More on BPS States in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory on $\mathbf{R} \times \mathbf{S}^3$

Shuichi Yokoyama

*Department of Theoretical Physics, Tata Institute of Fundamental Research,
Homi Bhabha Road, Mumbai 400005, India*

E-mail: yokoyama@theory.tifr.res.in

ABSTRACT: We perform a systematic analysis on supersymmetric states in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) on $\mathbf{R} \times \mathbf{S}^3$. We find a new set of 1/16 BPS equations and determine the precise configuration of the supersymmetric states by solving all 1/16 BPS equations when they are valued in Cartan subalgebra of a gauge group and the fermionic fields vanish. We also determine the number of supersymmetries preserved by the supersymmetric states varying the parameters of the BPS solutions. As a byproduct we present the complete set of such supersymmetric states in $\mathcal{N} = 8$ SYM on $\mathbf{R} \times \mathbf{S}^2$ by carrying out dimensional reduction.

KEYWORDS: Supersymmetric gauge theory, Extended Supersymmetry

ARXIV EPRINT: [1406.6694](https://arxiv.org/abs/1406.6694)

Contents

1	Introduction	1
2	$\mathcal{N} = 4$ SYM on $\mathbf{R} \times \mathbf{S}^3$	2
3	BPS states in the $\mathcal{N} = 4$ SYM	4
3.1	Spherically symmetric BPS states	5
3.2	Non-spherical BPS states	7
3.2.1	BPS states preserving $\epsilon_4^{(+)}$	7
3.2.2	BPS states preserving $\epsilon_1^{(-)}$	13
3.3	Counting of supersymmetries	17
4	BPS states in $\mathcal{N} = 8$ SYM on $\mathbf{R} \times \mathbf{S}^2$	21
5	Discussion	24
A	Convention	26
B	Basics on $\mathbf{S}^3$	27
B.1	The scalar spherical harmonics on $\mathbf{S}^3$	29
B.2	Reduction to $\mathbf{S}^2$	30

1 Introduction

$\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) in four dimensions has been a fundamental theoretical tool to extract important lessons of duality. In particular $\mathcal{N} = 4$ SYM enjoys S-duality [1, 2], which is a generalization of the electro-magnetic duality [3, 4], and also exhibits a dual description of type IIB supergravity on $\mathbf{AdS}_5 \times \mathbf{S}^5$ under a particular limit [5]. For precise investigation of these kinds of strong-weak duality, supersymmetry often plays important roles.

Since maximally supersymmetric completion of Yang-Mills theory gives rise to full quantum conformal invariance in the flat space [6–9], $\mathcal{N} = 4$ SYM can be studied by mapping the system onto $\mathbf{R} \times \mathbf{S}^3$ by a conformal transformation, which makes the system free from infra-red divergence and discretizes the spectrum of the system with the mass gap of order the inverse radius of the three sphere. In particular discrete spectrum of the supersymmetric (or BPS) states of the theory has attracted a great deal of attention in the context of $\mathbf{AdS}_5/\mathbf{CFT}_4$ duality, which can be tested by confirming match of supersymmetric spectra in both sides [10, 11].

One aspect in the study of BPS spectrum is counting BPS states. (Other works related to $\mathcal{N} = 4$ SYM on $\mathbf{R} \times \mathbf{S}^3$ are found, for instance, in [12–15].) In particular an important

tool to study BPS spectrum is a (superconformal) index [16], which encodes BPS spectrum as a form of polynomial. A superconformal index of the $\mathcal{N} = 4$ SYM has been computed exactly in arbitrary $SU(N)$ gauge group [17]. (See also [18–20].) Under the large N limit with the charges kept finite the index of the $\mathcal{N} = 4$ SYM precisely matches that of multi-particle states of supergravity multiplet in type IIB supergravity on $AdS_5 \times S^5$ [17]. This result is reasonable in the sense that other supersymmetric objects such as (dual) BPS giant gravitons [21–23] and BPS black holes [24–26] in the dual geometry have much larger charges, which are of order N , N^2 respectively. Counting of supersymmetric states with such large charges in the $\mathcal{N} = 4$ SYM has also been done as a purely mathematical problem by recasting BPS states as cohomology of a supercharge [27]. As a result, an index of 1/8 BPS (dual) giant gravitons [28, 29] has been reproduced from the $\mathcal{N} = 4$ SYM, whereas that of 1/16 supersymmetric black holes has not. 1/16 BPS states are less understood and expected to have a key to make more progress in AdS_5/CFT_4 duality. (See [30] for a recent study in this direction.)

Under the circumference this paper is aimed at carrying out a systematic analysis on 1/16 BPS states of the $\mathcal{N} = 4$ SYM. The goal of this paper is to clarify basic information on the BPS states of the $\mathcal{N} = 4$ SYM such as the 1/16 BPS equations, the BPS configuration and preserved supersymmetries by setting assumptions that the fermionic fields vanish and that the fields are valued in Cartan subalgebra of a gauge group.

The rest of this paper is organized in the following way. In section 2 we review basics of $\mathcal{N} = 4$ SYM explaining our convention. In section 3 we carry out a systematic analysis on supersymmetric states of the $\mathcal{N} = 4$ SYM by setting the fermionic fields to zero. In section 3.1 we study the spherical symmetric BPS states for warm-up. In section 3.2 we study supersymmetric states with angular momenta, which contain 1/16 BPS states. In section 3.3 we count the number of preserved supersymmetries of the BPS states determined in section 3.2. In section 4 we derive similar information of BPS states in $\mathcal{N} = 8$ SYM on $\mathbf{R} \times \mathbf{S}^2$ by performing dimensional reduction. Section 5 is dedicated to discussion and future works. Appendix contains our convention (A), and basic information on $\mathbf{S}^3$ (B) including construction of the scalar spherical harmonics on $\mathbf{S}^3$ (B.1) and reduction of information of $\mathbf{S}^3$ to that of $\mathbf{S}^2$ (B.2).

2 $\mathcal{N} = 4$ SYM on $\mathbf{R} \times \mathbf{S}^3$

In this section we review basics of $\mathcal{N} = 4$ SYM on $\mathbf{R} \times \mathbf{S}^3$ so as to be self-contained. The $\mathcal{N} = 4$ SYM consists of six real scalars, gauge field and their supersymmetric partners, all of which are valued in the adjoint representation of a gauge group.¹ This theory has $PSU(2, 2|4)$ global symmetry, and especially $SO(6)_R \simeq SU(4)_R$ R-symmetry. The fermionic fields are in the anti-fundamental representation of $SU(4)_R$, which we denote by λ_A , and the six scalar fields form the anti-symmetric representation of $SU(4)_R$, which is denoted by X^{AB} satisfying

$$X^{AB} = -X^{BA}, \quad (X^{AB})^\dagger = \frac{1}{2} \varepsilon_{ABCD} X^{CD} =: X_{AB}, \quad (2.1)$$

¹In this paper we do not need to specify a gauge group.

where $A, B = 1, 2, 3, 4$. The action of $\mathcal{N} = 4$ SYM on $\mathbf{R} \times \mathbf{S}^3$ is given by

$$S = \frac{1}{g^2} \int dt r^3 d^3\Omega \operatorname{Tr} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu X_{AB} D^\mu X^{AB} + \frac{1}{4} [X_{AB}, X_{CD}] [X^{AB}, X^{CD}] - \frac{1}{2r^2} X_{AB} X^{AB} + i(\lambda_A)^\dagger \gamma^\mu D_\mu \lambda_A + \lambda_A [X^{AB}, \lambda_B] + (\lambda_A)^\dagger [X_{AB}, (\lambda_B)^\dagger] \right] \quad (2.2)$$

where g is a gauge coupling, r is the radius of the three sphere, $d^3\Omega = \frac{1}{8} d\theta \sin\theta d\phi d\psi$ is the volume form of the unit three sphere, D_μ is the covariant derivative with respect to gauge and space index, which acts on the fields as follows.

$$D_\mu X^{AB} = \partial_\mu X^{AB} + i[A_\mu, X^{AB}] \quad (2.3)$$

$$D_\mu \lambda_A = \partial_\mu \lambda_A + \frac{1}{4} \omega_{\mu, \nu\rho} \gamma^{\nu\rho} \lambda_A + i[A_\mu, \lambda_A] \quad (2.4)$$

where $\omega_{\mu, \nu\rho}$ is the one form connection of the space-time. See appendix B for more details.

For later convenience we present equations of motion and conserved charges in the bosonic part. These can be computed in the standard way. Equations of motion for the gauge field and complex scalar fields in the bosonic part are

$$D_\mu F^{\mu\nu} - i[X_{AB}, D^\nu X^{AB}] = 0, \quad (2.5)$$

$$D^\mu D_\mu X^{AB} + [X_{CD}, [X^{AB}, X^{CD}]] - \frac{1}{r^2} X^{AB} = 0. \quad (2.6)$$

$SU(4)_R$ R-symmetry charge is

$$R^C{}_A = i \int d^3\Omega \frac{1}{g^2} \operatorname{Tr} (-X^{CB} D^0 X_{AB} + D^0 X^{CB} X_{AB}). \quad (2.7)$$

The energy is

$$H = \int d^3\Omega \frac{1}{g^2} \operatorname{Tr} \left(\frac{1}{2} (F^{0i})^2 + \frac{1}{4} (F^{ij})^2 + \frac{1}{2} |D_0 X_{AB}|^2 + \frac{1}{2} |D_i X^{AB}|^2 + \frac{1}{4} |[X_{AB}, X_{CD}]|^2 + \frac{1}{2r^2} |X_{AB}|^2 \right) \quad (2.8)$$

where $|A|^2 = AA^\dagger$. The angular momentum is

$$J^i = \int d^3\Omega \frac{1}{g^2} \operatorname{Tr} \left(F^0{}_\mu F^{i\mu} + \frac{1}{2} (D^0 X_{AB} D^i X^{AB} + D^i X_{AB} D^0 X^{AB}) \right) \quad (2.9)$$

where $i = \theta, \phi, \psi$.

The action (2.2) is invariant under the following supersymmetry transformation rule.

$$\begin{aligned} \Delta_\epsilon A_\mu &= i((\epsilon_A)^* \gamma_\mu \lambda_A + \epsilon_A \gamma_\mu (\lambda_A)^\dagger), \\ \Delta_\epsilon X^{AB} &= i(-\varepsilon^{ABCD} \epsilon_C \lambda_D - (\epsilon_A)^* (\lambda_B)^\dagger + (\epsilon_B)^* (\lambda_A)^\dagger), \\ \Delta_\epsilon \lambda_A &= \frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \epsilon_A - 2D_\mu X_{AB} \gamma^\mu (\epsilon_B)^* - 2i[X_{AB}, X^{BC}] \epsilon_C - X_{AB} \gamma^\mu \nabla_\mu (\epsilon_B)^*, \end{aligned} \quad (2.10)$$

where ∇_μ is the covariant derivative for spin and ϵ_A is a supersymmetry parameter valued in Grassmann number, which is given by a conformal Killing spinor on $\mathbf{R} \times \mathbf{S}^3$

$$\nabla_\mu \epsilon_A^{(\pm)} = \pm \frac{i}{2r} \gamma_\mu \gamma^t \epsilon_A^{(\pm)}. \quad (2.11)$$

Let us solve this Killing spinor equation briefly. For $\epsilon_A^{(+)}$, (2.11) reduces to

$$\partial_t \epsilon_A^{(+)} = \frac{i}{2r} \epsilon_A^{(+)}, \quad \partial_\psi \epsilon_A^{(+)} = \frac{i}{2} \gamma_3^t \epsilon_A^{(+)}, \quad \partial_\theta \epsilon_A^{(+)} = 0, \quad \partial_\phi \epsilon_A^{(+)} = 0. \quad (2.12)$$

This can be solved as follows.

$$\epsilon_A^{(+)} = e^{\frac{i}{2r}t} e^{-\frac{i}{2}\gamma_{30}\psi} \eta_A^{(+)} \quad (2.13)$$

where $\eta_A^{(+)}$ is a constant spinor. On the other hand, for $\epsilon_A^{(-)}$, (2.11) becomes

$$\partial_t \epsilon_A^{(-)} = -\frac{i}{2r} \epsilon_A^{(-)}, \quad \nabla_\theta^{\mathbf{S}^2} \epsilon_A^{(-)} = -\frac{i}{r} \gamma_\theta^t \epsilon_A^{(-)}, \quad \nabla_\phi^{\mathbf{S}^2} \epsilon_A^{(-)} = -\frac{i}{r} \gamma_\phi^t \epsilon_A^{(-)}, \quad \partial_\psi \epsilon_A^{(-)} = 0, \quad (2.14)$$

where $\nabla_\mu^{\mathbf{S}^2}$ is the spin covariant derivative on $\mathbf{S}^2$. More explicitly, the covariant derivatives act on a spinor as follows.

$$\nabla_\theta \epsilon_A^{(-)} = \partial_\theta \epsilon_A^{(-)}, \quad \nabla_\phi \epsilon_A^{(-)} = \partial_\phi \epsilon_A^{(-)} + \frac{1}{2} \cos \theta \gamma^{21} \epsilon_A^{(-)}. \quad (2.15)$$

The solution of Killing spinor equation is

$$\epsilon_A^{(-)} = e^{-\frac{i}{2r}t} e^{-\frac{i}{2}\gamma_{01}\theta} e^{-\frac{1}{2}\gamma_{21}\phi} \eta_A^{(-)} \quad (2.16)$$

where $\eta_A^{(-)}$ is a constant spinor.

3 BPS states in the $\mathcal{N} = 4$ SYM

In this section we perform a systematic analysis on BPS states of $\mathcal{N} = 4$ SYM on $\mathbf{R} \times \mathbf{S}^3$. That is we study a condition of the fields such that the supersymmetry variation given by (2.10) vanishes for a certain Killing spinor. In this paper we consider a situation where the gaugino fields vanish. In this case we have only to study conditions of the bosonic fields for the supersymmetry transformation of gaugino to vanish. By using the Killing spinor equation (2.11) the supersymmetry variation of gaugino can be written as

$$\Delta_{\epsilon^{(\pm)}} \lambda_A = \frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \epsilon_A^{(\pm)} - 2i [X_{AB}, X^{BC}] \epsilon_C^{(\pm)} + \left(-2D_0 X_{AB} \pm X_{AB} \frac{2i}{r} \right) \gamma^t (\epsilon_B^{(\pm)})^* - 2D_i X_{AB} \gamma^i (\epsilon_B^{(\pm)})^* \quad (3.1)$$

where $i = \theta, \phi, \psi$.

There are two kinds of BPS states on the sphere: one is spherically symmetric, the other is not. This separation is achieved by studying whether a preserved Killing spinor of a BPS state is projected or not by a certain operator, which will turn out to break spherical symmetry. In this paper we perform projection for a constant spinor in Killing spinors (2.13), (2.16) by using γ_0 .²

²The (idempotent) projective operator associated with γ_0 is $\mathcal{P}_0 = \frac{1 \pm i\gamma_0}{2}$.

3.1 Spherically symmetric BPS states

In this subsection we study spherically symmetric BPS solutions for warm-up. For this purpose we study a BPS condition which preserves at least one Killing spinor without any projection. As such a preserved Killing spinor we first choose $\epsilon_4^{(+)}$, where $4 = 1, 2, 3, 4$. We use German letters $\mathbf{1}, 2, 3, 4$ as symbols representing $1, 2, 3, 4$ in a way that there exists a permutation σ such that $\mathbf{i} = \sigma(i)$, where $\mathbf{i} = \mathbf{1}, 2, 3, 4$, $i = 1, 2, 3, 4$. For the supersymmetry variation of gaugino to vanish with nonzero $\epsilon_4^{(+)}$, the fields are required to satisfy

$$F_{\mu\nu} = 0, \quad [X_{AB}, X^{B4}] = 0, \quad -2D_0 X_{A4} + X_{A4} \frac{2i}{r} = 0, \quad D_i X_{A4} = 0 \quad (3.2)$$

for $A = \mathbf{1}, 2, 3, 4$, since $\gamma^\mu \epsilon_4^{(+)}$ and $(\gamma^\mu \epsilon_4^{(+)})^*$ are linearly independent for all μ . This is the 1/8 BPS condition which preserves a Killing spinor $\epsilon_4^{(+)}$. As asserted, the last equation in (3.2) constrains BPS states to be spherically symmetric.

Let us study this BPS condition when the fields take values in Cartan subalgebra of the gauge group. In this case these become

$$F_{\mu\nu} = 0, \quad 2\partial_0 X_{A4} = X_{A4} \frac{2i}{r}, \quad \partial_i X_{A4} = 0. \quad (3.3)$$

This can be easily solved as

$$X_{A4} = x_{A4}^{(+)} e^{i\frac{t}{r}} \quad (3.4)$$

where x_{A4} are integral constants valued in complex number. Note that there is no nontrivial flat connection on $\mathbf{R} \times \mathbf{S}^3$. Other components are determined by using the relation (2.1) as

$$X_{12} = X_{34}^\dagger = x_{34}^{(+)\dagger} e^{-i\frac{t}{r}}, \quad X_{13} = -X_{24}^\dagger = -x_{24}^{(+)\dagger} e^{-i\frac{t}{r}}, \quad X_{23} = -X_{14}^\dagger = -x_{14}^{(+)\dagger} e^{-i\frac{t}{r}}. \quad (3.5)$$

This is a general 1/8 BPS solution which preserves a Killing spinor $\epsilon_4^{(+)}$.

Let us study a case for this BPS solution to preserve another Killing spinor. One will soon notice that $\epsilon_4^{(-)}$ is broken unless all fields are trivial. So let us set it to zero to study a nontrivial BPS solution. Under the BPS condition (3.2) the supersymmetry variation of gaugino becomes

$$\begin{aligned} \Delta_{\epsilon^{(\pm)}} \lambda_1 &= \left(X_{12} \frac{2i}{r} \pm X_{12} \frac{2i}{r} \right) \gamma^t (\epsilon_2^{(\pm)})^* + \left(X_{13} \frac{2i}{r} \pm X_{13} \frac{2i}{r} \right) \gamma^t (\epsilon_3^{(\pm)})^*, \\ \Delta_{\epsilon^{(\pm)}} \lambda_2 &= \left(X_{21} \frac{2i}{r} \pm X_{21} \frac{2i}{r} \right) \gamma^t (\epsilon_1^{(\pm)})^* + \left(X_{23} \frac{2i}{r} \pm X_{23} \frac{2i}{r} \right) \gamma^t (\epsilon_3^{(\pm)})^*, \\ \Delta_{\epsilon^{(\pm)}} \lambda_3 &= \left(X_{31} \frac{2i}{r} \pm X_{31} \frac{2i}{r} \right) \gamma^t (\epsilon_1^{(\pm)})^* + \left(X_{32} \frac{2i}{r} \pm X_{32} \frac{2i}{r} \right) \gamma^t (\epsilon_2^{(\pm)})^*, \\ \Delta_{\epsilon^{(\pm)}} \lambda_4 &= \left(-X_{41} \frac{2i}{r} \pm X_{41} \frac{2i}{r} \right) \gamma^t (\epsilon_1^{(\pm)})^* + \left(-X_{42} \frac{2i}{r} \pm X_{42} \frac{2i}{r} \right) \gamma^t (\epsilon_2^{(\pm)})^* \\ &\quad + \left(-X_{43} \frac{2i}{r} \pm X_{43} \frac{2i}{r} \right) \gamma^t (\epsilon_3^{(\pm)})^*. \end{aligned} \quad (3.6)$$

In order for these to vanish, it is required to simultaneously satisfy

$$(\epsilon_1^{(+)})^* = 0 \quad \text{or} \quad (\epsilon_1^{(+)})^* \neq 0, X_{34} = X_{24} = 0, \quad (3.7)$$

$$(\epsilon_2^{(+)})^* = 0 \quad \text{or} \quad (\epsilon_2^{(+)})^* \neq 0, X_{34} = X_{14} = 0, \quad (3.8)$$

$$(\epsilon_3^{(+)})^* = 0 \quad \text{or} \quad (\epsilon_3^{(+)})^* \neq 0, X_{24} = X_{14} = 0, \quad (3.9)$$

$$(\epsilon_1^{(-)})^* = 0 \quad \text{or} \quad (\epsilon_1^{(-)})^* \neq 0, X_{41} = 0, \quad (3.10)$$

$$(\epsilon_2^{(-)})^* = 0 \quad \text{or} \quad (\epsilon_2^{(-)})^* \neq 0, X_{42} = 0, \quad (3.11)$$

$$(\epsilon_3^{(-)})^* = 0 \quad \text{or} \quad (\epsilon_3^{(-)})^* \neq 0, X_{43} = 0, \quad (3.12)$$

where we used the relation (2.1). Therefore the 1/8 BPS solution has enhanced supersymmetry in the following situation.

1. One complex scalar field is trivial: $X_{14} = 0$. In this case, the Killing spinor $\epsilon_1^{(-)}$ is also preserved and the solution (3.4) becomes 1/4 BPS.
2. Two scalar fields are trivial: $X_{14} = X_{24} = 0$. In this case, the Killing spinors $\epsilon_1^{(-)}, \epsilon_2^{(-)}, \epsilon_3^{(+)}$ are also conserved and the solution (3.4) becomes 1/2 BPS.

In the same way we can obtain a BPS condition and solution which preserve $\epsilon_4^{(-)}$. The BPS condition to conserve $\epsilon_4^{(-)}$ is

$$F_{\mu\nu} = 0, \quad [X_{AB}, X^{B4}] = 0, \quad 2D_0 X_{A4} - X_{A4} \frac{2i}{r} = 0, \quad D_i X_{A4} = 0. \quad (3.13)$$

When the fields take values in Cartan subalgebra of the gauge group, this reduces to

$$F_{\mu\nu} = 0, \quad 2\partial_0 X_{A4} + X_{A4} \frac{2i}{r} = 0, \quad \partial_i X_{A4} = 0. \quad (3.14)$$

The equations of the complex scalar field can be easily solved as

$$X_{A4} = x_{A4}^{(-)} e^{-i\frac{t}{r}}. \quad (3.15)$$

This is a general 1/8 BPS solution which preserves $\epsilon_4^{(-)}$. This BPS solution has enhanced supersymmetry in the following situation.

1. One complex scalar field is trivial: $X_{14} = 0$. In this case, the Killing spinor $\epsilon_1^{(+)}$ is also conserved, and the solution becomes 1/4 BPS.
2. Two scalar fields are trivial: $X_{14} = X_{24} = 0$. In this case, the Killing spinors $\epsilon_1^{(+)}, \epsilon_2^{(+)}, \epsilon_3^{(-)}$ are also unbroken, and the solution becomes 1/2 BPS.

We summarize the result in the following table.

Let us compute conserved charges (2.8), (2.9), (2.7) under the BPS conditions.

$$H = \frac{1}{g^2} \int d^3\Omega \frac{4}{r^2} \text{Tr} |X_{A4}|^2, \quad (3.16)$$

$$P_i = 0, \quad (3.17)$$

$$R^4 = \int d^3\Omega \frac{2}{g^2} \frac{\mp 1}{r} \text{Tr} |X_{A4}|^2, \quad (3.18)$$

where the upper sign is for the BPS condition given by (3.2) and the lower for that of (3.13). We can show that $\sum_{A=1}^4 R^A = 0$. We can easily see the BPS relation of conserved charge as

$$rH = \mp 2R^4 = \pm 2 \sum_{A=1}^3 R^A. \quad (3.19)$$

BPS solutions.	Preserved Killing spinors.	Number of SUSY.
$X_{A4} = x_{A4}^{(\pm)} e^{\pm i \frac{t}{r}}$ for $A = 1, 2, 3$.	$\epsilon_4^{(\pm)}$.	4 ($\frac{1}{8}$ BPS)
$X_{A4} = x_{A4}^{(\pm)} e^{\pm i \frac{t}{r}}$ for $A = 2, 3$, $X_{14} = 0$.	$\epsilon_1^{(\mp)}, \epsilon_4^{(\pm)}$.	8 ($\frac{1}{4}$ BPS)
$X_{34} = x_{34}^{(\pm)} e^{\pm i \frac{t}{r}}$, $X_{14} = X_{24} = 0$.	$\epsilon_1^{(\mp)}, \epsilon_2^{(\mp)}, \epsilon_3^{(\pm)}, \epsilon_4^{(\pm)}$.	16 ($\frac{1}{2}$ BPS)
$X_{14} = X_{24} = X_{34} = 0$.	$\epsilon_1^{(\pm)}, \epsilon_2^{(\pm)}, \epsilon_3^{(\pm)}, \epsilon_4^{(\pm)}$.	32 (Unique vacuum)

Table 1. We list spherically symmetric BPS solutions, preserved Killing spinors and its number. Signature in multi-column has to be chosen with order respected except the last column.

	H	J_ϕ	J_ψ	R^1_1	R^2_2	R^3_3	R^4_4
$\epsilon_4^{(+)}$	$-\frac{1}{2r}$	0	$-\frac{w^{(+)}}{2}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{3}{4}$

Table 2. Charges of the Killing spinor $\epsilon_4^{(+)}$ are shown.

3.2 Non-spherical BPS states

3.2.1 BPS states preserving $\epsilon_4^{(+)}$

In this subsection we study supersymmetric states with angular momenta. To this end we study supersymmetric states which preserves a Killing spinor projected by a certain operator which breaks spherical symmetry. We first choose $\epsilon_4^{(+)}$ as such a preserved Killing spinor. We carry out projection for a constant spinor $\eta_4^{(+)}$ such that

$$\gamma_0 \eta_4^{(+)} = i w^{(+)} \eta_4^{(+)} \quad (3.20)$$

where $w^{(+)} = \pm 1$.³ For this projected constant spinor, the Killing spinor $\epsilon_4^{(+)}$ given by (2.13) is evaluated as

$$\epsilon_4^{(+)} = e^{\frac{i}{2r} t} e^{i \frac{w^{(+)}}{2} \psi} \eta_4^{(+)} \quad (3.21)$$

Let us fix charges of this Killing spinor. The energy, the angular momentum of ϕ direction, and that of ψ direction are evaluated as eigenvalues of the operators $H := i \partial_t$, $J_\phi := \hat{L}_3 = -i \partial_\phi$, and $J_\psi := \hat{R}_3 = i \partial_\psi$, respectively, where $\hat{L}_i, \hat{R}_i$ are defined in appendix B. The charge assignments are summarized in table 2. Therefore the BPS condition of charges associated with the Killing spinor $\epsilon_4^{(+)}$ is

$$rH = -2R^4_4 + 2w^{(+)} J_\psi \quad (3.22)$$

Note that the relative factors match that obtained from $psu(2, 2|4)$ superconformal algebra in the standard normalization [17, 31].

³In terms of a Killing spinor the projection is given by $\Gamma_0 \epsilon_4^{(+)} = i w^{(+)} \epsilon_4^{(+)}$ where $\Gamma_0 = U \gamma_0 U^{-1}$ with $\epsilon_4^{(+)} = U \eta_4^{(+)}$, and the associated projective operator is given by $\mathcal{P} = U \mathcal{P}_0 U^{-1}$.

In order to study a BPS condition preserving $\epsilon_4^{(+)}$, it is convenient to divide the supersymmetric transformation of gaugino into two parts such that

$$\Delta_{\epsilon^{(\pm)}}\lambda_C = \Delta_{(\pm)}\lambda_C + \bar{\Delta}_{(\pm)}\lambda_C \quad (3.23)$$

where

$$\Delta_{(\pm)}\lambda_A = \frac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon_A^{(\pm)} - 2i[X_{AB}, X^{BC}]\epsilon_C^{(\pm)}, \quad (3.24)$$

$$\bar{\Delta}_{(\pm)}\lambda_A = \left(-2D_0X_{AB} \pm X_{AB}\frac{2i}{r}\right)\gamma^t(\epsilon_B^{(\pm)})^* - 2D_iX_{AB}\gamma^i(\epsilon_B^{(\pm)})^*. \quad (3.25)$$

Let us extract the terms containing $\eta_4^{(+)}$ from these. By using (3.20) and (3.21), we find

$$\begin{aligned} \Delta_{(+)}\lambda_A &= e^{\frac{i\mu}{4}t}e^{i\frac{w^{(+)}}{2}\psi}\left[iw^{(+)}\delta_A^4(-F^{01} + iF^{23} + iw^{(+)}F^{02} - w^{(+)}F^{13})\gamma_1\eta_4^{(+)} \right. \\ &\quad \left. + \{iw^{(+)}\delta_A^4(iF^{03} + F^{12}) + (-2i[X_{AB}, X^{B4}])\}\eta_4^{(+)}\right] + \dots \end{aligned} \quad (3.26)$$

$$\begin{aligned} \bar{\Delta}_{(+)}\lambda_A &= e^{-\frac{i\mu}{4}t}e^{i\frac{-w^{(+)}}{2}\psi}\left[\left(\left(-2D_0X_{A4} + X_{A4}\frac{2i}{r}\right)(iw^{(+)}) - 2iD_3X_{A4}\right)(\eta_4^{(+)})^* \right. \\ &\quad \left. + (-2D_1X_{A4} - 2iD_2X_{A4}w^{(+)})\gamma^1(\eta_4^{(+)})^*\right] + \dots \end{aligned} \quad (3.27)$$

where the ellipses represent the other terms which do not contain $\eta_4^{(+)}$. In order for $\Delta_{\epsilon^{(\pm)}}\lambda_C$ to vanish with $\eta_4^{(+)}$ nonzero, it is necessary to satisfy

$$\begin{aligned} -F^{01} + iF^{23} + iw^{(+)}F^{02} - w^{(+)}F^{13} &= 0, \\ iw^{(+)}\delta_A^4(iF^{03} + F^{12}) - 2i[X_{AB}, X^{B4}] &= 0, \\ \left(-2D_0X_{A4} + X_{A4}\frac{2i}{r}\right)(iw^{(+)}) - 2iD_3X_{A4} &= 0, \\ -2D_1X_{A4} - 2iD_2X_{A4}w^{(+)} &= 0, \end{aligned}$$

for arbitrary A , since $\eta_4^{(+)}, \gamma_1\eta_4^{(+)}, (\eta_4^{(+)})^*, \gamma_1(\eta_4^{(+)})^*$ are linearly independent under the projection (3.20). By using the fact that all the components of $F^{\mu\nu}$ are real, these can be simplified as

$$\begin{aligned} F^{01} &= -w^{(+)}F^{13}, \quad F^{23} = -w^{(+)}F^{02}, \\ w^{(+)}\delta_A^4(iF^{03} + F^{12}) - 2[X_{AB}, X^{B4}] &= 0, \\ \left(D_0 - \frac{i}{r} - w^{(+)}D_3\right)X_{A4} &= 0, \\ (D_1 + iw^{(+)}D_2)X_{A4} &= 0. \end{aligned} \quad (3.28)$$

This is the 1/16 BPS condition preserving $\epsilon_4^{(+)}$ projected by (3.20). This result is essentially the same as 1/16 BPS equations derived in [27], (4.9) and (4.10), in a different way by making the energy density complete square and looking for configurations to saturate the Bogomol'nyi bound.

Let us solve these BPS conditions when the fields are valued in Cartan subalgebra of a gauge group. In this case these BPS equations boil down to

$$F^{12} = F^{03} = 0, \quad F^{13} = -w^{(+)}F^{01}, \quad F^{02} = -w^{(+)}F^{23}, \quad (3.29)$$

and

$$\left(\partial_0 - \frac{i}{r} - w^{(+)}\partial_3\right)X_{A4} = 0, \quad (\partial_1 + iw^{(+)}\partial_2)X_{A4} = 0 \quad (3.30)$$

for $A = 1, 2, 3$. Note that by restricting our interest to the Cartan part the gauge sector and the matter one decouple so that we can solve each sector independently.⁴

First let us solve the matter BPS equation. The equation of motion of the matter fields (2.6) suggests that solutions thereof are expanded by $\mathbf{S}^3$ scalar spherical harmonics. We denote the scalar spherical harmonics by Y_{s,l_3,r_3} with non-negative half integer s and two half integers l_3, r_3 whose moduli are bounded above by s . These quantum numbers are Cartan charges of the representation of the spherical harmonics of $su(2)_L \times su(2)_R$ acting on $\mathbf{S}^3$.

$$\hat{L}^2 Y_{s,l_3,r_3} = \hat{R}^2 Y_{s,l_3,r_3} = s(s+1)Y_{s,l_3,r_3}, \quad (3.31)$$

$$\hat{L}_3 Y_{s,l_3,r_3} = l_3 Y_{s,l_3,r_3}, \quad \hat{R}_3 Y_{s,l_3,r_3} = r_3 Y_{s,l_3,r_3}, \quad (3.32)$$

where $\hat{L}_i, \hat{R}_i$ are differential operators generating $su(2)_L \times su(2)_R$ algebra. See appendix B for more details. By using these operators the second equation in (3.30) can be written as

$$\hat{R}_{w^{(+)}} X_{A4} = 0 \quad (3.33)$$

where $\hat{R}_{w^{(+)}} = \hat{R}_{\pm}$ when $w^{(+)} = \pm 1$ respectively, and $\hat{R}_{\pm}$ is given by (B.16). Therefore BPS solutions are expanded by the spherical harmonics of highest (or lowest) weight of $su(2)_R$ for $w^{(+)} = +1$ (or -1).

$$X_{A4} = \sum_{s \geq 0, |l_3| \leq s} x_{A4}^{s,l_3}(t) Y_{s,l_3,w^{(+)}s}(\theta, \phi, \psi) \quad (3.34)$$

where $x_{A4}^{s,l_3}(t)$ are unknown functions of time, which is determined from the first equation in (3.30). Plugging the above into the first equation in (3.30) gives

$$\partial_0 x_{A4}^{s,l_3}(t) = \frac{i}{r} x_{A4}^{s,l_3}(t) - \frac{2}{r} w^{(+)} (-i w^{(+)} s) x_{A4}^{s,l_3}(t). \quad (3.35)$$

This equation is easily solved as

$$x_{A4}^{s,l_3}(t) = x_{A4}^{s,l_3} e^{i \frac{2s+1}{r} t} \quad (3.36)$$

where x_{A4}^{s,l_3} is an integral constant valued in complex number.⁵ As a result we obtain

$$X_{A4} = \sum_{s \geq 0, |l_3| \leq s} x_{A4}^{s,l_3} e^{i \frac{2s+1}{r} t} Y_{s,l_3,w^{(+)}s}(\theta, \phi, \psi). \quad (3.37)$$

⁴This is not the case in the non-abelian sector [27].

⁵More rigorously or mathematically speaking, x_{A4}^{s,l_3} is a constant valued in Cartan subalgebra of a gauge group over the complex number. But we often abuse such a variable as a coefficient of a Cartan generator by which the variable is expanded.

This is a general 1/16 BPS solution of the complex scalar fields preserving the Killing spinor $\epsilon_4^{(+)}$ with projection (3.20). Note that this satisfies the equation of motion of the scalar field (2.6).

Let us move on to determining BPS solutions of gauge sector. For this purpose we substitute the BPS condition (3.29) into the equations of motion of the field strength (2.5) and one of the Bianchi identities, which are in our situation given by $\nabla_\mu F^{\mu\nu} = 0$, $\nabla_1 F^{23} + \nabla_2 F^{31} + \nabla_3 F^{12} = 0$, respectively. Under the BPS condition (3.29) the equations of motion reduce to the following three equations

$$\partial_t F^{01} + w^{(+)} \partial_3 F^{01} = 0, \quad (3.38)$$

$$\partial_t F^{23} + w^{(+)} \partial_3 F^{32} = 0, \quad (3.39)$$

$$\partial_1(w^{(+)} F^{01}) - \partial_2 F^{23} + \frac{2}{r} \cot \theta (w^{(+)} F^{01}) = 0, \quad (3.40)$$

and the Bianchi identity becomes

$$\partial_1(\sin \theta F^{23}) + w^{(+)} \partial_2(\sin \theta F^{01}) = 0. \quad (3.41)$$

After multiplying $\sin \theta$ to both sides in (3.40), we can write it as

$$\partial_1(\sin \theta F^{01}) - w^{(+)} \partial_2(\sin \theta F^{23}) = 0. \quad (3.42)$$

A general solution of (3.41) and (3.42) is given by

$$\begin{aligned} \sin \theta F^{01} &= \sum_{s \geq \frac{1}{2}, |l_3| \leq s} (B^{s,l_3}(t) \text{Re}[Y_{s,l_3,w^{(+)}s}(\theta, \phi, \psi)] - A^{s,l_3}(t) \text{Im}[Y_{s,l_3,w^{(+)}s}(\theta, \phi, \psi)]) \\ \sin \theta F^{23} &= \sum_{s \geq \frac{1}{2}, |l_3| \leq s} (A^{s,l_3}(t) \text{Re}[Y_{s,l_3,w^{(+)}s}(\theta, \phi, \psi)] + B^{s,l_3}(t) \text{Im}[Y_{s,l_3,w^{(+)}s}(\theta, \phi, \psi)]) \end{aligned} \quad (3.43)$$

where $A^{s,l_3}(t), B^{s,l_3}(t)$ are unknown functions of time. The reason is as follows. First we recall that $Y_{s,l_3,w^{(+)}s}$ is annihilated by the operator $\hat{R}_{w^{(+)}}$. By dividing $Y_{s,l_3,w^{(+)}s}$ into the real and imaginary part, $Y_{s,l_3,w^{(+)}s} = u + iv$, we can rewrite $\hat{R}_{w^{(+)}} Y_{s,l_3,w^{(+)}s} = 0$ as

$$\partial_1 u - w^{(+)} \partial_2 v = 0, \quad \partial_1 v + w^{(+)} \partial_2 u = 0. \quad (3.44)$$

This is equivalent to

$$\partial_1(Au + Bv) + w^{(+)} \partial_2(Bu - Av) = 0, \quad \partial_1(Bu - Av) - w^{(+)} \partial_2(Au + Bv) = 0 \quad (3.45)$$

where A, B are arbitrary constants independent of θ, ϕ, ψ . This implies that the equations (3.41) and (3.42) are solved by $\sin \theta F^{23} = Au + Bv$, $\sin \theta F^{01} = Bu - Av$. Superposing all modes labeled by l, l_3 gives a general solution (3.43), as asserted.

The time dependence of the field strength is determined by (3.38), (3.39). The result is

$$\sin \theta F^{01} = \sum_{s \geq \frac{1}{2}, |l_3| \leq s} a_{s,l_3}^{(+)} \text{Re} \left[Y_{s,l_3,w^{(+)}s}(\theta, \phi, \psi - w^{(+)} \frac{2}{r} t + \alpha_{s,l_3}^{(+)}) \right] \quad (3.46)$$

$$\sin \theta F^{23} = \sum_{s \geq \frac{1}{2}, |l_3| \leq s} a_{s,l_3}^{(+)} \text{Im} \left[Y_{s,l_3,w^{(+)}s}(\theta, \phi, \psi - w^{(+)} \frac{2}{r} t + \alpha_{s,l_3}^{(+)}) \right] \quad (3.47)$$

where $a_{s,l_3}^{(+)}, \alpha_{s,l_3}^{(+)}$ are integral constants taking real values with the range $a_{s,l_3}^{(+)} \geq 0, 0 \leq \alpha_{s,l_3}^{(+)} < 2\pi$. Plugging the explicit expression of the spherical harmonics given by (B.24) into these, we find the general 1/16 BPS solution of the field strength as

$$F^{01} = \sum_{s \geq \frac{1}{2}, |l_3| \leq s} a_{s,l_3}^{(+)} c_{s,l_3} \tan^{w^{(+)}l_3} \left(\frac{\theta}{2} \right) \sin^{s-1} \theta \cos(l_3 \phi + s \left(\frac{2}{r} t - w^{(+)} \psi \right) + \alpha_{s,l_3}^{(+)}) \quad (3.48)$$

$$F^{23} = \sum_{s \geq \frac{1}{2}, |l_3| \leq s} a_{s,l_3}^{(+)} c_{s,l_3} \tan^{w^{(+)}l_3} \left(\frac{\theta}{2} \right) \sin^{s-1} \theta \sin(l_3 \phi + s \left(\frac{2}{r} t - w^{(+)} \psi \right) + \alpha_{s,l_3}^{(+)}) \quad (3.49)$$

where c_{s,l_3} is given by (B.25).⁶ Other components of the field strength are determined by (3.29).

Let us emphasize that it is possible to turn on BPS excitation for the gauge field independently from the matter sector in the abelian sector. This behavior is different from the non-abelian sector [27].⁷ We also note that it is not required to use the vector spherical harmonics to solve the BPS equation for the gauge field.

The moduli space of the 1/16 BPS solution is given by

$$x_{14}^{s,l_3}, x_{24}^{s,l_3}, x_{34}^{s,l_3} \in \mathbf{C}, a_{s,l_3}^{(+)} \geq 0, 0 \leq \alpha_{s,l_3}^{(+)} < 2\pi \quad (3.50)$$

for $s \geq 0, |l_3| \leq s$ for x_{A4}^{s,l_3} , $s \geq \frac{1}{2}, |l_3| \leq s$ for $a_{s,l_3}^{(+)}$. Note that we do not take into account the flux quantization condition here and hereafter, which would make the moduli space quantized in a certain manner.

Let us compute conserved charges (2.8), (2.9) and (2.7). We first simplify them by using the BPS solution (3.29), (3.30). As a result the conserved charges are written in terms of $X_{A4}, \partial_2 X_{A4}, \partial_3 X_{A4}, F^{01}, F^{23}$. We present results by separating the matter sector and gauge sector. The conserved charges in the matter sector are simplified as

$$H|_{F_{\mu\nu}=0} = \int d^3\Omega \frac{4}{g^2} \text{Tr} \left(\frac{1}{r^2} |X_{A4}|^2 - w^{(+)} \frac{i}{r} X_{A4} \partial_3 X^{A4} + |\partial_2 X_{A4}|^2 + |\partial_3 X_{A4}|^2 \right), \quad (3.51)$$

$$J_\theta|_{F_{\mu\nu}=0} = 0, \quad (3.52)$$

$$J_\phi|_{F_{\mu\nu}=0} = \int d^3\Omega \frac{2}{g^2} \text{Tr} \left(- \left(\frac{i}{r} X_{A4} - w^{(+)} \partial_3 X_{A4} \right) \partial_\phi X^{A4} + (c.c.) \right), \quad (3.53)$$

$$J_\psi|_{F_{\mu\nu}=0} = \int d^3\Omega \frac{2}{g^2} \text{Tr} \left(- \left(\frac{i}{r} X_{A4} - w^{(+)} \partial_3 X_{A4} \right) \partial_\psi X^{A4} + (c.c.) \right), \quad (3.54)$$

$$R^4_4|_{F_{\mu\nu}=0} = i \int d^3\Omega \frac{2}{g^2} \text{Tr} \left(\frac{i}{r} |X_{4B}|^2 - w^{(+)} X^{4B} \partial_3 X_{4B} \right), \quad (3.55)$$

⁶Here we exclude the mode of $s = 0$, which formally satisfies the 1/16 BPS equations but turns out to be non-normalizable.

⁷The author would like to thank S.Kim for pointing this out.

where (c.c.) means the complex conjugation of the first term. The gauge field part is the following.

$$H|_{X^{AB}=0} = \int d^3\Omega \frac{1}{g^2} \text{Tr} \left((F^{01})^2 + (F^{23})^2 \right), \quad (3.56)$$

$$J_\theta|_{X^{AB}=0} = 0, \quad (3.57)$$

$$J_\phi|_{X^{AB}=0} = w^{(+)} \frac{r}{2g^2} \int d^3\Omega \cos \theta \text{Tr} \left((F^{01})^2 + (F^{23})^2 \right), \quad (3.58)$$

$$J_\psi|_{X^{AB}=0} = w^{(+)} \frac{r}{2g^2} \int d^3\Omega \text{Tr} \left((F^{01})^2 + (F^{23})^2 \right), \quad (3.59)$$

$$R^4_4|_{X^{AB}=0} = 0. \quad (3.60)$$

Note that at this stage we see the BPS relation of charges given by (3.22) in the gauge sector.

Let us compute conserved charges using the BPS solution (3.37), (3.48), (3.49). The matter part is the following.⁸

$$H|_{F_{\mu\nu}=0} = \frac{8}{r^2 g^2} \sum_{s \geq 0, |l_3| \leq s} (2s+1)^2 \text{Tr} |x_{A4}^{s,l_3}|^2, \quad (3.64)$$

$$J_\phi|_{F_{\mu\nu}=0} = \frac{-4}{r g^2} \sum_{s \geq 0, |l_3| \leq s} (2s+1) l_3 \text{Tr} |x_{A4}^{s,l_3}|^2, \quad (3.65)$$

$$J_\psi|_{F_{\mu\nu}=0} = \frac{4}{r g^2} w^{(+)} \sum_{s \geq 0, |l_3| \leq s} (2s+1) s \text{Tr} |x_{A4}^{s,l_3}|^2, \quad (3.66)$$

$$R^4_4|_{F_{\mu\nu}=0} = -\frac{2}{r g^2} \sum_{s \geq 0, |l_3| \leq s} (2s+1) \text{Tr} |x_{A4}^{s,l_3}|^2. \quad (3.67)$$

In the matter sector we also find the BPS relation given by (3.22).

The gauge field part is computed as follows.

$$H|_{X_{AB}=0} = w^{(+)} \frac{2}{r} J_\psi|_{X_{AB}=0} = \frac{1}{g^2} \sum_{s \geq \frac{1}{2}, |l_3| \leq s} \frac{s(2s+1)}{2(s^2 - l_3^2)} \text{Tr} (a_{s,l_3}^{(+)})^2, \quad (3.68)$$

$$J_\phi|_{X_{AB}=0} = \frac{r}{2g^2} \sum_{s \geq \frac{1}{2}, |l_3| \leq s} \frac{l_3(2s+1)}{2(l_3^2 - s^2)} \text{Tr} (a_{s,l_3}^{(+)})^2. \quad (3.69)$$

Note that the energy and momenta become divergent when the modes with $|l_3| = s$ are nonzero. Although it is not clear whether the solutions with divergent energy are

⁸For this computation we used formulas such that

$$\int d^3\Omega \frac{1}{\sin^2 \theta} |Y_{s,l_3,w^{(+)}s}|^2 = \frac{s(1+2s)}{2(-l_3^2 + s^2)}, \quad (3.61)$$

$$\int d^3\Omega \frac{\cot \theta}{\sin \theta} |Y_{s,l_3,w^{(+)}s}|^2 = \frac{w^{(+)} l_3 (1+2s)}{2(l_3^2 - s^2)}, \quad (3.62)$$

$$\int d^3\Omega \cot^2 \theta |Y_{s,l_3,w^{(+)}s}|^2 = -\frac{2l_3^2 + s}{2(l_3^2 - s^2)}, \quad (3.63)$$

where $|l_3| < s$. In the main text, we also used these formulas formally at $|l_3| = s$.

	H	J_ϕ	J_ψ	R^1_1	R^2_2	R^3_3	R^4_4
$(\epsilon_1^{(-)})^*$	$-\frac{1}{2r}$	$\frac{w^{(-)}}{2}$	0	$-\frac{3}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

Table 3. Charges of the Killing spinor $(\epsilon_1^{(-)})^*$ are shown.

physically important, they may have dual counterparts in the gravity side, which has a singular configuration with such a divergent energy. We would like to leave a problem to clarify physical importance of these solutions to future works.

3.2.2 BPS states preserving $\epsilon_1^{(-)}$

Next we study BPS states preserving a Killing spinor $\epsilon_1^{(-)}$ whose constant spinor $\eta_1^{(-)}$ is projected in such a way that

$$\gamma_0 \eta_1^{(-)} = i w^{(-)} \eta_1^{(-)} \quad (3.70)$$

where $w^{(-)} = \pm 1$. Under this projection the Killing spinor $\epsilon_1^{(-)}$ given by (2.16) is evaluated as

$$\epsilon_1^{(-)} = e^{-\frac{i}{2r}t} e^{\frac{-i w^{(-)} \phi}{2}} (c_\theta - w^{(-)} \gamma_1 s_\theta) \eta_1^{(-)} \quad (3.71)$$

where we used notation such that

$$c_\theta = \cos \frac{\theta}{2}, \quad s_\theta = \sin \frac{\theta}{2}. \quad (3.72)$$

We determine charges of this Killing spinor $(\epsilon_1^{(-)})^*$ to find out the BPS relation of charges in table 3.⁹ Therefore the BPS relation of charges associated with the Killing spinor $\epsilon_1^{(-)}$ is

$$rH = 2R^1_1 + 2w^{(-)} J_\phi. \quad (3.73)$$

To determine a BPS condition preserving $\epsilon_1^{(-)}$, we extract the terms containing $\eta_1^{(-)}$ from (3.24), (3.25) as done previously. The results are as follows.

$$\begin{aligned} \Delta_{(-)} \lambda_A &= e^{-\frac{i\mu}{4}t} e^{\frac{-i w^{(-)} \phi}{2}} \times \\ &\times \left[\{ (-i w^{(-)} c_\theta) (F^{01} - i F^{23}) - c_\theta (F^{02} + i F^{13}) + s_\theta (F^{03} - i F^{12}) - 2i [X_{AB}, X^{B1}] (-w^{(-)} s_\theta) \} \gamma_1 \eta_1^{(-)} \right. \\ &\quad \left. + \{ -i s_\theta (F^{01} - i F^{23}) + s_\theta w^{(-)} (F^{02} + i F^{13}) + c_\theta w^{(-)} (F^{03} - i F^{12}) - 2i [X_{AB}, X^{B1}] c_\theta \} \eta_1^{(-)} \right] + \dots \\ \bar{\Delta}_{(-)} \lambda_A &= e^{\frac{i\mu}{4}t} e^{\frac{i w^{(-)} \phi}{2}} \times \\ &\times \left[\left\{ \left(-2D_0 X_{A1} - X_{A1} \frac{2i}{r} \right) (i w^{(-)} c_\theta) - 2D_1 X_{A1} (-w^{(-)} s_\theta) - 2D_2 X_{A1} (i s_\theta) - 2D_3 X_{A1} (i c_\theta) \right\} (\eta_1^{(-)})^* \right. \\ &\quad \left. + \left\{ \left(-2D_0 X_{A1} - X_{A1} \frac{2i}{r} \right) (i s_\theta) - 2D_1 X_{A1} (c_\theta) - 2D_2 X_{A1} (i w^{(-)} c_\theta) - 2D_3 X_{A1} (-i w^{(-)} s_\theta) \right\} \gamma_1 (\eta_1^{(-)})^* \right] \\ &+ \dots \end{aligned}$$

⁹The reason why we determine charges of $(\epsilon_1^{(-)})^*$ instead of those of $\epsilon_1^{(-)}$ is because $(\epsilon_1^{(-)})^*$ has the negative energy, which has a corresponding supersymmetry charge, while $\epsilon_1^{(-)}$ has the positive energy, which corresponds to a special superconformal charge. We fix a BPS relation of charges for a particular supersymmetry charge.

where the ellipses describe the other terms than $\eta_1^{(-)}$. In order for these to vanish with $\eta_1^{(-)}$ nonzero, the fields are required to satisfy

$$\begin{aligned} & \{ -is_\theta(F^{01} - iF^{23}) + s_\theta w^{(-)}(F^{02} + iF^{13}) + c_\theta w^{(-)}(F^{03} - iF^{12}) \} \delta_A^1 - 2i[X_{AB}, X^{B1}]c_\theta = 0, \\ & \{ (-iw^{(-)}c_\theta)(F^{01} - iF^{23}) - c_\theta(F^{02} + iF^{13}) + s_\theta(F^{03} - iF^{12}) \} \delta_A^1 - 2i[X_{AB}, X^{B1}](-w^{(-)}s_\theta) = 0, \\ & \left(-2D_0X_{A1} - X_{A1}\frac{2i}{r} \right) (iw^{(-)}c_\theta) - 2D_1X_{A1}(-w^{(-)}s_\theta) - 2D_2X_{A1}(is_\theta) - 2D_3X_{A1}(ic_\theta) = 0, \\ & \left(-2D_0X_{A1} - X_{A1}\frac{2i}{r} \right) (is_\theta) - 2D_1X_{A1}(c_\theta) - 2D_2X_{A1}(iw^{(-)}c_\theta) - 2D_3X_{A1}(-iw^{(-)}s_\theta) = 0, \end{aligned}$$

for all $A = 1, 2, 3, 4$, since $\eta_1^{(-)}, \gamma_1\eta_1^{(-)}, (\eta_1^{(-)})^*, \gamma_1(\eta_1^{(-)})^*$ are linearly independent under the projection (3.70). Using the fact that all the components of $F^{\mu\nu}$ are real, we can simplify these BPS equations as follows.

$$\begin{aligned} F^{02} &= -w^{(-)}F^{23}\sec\theta + F^{03}\tan\theta, \quad F^{13} = -w^{(-)}F^{01}\sec\theta - F^{12}\tan\theta, \\ [X_{AB}, X^{B1}] &= \frac{1}{2i} \{ w^{(-)}\sec\theta(F^{03} - iF^{12}) - (F^{23} + iF^{01})\tan\theta \} \delta_A^1, \\ D_0X_{A1} &= -\frac{i}{r}X_{A1} - w^{(-)}\frac{2}{r}D_\phi X_{A1}, \\ D_1X_{A1} &= iw^{(-)}(-\cos\theta D_2X_{A1} + \sin\theta D_3X_{A1}). \end{aligned} \tag{3.74}$$

These are the other set of 1/16 BPS equations, which is a new result in this paper.

Let us solve these BPS equations when the fields take values in Cartan subalgebra of a gauge group. In this case these BPS equations are further simplified as

$$F^{02} = -w^{(-)}\cos\theta F^{23}, \quad F^{03} = w^{(-)}\sin\theta F^{23}, \tag{3.75}$$

$$F^{12} = -w^{(-)}\sin\theta F^{01}, \quad F^{13} = -w^{(-)}\cos\theta F^{01}, \tag{3.76}$$

$$\left(\partial_0 + \frac{i}{r} + \frac{2}{r}w^{(-)}\partial_\phi \right) X_{A1} = 0, \tag{3.77}$$

$$(\partial_1 - iw^{(-)}(-\cos\theta\partial_2 + \sin\theta\partial_3))X_{A1} = 0. \tag{3.78}$$

Let us first solve the matter BPS equation. By noticing that (3.78) can be written as $\hat{L}_{w^{(-)}}X_{A1} = 0$, where $\hat{L}_{w^{(-)}} = \hat{L}_\pm$ when $w^{(-)} = \pm 1$, the BPS solution of the matter fields are expanded by the spherical harmonics of highest (or lowest) weight of $su(2)_L$ for $w^{(-)} = +1$ (or -1).

$$X_{A1} = \sum_{s \geq 0, |r_3| \leq s} x_{A1}^{s,r_3}(t) Y_{s,w^{(-)}s,r_3}(\theta, \phi, \psi) \tag{3.79}$$

where $x_{A1}^{s,r_3}(t)$ represents time-dependence of the scalar field, which is determined from (3.77).

$$\partial_0 x_{A1}^{s,r_3}(t) = \frac{-i}{r} x_{A1}^{s,r_3}(t) - \frac{2}{r} w^{(-)}(iw^{(-)}s) x_{A1}^{s,r_3}(t). \tag{3.80}$$

This equation is easily solved as

$$x_{A1}^{s,r_3}(t) = x_{A1}^{s,r_3} e^{-i\frac{2s+1}{r}t} \tag{3.81}$$

where x_{A1}^{s,r_3} is an integral constant. As a result we obtain BPS solutions

$$X_{A1} = \sum_{s \geq 0, |r_3| \leq s} x_{A1}^{s,r_3} e^{-i \frac{2s+1}{r} t} Y_{s,w(-)s,r_3}(\theta, \phi, \psi). \quad (3.82)$$

This is a general 1/16 BPS solution which preserves $\epsilon_1^{(-)}$ projected by (3.20). It is not difficult to check that this satisfies the equation of motion of the scalar field (2.6).

Let us move on to determining the BPS solution of the gauge sector combining the equations of motion (2.5) and Bianchi identity. In the current situation they are given by $\nabla_\mu F^{\mu\nu} = 0$ and $\nabla_1 F^{23} + \nabla_2 F^{31} + \nabla_3 F^{12} = 0$, respectively. Under the BPS conditions (3.75), (3.76) the equations of motion reduce to the following three equations

$$\partial_t F^{01} = -w^{(-)} \frac{2}{r} \partial_\phi F^{01}, \quad (3.83)$$

$$\partial_t F^{23} = -w^{(-)} \frac{2}{r} \partial_\phi F^{23}, \quad (3.84)$$

$$\partial_3 F^{23} = -w^{(-)} (\partial_t (\cos \theta F^{23}) + \partial_1 (\sin \theta F^{01})), \quad (3.85)$$

and the Bianchi identity becomes

$$\left(-\frac{1}{\sin \theta} \partial_\psi + \cot \theta \partial_\phi \right) (\sin \theta F^{01}) = -w^{(-)} \partial_\theta (\sin \theta F^{23}). \quad (3.86)$$

We can solve (3.85) and (3.86) by noticing the fact that they can be rewritten as $\hat{L}_{w^{(-)}} G = 0$, where $G = \sin \theta F^{01} + i \sin \theta F^{23}$. Therefore this solution is given by the spherical harmonics of the highest (or lowest) weight of $su(2)_L$ algebra.

$$G = \sum_{s \geq \frac{1}{2}, |l_3| \leq s} C_{s,r_3}^{(-)}(t) Y_{s,w^{(-)}s,r_3}(\theta, \phi, \psi) \quad (3.87)$$

where $C_{s,r_3}^{(-)}(t)$ is an unknown function of time, which can be easily determined from the other BPS equations (3.83), (3.84).

$$\partial_t C_{s,r_3}^{(-)}(t) = -i \frac{2}{r} s C_{s,r_3}^{(-)}(t) \quad (3.88)$$

which is solved as

$$C_{s,r_3}^{(-)}(t) = C_{s,r_3}^{(-)} e^{-i \frac{2}{r} s t} = a_{s,r_3}^{(-)} e^{-i \frac{2}{r} s t + i \alpha_{s,r_3}^{(-)}} \quad (3.89)$$

where $a_{s,r_3}^{(-)}, \alpha_{s,r_3}^{(-)}$ are integral constants with the range $a_{s,r_3}^{(-)} \geq 0$, $0 \leq \alpha_{s,r_3}^{(-)} < 2\pi$. Plugging this back into the above gives

$$G = \sum_{s \geq \frac{1}{2}, |l_3| \leq s} a_{s,r_3}^{(-)} e^{-i \frac{2}{r} s t + i \alpha_{s,r_3}^{(-)}} Y_{s,w^{(-)}s,r_3}(\theta, \phi, \psi). \quad (3.90)$$

Plugging the polar coordinates expression of the spherical harmonics given by (B.27) we find the 1/16 BPS solutions of the field strength

$$F^{01} = \sum_{s \geq \frac{1}{2}, |r_3| \leq s} a_{s,r_3}^{(-)} c_{s,r_3} \tan^{w^{(-)}r_3} \left(\frac{\theta}{2} \right) \sin^{s-1} \theta \cos \left(-r_3 \psi + s \left(w^{(-)} \phi - \frac{2}{r} t \right) + \alpha_{s,r_3}^{(-)} \right), \quad (3.91)$$

$$F^{23} = \sum_{s \geq \frac{1}{2}, |r_3| \leq s} a_{s,r_3}^{(-)} c_{s,r_3} \tan^{w^{(-)}r_3} \left(\frac{\theta}{2} \right) \sin^{s-1} \theta \sin \left(-r_3 \psi + s \left(w^{(-)} \phi - \frac{2}{r} t \right) + \alpha_{s,r_3}^{(-)} \right), \quad (3.92)$$

where c_{s,r_3} is given by (B.28).¹⁰ Other components can be obtained from (3.75), (3.76).

The moduli space of the 1/16 BPS solution is given by

$$x_{12}^{s,r_3}, x_{12}^{s,r_3}, x_{13}^{s,r_3} \in \mathbf{C}, \quad a_{s,r_3}^{(-)} \geq 0, \quad 0 \leq \alpha_{s,r_3}^{(-)} < 2\pi \quad (3.93)$$

for $s \geq 0, |r_3| \leq s$ for x_{A4}^{s,r_3} , $s \geq \frac{1}{2}, |r_3| \leq s$ for $a_{s,r_3}^{(+)}$. Here we do not take into account the flux quantization condition.

We observe that the BPS solutions preserving $\epsilon^{(-)}$ are mapped to those preserving $\epsilon^{(+)}$ by exchanging the roles of $SU(2)_L$ and $SU(2)_R$, or the coordinates ϕ and ψ . It does not seem a priori clear why this should happen, because, for example, there is no such a simple relation between the Killing spinors $\epsilon^{(+)}$ and $\epsilon^{(-)}$. It may be better to be conservative to say that we have found a set of Killing spinors and projection conditions to give such simple results. We leave further discussion in the future.

Let us compute conserved charges (2.8), (2.9), (2.7) of the BPS solution. For this end we first simplify them by using the BPS conditions (3.75), (3.76), (3.77), (3.78). The results of the matter part are the following.

$$H|_{F_{\mu\nu}=0} = \frac{4}{g^2} \int d^3\Omega \operatorname{Tr} \left(\frac{1}{r^2} |X_{A1}|^2 + |\partial_2 X_{A1}|^2 + |\partial_3 X_{A1}|^2 + \frac{2i}{r^2} w^{(-)} X_{A1} \partial_\phi X^{A1} \right), \quad (3.94)$$

$$J_\theta|_{F_{\mu\nu}=0} = 0, \quad (3.95)$$

$$J_\phi|_{F_{\mu\nu}=0} = \int d^3\Omega \frac{1}{g^2} \operatorname{Tr} \left(-2 \left(-\frac{i}{r} X_{A1} - w^{(-)} \frac{2}{r} D_\phi X_{A1} \right) D_\phi X^{A1} + (c.c.) \right),$$

$$J_\psi|_{F_{\mu\nu}=0} = \int d^3\Omega \frac{1}{g^2} \operatorname{Tr} \left(-2 \left(-\frac{i}{r} X_{A1} - w^{(-)} \frac{2}{r} D_\phi X_{A1} \right) D_\psi X^{A1} + (c.c.) \right), \quad (3.96)$$

$$R^1|_{F_{\mu\nu}=0} = i \int d^3\Omega \frac{2}{g^2} \operatorname{Tr} \left(-\frac{i}{r} |X_{1A}|^2 - w^{(-)} \frac{2}{r} X^{1A} D_\phi X_{A1} \right). \quad (3.97)$$

The gauge sector is as follows.

$$H|_{X_{AB}=0} = \int d^3\Omega \frac{1}{g^2} \operatorname{Tr} \left((F^{01})^2 + (F^{23})^2 \right), \quad (3.98)$$

$$J_\theta|_{X_{AB}=0} = 0, \quad (3.99)$$

$$J_\phi|_{X_{AB}=0} = w^{(-)} \frac{r}{2g^2} \int d^3\Omega \operatorname{Tr} \left((F^{01})^2 + (F^{23})^2 \right), \quad (3.100)$$

$$J_\psi|_{X_{AB}=0} = w^{(-)} \frac{r}{2g^2} \int d^3\Omega \cos \theta \operatorname{Tr} \left((F^{01})^2 + (F^{23})^2 \right), \quad (3.101)$$

with $R^1|_{X_{AB}=0} = 0$. Note that at this stage we see the BPS relation of charges given by (3.73) in the gauge sector.

¹⁰We exclude the mode with $s = 0$.

Let us compute these conserved charges under the BPS solution (3.82). (3.91), (3.92). The matter parts are¹¹

$$H|_{F_{\mu\nu}=0} = \frac{8}{r^2 g^2} \sum_{s \geq 0, |r_3| \leq s} (2s+1)^2 \text{Tr} |x_{A1}^{s,r_3}|^2, \quad (3.105)$$

$$J_\phi|_{F_{\mu\nu}=0} = \frac{4}{r g^2} w^{(-)} \sum_{s \geq 0, |r_3| \leq s} (2s+1)s \text{Tr} |x_{A1}^{s,r_3}|^2, \quad (3.106)$$

$$J_\psi|_{F_{\mu\nu}=0} = \frac{4}{r g^2} \sum_{s \geq 0, |r_3| \leq s} -(2s+1)r_3 \text{Tr} |x_{A1}^{s,r_3}|^2, \quad (3.107)$$

$$R^1{}_1|_{F_{\mu\nu}=0} = \frac{2}{r g^2} \sum_{s \geq 0, |r_3| \leq s} (2s+1) \text{Tr} |x_{A1}^{s,r_3}|^2, \quad (3.108)$$

which satisfies the BPS relation of charges given by (3.73). The gauge field part is the following.

$$H|_{X_{AB}=0} = w^{(-)} \frac{2}{r} J_\phi|_{X_{AB}=0} = \frac{1}{g^2} \sum_{s \geq \frac{1}{2}, |r_3| \leq s} \frac{s(1+2s)}{2(s^2 - r_3^2)} \text{Tr} (a_{s,r_3}^{(-)})^2, \quad (3.109)$$

$$J_\psi|_{X_{AB}=0} = \frac{r}{2g^2} \sum_{s \geq \frac{1}{2}, |r_3| \leq s} \frac{r_3(1+2s)}{2(r_3^2 - s^2)} \text{Tr} (a_{s,r_3}^{(-)})^2. \quad (3.110)$$

Note that the modes with $|r_3| = s$ give divergent contribution to the energy and momenta.

3.3 Counting of supersymmetries

In this subsection we count number of supersymmetries preserved by BPS solutions constructed in the previous subsections. Let us count the number of supersymmetries of the BPS solutions given by (3.37), (3.48), (3.49), which preserves at least $\eta_4^{(+)}$ projected by (3.20). For convenience we write them down here again.

$$X_{A4} = \sum_{s \geq 0, |l_3| \leq s} x_{A4}^{s,l_3} e^{i \frac{2s+1}{r} t} Y_{s,l_3,w^{(+)}s}(\theta, \phi, \psi), \quad (3.111)$$

$$F^{01} = \sum_{s \geq \frac{1}{2}, |l_3| \leq s} a_{s,l_3}^{(+)} c_{s,l_3} \tan^{w^{(+)}l_3} \left(\frac{\theta}{2} \right) \sin^{s-1} \theta \cos \left(l_3 \phi + s \left(\frac{2}{r} t - w^{(+)} \psi \right) + \alpha_{s,l_3}^{(+)} \right), \quad (3.112)$$

$$F^{23} = \sum_{s \geq \frac{1}{2}, |l_3| \leq s} a_{s,l_3}^{(+)} c_{s,l_3} \tan^{w^{(+)}l_3} \left(\frac{\theta}{2} \right) \sin^{s-1} \theta \sin \left(l_3 \phi + s \left(\frac{2}{r} t - w^{(+)} \psi \right) + \alpha_{s,l_3}^{(+)} \right), \quad (3.113)$$

¹¹We note useful formulas

$$\int d^3\Omega \frac{1}{\sin^2 \theta} |Y_{s,w^{(-)}s,r_3}|^2 = \frac{s(1+2s)}{2(-r_3^2 + s^2)}, \quad (3.102)$$

$$\int d^3\Omega \frac{\cot \theta}{\sin \theta} |Y_{s,w^{(-)}s,r_3}|^2 = \frac{w^{(-)}r_3(1+2s)}{2(r_3^2 - s^2)}, \quad (3.103)$$

$$\int d^3\Omega \cot^2 \theta |Y_{s,w^{(-)}s,r_3}|^2 = -\frac{2r_3^2 + s}{2(r_3^2 - s^2)}, \quad (3.104)$$

where $|r_3| < s$. We however used these formulas formally at $|r_3| = s$ in the main text.

for arbitrary $A = 1, 2, 3$. To exclude the case of trivial angular momenta, we consider the case whether $x_{C4}^{s,l_3} \neq 0$ or $a_{s,l_3}^{(+)} \neq 0$ for some positive s and l_3 .

First let us consider the case where $a_{s,l_3}^{(+)} \neq 0$ for some positive s and l_3 . Under this situation we consider the BPS solution which also preserves $\eta_3^{(+)}$ satisfying $\gamma_0 \eta_3^{(+)} = i w_3^{(+)} \eta_3^{(+)}$. Then the form of the BPS solution gets another constraint such that

$$X_{A3} = \sum_{s \geq 0, |l_3| \leq s} x_{A3}^{s,l_3} e^{i \frac{2s+1}{r} t} Y_{s,l_3,w_3^{(+)}s}(\theta, \phi, \psi), \quad (3.114)$$

$$F^{01} = \sum_{s \geq \frac{1}{2}, |l_3| \leq s} a_{s,l_3}^{(+)} c_{s,l_3} \tan^{w_3^{(+)}l_3} \left(\frac{\theta}{2} \right) \sin^{s-1} \theta \cos \left(l_3 \phi + s \left(\frac{2}{r} t - w_3^{(+)} \psi \right) + \alpha_{s,l_3}^{(+)} \right), \quad (3.115)$$

$$F^{23} = \sum_{s \geq \frac{1}{2}, |l_3| \leq s} a_{s,l_3}^{(+)} c_{s,l_3} \tan^{w_3^{(+)}l_3} \left(\frac{\theta}{2} \right) \sin^{s-1} \theta \sin \left(l_3 \phi + s \left(\frac{2}{r} t - w_3^{(+)} \psi \right) + \alpha_{s,l_3}^{(+)} \right), \quad (3.116)$$

for $A = 1, 2, 4$. Under the assumption, matching of (3.112), (3.113) with (3.115), (3.116) requires the signature of the projection for $\eta_3^{(+)}$, $\eta_4^{(+)}$ to be the same, $w_3^{(+)} = w^{(+)}$. Matching of the matter solution (3.111), (3.114) demands $X_{14} = X_{24} = 0$ since $X_{14}^\dagger = -X_{23}$, $X_{24}^\dagger = X_{13}$ from (2.1). Under this situation supersymmetries specified by $\eta_1^{(+)}$ and $\eta_2^{(+)}$ are broken unless $X_{34}^\dagger = 0$. In other words, the matter becomes trivial if we keep three supersymmetries of the form $\eta_A^{(+)}$. Can the BPS solution preserve $\eta_A^{(-)}$ supersymmetry concurrently? Let us study this possibility. Keeping also $\eta_4^{(-)}$ makes the matter fields completely trivial and gives constraint for the field strength so that the highest or lowest weight modes in terms of $su(2)_L$ remain: $a_{s,l_3}^{(+)} = 0$ for $\forall s > 0, l_3 \neq -w^{(-)}s$ for fixed $w^{(-)}$. To preserve $\eta_1^{(-)}$ instead of $\eta_4^{(-)}$, the matter field X_{14} has to vanish and the other matter fields have only the modes which are also the highest or lowest weight states of $su(2)_L$. In this case projected $\eta_1^{(-)}$ is preserved. The condition to keep $\eta_2^{(-)}$, $\eta_3^{(-)}$ are the same with that of $\eta_1^{(-)}$ by replacing 1 by 2, 3 respectively. When all scalar fields become trivial, all Killing spinors $\eta_A^{(+)}$ are preserved for all A . In this case, supersymmetry is enhanced when the gauge field strength is expanded by the mode of the highest or lowest weight representation of $su(2)_L$ as well so that $a_{s,l_3}^{(+)} = 0$ for $\forall s > 0, l_3 \neq -w^{(-)}s$ for fixed $w^{(-)}$. In this case the Killing spinors of the form $\eta_A^{(-)}$ are also conserved. We summarize this result in table 4.

Next we consider a case where $x_{34}^{s,l_3} \neq 0$ for some positive s and l_3 . This assumption restricts us to two case to study BPS solutions to preserve another Killing spinor as (i) $\eta_3^{(+)}$, (ii) $\eta_1^{(-)}$. They are all projected by γ_0 . Note that the assumption excludes the cases $\eta_2^{(+)}$ (or $\eta_1^{(+)}$), $\eta_3^{(-)}$, $\eta_4^{(-)}$, which result in $X_{34} = 0$. (i) Let us first study BPS solutions to preserves another supersymmetry specified by $\eta_3^{(+)}$ which satisfies $\gamma_0 \eta_3^{(+)} = i w_3^{(+)} \eta_3^{(+)}$. We study the BPS solution of the matter fields since the constraint of the field strength can be discussed in the same way as above. As in the above case, the matter BPS solution should also be of the form such as (3.114). Due to the assumption, matching of (3.111) and (3.114) requires the signature of the projection for $\eta_3^{(+)}$, $\eta_4^{(+)}$ to be the same, $w_3^{(+)} = w^{(+)}$, and $X_{14} = X_{24} = 0$ from (2.1). It is not possible to preserve $\eta_1^{(+)}$ or $\eta_2^{(+)}$, which would make all matter fields trivial, though it is possible to conserve $\eta_1^{(-)}$ projected by γ_0 , in which case the non-trivial

Parameter region.	Preserved Killing spinors.	Number of SUSY.
$x_{14}^{s,l_3}, x_{24}^{s,l_3}, x_{34}^{s,l_3}, a_{s,l_3}^{(+)}$: generic.	$\eta_4^{(+)}$ with $\gamma_0 \eta_4^{(+)} = iw^{(+)} \eta_4^{(+)}$.	2 ($\frac{1}{16}$ BPS)
$x_{14}^{s,l_3} = x_{24}^{s,l_3} = 0$ for $\forall s, l_3$.	$\eta_3^{(+)}, \eta_4^{(+)}$ with $\gamma_0 \eta_A^{(+)} = iw^{(+)} \eta_A^{(+)}$.	4 ($\frac{1}{8}$ BPS)
$x_{24}^{s,l_3}, x_{34}^{s,l_3}, a_{s,l_3}^{(+)} = 0$ for $\forall s > 0, l_3 \neq -w^{(-)}s$, $x_{14}^{s,l_3} = 0$ for $\forall s, l_3$.	$\eta_1^{(-)}, \eta_4^{(+)}$ with $\gamma_0 \eta_A^{(\pm)} = iw^{(\pm)} \eta_A^{(\pm)}$.	4 ($\frac{1}{8}$ BPS)
$x_{34}^{s,l_3}, a_{s,l_3}^{(+)} = 0$ for $\forall s > 0, l_3 \neq -w^{(-)}s$, $x_{14}^{s,l_3} = x_{24}^{s,l_3} = 0$ for $\forall s, l_3$.	$\eta_1^{(-)}, \eta_2^{(-)}, \eta_3^{(+)}, \eta_4^{(+)}$ with $\gamma_0 \eta_A^{(\pm)} = iw^{(\pm)} \eta_A^{(\pm)}$.	8 ($\frac{1}{4}$ BPS)
$x_{14}^{s,l_3} = x_{24}^{s,l_3} = x_{34}^{s,l_3} = 0$ for $\forall s, l_3$.	$\eta_1^{(+)}, \eta_2^{(+)}, \eta_3^{(+)}, \eta_4^{(+)}$ with $\gamma_0 \eta_A^{(+)} = iw^{(+)} \eta_A^{(+)}$.	8 ($\frac{1}{4}$ BPS)
$a_{s,l_3}^{(+)} = 0$ for $\forall s > 0, l_3 \neq -w^{(-)}s$, $x_{14}^{s,l_3} = x_{24}^{s,l_3} = x_{34}^{s,l_3} = 0$ for $\forall s, l_3$.	$\eta_1^{(\pm)}, \eta_2^{(\pm)}, \eta_3^{(\pm)}, \eta_4^{(\pm)}$ with $\gamma_0 \eta_A^{(\pm)} = iw^{(\pm)} \eta_A^{(\pm)}$.	16 ($\frac{1}{2}$ BPS)

Table 4. We list the number of supersymmetry of BPS solutions (3.37), (3.48), (3.49) to preserve $\eta_4^{(+)}$ with $\gamma_0 \eta_4^{(+)} = iw^{(+)} \eta_4^{(+)}$ at each point of moduli space with $a_{s,l_3}^{(+)} \neq 0$ for $\exists s > 0, l_3$.

Parameter region.	Preserved Killing spinors.	Number of SUSY.
$x_{14}^{s,l_3}, x_{24}^{s,l_3}, x_{34}^{s,l_3}, a_{s,l_3}^{(+)}$: generic.	$\eta_4^{(+)}$ with $\gamma_0 \eta_4^{(+)} = iw^{(+)} \eta_4^{(+)}$.	2 ($\frac{1}{16}$ BPS)
$x_{14}^{s,l_3} = x_{24}^{s,l_3} = 0$ for $\forall s, l_3$.	$\eta_3^{(+)}, \eta_4^{(+)}$ with $\gamma_0 \eta_A^{(+)} = iw^{(+)} \eta_A^{(+)}$.	4 ($\frac{1}{8}$ BPS)
$x_{24}^{s,l_3}, x_{34}^{s,l_3}, a_{s,l_3}^{(+)} = 0$ for $\forall s > 0, l_3 \neq -w^{(-)}s$, $x_{14}^{s,l_3} = 0$ for $\forall s, l_3$.	$\eta_1^{(-)}, \eta_4^{(+)}$ with $\gamma_0 \eta_A^{(\pm)} = iw^{(\pm)} \eta_A^{(\pm)}$.	4 ($\frac{1}{8}$ BPS)
$x_{34}^{s,l_3} = 0$ for $\forall s > 0, l_3 \neq -w^{(-)}s$ $x_{24}^{s,l_3}, a_{s,l_3}^{(+)} = 0$ for $\forall s > 0, l_3$; $x_{14}^{s,l_3} = 0$ for $\forall s, l_3$.	$\eta_1^{(-)}, \eta_4^{(+)}$ with $\gamma_0 \eta_4^{(+)} = iw^{(+)} \eta_4^{(+)}$.	6 ($\frac{3}{16}$ BPS)
$x_{34}^{s,l_3}, a_{s,l_3}^{(+)} = 0$ for $\forall s > 0, l_3 \neq -w^{(-)}s$, $x_{14}^{s,l_3} = x_{24}^{s,l_3} = 0$ for $\forall s, l_3$.	$\eta_1^{(-)}, \eta_2^{(-)}, \eta_3^{(+)}, \eta_4^{(+)}$ with $\gamma_0 \eta_A^{(\pm)} = iw^{(\pm)} \eta_A^{(\pm)}$.	8 ($\frac{1}{4}$ BPS)

Table 5. We list the number of supersymmetries of BPS solutions (3.37), (3.48), (3.49) to preserve $\eta_4^{(+)}$ with $\gamma_0 \eta_4^{(+)} = iw^{(+)} \eta_4^{(+)}$ at each point of moduli space with $x_{34}^{s,l_3} \neq 0$ for $\exists s > 0, l_3$.

scalar field X_{34} and the field strength have to also be expanded by the highest weight states of $su(2)_L$: $x_{34}^{s,l_3}, a_{s,l_3}^{(+)} = 0$ for $l_3 \neq -w^{(-)}s$. This solution automatically has supersymmetry specified by $\eta_2^{(-)}$ as well. (ii) Let us consider the case for the BPS solution to preserves $\eta_1^{(-)}$ which satisfies $\gamma_0 \eta_1^{(-)} = iw^{(-)} \eta_1^{(-)}$. This demands that $X_{14} = 0$ and $x_{24}^{s,l_3}, x_{34}^{s,l_3}, a_{s,l_3}^{(+)} = 0$ for $l_3 \neq -w^{(-)}s$. Notice that supersymmetry is enhanced when $x_{24}^{s,l_3}, a_{s,l_3}^{(+)} = 0$ for all $s > 0$, in which case $\eta_1^{(-)}$ is preserved without any projection. Furthermore in order to preserve $\eta_2^{(-)}$ in addition it is necessary for X_{24} to vanish. In this case supersymmetry is enhanced so that $\eta_3^{(+)}$ is also preserved. We summarize this result in table 5.

Parameter region.	Preserved Killing spinors.	Number of SUSY.
$x_{12}^{s,r_3}, x_{13}^{s,r_3}, x_{14}^{s,r_3}, a_{s,r_3}^{(-)}$: generic.	$\eta_1^{(-)}$ with $\gamma_0 \eta_1^{(-)} = i w^{(-)} \eta_1^{(-)}$.	2 ($\frac{1}{16}$ BPS)
$x_{13}^{s,r_3} = x_{14}^{s,r_3} = 0$ for $\forall s, r_3$.	$\eta_1^{(-)}, \eta_2^{(-)}$ with $\gamma_0 \eta_A^{(-)} = i w^{(-)} \eta_A^{(-)}$.	4 ($\frac{1}{8}$ BPS)
$x_{12}^{s,r_3}, x_{13}^{s,r_3}, a_{s,r_3}^{(-)} = 0$ for $\forall s, r_3 \neq -w^{(+)}s$, $x_{14}^{s,r_3} = 0$ for $\forall s, r_3$.	$\eta_1^{(-)}, \eta_4^{(+)}$ with $\gamma_0 \eta_A^{(\pm)} = i w^{(\pm)} \eta_A^{(\pm)}$.	4 ($\frac{1}{8}$ BPS)
$x_{12}^{s,r_3}, x_{13}^{s,r_3}, x_{14}^{s,r_3} = 0$ for $\forall s > 0, r_3$, $a_{s,r_3}^{(-)} = 0$ for $\forall s, r_3$.	$\eta_1^{(-)}$.	4 ($\frac{1}{8}$ BPS)
$x_{12}^{s,r_3} = 0$ for $\forall s, r_3 \neq -w^{(+)}s$ $x_{13}^{s,r_3}, a_{s,r_3}^{(-)} = 0$ for $\forall s > 0, l_3$; $x_{14}^{s,r_3} = 0$ for $\forall s, l_3$.	$\eta_1^{(-)}, \eta_4^{(+)}$, with $\gamma_0 \eta_1^{(-)} = i w^{(-)} \eta_1^{(-)}$.	6 ($\frac{3}{16}$ BPS)
$x_{12}^{s,r_3}, a_{s,r_3}^{(-)} = 0$ for $\forall s, r_3 \neq -w^{(+)}s$, $x_{13}^{s,r_3} = x_{14}^{s,r_3} = 0$ for $\forall s, r_3$.	$\eta_1^{(-)}, \eta_2^{(-)}, \eta_3^{(+)}, \eta_4^{(+)}$ with $\gamma_0 \eta_A^{(\pm)} = i w^{(\pm)} \eta_A^{(\pm)}$.	8 ($\frac{1}{4}$ BPS)
$x_{12}^{s,r_3} = x_{13}^{s,r_3} = x_{14}^{s,r_3} = 0$ for $\forall s, r_3$.	$\eta_1^{(-)}, \eta_2^{(-)}, \eta_3^{(-)}, \eta_4^{(-)}$ with $\gamma_0 \eta_A^{(-)} = i w^{(-)} \eta_A^{(-)}$.	8 ($\frac{1}{4}$ BPS)
$x_{12}^{s,r_3} = x_{13}^{s,r_3} = 0$ for $\forall s > 0, r_3$, $x_{14}^{s,r_3} = a_{s,r_3}^{(-)} = 0$ for $\forall s, r_3$.	$\eta_1^{(-)}, \eta_4^{(+)}$.	8 ($\frac{1}{4}$ BPS)
$a_{s,r_3}^{(-)} = 0$ for $\forall s, r_3 \neq -w^{(+)}s$, $x_{12}^{s,r_3} = x_{13}^{s,r_3} = x_{14}^{s,r_3} = 0$ for $\forall s, r_3$.	$\eta_1^{(\pm)}, \eta_2^{(\pm)}, \eta_3^{(\pm)}, \eta_4^{(\pm)}$ with $\gamma_0 \eta_A^{(\pm)} = i w^{(\pm)} \eta_A^{(\pm)}$.	16 ($\frac{1}{2}$ BPS)
$x_{12}^{s,r_3} = 0$ for $\forall s > 0, r_3$, $x_{13}^{s,r_3} = x_{14}^{s,r_3} = a_{s,r_3}^{(-)} = 0$ for $\forall s, r_3$.	$\eta_1^{(-)}, \eta_2^{(-)}, \eta_3^{(+)}, \eta_4^{(+)}$.	16 ($\frac{1}{2}$ BPS)
$x_{12}^{s,r_3} = x_{13}^{s,r_3} = x_{14}^{s,r_3} = a_{s,r_3}^{(-)} = 0$ for $\forall s, r_3$.	$\eta_1^{(\pm)}, \eta_2^{(\pm)}, \eta_3^{(\pm)}, \eta_4^{(\pm)}$.	32 (Unique vacuum)

Table 6. We list the number of supersymmetry of BPS solutions (3.82), (3.91), (3.92) which preserve $\eta_1^{(-)}$ with $\gamma_0 \eta_1^{(-)} = i w^{(-)} \eta_1^{(-)}$ at each point of moduli space.

Similarly we can count the number of supersymmetries of the BPS solution which preserves at least $\eta_1^{(-)}$ with projection (3.70). They are given by (3.82), (3.91), (3.92), which we write down again for convenience.

$$X_{A1} = \sum_{s \geq 0, |r_3| \leq s} x_{A1}^{s,r_3} e^{-i \frac{2s+1}{r} t} Y_{s, w_3^{(-)} s, r_3}(\theta, \phi, \psi), \quad (3.117)$$

$$F^{01} = \sum_{s \geq \frac{1}{2}, |r_3| \leq s} a_{s,r_3}^{(-)} c_{s,r_3} \tan^{w^{(-)} r_3} \left(\frac{\theta}{2} \right) \sin^{s-1} \theta \cos \left(-r_3 \psi + s \left(w^{(-)} \phi - \frac{2}{r} t \right) + \alpha_{s,r_3}^{(-)} \right), \quad (3.118)$$

$$F^{23} = \sum_{s \geq \frac{1}{2}, |r_3| \leq s} a_{s,r_3}^{(-)} c_{s,r_3} \tan^{w^{(-)} r_3} \left(\frac{\theta}{2} \right) \sin^{s-1} \theta \sin \left(-r_3 \psi + s \left(w^{(-)} \phi - \frac{2}{r} t \right) + \alpha_{s,r_3}^{(-)} \right). \quad (3.119)$$

Discussion how to count the number of supersymmetries can be done in a parallel way, so we only present results of the table corresponding to table 1, table 4, table 5 by table 6.

4 BPS states in $\mathcal{N} = 8$ SYM on $\mathbf{R} \times \mathbf{S}^2$

In this section we investigate supersymmetric states in $\mathcal{N} = 8$ SYM on $\mathbf{R} \times \mathbf{S}^2$ by carrying out dimensional reduction of $\mathcal{N} = 4$ SYM on $\mathbf{R} \times \mathbf{S}^3$ so that the Hopf fiber direction ψ is degenerated. We present basic results obtained by this dimensional reduction for this paper to be self-contained. From the metric of $\mathbf{R} \times \mathbf{S}^3$ we obtain that of $\mathbf{R} \times \mathbf{S}^2$ as

$$ds_{\mathbf{R} \times \mathbf{S}^2}^2 = -dt^2 + \mu^{-2}(d\theta^2 + \sin^2 \theta d\phi^2) \quad (4.1)$$

where we set $\mu = \frac{2}{r}$, which is the inverse radius of $\mathbf{S}^2$. For more detail, see appendix B.2. Dimensional reduction for fields can be achieved by truncating the fields to leave the zero modes of the Hopf fiber direction. $\partial_\psi \Phi = 0$, or $\partial_3 \Phi = 0$, where Φ is any four dimensional field. Accordingly the four dimensional gauge field is separated into the three dimensional one and a scalar field in a way that

$$A_i = a_i, \quad A_3 = \phi, \quad (4.2)$$

where $i = 0, 1, 2$. Thus the gauge field strength become

$$F_{01} = f_{01}, \quad F_{02} = f_{02}, \quad F_{\mu 3} = D_\mu \phi, \quad F_{12} = f_{12} - \mu \phi, \quad (4.3)$$

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + i[a_\mu, a_\nu]$, $D_\mu \phi = \partial_\mu \phi + i[a_\mu, \phi]$ with $\mu, \nu = t, \theta, \phi$. Note that the local Lorentz indices $0, 1, 2$ and the global indices t, θ, ϕ are now transformed to each other by using the transition function of $\mathbf{R} \times \mathbf{S}^2$ given by (B.33).

Performing this dimensional reduction to the $\mathcal{N} = 4$ SYM action given by (2.2) we obtain the action of $\mathcal{N} = 8$ SYM on $\mathbf{R} \times \mathbf{S}^2$

$$\begin{aligned} S = \frac{1}{g_3^2} \int dt \frac{d^2 \Omega}{\mu^2} \text{Tr} & \left[-\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{1}{2} D_\mu \phi D^\mu \phi + \mu \phi f_{12} - \frac{1}{2} \mu^2 \phi^2 \right. \\ & - \frac{1}{2} D_\mu X_{AB} D^\mu X^{AB} - \frac{\mu^2}{8} X_{AB} X^{AB} + \frac{1}{2} [X_{AB}, \phi] [X^{AB}, \phi] + \frac{1}{4} [X_{AB}, X_{CD}] [X^{AB}, X^{CD}] \\ & \left. + i(\psi_A)^\dagger \gamma^\mu D_\mu \psi_A + \frac{\mu}{4} (\psi_A)^\dagger \rho^t \psi_A + i(\psi_A)^\dagger [\phi, \psi_A] + (\psi_A)^\dagger [X_{AB}, (\psi_B)^\dagger] + \psi_A [X^{AB}, \psi_B] \right] \end{aligned} \quad (4.4)$$

where we set $d^2 \Omega = d\theta \sin \theta d\phi$, $\lambda_A = \psi_A$, $\frac{1}{g_3^2} = \frac{1}{g^2} \frac{4\pi}{\mu}$, and D_μ is the covariant derivative in terms of gauge and spin indices in the three dimension.

Equations of motion for the gauge field and the scalar fields in the bosonic part are

$$D_\mu f^{\mu\nu} - i[\phi, D^\nu \phi] - i[X_{AB}, D^\nu X^{AB}] + \mu \gamma^{0\mu\nu} D_\mu \phi = 0, \quad (4.5)$$

$$D^\mu D_\mu \phi + \mu f_{12} - \mu^2 \phi^2 + [X_{AB}, [\phi, X^{AB}]] = 0, \quad (4.6)$$

$$D^\mu D_\mu X^{AB} - \frac{\mu^2}{4} X^{AB} + [\phi, [X^{AB}, \phi]] + [X_{CD}, [X^{AB}, X^{CD}]] = 0. \quad (4.7)$$

Conserved charges of this theory can be obtained by dimensional reduction. $\text{SU}(4)_R$ R-symmetry charge of the bosonic part is

$$R^C{}_A = i \int d^2 \Omega \frac{1}{g_3^2} \text{Tr} \left(-X^{CB} D^0 X_{AB} + D^0 X^{CB} X_{AB} \right). \quad (4.8)$$

The energy is

$$H = \int d^2\Omega \frac{1}{g_3^2} \text{Tr} \left(\frac{1}{2}(f^{0i})^2 + \frac{1}{4}(f^{ij})^2 + \frac{1}{2}|D_0\phi|^2 + \frac{1}{2}|\vec{D}\phi|^2 + \frac{1}{2}|D_0X_{AB}|^2 + \frac{1}{2}|\vec{D}X^{AB}|^2 \right. \\ \left. - \mu\phi f_{12} + \frac{1}{2}\mu^2\phi^2 + \frac{1}{2}|[X_{AB}, \phi]|^2 + \frac{1}{4}|[X_{AB}, X_{CD}]|^2 + \frac{\mu^2}{8}|X_{AB}|^2 \right). \quad (4.9)$$

The angular momentums are

$$P^i = \int d^2\Omega \frac{1}{g_3^2} \text{Tr} \left(f^0{}_\mu f^{i\mu} + D^0\phi D^i\phi + \frac{1}{2}(D^0X_{AB}D^iX^{AB} + D^iX_{AB}D^0X^{AB}) \right) \quad (4.10)$$

where $i = \theta, \phi$.

Under the truncation supersymmetries specified by $\epsilon_A^{(+)}$ are all broken, since they are dependent on the Hopf fiber direction ψ . The other supersymmetries specified by $\epsilon_A^{(-)}$ are all preserved, which we denote by ξ_A . The determining equation of ξ_A comes from (2.14), which is now given by

$$\partial_t \xi_A = -\frac{i\mu}{4}\xi_A, \quad \nabla_\theta^{\mathbf{S}^2} \xi_A = -\frac{i\mu}{2}\rho_\theta{}^t \xi_A, \quad \nabla_\phi^{\mathbf{S}^2} \xi_A = -\frac{i\mu}{2}\rho_\phi{}^t \xi_A, \quad (4.11)$$

and the solution is

$$\xi_A = e^{-i\frac{\mu}{4}t} e^{-\frac{i}{2}\rho_{01}\theta} e^{-\frac{i}{2}\rho_{21}\phi} \eta_A \quad (4.12)$$

where η_A is a constant spinor. By using this the supersymmetry transformation rule of $\mathcal{N} = 8$ SYM on $\mathbf{R} \times \mathbf{S}^2$ is obtained as

$$\begin{aligned} \Delta_\xi A_\mu &= i((\xi_A)^* \rho_\mu \psi_A - (\psi_A)^\dagger \rho_\mu \xi_A), \\ \Delta_\xi \phi &= (\xi_A)^* \psi_A - (\psi_A)^\dagger \xi_A, \\ \Delta_\xi X^{AB} &= i(-\varepsilon^{ABCD} \xi_C \psi_D - (\xi_A)^* (\psi_B)^\dagger + (\xi_B)^* (\psi_A)^\dagger), \\ \Delta_\xi \psi_A &= \left(-iD_\mu \phi \rho^\mu + \frac{1}{2} f_{\mu\nu} \rho^{\mu\nu} - \mu \phi \rho^{12} \right) \xi_A - 2(D_\mu X_{AB} \rho^\mu - [\phi, X_{AB}]) (\xi_B)^* \\ &\quad - 2i[X_{AB}, X^{BC}] \xi_C - X_{AB} \left(\rho^\mu \nabla_\mu^{\mathbf{S}^2} - \frac{i\mu}{4} \rho^{12} \right) (\xi_B)^*. \end{aligned} \quad (4.13)$$

Note that due to this reduction the global symmetry reduces from $\text{PSU}(2, 2|4)$ to $\text{PSU}(2|4)$.

BPS states of $\mathcal{N} = 8$ SYM on $\mathbf{R} \times \mathbf{S}^2$ can be studied in the same manner as the case of $\mathcal{N} = 4$ SYM on $\mathbf{R} \times \mathbf{S}^3$ in section 3.2.2. Especially calculation in section 3.2.2 can be applied to this case as it is by exchanging the fields from four dimension to three dimension as discussed above. For this reason we do not repeat the similar calculation, and we present only relevant results.

Let us study BPS states of $\mathcal{N} = 8$ SYM on $\mathbf{R} \times \mathbf{S}^2$. For this purpose we study a BPS state which preserves a Killing spinor ϵ_1 whose constant spinor η_1 is projected in such a way that

$$\gamma_0 \eta_1 = iw \eta_1 \quad (4.14)$$

where $w = \pm 1$ and $1 = 1, 2, 3, 4$. The BPS relation of charges associated with the Killing spinor ϵ_1 is

$$\mu^{-1} H = R^1{}_1 + w J_\phi. \quad (4.15)$$

By performing the dimensional reduction to (3.74), we find

$$\begin{aligned}
 f^{02} &= -wD^2\phi \sec\theta + D^0\phi \tan\theta, \quad D^1\phi = -wf^{01} \sec\theta - (f_{12} - \mu\phi) \tan\theta, \\
 [X_{AB}, X^{B1}] &= \frac{1}{2i} \{w \sec\theta (D^0\phi - i(f_{12} - \mu\phi)) - (D^2\phi + if^{01}) \tan\theta\} \delta_A^1, \\
 D_0 X_{A1} &= -\frac{i\mu}{2} X_{A1} - w\mu D_\phi X_{A1}, \\
 D_1 X_{A1} &= iw(-\cos\theta D_2 X_{A1} + \sin\theta [i\phi, X_{A1}]),
 \end{aligned} \tag{4.16}$$

where $A = 1, 2, 3, 4$. These are the most general 1/8 BPS conditions of $\mathcal{N} = 8$ SYM on $\mathbf{R} \times \mathbf{S}^2$ with the fermionic fields trivial, which preserves ϵ_1 with $\mathbf{1} = 1, 2, 3, 4$.

Let us show a general solution of these 1/8 BPS equations when the fields take values in Cartan subalgebra of a gauge group, in which case these BPS equations reduce to

$$\begin{aligned}
 f^{02} &= -w \cos\theta \partial^2 \phi, & \partial_t \phi &= -w\mu \partial_\phi \phi, \\
 f_{12} &= \mu\phi - w \sin\theta f^{01}, & \partial^1 \phi &= -w \cos\theta f^{01},
 \end{aligned} \tag{4.17}$$

$$\left(\partial_0 + \frac{i\mu}{2} + \mu w \partial_\phi \right) X_{A1} = 0, \quad \left(\partial_1 + iw \cos\theta \partial_2 \right) X_{A1} = 0. \tag{4.18}$$

The gauge field strength has to be determined by combining with the equation of motion (4.5), which is now given by $\partial_\mu f^{\mu\nu} + \mu \gamma^{0\mu\nu} \partial_\mu \phi = 0$.¹² Under the BPS condition (4.17) this becomes

$$\partial_t f^{01} = -w\mu \partial_\phi f^{01}, \quad \mu \partial_\phi^2 \phi = \tan\theta \partial_\theta (\sin\theta f^{01}). \tag{4.19}$$

By carrying out dimensional reduction for (3.82), (3.91), (3.92), we find

$$X_{A1} = \sum_{s \geq 0} x_{A1}^s e^{-i\mu(s+\frac{1}{2})t} Y_{s,ws}(\theta, \phi), \tag{4.20}$$

$$f^{01} = \sum_{s \geq \frac{1}{2}} a_s c_s \sin^{s-1} \theta \cos(s(w\phi - \mu t) + \alpha_s), \tag{4.21}$$

$$\partial^2 \phi = \sum_{s \geq \frac{1}{2}} a_s c_s \sin^{s-1} \theta \sin(s(w\phi - \mu t) + \alpha_s), \tag{4.22}$$

where x_{A1}^s, a_s, α_s are integral constants with the range $x_{A1}^s \in \mathbf{C}, a_s \geq 0, 0 \leq \alpha_s < 2\pi$ for all $A = 2, 3, 4; s \geq \frac{1}{2}$, and $Y_{s,ws}(\theta, \phi), c_s$ are given in appendix B.2. Thus the real adjoint scalar field ϕ is determined as

$$\phi = \phi_0 - \frac{w}{\mu} \sum_{s \geq \frac{1}{2}} \frac{a_s}{s} c_s \sin^s \theta \cos(s(w\phi - \mu t) + \alpha_s), \tag{4.23}$$

where ϕ_0 is another integral constant taking real values, which parametrizes the vacua of the theory.¹³ The other components of field strength can be obtained from (4.17). This is

¹²Different from the four dimensional case, the Bianchi identity becomes redundant in the three dimension.

¹³If the flux quantization condition is taken into account, not only x_{A1}^s, a_s but also ϕ_0 are quantized in a suitable way.

a set of general 1/8 BPS solutions of $\mathcal{N} = 8$ SYM on $\mathbf{R} \times \mathbf{S}^2$.¹⁴ Note that this result is consistent with the BPS solutions obtained in [32], where BPS solutions of the $\mathcal{N} = 8$ SYM were obtained from those of $\mathcal{N} = 6$ Chern-Simons (or ABJM) theory defined on $\mathbf{R} \times \mathbf{S}^2$ by performing a particular scaling limit from a half BPS solution of ABJM.

We present the values of conserved charges under the BPS solution (4.20), (4.21), (4.23).

$$H = \frac{1}{g_3^2} \left(2\mu^2 \sum_{s \geq 0} (2s+1)^2 \text{Tr} |x_{A1}^s|^2 + \sum_{s \geq \frac{1}{2}} \frac{(1+2s)}{2s} \text{Tr} (a_s)^2 \right), \quad (4.26)$$

$$J_\theta = 0, \quad (4.27)$$

$$J_\phi = \frac{w}{g_3^2} \left(\sum_{s \geq 0} 2\mu(2s+1)s \text{Tr} |x_{A1}^s|^2 + \mu^{-1} \sum_{s \geq \frac{1}{2}} \frac{(1+2s)}{2s} \text{Tr} (a_s)^2 \right), \quad (4.28)$$

$$R^1_{-1} = \frac{\mu}{g_3^2} \sum_{s \geq 0} (2s+1) \text{Tr} |x_{A1}^s|^2, \quad (4.29)$$

which satisfies the BPS relation of charges given by (4.15).

The number of supersymmetries preserved by the 1/8 BPS solution at each point of moduli space is given in table 7. In particular this theory has degenerate vacua, which are parametrized by vacuum expectation values of the real scalar field denoted by ϕ_0 .¹⁵

5 Discussion

We have done a systematic analysis of supersymmetric states in $\mathcal{N} = 4$ SYM on $\mathbf{R} \times \mathbf{S}^3$ by setting the gaugino fields to zero. As a result we have found two sets of 1/16 BPS conditions and we have solved them completely when the bosonic fields are valued in Cartan subalgebra of a gauge group. We have precisely counted the number of supersymmetries preserved by the BPS solutions varying the parameters of the solution. We have also obtained the most general 1/8 BPS solution of $\mathcal{N} = 8$ SYM on $\mathbf{R} \times \mathbf{S}^2$ with the precise number of supersymmetries under the same assumptions by performing dimensional reduction.

In this paper we have solved the 1/16 BPS solutions assuming that they are valued in Cartan subalgebra of a gauge group. We pointed out that in the Cartan part the gauge field and the matter ones decouple and they can be independently excited preserving supersymmetry. It would be interesting to study more general BPS solutions by relaxing this

¹⁴If we include the non-normalizable mode, the general solution (4.21), (4.23) becomes

$$f^{01} = \frac{a'_0}{\sin \theta} + \sum_{s \geq \frac{1}{2}} a_s c_s \sin^{s-1} \theta \cos(s(w\phi - \mu t) + \alpha_s), \quad (4.24)$$

$$\phi = \phi_0 - \frac{w}{\mu} \left(a'_0 \log \sin \theta + \sum_{s \geq \frac{1}{2}} \frac{a_s}{s} c_s \sin^s \theta \cos(s(w\phi - \mu t) + \alpha_s) \right). \quad (4.25)$$

¹⁵The degeneracy of vacua in $\mathcal{N} = 8$ SYM on $\mathbf{R} \times \mathbf{S}^2$ can be encoded in non-trivial holonomy in the Hopf fiber direction in $\mathcal{N} = 4$ SYM on $\mathbf{R} \times \mathbf{S}^3/\mathbf{Z}_k$, where the Hopf fiber direction is orbifolded [33, 34].

Parameter region.	Preserved Killing spinors.	Number of SUSY.
$x_{12}^s, x_{13}^s, x_{14}^s, a_s, \phi_0$: generic.	η_1 with $\gamma_0 \eta_1 = i w \eta_1$.	2 ($\frac{1}{8}$ BPS)
$x_{13}^s = x_{14}^s = 0$ for $\forall s$.	η_1, η_2 with $\gamma_0 \eta_A = i w \eta_A$.	4 ($\frac{1}{4}$ BPS)
$x_{12}^s, x_{13}^s, x_{14}^s = 0$ for $\forall s > 0$, $a_s = 0$ for $\forall s, r_3$.	η_1 .	4 ($\frac{1}{4}$ BPS)
$x_{12}^s = x_{13}^s = x_{14}^s = 0$ for $\forall s$.	$\eta_1, \eta_2, \eta_3, \eta_4$ with $\gamma_0 \eta_A = i w \eta_A$.	8 ($\frac{1}{2}$ BPS)
$x_{12}^s = 0$ for $\forall s > 0$, $x_{13}^s = x_{14}^s = a_s = 0$ for $\forall s$.	η_1, η_2 .	8 ($\frac{1}{2}$ BPS)
ϕ_0 : generic, $x_{12}^s = x_{13}^s = x_{14}^s = a_s = 0$ for $\forall s$.	$\eta_1, \eta_2, \eta_3, \eta_4$.	16 (Vacua)

Table 7. We list the number of supersymmetry of BPS solutions (4.20), (4.21), (4.23) which preserve η_1 with $\gamma_0 \eta_1 = i w \eta_1$ at each point of moduli space.

assumption. In this case one has to solve non-linear differential equations given by (3.28) or (3.74), which are much more complicated and technically much harder to solve. Therefore it will become important to reduce the problem to a simpler one by restricting attention to a special subsector as performed in [27].

It would also be interesting to compare BPS solutions on $\mathbf{R} \times \mathbf{S}^3$ preserving $\epsilon^{(+)}$ in this paper and those in non-abelian sector on $\mathbf{R}^4$ by setting the gauge coupling constant to zero. The former ones are expected to be obtained from the latter ones via Wick rotation and conformal transformation described in [27]. (For those conserving $\epsilon^{(-)}$ in this paper, it will need an extra coordinate transformation such as exchange between ϕ and ψ .) This is non-trivial because the former ones are expressed in terms of trigonometric functions with multiple angles and given by the gauge field strength in the gauge sector whereas the latter ones are in terms of powers of trigonometric functions and the gauge potential in the gauge sector.

An important problem is to clarify a relation between 1/16 BPS states in the $\mathcal{N} = 4$ SYM and 1/16 BPS objects in type IIB supergravity on $\mathbf{AdS}_5 \times \mathbf{S}^5$. It would be interesting to find out the counterparts of the BPS states constructed in this paper, especially those in which only electromagnetic field is turned on. Since charges of the supersymmetric states found in this paper can be of order N , the corresponding objects in the dual geometry can be supersymmetric (dual) giant gravitons. General 1/16 BPS (dual) giant gravitons have been constructed in [35] by using the same technique to construct a general 1/8 BPS giant graviton [36], where the configuration of 1/8 BPS giant gravitons of energy E is realized by the intersection of $\mathbf{S}^5$ and the zero locus of a polynomial of the form

$$\sum_{n_1+n_2+n_3=E/R} c_{n_1 n_2 n_3} e^{-iEt} z_1^{n_1} z_2^{n_2} z_3^{n_3} \quad (5.1)$$

where R is the radius of AdS_5 , z_1, z_2, z_3 are coordinates of $\mathbf{C}^3$ into which $\mathbf{S}^5$ is embedded. (See also [28, 37–43].) Supersymmetric black holes have also been found in [24–26], which have turned out to be 1/16 BPS. Reproducing behaviors of these objects from the $\mathcal{N} = 4$ SYM side is an important issue in $\text{AdS}_5/\text{CFT}_4$ duality.

Another interesting direction is to study the $\mathcal{N} = 4$ SYM on $\mathbf{R} \times \mathbf{S}^3/\mathbf{Z}_k$ by orbifolding the Hopf fiber direction so that the $su(2)_R$ algebra is broken and the global symmetry algebra of the system becomes $psu(2|4)$. Study of a class of the theories with $psu(2|4)$ symmetry is interesting because a family of BPS solutions with $psu(2|4)$ symmetry (called bubbling geometries) and BPS objects in ten or eleven dimensional supergravity theories has extensively studied in [33, 34, 44, 45]. It would be attracting to pursue the correspondence of BPS spectra of both sides more. (See [13, 46–49] for the study of this direction.)

It would also be fascinating to study BPS states in other supersymmetric gauge theories defined on $\mathbf{R} \times \mathbf{S}^n$ as done in this paper, for example SYM on $\mathbf{R} \times \mathbf{S}^4$ constructed recently in [50]. (The case of $\mathcal{N} = 6$ Chern-Simons (ABJM) theory on $\mathbf{R} \times \mathbf{S}^2$ was done in [32].)

We leave these issues to future works.

Acknowledgments

The author would like to thank S. Kim and S. Minwalla for useful comments for the first version of this paper.

A Convention

In this appendix we collect our convention used in this paper. In the flat space-time, the metric is given by $g_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and $\text{SO}(1, 3)$ gamma matrices are realized by

$$\gamma_\mu = \begin{pmatrix} 0 & \rho_\mu \\ \rho_\mu & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \quad (\text{A.1})$$

Here $\rho_0 = i\sigma_2, \rho_1 = \sigma_1, \rho_2 = \sigma_3$, where σ_i is the Pauli matrices satisfying $\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$ so that $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. Note that ρ_μ becomes $\text{SO}(1, 2)$ gamma matrices.

Four-dimensional spinors in this paper are all Weyl spinors, so they have two complex components. However we always suppress spinor indices for simplicity. In a fermionic bilinear, spinor indices are contracted in the southwest-northeast manner. For example,

$$\psi^\dagger \gamma_\mu \chi = \psi^\dagger_{\dot{\alpha}} \gamma_\mu^{\dot{\alpha}}{}_{\beta} \chi^\beta = C_{\dot{\alpha}\gamma} \psi^{\dagger\dot{\gamma}} \gamma_\mu^{\dot{\gamma}}{}_{\beta} \chi^\beta \quad (\text{A.2})$$

where $C_{\dot{\alpha}\gamma}$ is a certain charge conjugation matrix and undotted (dotted) indices represent Weyl spinor ones of chirality plus (minus). Note that $\gamma_3^{\dot{\alpha}}{}_{\beta} = -i\delta^{\alpha}{}_{\beta}$. We also use notation such that

$$\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu). \quad (\text{A.3})$$

B Basics on $\mathbf{S}^3$

In this appendix we collect basics on $\mathbf{S}^3$ used in this paper. It is convenient for us to realize $\mathbf{S}^3$ by SU(2) group. Any SU(2) element denoted by g can be parametrized by using the polar coordinates of $\mathbf{S}^3$ as

$$g = e^{-i\frac{\phi}{2}\sigma_3} e^{-i\frac{\theta}{2}\sigma_2} e^{-i\frac{\psi}{2}\sigma_3} = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{(\phi+\psi)}{2}} & -\sin\frac{\theta}{2}e^{\frac{i(-\phi+\psi)}{2}} \\ \sin\frac{\theta}{2}e^{-i\frac{(\phi+\psi)}{2}} & \cos\frac{\theta}{2}e^{\frac{i(\phi+\psi)}{2}} \end{pmatrix} \quad (\text{B.1})$$

where the parameter region is $0 \leq \theta < \pi, 0 \leq \phi < 2\pi, 0 \leq \psi < 4\pi$. On the other hand, any point on $\mathbf{S}^3$ with radius r can be embedded into $\mathbf{C}^2$ by using SU(2) elements

$$z^i = r g^{i1} = r \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{(\phi+\psi)}{2}} \\ \sin\frac{\theta}{2}e^{-i\frac{(-\phi+\psi)}{2}} \end{pmatrix} \quad (\text{B.2})$$

where z^1, z^2 are complex coordinates of $\mathbf{C}^2$. Therefore the metric of $\mathbf{S}^3$ with radius r is given by

$$ds_{\mathbf{S}^3}^2 = |dz^1|^2 + |dz^2|^2 = \frac{r^2}{4}(d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2) \quad (\text{B.3})$$

where we used (B.2) to obtain the second equation. The metric of $\mathbf{R} \times \mathbf{S}^3$ is

$$ds_{\mathbf{R} \times \mathbf{S}^3}^2 = -dt^2 + ds_{\mathbf{S}^3}^2. \quad (\text{B.4})$$

Therefore a local orthonormal frame is given by

$$e^0 = dt, \quad e^1 = \frac{r}{2}d\theta, \quad e^2 = \frac{r}{2}\sin\theta d\phi, \quad e^3 = \frac{r}{2}(d\psi + \cos\theta d\phi). \quad (\text{B.5})$$

Vielbein of this geometry reads

$$e_t^0 = 1, \quad e_\theta^1 = \frac{r}{2}, \quad e_\phi^2 = \frac{r}{2}\sin\theta, \quad e_\phi^3 = \frac{r}{2}\cos\theta, \quad e_\psi^3 = \frac{r}{2}. \quad (\text{B.6})$$

The inverse of vielbein is

$$e_0^t = 1, \quad e_1^\theta = \frac{2}{r}, \quad e_2^\phi = \frac{2}{r\sin\theta}, \quad e_2^\psi = \frac{2}{r}(-\cot\theta), \quad e_3^\psi = \frac{2}{r}. \quad (\text{B.7})$$

We interchange local Lorentz indices and global space-time ones by using these.

The 1-form connection of this geometry can be determined as follows.

$$\omega_{21} = \frac{2}{r}\left(\cot\theta e_2 - \frac{1}{2}e_3\right), \quad \omega_{13} = \frac{1}{r}e_2, \quad \omega_{23} = -\frac{1}{r}e_1. \quad (\text{B.8})$$

Other components are trivial. This reads

$$\omega_{2,21} = \frac{2}{r}\cot\theta, \quad \omega_{3,21} = -\frac{1}{r}, \quad \omega_{2,13} = \frac{1}{r}, \quad \omega_{1,23} = -\frac{1}{r}. \quad (\text{B.9})$$

The partial derivatives with the local Lorentz indices and those with the global coordinates are related by

$$\partial_1 = \frac{2}{r}\partial_\theta, \quad \partial_2 = \frac{2}{r}\left(\frac{1}{\sin\theta}\partial_\phi - \cot\theta\partial_\psi\right), \quad \partial_3 = \frac{2}{r}\partial_\psi, \quad (\text{B.10})$$

$$\partial_\theta = \frac{r}{2}\partial_\theta, \quad \partial_\phi = \frac{r}{2}(\sin\theta\partial_2 + \cos\theta\partial_3), \quad \partial_\psi = \frac{r}{2}\partial_3. \quad (\text{B.11})$$

A benefit to realize $\mathbf{S}^3$ as $SU(2)$ group is that $so(4)$ Killing action on $\mathbf{S}^3$ is realized by the $su(2)_L \times su(2)_R$ algebraic action.

$$\hat{L}_a g = -\frac{1}{2}\sigma_a g, \quad \hat{R}_a g = \frac{1}{2}g\sigma_a. \quad (\text{B.12})$$

Thanks to this definition we can easily show that

$$[\hat{L}_a, \hat{L}_b] = i\epsilon_{abc}\hat{L}_c, \quad [\hat{R}_a, \hat{R}_b] = i\epsilon_{abc}\hat{R}_c, \quad [\hat{L}_a, \hat{R}_b] = 0. \quad (\text{B.13})$$

The explicit coordinate expressions of these operators are given by

$$\begin{aligned} \hat{L}_1 &= i(\sin\phi\partial_\theta - \csc\theta\cos\phi\partial_\psi + \cot\theta\cos\phi\partial_\phi), \\ \hat{L}_2 &= i(-\cos\phi\partial_\theta - \csc\theta\sin\phi\partial_\psi + \cot\theta\sin\phi\partial_\phi), \\ \hat{L}_3 &= -i\partial_\phi, \end{aligned} \quad (\text{B.14})$$

$$\begin{aligned} \hat{R}_1 &= i(\sin\psi\partial_\theta + \cot\theta\cos\psi\partial_\psi - \csc\theta\cos\psi\partial_\phi), \\ \hat{R}_2 &= i(\cos\psi\partial_\theta - \cot\theta\sin\psi\partial_\psi + \csc\theta\sin\psi\partial_\phi), \\ \hat{R}_3 &= i\partial_\psi. \end{aligned} \quad (\text{B.15})$$

For later convenience to construct the spherical harmonics, we define

$$\hat{L}_\pm = \hat{L}_1 \pm i\hat{L}_2, \quad \hat{R}_\pm = \hat{R}_1 \pm i\hat{R}_2. \quad (\text{B.16})$$

Then we can show that

$$[\hat{L}_3, \hat{L}_\pm] = \pm\hat{L}_\pm, \quad [\hat{L}_+, \hat{L}_-] = 2\hat{L}_3, \quad (\text{B.17})$$

$$[\hat{R}_3, \hat{R}_\pm] = \pm\hat{R}_\pm, \quad [\hat{R}_+, \hat{R}_-] = 2\hat{R}_3, \quad (\text{B.18})$$

and

$$\hat{L}_\pm = e^{\pm i\phi} \left(i\cot\theta\partial_\phi \pm \partial_\theta - i\frac{1}{\sin\theta}\partial_\psi \right), \quad (\text{B.19})$$

$$\hat{R}_\pm = e^{\mp i\psi} \left(i\cot\theta\partial_\psi \mp \partial_\theta - i\frac{1}{\sin\theta}\partial_\phi \right). \quad (\text{B.20})$$

Note that

$$\hat{L}^2 = \hat{R}^2 = \partial_\theta^2 + \cot\theta\partial_\theta + \frac{1}{\sin^2\theta}(\partial_\phi^2 + \partial_\psi^2 - 2\cos\theta\partial_\phi\partial_\psi) \quad (\text{B.21})$$

which is the same as the Laplacian of $\mathbf{S}^3$ with radius $r = 2$ acting on a scalar field.

B.1 The scalar spherical harmonics on $\mathbf{S}^3$

In this subsection we construct the scalar spherical harmonics on $\mathbf{S}^3$. Since the spherical harmonics are roots of the $su(2)_L \times su(2)_R$ algebra, we can construct them by the standard algebraic method.

$$\begin{aligned}\hat{L}^2 Y_{s,l_3,r_3} &= \hat{R}^2 Y_{s,l_3,r_3} = s(s+1) Y_{s,l_3,r_3}, \\ \hat{L}_3 Y_{s,l_3,r_3} &= l_3 Y_{s,l_3,r_3}, \quad \hat{R}_3 Y_{s,l_3,r_3} = r_3 Y_{s,l_3,r_3},\end{aligned}\tag{B.22}$$

where s is non-negative half integer and $|l_3|, |r_3| \leq s$.¹⁶ Firstly we determine the spherical harmonics of the highest or lowest weight of $su(2)_R$ by solving $R_{\pm} Y_{s,l_3,\pm s} = 0$ as well, which is given in the polar coordinates by

$$\left(\partial_{\theta} \mp \frac{1}{\sin \theta} l_3 - s \cot \theta \right) y_{s,l_3,\pm s}(\theta) = 0\tag{B.23}$$

where $Y_{s,l_3,\pm s} = y_{s,l_3,\pm s}(\theta) e^{i(l_3\phi \mp s\psi)}$. The solution is given by

$$Y_{s,l_3,\pm s} = c_{s,l_3} \tan^{\pm l_3} \left(\frac{\theta}{2} \right) \sin^s \theta e^{i(l_3\phi \mp s\psi)},\tag{B.24}$$

where c_{s,l_3} is a normalization constant given by

$$c_{s,l_3} = \frac{1}{\pi} \sqrt{\frac{\Gamma(2s+2)}{2^{2s+1} \Gamma(l_3+s+1) \Gamma(-l_3+s+1)}}\tag{B.25}$$

which we determine by demanding $\int d^3\Omega |Y_{s,l_3,\pm s}|^2 = 1$.

In the same way, we can construct the spherical harmonics of the highest or lowest weight of $su(2)_L$ by solving $L_{\pm} Y_{s,\pm s,r_3} = 0$, which is

$$\left(\partial_{\theta} \mp \frac{1}{\sin \theta} r_3 - s \cot \theta \right) y_{s,\pm s,r_3}(\theta) = 0\tag{B.26}$$

where $Y_{s,\pm s,r_3} = y_{s,\pm s,r_3}(\theta) e^{i(-r_3\psi \pm s\phi)}$. The solution thereof is given by

$$Y_{s,\pm s,r_3} = c_{s,r_3} \tan^{\pm r_3} \left(\frac{\theta}{2} \right) \sin^s \theta e^{i(-r_3\psi \pm s\phi)},\tag{B.27}$$

where c_{s,r_3} is determined to satisfy $\int d^3\Omega |Y_{s,\pm s,r_3}|^2 = 1$ so that

$$c_{s,r_3} = \frac{1}{\pi} \sqrt{\frac{\Gamma(2s+2)}{2^{2s+1} \Gamma(r_3+s+1) \Gamma(-r_3+s+1)}}.\tag{B.28}$$

A general scalar spherical harmonics, Y_{s,l_3,r_3} , can be obtained by acting lowering operators suitably.

$$Y_{s,l_3,r_3} = \left(\prod_{n=l_3+1}^s \frac{\hat{L}_-}{\sqrt{s(s+1) - n(n-1)}} \right) Y_{s,s,r_3}\tag{B.29}$$

where we normalized the spherical harmonics to satisfy the orthonormal relation

$$\int d^3\Omega (Y_{s',l'_3,r'_3})^* Y_{s,l_3,r_3} = \delta_{ss'} \delta_{l_3 l'_3} \delta_{r_3 r'_3}.\tag{B.30}$$

¹⁶Solutions of these equations (B.22) may be already known as the Wigner D-matrices. But we construct all of them for this paper to be self-contained.

B.2 Reduction to $\mathbf{S}^2$

In this subsection we extract information on $\mathbf{S}^2$ from the results obtained in the previous subsections by degenerating the Hopf fiber direction ψ of $\mathbf{S}^3$. The metric of $\mathbf{R} \times \mathbf{S}^2$ is

$$ds_{\mathbf{R} \times \mathbf{S}^2}^2 = -dt^2 + \frac{1}{\mu^2}(d\theta^2 + \sin^2 \theta d\phi^2) \quad (\text{B.31})$$

where μ is the inverse radius of $\mathbf{S}^2$, which is related to the $\mathbf{S}^3$ radius by $\mu = \frac{2}{r}$. A local orthonormal frame is

$$e^0 = dt, \quad e^1 = \mu^{-1} d\theta, \quad e^2 = \mu^{-1} \sin \theta d\phi. \quad (\text{B.32})$$

From this we find vielbein of $\mathbf{R} \times \mathbf{S}^2$ as

$$\begin{aligned} e_t^0 &= 1, & e_\theta^1 &= \mu^{-1}, & e_\phi^2 &= \mu^{-1} \sin \theta, \\ e_0^t &= 1, & e_1^\theta &= \mu, & e_2^\phi &= \frac{\mu}{\sin \theta}. \end{aligned} \quad (\text{B.33})$$

One-form connection is

$$\omega_{21}^{\mathbf{S}^2} = \mu \cot \theta e^2. \quad (\text{B.34})$$

Other components are zero.

$so(3)$ Killing actions on $\mathbf{S}^2$ are given by $\hat{L}_i$ with ∂_ψ eliminated. It is automatic that $\hat{L}_i$ restricted on $\mathbf{S}^2$ form $su(2)$ algebra. By using the operators $\hat{L}_i$ the $\mathbf{S}^2$ spherical harmonics are defined as root vectors of the $su(2)$ algebra.

$$\hat{L}^2 Y_{s,s_3} = s(s+1) Y_{s,s_3}, \quad \hat{L}_3 Y_{s,s_3} = l_3 Y_{s,s_3}, \quad (\text{B.35})$$

where s is non-negative half integer and $|s_3| \leq s$. The $\mathbf{S}^2$ spherical harmonics can be obtained from $\mathbf{S}^3$ spherical harmonics by taking a zero mode of ψ direction.

$$Y_{s,s_3} = \sqrt{\frac{\pi}{2}} Y_{s,s_3,r_3=0}. \quad (\text{B.36})$$

For instance, the $\mathbf{S}^2$ spherical harmonics of the highest or lowest weight are given by

$$Y_{s,\pm s} = c_s \sin^s \theta e^{\pm i s \phi}, \quad (\text{B.37})$$

where $c_s = \sqrt{\frac{\pi}{2}} c_{s,0} = \sqrt{\frac{\Gamma(2s+2)}{\pi 2^{2s+2} \Gamma(s+1)^2}}$. A general scalar spherical harmonics is given by

$$Y_{s,s_3} = \left(\prod_{n=s_3+1}^s \frac{\hat{L}_-}{\sqrt{s(s+1) - n(n-1)}} \right) Y_{s,s}. \quad (\text{B.38})$$

These $\mathbf{S}^2$ spherical harmonics satisfy the orthonormal relation

$$\int d^2 \Omega (Y_{s',s'_3})^* Y_{s,s_3} = \delta_{ss'} \delta_{s_3 s'_3} \quad (\text{B.39})$$

where $d^2 \Omega = d\theta \sin \theta d\phi$.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] H. Osborn, *Topological charges for $N = 4$ supersymmetric gauge theories and monopoles of spin 1*, *Phys. Lett. B* **83** (1979) 321 [[INSPIRE](#)].
- [2] A. Sen, *Dyon-monopole bound states, selfdual harmonic forms on the multi-monopole moduli space and $SL(2, \mathbb{Z})$ invariance in string theory*, *Phys. Lett. B* **329** (1994) 217 [[hep-th/9402032](#)] [[INSPIRE](#)].
- [3] P. Goddard, J. Nuyts and D.I. Olive, *Gauge theories and magnetic charge*, *Nucl. Phys. B* **125** (1977) 1 [[INSPIRE](#)].
- [4] C. Montonen and D.I. Olive, *Magnetic monopoles as gauge particles?*, *Phys. Lett. B* **72** (1977) 117 [[INSPIRE](#)].
- [5] J.M. Maldacena, *The large- N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113 [[hep-th/9711200](#)] [[INSPIRE](#)].
- [6] M.F. Sohnius and P.C. West, *Conformal invariance in $N = 4$ supersymmetric Yang-Mills theory*, *Phys. Lett. B* **100** (1981) 245 [[INSPIRE](#)].
- [7] L. Brink, O. Lindgren and B.E.W. Nilsson, *$N = 4$ Yang-Mills theory on the light cone*, *Nucl. Phys. B* **212** (1983) 401 [[INSPIRE](#)].
- [8] P.S. Howe, K.S. Stelle and P.K. Townsend, *Miraculous ultraviolet cancellations in supersymmetry made manifest*, *Nucl. Phys. B* **236** (1984) 125 [[INSPIRE](#)].
- [9] N. Seiberg, *Supersymmetry and nonperturbative β -functions*, *Phys. Lett. B* **206** (1988) 75 [[INSPIRE](#)].
- [10] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [[hep-th/9802150](#)] [[INSPIRE](#)].
- [11] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [[hep-th/9802109](#)] [[INSPIRE](#)].
- [12] E. Witten, *Anti-de Sitter space, thermal phase transition and confinement in gauge theories*, *Adv. Theor. Math. Phys.* **2** (1998) 505 [[hep-th/9803131](#)] [[INSPIRE](#)].
- [13] K. Okuyama, *$N = 4$ SYM on $R \times S^3$ and PP wave*, *JHEP* **11** (2002) 043 [[hep-th/0207067](#)] [[INSPIRE](#)].
- [14] D. Berenstein, *Large- N BPS states and emergent quantum gravity*, *JHEP* **01** (2006) 125 [[hep-th/0507203](#)] [[INSPIRE](#)].
- [15] M. Berkooz and D. Reichmann, *Weakly renormalized near $1/16$ SUSY Fermi liquid operators in $N = 4$ SYM*, *JHEP* **10** (2008) 084 [[arXiv:0807.0559](#)] [[INSPIRE](#)].
- [16] E. Witten, *Constraints on supersymmetry breaking*, *Nucl. Phys. B* **202** (1982) 253 [[INSPIRE](#)].
- [17] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [[hep-th/0510251](#)] [[INSPIRE](#)].
- [18] C. Romelsberger, *Counting chiral primaries in $N = 1$, $D = 4$ superconformal field theories*, *Nucl. Phys. B* **747** (2006) 329 [[hep-th/0510060](#)] [[INSPIRE](#)].

- [19] Y. Imamura, *Relation between the 4d superconformal index and the S^3 partition function*, *JHEP* **09** (2011) 133 [[arXiv:1104.4482](#)] [[INSPIRE](#)].
- [20] S. Nawata, *Localization of $N = 4$ superconformal field theory on $S^1 \times S^3$ and index*, *JHEP* **11** (2011) 144 [[arXiv:1104.4470](#)] [[INSPIRE](#)].
- [21] J. McGreevy, L. Susskind and N. Toumbas, *Invasion of the giant gravitons from anti-de Sitter space*, *JHEP* **06** (2000) 008 [[hep-th/0003075](#)] [[INSPIRE](#)].
- [22] M.T. Grisaru, R.C. Myers and O. Tafjord, *SUSY and goliath*, *JHEP* **08** (2000) 040 [[hep-th/0008015](#)] [[INSPIRE](#)].
- [23] A. Hashimoto, S. Hirano and N. Itzhaki, *Large branes in AdS and their field theory dual*, *JHEP* **08** (2000) 051 [[hep-th/0008016](#)] [[INSPIRE](#)].
- [24] J.B. Gutowski and H.S. Reall, *General supersymmetric AdS_5 black holes*, *JHEP* **04** (2004) 048 [[hep-th/0401129](#)] [[INSPIRE](#)].
- [25] J.B. Gutowski and H.S. Reall, *Supersymmetric AdS_5 black holes*, *JHEP* **02** (2004) 006 [[hep-th/0401042](#)] [[INSPIRE](#)].
- [26] H.K. Kunduri, J. Lucietti and H.S. Reall, *Supersymmetric multi-charge AdS_5 black holes*, *JHEP* **04** (2006) 036 [[hep-th/0601156](#)] [[INSPIRE](#)].
- [27] L. Grant, P.A. Grassi, S. Kim and S. Minwalla, *Comments on 1/16 BPS quantum states and classical configurations*, *JHEP* **05** (2008) 049 [[arXiv:0803.4183](#)] [[INSPIRE](#)].
- [28] G. Mandal and N.V. Suryanarayana, *Counting 1/8-BPS dual-giants*, *JHEP* **03** (2007) 031 [[hep-th/0606088](#)] [[INSPIRE](#)].
- [29] I. Biswas, D. Gaiotto, S. Lahiri and S. Minwalla, *Supersymmetric states of $N = 4$ Yang-Mills from giant gravitons*, *JHEP* **12** (2007) 006 [[hep-th/0606087](#)] [[INSPIRE](#)].
- [30] C.-M. Chang and X. Yin, *1/16 BPS states in $N = 4$ super-Yang-Mills theory*, *Phys. Rev. D* **88** (2013) 106005 [[arXiv:1305.6314](#)] [[INSPIRE](#)].
- [31] F.A. Dolan and H. Osborn, *On short and semi-short representations for four-dimensional superconformal symmetry*, *Annals Phys.* **307** (2003) 41 [[hep-th/0209056](#)] [[INSPIRE](#)].
- [32] B. Ezhuthachan, S. Shimasaki and S. Yokoyama, *BPS solutions in ABJM theory and maximal super Yang-Mills on $R \times S^2$* , *JHEP* **12** (2011) 048 [[arXiv:1107.3545](#)] [[INSPIRE](#)].
- [33] H. Lin, O. Lunin and J.M. Maldacena, *Bubbling AdS space and 1/2 BPS geometries*, *JHEP* **10** (2004) 025 [[hep-th/0409174](#)] [[INSPIRE](#)].
- [34] H. Lin and J.M. Maldacena, *Fivebranes from gauge theory*, *Phys. Rev. D* **74** (2006) 084014 [[hep-th/0509235](#)] [[INSPIRE](#)].
- [35] S. Kim and K.-M. Lee, *1/16-BPS black holes and giant gravitons in the $AdS_5 \times S^5$ space*, *JHEP* **12** (2006) 077 [[hep-th/0607085](#)] [[INSPIRE](#)].
- [36] A. Mikhailov, *Giant gravitons from holomorphic surfaces*, *JHEP* **11** (2000) 027 [[hep-th/0010206](#)] [[INSPIRE](#)].
- [37] G. Mandal, *Fermions from half-BPS supergravity*, *JHEP* **08** (2005) 052 [[hep-th/0502104](#)] [[INSPIRE](#)].
- [38] S. Kim and K.-M. Lee, *BPS electromagnetic waves on giant gravitons*, *JHEP* **10** (2005) 111 [[hep-th/0502007](#)] [[INSPIRE](#)].

- [39] A. Dhar, G. Mandal and M. Smedback, *From gravitons to giants*, *JHEP* **03** (2006) 031 [[hep-th/0512312](#)] [[INSPIRE](#)].
- [40] A. Basu and G. Mandal, *Dual giant gravitons in $AdS_m \times Y^n$ (Sasaki-Einstein)*, *JHEP* **07** (2007) 014 [[hep-th/0608093](#)] [[INSPIRE](#)].
- [41] A. Sinha and J. Sonner, *Black hole giants*, *JHEP* **08** (2007) 006 [[arXiv:0705.0373](#)] [[INSPIRE](#)].
- [42] M. Ali-Akbari and M.M. Sheikh-Jabbari, *Electrified BPS giants: BPS configurations on giant gravitons with static electric field*, *JHEP* **10** (2007) 043 [[arXiv:0708.2058](#)] [[INSPIRE](#)].
- [43] S.K. Ashok and N.V. Suryanarayana, *Supersymmetric electromagnetic waves on giants and dual-giants*, *JHEP* **05** (2012) 074 [[arXiv:1004.0098](#)] [[INSPIRE](#)].
- [44] O. Lunin, *1/2-BPS states in M-theory and defects in the dual CFTs*, *JHEP* **10** (2007) 014 [[arXiv:0704.3442](#)] [[INSPIRE](#)].
- [45] O. Lunin, *Brane webs and 1/4-BPS geometries*, *JHEP* **09** (2008) 028 [[arXiv:0802.0735](#)] [[INSPIRE](#)].
- [46] N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, *BMN gauge theory as a quantum mechanical system*, *Phys. Lett. B* **558** (2003) 229 [[hep-th/0212269](#)] [[INSPIRE](#)].
- [47] N. Kim, T. Klose and J. Plefka, *Plane wave matrix theory from $N = 4$ super Yang-Mills on $R \times S^3$* , *Nucl. Phys. B* **671** (2003) 359 [[hep-th/0306054](#)] [[INSPIRE](#)].
- [48] G. Ishiki, Y. Takayama and A. Tsuchiya, *$N = 4$ SYM on $R \times S^3$ and theories with 16 supercharges*, *JHEP* **10** (2006) 007 [[hep-th/0605163](#)] [[INSPIRE](#)].
- [49] G. Ishiki, S. Shimasaki, Y. Takayama and A. Tsuchiya, *Embedding of theories with $SU(2|4)$ symmetry into the plane wave matrix model*, *JHEP* **11** (2006) 089 [[hep-th/0610038](#)] [[INSPIRE](#)].
- [50] J. Kim, S. Kim, K. Lee and J. Park, *Super-Yang-Mills theories on $S^4 \times R$* , *JHEP* **08** (2014) 167 [[arXiv:1405.2488](#)] [[INSPIRE](#)].