-Supported Program for the Strategic Research Foundation at Private Universities “Topological Science” (Grant No. S1511006).

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