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Physics of F-theory compactifications without section

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Abstract: We study the physics of F-theory compactifications on genus-one fibrations without section by using an M-theory dual description. The five-dimensional action obtained by considering M-theory on a Calabi-Yau threefold is compared with a six-dimensional F-theory effective action reduced on an additional circle. We propose that the six-dimensional effective action of these setups admits geometrically massive U(1) vectors with a charged hypermultiplet spectrum. The absence of a section induces NS-NS and R-R three-form fluxes in F-theory that are non-trivially supported along the circle and induce a shift-gauging of certain axions with respect to the Kaluza-Klein vector. In the five-dimensional effective theory the Kaluza-Klein vector and the massive U(1)s combine into a linear combination that is massless. This U(1) is identified with the massless U(1) corresponding to the multi-section of the Calabi-Yau threefold in M-theory. We confirm this interpretation by computing the one-loop Chern-Simons terms for the massless vectors of the five-dimensional setup by integrating out all massive states. A closed formula is found that accounts for the hypermultiplets charged under the massive U(1)s.

Keywords: F-Theory, Superstring Vacua, String Duality

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Contents

1 Introduction

F-theory, as introduced in [1], provides a beautiful geometric reformulation of Type IIB string theory with varying string coupling. Not only has it been explored from a formal perspective, but, more recently, it has also found exciting applications to realistic model building, starting with [2–5]. The underlying idea of F-theory is to identify the complexified string coupling τ of Type IIB string theory with the complex structure of an auxiliary two-torus. Such an interpretation is motivated by the existence of the non-perturbative $SL(2, \mathbb{Z})$ symmetry of Type IIB. Remarkably, this construction extends to situations in which τ depends non-trivially on the space-time coordinates of the Type IIB background. One can thus consider backgrounds in which the T^2 is fibered over some compact base manifold. If the effective theory is to be supersymmetric the entire T^2 fibration X must be a Calabi-Yau manifold.

So far, most of the literature has focused on a subclass of T^2 fibrations X that are simpler to analyze. Namely, it has largely been assumed that X has a section, that is, a global meromorphic embedding of the base into the total space of the fibration; or equivalently, a canonical choice of point in the fiber well defined everywhere (except possibly at some lower-dimensional loci in the base where the fiber degenerates). All such fibrations can be birationally transformed [6] into a *Weierstrass model* of the form

$$y^2 = x^3 + fxz^4 + gz^6 \tag{1.1}$$

with (x, y, z) coordinates of a $\mathbb{P}^{2,3,1}$, and f, g functions on the base of the fibration. A canonical section is provided by $z = 0$. As pointed out by Witten in [7], this subclass of models is physically simpler to treat, because the existence of a section implies the absence of certain fluxes, as we will explain in more detail later on. Geometrically, the restriction to Weierstrass models facilitated model building with non-Abelian gauge symmetries, as the widely used algorithm of [8] (see also [9, 10] for later extensions) could be applied directly to models with Weierstrass form.

We emphasize, however, that while the assumption of having a section simplifies the analysis, it is in no way necessary for the consistency of the physics, or the existence of an F-theory limit. In fact, it is very easy to construct T^2 fibrations with no section that serve as natural backgrounds for F-theory and we analyze explicitly various examples below. For completeness, let us also note that the approach taken by [11, 12] provides a convenient and more general way of generating non-Abelian gauge symmetries also for models without section.

Based on this observation, in this paper we want to explore the physics of F-theory backgrounds $\mathcal{X}$ in which the T^2 does *not* have a section, and thus no Weierstrass model. This case remains basically unexplored, with the exception of the recent works [13, 14] (which appeared while this work was in progress), and some remarks in [7] that will play a role in our analysis below. We will focus on the formal aspects of this class of F-theory backgrounds, uncovering some interesting characteristics of the resulting effective field theories.

We will argue that a *massive* U(1) symmetry in the resulting six-dimensional theory coming from F-theory on $\mathcal{X}$ plays an essential role in a proper understanding of the theory. In fact, one of the important results in this paper is a proposal for a method of computing the massless and part of the massive spectrum of F-theory on a fibration $\mathcal{X}$ without section. We will test this proposal in a particular class of examples where the origin and properties of this massive U(1) are particularly transparent — namely, examples where $\mathcal{X}$ is obtained from a conifold transition from a Calabi-Yau threefold $\mathbb{X}$ with *two* sections. Note that massive U(1)s in F-theory have recently been investigated in [15–17].

In fact, for the cases studied in detail in this paper there exist both geometrical and physical reasons for why the Calabi-Yau manifolds $\mathcal{X}$ with bi-section are naturally related to fibrations $\mathbb{X}$ with two independent sections. Geometrically, by transitioning to a different manifold $\mathbb{X}$ the bi-section can be split into two independent sections. Physically, the massive U(1) becomes massless in that limit. Recently, the study of massless U(1) gauge symmetries in global F-theory compactifications has been a heavily investigated topic. Geometrically, the number of the Abelian gauge fields corresponds to the rank of the Mordell-Weil group of the fibration. As the Mordell-Weil group is generated by the sections, there is a direct correspondence between the number of independent sections and the number of U(1) generators. Let us note here that starting with the U(1)-restricted models of [15], continued by a systematic six-dimensional analysis of single U(1) models [18] and extended to more general treatments of multiple U(1) factors [19–27] both with holomorphic and non-holomorphic sections [21, 22, 28] a variety of methods has been developed that we will draw from in order to analyze the properties of our specific models.

However, in order to study the effective physics of the F-theory compactifications without sections, it is most useful to employ the M- to F-theory limit. One can define F-theory on a T^2 fibered manifold X as M-theory compactified on X in the limit where the size of the T^2 fiber goes to 0. When the T^2 is small, but of finite size, F-theory is compactified on $X \times S^1$, with the size of the S^1 inversely proportional to the area of the T^2 fiber (so in the strict F-theory limit the S^1 decompactifies). Much of the subtle behavior of F-theory on manifolds $\mathcal{X}$ without a section can be best understood by taking the S^1 to have finite size. For concreteness, in this paper we take $\dim_{\mathbb{C}}(\mathcal{X}) = 3$, so F-theory on $\mathcal{X}$ gives a six-dimensional theory. Further compactification on an S^1 gives a five-dimensional theory, which can be alternatively obtained by compactifying M-theory on $\mathcal{X}$. Matching the two five-dimensional theories then allows one to identify geometric quantities of $\mathcal{X}$ with physical observables of the effective F-theory physics [29, 30].

We have organized this paper as follows. Section 2 contains a general discussion of the six-dimensional theories arising from F-theory compactifications on T^2-fibered Calabi-Yau threefolds with no section. Section 3 then describes the reduction of these theories down to five dimensions by compactification on a circle. A number of subtleties arise, which we solve. This general discussion is then illustrated in section 4 in a number of examples. Since there are a number of different actors in play in our construction, we have summarized the outline of our discussion in figure 1 for the convenience of the reader.

2 Six-dimensional action of F-theory on multi-section threefolds

In this section we introduce the six-dimensional effective theories that we claim to arise in F-theory compactifications on a genus-one fibered Calabi-Yau threefold $\mathcal{X}$ with a multi-section. To begin with, we recall in subsection 2.1 the effective theory of an F-theory compactification on a manifold $\mathbb{X}$ with two sections. This theory will admit a massless Abelian gauge field $\hat{A}^1$, where the hat indicates here and in the following that we are dealing with a field in a six-dimensional space-time. In contrast, we explain in subsection 2.2 that the compactification on $\mathcal{X}$ yields a U(1) gauge field $\hat{A}^1$ made massive by a Stückelberg mechanism. For simplicity, we will restrict ourselves to scenarios with a single Abelian gauge field and no non-Abelian gauge symmetry. In geometric terms this amounts to assuming that $\mathcal{X}$ has a bi-section, i.e. a multi-section of rank two, and no non-Abelian singularities. We discuss the first row in figure 1 and thus establish figure 2.

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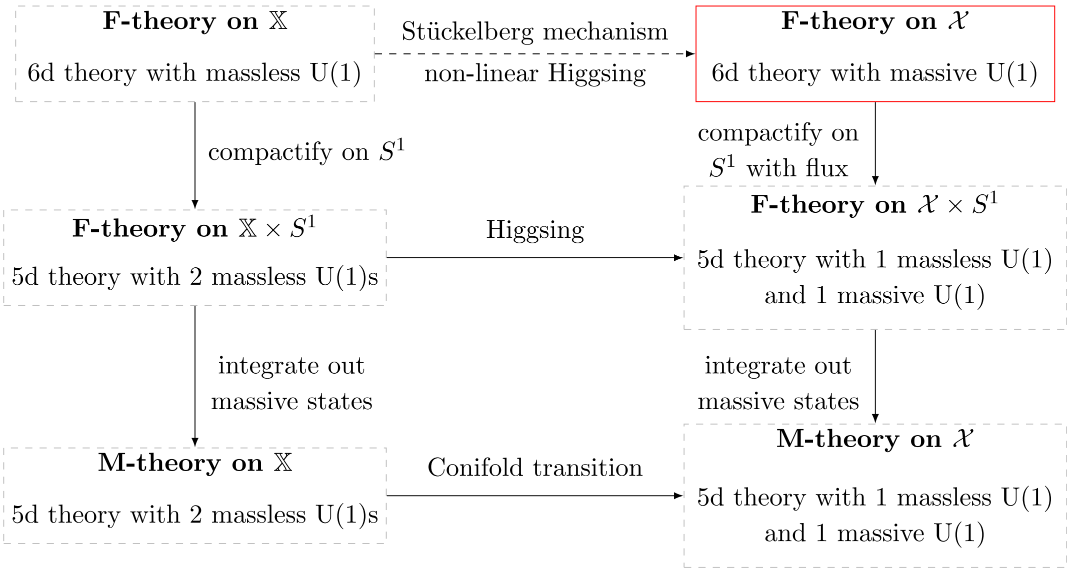

Figure 1. Overview of our discussion. The object of interest in the top-right corner, corresponding to the six-dimensional theories coming from F-theory on a space without section $\mathcal{X}$. In the examples we will discuss explicitly these compactifications are closely related (by making some fields massive) to F-theory on spaces with section $\mathbb{X}$, giving the six-dimensional theories in the top-left corner. Compactification of these theories on S^1 gives two five-dimensional theories, in the middle row, which can also be obtained by M-theory on the corresponding Calabi-Yau threefolds (shown in the bottom row). The five-dimensional theories are related by Higgsing, or equivalently, by conifold transitions in M-theory.

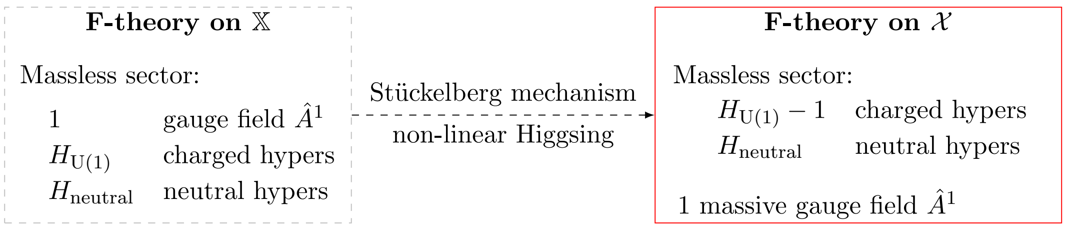

Figure 2. Six-dimensional effective theories with a massless and massive U(1) gauge field.

2.1 Review of massless U(1) in F-theory

In order to set the stage for our considerations of a massive U(1), let us first recall the simpler situation in which the U(1) is massless. Six-dimensional effective theories with a massless U(1) arise when considering F-theory on a manifold with two sections $\mathbb{X}$. One of these sections is identified with the massless U(1) while the second section, the zero section, corresponds to the Kaluza-Klein vector in the F-theory to M-theory reduction as we recall in section 3. The effective theory for F-theory compactifications with multiple sections was studied in detail in [22]. The spectrum of the six-dimensional theory consists of T tensor multiplets and V vector multiplets with

$$T = h^{1,1}(B_2) - 1\,, \qquad V = h^{1,1}(\mathbb{X}) - h^{1,1}(B_2) - 1 = 1\,, \tag{2.1}$$

where we have considered, for simplicity, that $\mathbb{X}$ induces no non-Abelian gauge symmetries. The base of the elliptic fibration is denoted by B_2. The vector multiplet contains precisely the massless U(1) vector $\hat{A}^1$. In addition to these multiplets the theory will generally contain a number of hypermultiplets H. Generally, one can split

$$H = H_{\text{neutral}} + H_{\text{charged}} \,, \qquad H_{\text{neutral}} = h^{2,1}(\mathbb{X}) + 1 \tag{2.2}$$

and if there are no non-Abelian gauge symmetries

$$H_{\text{charged}} = H_{\text{U}(1)} \,, \tag{2.3}$$

where $H_{\text{U}(1)}$ counts the number of hypermultiplets charged under $\hat{A}^1$. Recall that the cancellation of six-dimensional pure gravitational anomalies requires the relation

$$H - V = 273 - 29T \,. \tag{2.4}$$

In addition one has to cancel the gauge and mixed anomalies. In order to do that one can employ a generalized Green-Schwarz mechanism [31–33] induced by a coupling

$$S_{\text{GS}} = -\frac{1}{2}\int \Omega_{\alpha\beta}\hat{B}^\alpha \wedge \left(\frac{1}{2}a^\beta \,\text{Tr}(\hat{\mathcal{R}} \wedge \hat{\mathcal{R}}) + 2b^\alpha \hat{F}^1 \wedge \hat{F}^1\right), \tag{2.5}$$

where $\hat{\mathcal{R}}$ is the six-dimensional curvature two-form and $\hat{F}^1$ is the field strength of the U(1) vector $\hat{A}^1$. The tensors $\hat{B}^\alpha$, $\alpha = 1, \dots, T+1$ arise from the T tensor multiplets and the gravity multiplet, and the symmetric constant matrix $\Omega_{\alpha\beta}$ and the constant vectors (a^α, b^α) are crucial to determine the couplings of the six-dimensional supergravity theory.

Finally, recall that both (a^α, b^α) and $\Omega_{\alpha\beta}$ are naturally determined by the topology of the compactification manifold $\mathbb{X}$ as

$$a^\alpha = -\Omega^{\alpha\beta}\Big(D_\beta \cdot [\pi^* c_1(B_2)]\Big)_{B_2}, \quad b^\alpha = -\Omega^{\alpha\beta}\left(D^2_{\text{U}(1)} \cdot D_\beta\right), \quad \Omega_{\alpha\beta} = (D_\alpha \cdot D_\beta)_{B_2}\,, \tag{2.6}$$

where we have denoted by D_α the divisors inside $\mathbb{X}$ that are obtained by fibering the genus-one curve over a divisor in the base B_2 and write $[\pi^* c_1(B_2)]$ for the Poincaré-dual of the first Chern class of B_2 pulled back to $\mathbb{X}$. Furthermore, we take $\Omega^{\alpha\beta}$ to be the inverse of $\Omega_{\alpha\beta}$. $D_{\text{U}(1)}$ is the divisor in $\mathbb{X}$ obtained from the U(1) seven-brane divisor in the base.

2.2 Massive U(1) and the Stückelberg mechanism

Let us now turn to the compactifications most relevant to this work and consider F-theory on the space $\mathcal{X}$ with bi-section. We propose that in this case one finds a massive U(1) vector multiplet that can be described by a massless U(1) vector multiplet coupled to a hypermultiplet by a Stückelberg mechanism. In addition to this non-linearly charged hypermultiplet, $H_{\text{U}(1)} - 1$ matter hypermultiplets will be part of the six-dimensional effective theory. In the next section we will use the dual M-theory picture in order to argue for the correctness of this proposal. We will also determine the total number of charged hypermultiplets $H_{\text{U}(1)}$ and their charges.

Let us denote the scalars in the $H_{\mathrm{U}(1)} - 1$ linearly charged matter hypermultiplets by h^s. The additional non-linearly charged hypermultiplet contains an axion c with shift symmetry gauged under $\hat{A}^1$. In summary, one has[1]

$$\hat{\mathcal{D}}c = dc + m\hat{A}^1\,, \qquad \hat{\mathcal{D}}h^s = dh^s + q^s\,\hat{A}^1 h^s\,, \tag{2.7}$$

where q^s is the charge of the state h^s. In other words, the theory differs from the one introduced in the previous subsection 2.1 due to the gauging of the shift symmetry of c parametrized by m. More details on the difference between the non-linear Higgs mechanism induced by the coupling to c and a linear Higgs mechanism are discussed in [34].

After gauge fixing the U(1) gauge symmetry, the kinetic term $|\hat{\mathcal{D}}c|^2$ of the axion c becomes a mass term for $\hat{A}^1$, which is proportional to m^2. Hence, the U(1) can become massive by "eating" the axion c. In F-theory the shift gauging (2.7) can arise from a geometric Stückelberg mechanism [16]. More precisely, if the seven-brane action induces a six-dimensional coupling

$$S_{\mathrm{St}} = \int_{M^{5,1}} m\, c_4 \wedge \hat{F}^1\,, \tag{2.8}$$

then the four-form c_4 can be dualized into the axion c to obtain the gauging (2.7).

For D7-branes at weak coupling the effective coupling (2.8) arises indeed from a non-trivial Chern-Simons coupling $\int_{\mathcal{M}_8} C_6 \wedge F$, where C_6 is the R-R six-form of Type IIB string theory, and $\mathcal{M}_8 = M^{5,1} \times \mathcal{C}^{\mathrm{D7}}$ is the eight-dimensional subspace wrapped by the D7-brane and its orientifold image [35]. Comparing (2.8) with these Chern-Simons terms one finds $m\, c_4 = \int_{\mathcal{C}^{\mathrm{D7}}} C_6$, which determines m as an intersection number at weak string coupling. Since the axion c is the dual of c_4 in six dimensions, it arises in the expansion of the R-R two-from C_2 as

$$C_2 = c\,\tilde{\omega}\,, \tag{2.9}$$

where $\tilde{\omega}$ is a (1,1)-form on the Type IIB covering space that is negative under the orientifold involution. Since there is no flux involved in this mechanism, it was termed geometric Stückelberg mechanism in [16]. It should be stressed that determining m in a general F-theory setting is more involved and we will return to this question in the later parts of the paper.

For completeness, let us consider the effective theory at an energy scale below the mass of the U(1). In order to obtain this theory we have to integrate out the massive vector multiplet containing A_1, which was obtained by a massless vector multiplet "eating" a massless hypermultiplet. In other words one finds

$$V \;\rightarrow\; V - 1\,, \qquad H \;\rightarrow\; H - 1\,, \tag{2.10}$$

consistent with (2.4). Furthermore, all hypermultiplets charged under the massive U(1) are neutral in the effective theory and one has

$$H_{\text{charged}} \;\rightarrow\; 0\,, \qquad H_{\text{neutral}} \rightarrow\; H_{\text{neutral}} + H_{\mathrm{U}(1)} - 1\;. \tag{2.11}$$

[1]Since the scalars c, h^s remain scalars without redefinition when compactifying the theory to five dimensions in section 3, we will slightly abuse notation and not put a hat on them to distinguish them from their five-dimensional counterparts.

While this theory is a valid effective theory at the massless level, we will see in the section 3 that it cannot be used in order to perform the F-theory to M-theory duality.

3 Fluxed S^1 reduction of the six-dimensional theory

In order to verify and further concretize the six-dimensional effective theory of subsection 2.2 obtained by compactifying F-theory on $\mathcal{X}$ one has to take a detour via M-theory. Therefore, our strategy, as depicted in figures 3 and 4, is to compactify the six-dimensional effective theories of subsections 2.1 and 2.2 on a circle and compare the resulting five-dimensional effective theory with M-theory reduced on $\mathbb{X}$ and $\mathcal{X}$, respectively. In subsection 3.1 we recall the circle reduction for $\mathbb{X}$ that yields two massless U(1)s in five dimensions. For the fibration $\mathcal{X}$ with a bi-section, however, it turns out that a circle reduction alone can never yield the correct match. In fact, we will argue in subsection 3.2 that it is crucial to include background fluxes for the gauged axion c in (2.7) in order to ever be able to match the effective theories. The effective theory obtained after circle reduction with fluxes is derived in subsection 3.3 and compared with the effective theory for $\mathcal{X}$. We stress that analyzing classical and one-loop Chern-Simons terms in the five-dimensional effective theories is crucial to establish the duality.

3.1 Massless U(1) on a circle and its M-theory dual

In this subsection we review the five-dimensional effective action obtained by compactifying an F-theory model with one massless U(1) on a circle. We also comment on the one-loop effective theory which one obtains by entering the Coulomb branch of the five-dimensional theory and integrating out all massive modes. This amounts to discussing the first column of figure 1, which we reproduce in more detail in figure 3.

The Kaluza-Klein ansatz for the six-dimensional metric is given by

$$ds^2_{(6)} = g_{\mu\nu}dx^\mu dx^\nu + r^2(dy - A^0)^2\,, \tag{3.1}$$

where r is the radius of the S^1 and A^0 is the Kaluza-Klein vector that will play a crucial role in the following. The U(1) vector A^1 reduces on a circle as

$$\hat{A}^1 = A^1 + \zeta(dy - A^0)\,, \tag{3.2}$$

with the vector A^1 and the scalar ζ forming the bosonic components of a five-dimensional vector multiplet. In addition, there are $T+1$ five-dimensional vectors A^α arising from six-dimensional tensors $\hat{B}^\alpha$ and $T+1$ scalars j^α satisfying one constraint $j^\alpha j^\beta \Omega_{\alpha\beta} = 1$. Note that in this section 3 all scalars including c, h^s live in a five-dimensional space-time.

Let us next package the reduced fields into five-dimensional vector multiplets and introduce the five-dimensional theory. To begin with, recall that the dynamics of the $T+2$ vector multiplets and the graviphoton are entirely specified in terms of a cubic potential

$$\mathcal{N} = \frac{1}{3!}k_{IJK}M^I M^J M^K\,. \tag{3.3}$$

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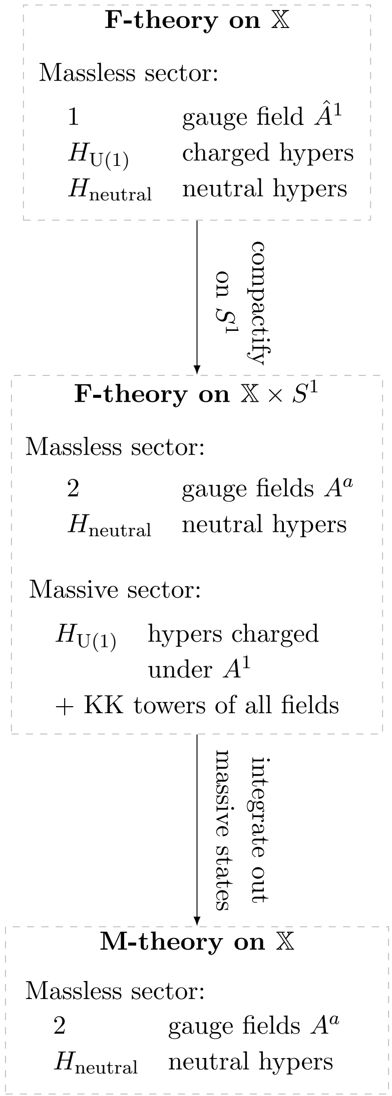

Figure 3. The different theories related to the resolved manifold $\mathbb{X}$ and their interrelations.

where k_{IJK} is a constant symmetric tensor. The potential $\mathcal{N}$ depends on the real coordinates M^I, $I = 0, \ldots, T+2$ and encodes a real special geometry of $\mathcal{N} = 2$ supergravity. The M^I combine with the vectors A^I of the theory. However, since the vector in the gravity multiplet is not accompanied by a scalar degree of freedom, the M^I have to satisfy one constraint. In fact, the $\mathcal{N} = 2$ scalar field space is identified with the hypersurface $\mathcal{N} \stackrel{!}{=} 1$. The gauge coupling function and the metric are obtained by evaluating the second derivative of $-\frac{1}{2}\log\mathcal{N}$ restricted to the constraint hypersurface. For completeness, let us give the M^I for the circle reduced setup:

$$M^0 = \frac{1}{2} r^{-4/3} , \quad M^1 = 2r^{-4/3}\zeta , \quad M^\alpha = 2r^{2/3}(j^\alpha + 2b^\alpha \zeta^2/r^2) . \tag{3.4}$$

The F-theory reduction with U(1)s was carried out in [22] and it was found that the cubic potential takes the form[2]

$$\mathcal{N}^{\rm F} = \frac{1}{2} M^0 \Omega_{\alpha\beta} M^\alpha M^\beta - \frac{1}{2} M^\alpha \Omega_{\alpha\beta} b^\beta M^1 M^1 \,. \tag{3.5}$$

It should be stressed that this is only the classical contribution with all charged hypermultiplets retained in the five-dimensional theory. As we will discuss below, equation (3.5) receives one-loop corrections from integrating out massive modes, such as the Kaluza-Klein states.

In the following we will mostly focus on the couplings of the vectors $A^I = (A^0, A^1, A^\alpha)$, as supersymmetry then also determines the vector multiplet couplings of the action. In particular, we will discuss the Chern-Simons action for the vectors

$$S_{\rm CS} = -\frac{1}{12} \int_{M^{4,1}} k_{IJK} A^I \wedge F^J \wedge F^K - \frac{1}{4} \int_{M^{4,1}} k_I A^I \wedge \mathrm{tr}(\mathcal{R} \wedge \mathcal{R}) \,, \tag{3.6}$$

where $F^I = dA^I$ and $\mathcal{R}$ is the five-dimensional curvature two-form. Classically, the Chern-Simons coefficients k_{IJK} can be read off from (3.5) as

$$k^{\rm class}_{0\alpha\beta} = \Omega_{\alpha\beta} \,, \qquad k^{\rm class}_{\alpha 11} = -\Omega_{\alpha\beta} b^\beta \,, \tag{3.7}$$

with all other classical triple couplings vanishing. In addition, one finds that k_I is classically given by

$$k^{\rm class}_{\alpha} = \Omega_{\alpha\beta} a^\beta \,, \qquad k^{\rm class}_{0} = 0 \,, \qquad k^{\rm class}_{1} = 0 \,, \tag{3.8}$$

with a^α as in (2.5).

In is important to stress that the classical theory with Chern-Simons terms (3.7) and (3.8) cannot be successfully compared with the M-theory reduction on the non-singular manifold $\mathbb{X}$. To make such a comparison, one first has to move to the five-dimensional Coulomb branch by giving the scalar ζ in the vector multiplet of the six-dimensional extra U(1) a vacuum expectation value. Furthermore, one has to integrate out all massive states. In general, the mass of a state $\mathbf{w}$ with Kaluza-Klein charge $\hat{n}$ is

$$m_{\mathbf{w}}(\hat{n}) = m^{\mathbf{w}}_{\rm CB} + \hat{n}\; m_{\rm KK} \,. \tag{3.9}$$

Note that the Coulomb branch mass $m^{\mathbf{w}}_{\rm CB}$ depends on the charges $\mathbf{w}_i$ of the state $\mathbf{w}$ via $m^{\mathbf{w}}_{\rm CB} = q_i(\mathbf{w})\zeta^i$. Integrating out a state causes the Chern-Simons terms of $A \wedge \mathrm{tr}\, \mathcal{R} \wedge \mathcal{R}$ and $A \wedge F \wedge F$ to shift according to [36][3]

$$k_{\Lambda\Sigma\Theta} \mapsto k_{\Lambda\Sigma\Theta} + c_{AFF}\; q_\Lambda q_\Sigma q_\Theta\, \mathrm{sign}(m) \tag{3.10}$$

$$k_{\Lambda} \mapsto k_{\Lambda} + c_{A\mathcal{R}\mathcal{R}}\; q_\Lambda\, \mathrm{sign}(m) \,, \tag{3.11}$$

respectively, where c_{AFF} and $c_{A\mathcal{R}\mathcal{R}}$ are constants depending on the sort of state integrated out. They were computed in [36] and are listed in table 1 in the conventions used here.

[2]We remark that there is an additional non-polynomial part acting as local counterterms in the five-dimensional action. As it does not take part in the match with M-theory, we omit it here and refer to [22, 30] for further information.

[3]The spin-1/2 case was first discussed in [37].

	spin-1/2 fermion	self-dual tensor $B_{\mu\nu}$	spin-3/2 fermion ψ_μ
c_{AFF}	$\frac{1}{2}$	-2	$\frac{5}{2}$
$c_{A\mathcal{RR}}$	-1	-8	19

Table 1. The different constant multipliers for the shifts of the Chern-Simons terms. Note that the individual multipliers may have to be multiplied by -1 depending on the chirality of the state.

To avoid clutter in the results for loop-corrections, let us introduce one more bit of notation, namely

$$l_{\mathbf{w}} \equiv \left\lfloor \frac{|m^{\mathbf{w}}_{\mathrm{CB}}|}{|m_{\mathrm{KK}}|} \right\rfloor , \tag{3.12}$$

the (floored) ratio of Coulomb branch mass and Kaluza-Klein mass of a state $\mathbf{w}$. We point out that $l_{\mathbf{w}}$ vanishes as long as the zero section of the compactification manifold is holomorphic. For non-holomorphic zero sections, however, a modified F-theory limit leads to important additional contributions [22] and in the examples studied in section 4 we will encounter such cases.

Keeping the number of vector multiplets general as V for the time being, one then finds that the loop-corrected Chern-Simons terms are

$$k_0 = \frac{1}{6}(H - V + 5T + 15) + \sum_{\mathcal{R}} H(\mathcal{R}) \sum_{\mathbf{w}\in\mathcal{R}} l_{\mathbf{w}}(l_{\mathbf{w}} + 1) \tag{3.13}$$

$$k_1 = \sum_{\mathcal{R}} H(\mathcal{R}) \sum_{\mathbf{w}\in\mathcal{R}} q_1(\mathbf{w})(2l_{\mathbf{w}} + 1)\,\mathrm{sign}(m^{\mathbf{w}}_{\mathrm{CB}}) , \tag{3.14}$$

and

$$k_{000} = \frac{1}{120}(H - V - T - 3) - \frac{1}{4}\sum_{\mathcal{R}} H(\mathcal{R}) \sum_{\mathbf{w}\in\mathcal{R}} l^2_{\mathbf{w}}(l_{\mathbf{w}} + 1)^2 \tag{3.15}$$

$$k_{001} = -\frac{1}{6}\sum_{\mathcal{R}} H(\mathcal{R}) \sum_{\mathbf{w}\in\mathcal{R}} q_1(\mathbf{w}) l_{\mathbf{w}}(l_{\mathbf{w}} + 1)(2l_{\mathbf{w}} + 1)\,\mathrm{sign}(m^{\mathbf{w}}_{\mathrm{CB}}) \tag{3.16}$$

$$k_{011} = -\frac{1}{12}\sum_{\mathcal{R}} H(\mathcal{R}) \sum_{\mathbf{w}\in\mathcal{R}} q_1(\mathbf{w})^2\,(1 + 6 l_{\mathbf{w}}(l_{\mathbf{w}} + 1)) \tag{3.17}$$

$$k_{111} = -\frac{1}{2}\sum_{\mathcal{R}} H(\mathcal{R}) \sum_{\mathbf{w}\in\mathcal{R}} q_1(\mathbf{w})^3 (2l_{\mathbf{w}} + 1)\,\mathrm{sign}(m^{\mathbf{w}}_{\mathrm{CB}}) \tag{3.18}$$

where, just as above, we have denoted the charge of the state $\mathbf{w}$ under the six-dimensional U(1) by $q_1(\mathbf{w})$ and sum over all matter representations $\mathcal{R}$ that are present in our theory. Note that since we are considering purely Abelian models, all representations $\mathcal{R}$ are one-dimensional and therefore contain only a single weight $\mathbf{w}$.

Finally, let us remark on how to use the loop-corrected Chern-Simons terms in order to compute the matter spectra of the associated F-theory model. As first explored in [38] and later refined in [22], one can make an ansatz for the matter spectrum, keeping the multiplicities general. Such an ansatz can for example be based on the curves found in the (relative) Mori cone of the Calabi-Yau, or, torically, the split induced by the top of the compactification manifold [23, 28]. Next, one uses that the matching of the M-theory and F-theory low-energy effective actions implies that the loop-corrected Chern-Simons terms must be given by simple topological intersection numbers in the M-theory geometry:

$$k_{IJK} = D_I \cdot D_J \cdot D_K \tag{3.19}$$

Here, just as before, the D_I are the divisors corresponding to the fields A^I in the usual manner.

For an explicit compactification manifold $\mathbb{X}$, we can therefore simply compute what the loop-corrected Chern-Simons terms of the five-dimensional theory must be by doing intersection theory on $\mathbb{X}$. Demanding that they match the formulas in (3.13)–(3.18) for the chosen ansatz one hence obtains a system of linear equations for the matter multiplicities. For all known examples in the literature, this system of equations has a unique solution.

3.2 Background flux and the M-theory to F-theory limit for multi-sections

In this subsection we argue that a simple circle reduction is not sufficient when considering F-theory on the Calabi-Yau threefold $\mathcal{X}$ with a multi-section. In order to do that we consider M-theory on $\mathcal{X}$ and dualize the setup step by step to obtain a Type IIB compactification.

To begin, we must consider the different structure of the Calabi-Yau metric in the case that the elliptically fibered space has, or does not have, a section. Let us denote by u^i the local (complex) coordinates on the base B_2 of $\mathcal{X}$ and by (x, y) local coordinates on the torus fiber. In the case that the fibration admits a section, it is possible to describe the base B_2 as a complex (algebraic) hypersurface within $\mathcal{X}$ given locally by a defining equation, $f(x, y, u) = 0$. This realization of B_2 as a hypersurface (in fact sub-manifold) of $\mathcal{X}$ makes it possible to use geodesics to define coordinates normal to B_2 within $\mathcal{X}$ consistently for each coordinate patch in B_2, and as a result the 3-fold metric takes a complex, Kähler version of Gaussian normal form [39, 40]. That is, the metric can be made block-diagonal with respect to the fiber/base with $g_{I5} = g_{I6} = 0$ for $I = 1, \ldots 4$ denoting base directions and $5, 6$ fiber directions.

By contrast, it was noted in [7] that in the case that $\mathcal{X}$ has multi-sections only, the base is no longer a submanifold of $\mathcal{X}$ and no such hypersurface description exists. As a result, there must exist *some* coordinate patch in B_2 for which the diagonalization described above fails and g_{I5} and/or $g_{I6} \neq 0$. Let us consider such a patch and over it, take a semi-flat approximation to the Calabi-Yau metric [41–43]. Away from any singular fibers the metric takes the local form

$$ds^2(\mathcal{X}) = g_{i\bar{j}}\, du^i d\bar{u}^{\bar{j}} + \frac{v^0}{\mathrm{Im}\tau} |X - \tau Y|^2 \,, \tag{3.20}$$

where at each point of B_2 one parametrizes the complex structure of the torus fiber by $\tau(u)$ and v^0 is the overall area of the T^2 fiber, which is constant over the base. The presence of off-diagonal (fiber/base) metric components are parametrized here by vectors $(\tilde{X}, \tilde{Y})$ on

B_2 in

$$X = dx + \tilde{X}\,, \qquad Y = dy + \tilde{Y}\,, \qquad K = \tilde{X} - \tau\tilde{Y}\,, \tag{3.21}$$

where we have introduced a complex vector K on B_2 in order to re-write the metric in complex coordinates. Defining $z = x - \tau y$, (3.20) takes the form

$$ds^2(\mathcal{X}) = g_{i\bar{\jmath}}\, du^i d\bar{u}^{\bar{\jmath}} + \frac{v^0}{\mathrm{Im}\tau}|dz - \frac{\mathrm{Im}z\, d\tau}{\mathrm{Im}\tau} + K|^2\ . \tag{3.22}$$

We locally define on $\mathcal{X}$ the two-form

$$\omega_0 = \frac{1}{\mathrm{Im}\tau}\left(dz - \frac{\mathrm{Im}z\, d\tau}{\mathrm{Im}\tau} + K\right)\wedge\left(d\bar{z} - \frac{\mathrm{Im}z\, d\bar{\tau}}{\mathrm{Im}\tau} + \bar{K}\right) = 2Y\wedge X \tag{3.23}$$

In terms of ω_0 the globally defined two-form on $\mathcal{X}$ is given by $J = J_{\mathrm{base}} + v^0\omega_0$. If K is a $(1,0)$ form then J is of type $(1,1)$ and we find compatibility of (3.20) with the complex structure [44]. Using that τ is holomorphic in the base coordinates it follows that $d(K/\mathrm{Im}\tau)$ and $d(\bar{K}/\mathrm{Im}\tau)$ are both $(1,1)$ forms. Together with the fact that

$$\frac{i(K-\bar{K})}{2\mathrm{Im}\tau} = \tilde{Y}\,, \qquad \frac{i(\bar{\tau}K - \tau\bar{K})}{2\mathrm{Im}\tau} = \tilde{X} \tag{3.24}$$

we obtain finally that $\langle d\tilde{X}\rangle$ and $\langle d\tilde{Y}\rangle$ are $(1,1)$ forms. In the following we will consider the case that

$$\langle d\tilde{X}\rangle = -n\tilde{\omega}\,, \qquad \langle d\tilde{Y}\rangle = 0\,, \tag{3.25}$$

where $\tilde{\omega}$ is an appropriately normalized (1,1) form on B_2, which has to be identified with the form appearing in (2.9). The ansatz (3.25) implies the presence of exactly one gauged axion c and has to be generalized accordingly for more involved situations. In this simplest setup, however, $\langle d\tilde{Y}\rangle$ has to vanish for the consistency of the effective theory.

In the following we consider M-theory on the space (3.20) and perform the M-theory to F-theory limit. The eleven-dimensional metric and M-theory three-form are expanded as

$$ds^2_{11} = ds^2_5 + ds^2(\mathcal{X})\,, \qquad C_3^M = B_2\wedge X + C_2\wedge Y + \frac{1}{2}A^0\wedge\omega_0 + \ldots\,, \tag{3.26}$$

where the dots indicate the expansion into further harmonic (1,1) forms of $\mathcal{X}$ irrelevant to the present discussion. We also expand $B_2 = b\tilde{\omega}$ and $C_2 = c\tilde{\omega}$ and compute

$$dC_3^M = db\wedge X\wedge\tilde{\omega} + b\,\tilde{\omega}^2 + (dc + nA^0)\wedge Y\wedge\tilde{\omega} + \frac{1}{2}F^0\wedge\omega_0 + \ldots\,, \tag{3.27}$$

where we have used $d\omega_0 = 2n\,Y\wedge\tilde{\omega}$. We note that the non-trivial background $\langle d\tilde{X}\rangle$ implies that the axion c is gauged by the vector A^0. Following the M-theory to F-theory duality, which we discuss next, one finds that with the expansion (3.26) the vector A^0 maps precisely to the Kaluza-Klein vector of the reduction from six to five dimensions.

Due to the presence of non-trivial $\tilde{X}, \tilde{Y}$ in (3.20) the standard M-theory to F-theory limit is modified (see [45] for a review). To fix an $\mathrm{SL}(2,\mathbb{Z})$ frame, let us pick an A-cycle and a B-cycle of the genus-one fiber with local coordinates x and y, respectively. In order to perform the duality we first go from M-theory to Type IIA by splitting the metric with respect to the A-cycle according to

$$ds_M^2 = e^{4\phi_{\rm IIA}/3}(dx + C_1^{\rm IIA})^2 + e^{-2\phi_{\rm IIA}/3} ds_{\rm IIA}^2 \, . \tag{3.28}$$

Comparing with (3.20) one finds the Type IIA R-R one-form $C_1^{\rm IIA}$ and metric $ds_{\rm IIA}^2$ to be

$$C_1^{\rm IIA} = {\rm Re}\,\tau\, dy + {\rm Re}\, K \, , \qquad ds_{\rm IIA}^2 = \sqrt{\frac{v^0}{{\rm Im}\tau}}\left(\frac{v^0}{{\rm Im}\tau}({\rm Im}\,\tau\, dy + {\rm Im}\, K)^2 + g_{i\bar{\jmath}}\, du^i d\bar{u}^{\bar{\jmath}}\right) \tag{3.29}$$

with $e^{4\phi_{\rm IIA}/3} = \frac{v}{{\rm Im}\tau}$. Using the T-duality rules along the B-cycle one encounters non-trivial NS-NS and R-R two-forms

$$C_2^{\rm IIB} = C_2 + \tilde{X} \wedge dy \, , \qquad B_2^{\rm IIB} = B_2 + \tilde{Y} \wedge dy \; . \tag{3.30}$$

The presence of non-trivial $C_2^{\rm IIB}$ and $B_2^{\rm IIB}$ in (3.30) implies that the F-theory reduction should include three-form fluxes

$$F_3 = \langle dC_2^{\rm IIB} \rangle = -n\, \tilde{\omega} \wedge dy \; . \tag{3.31}$$

We stress that this flux has one leg around the circle used to compactify six to five dimensions.

Let us now make contact with the discussion of subsection 2.2. After decompactifying the T-dualized Type IIB circle the scalars c, b are lifted to proper six-dimensional scalars. One can then reinterpret that flux (3.31). Compactifying the six-dimensional theory on a circle the flux n can be understood as a background of dc given by

$$\int_{S^1} \langle dc \rangle = n \; . \tag{3.32}$$

This implies that the standard circle reduction has to include this non-trivial background and we will explicitly perform this modified computation in the next subsection.

3.3 Fluxed circle reduction and M-theory comparison

Having motivated the inclusion of circle fluxes we are now in the position to compute the five-dimensional effective theory, that is, we proceed by discussing the second column of figure 1, reproduced in figure 4 with the relevant matter spectra included. In performing this reduction we include the circle fluxes

$$\int_{S^1} \langle dc \rangle = n \; . \tag{3.33}$$

Using the background metric (3.1) this implies that the kinetic term of the axion c reduces as

$$\mathcal{L}_c = G_{cc} |\hat{\mathcal{D}} c|^2 = G_{cc} |\mathcal{D} c|^2 \, , \tag{3.34}$$

where G_{cc} is the metric for the field c. In other words, the six-dimensional invariant derivative of the axion c given in (2.7) is replaced by

$$\mathcal{D} c = dc + m A^1 + n A^0 \, , \tag{3.35}$$

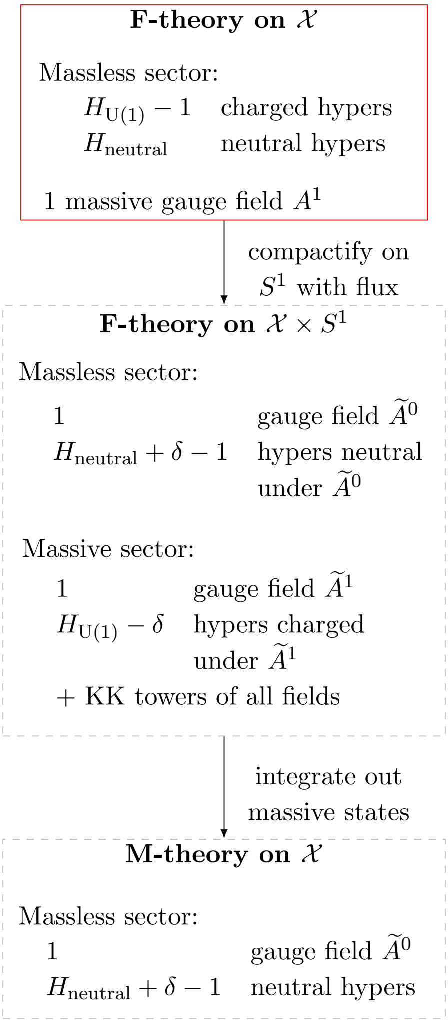

Figure 4. The different theories related to the deformed manifold $\mathcal{X}$ and their interrelations.

We stress that this modification only appears in the five-dimensional effective theory and mixes the reduced U(1) vector A^1 with the Kaluza-Klein vector A^0.

This implies that after absorbing the axion c the mass term in the five-dimensional theory reads

$$\mathcal{L}_{\mathrm{mass}} = G_{cc} |m A^1 + n A^0|^2 \,, \tag{3.36}$$

To evaluate the effective theory for the massless degrees of freedom only, we therefore first have to chose an appropriate basis of one massless vector field $\widetilde{A}^0$ and one massive vector field $\widetilde{A}^1$.

Starting with the two gauge fields A^0 and A^1, the most general transformation to a new basis of gauge fields $\widetilde{A}^0$ and $\widetilde{A}^1$ can be expressed as

$$\widetilde{A}^i = \frac{1}{a^2 + b^2} N^i{}_j \, A^j \,, \qquad N^i{}_j = \begin{pmatrix} b & -a \\ a & b \end{pmatrix} \,. \tag{3.37}$$

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Note that the orthogonality of the columns of $N^i{}_j$ guarantees that the kinetic terms of $\widetilde{A}^i$ remain diagonal under the transformation if they are already diagonal before. In the following we like to identify $\widetilde{A}^1$ with the massive U(1) with mass term (3.36). This implies that a, b in (3.37) are identified to be

$$a = n\,, \qquad b = m\,. \tag{3.38}$$

We also need to transform the charges $q_j(\mathbf{w})$ under the A^i of a state $\mathbf{w}$. The transformation (3.37) introduces new charges $\tilde{q}_i$ as

$$\tilde{q}_i = q_j\,(N^T)^j{}_i\,. \tag{3.39}$$

To compare the fluxed circle reduction to the M-theory reduction on $\mathcal{X}$ we thus rotate into the new basis $\widetilde{A}^i$ and then drop the couplings of the massive gauge field $\widetilde{A}^1$. As in section 3.1 we have to consistently integrate out all massive modes. The way to check that the reduction of the proposed six-dimensional F-theory action indeed matches we will identify the five-dimensional Chern-Simons terms. We note that the constant couplings k_{IJK} and k_I in (3.6) transform under the basis change (3.37) as

$$\begin{aligned} \tilde{k}_{ijk} &= k_{abc}\,(N^T)^a{}_i\,(N^T)^b{}_j\,(N^T)^c{}_k\,, & \tilde{k}_{ij\alpha} &= k_{ab\alpha}\,(N^T)^a{}_i\,(N^T)^b{}_j\,, \\ \tilde{k}_{i\alpha\beta} &= k_{a\alpha\beta}\,(N^T)^a{}_i\,, & \tilde{k}_i &= k_a\,(N^T)^a{}_i \end{aligned} \tag{3.40}$$

with $\tilde{k}_{\alpha\beta\gamma} = k_{\alpha\beta\gamma} = 0$ and $\tilde{k}_\alpha = k_\alpha$ as above. Using these expressions together with (3.37), (3.38), (3.7) and (3.8) we find the non-vanishing classical Chern-Simons terms for the massless five-dimensional gauge fields $(\widetilde{A}^0, A^\alpha)$ to be

$$\tilde{k}^{\text{class}}_{00\alpha} = -n^2\,\Omega_{\alpha\beta}b^\beta\,, \qquad \tilde{k}^{\text{class}}_{0\alpha\beta} = m\Omega_{\alpha\beta}\,, \tag{3.41}$$

$$\tilde{k}^{\text{class}}_{\alpha} = \Omega_{\alpha\beta}a^\beta\,. \tag{3.42}$$

Let us stress that $\tilde{k}^{\text{class}}_{00\alpha}$ is non-zero and depends on the classical coupling of the extra U(1). In contrast, if the $\widetilde{A}^0$ is only the Kaluza-Klein vector, one recalls from (3.7) that $k_{00\alpha} = 0$. The latter is indeed true for all models with multiple sections considered in the literature so far. Crucially, in the examples with multi-section this coupling no longer vanishes as we discuss below and show for specific examples in section 4.

The Chern-Simons terms induced by integrating out the massive states at one loop level are obtained from (3.40) using (3.15)–(3.14). For the triple coupling one finds for the massless gauge field $\widetilde{A}^0$ that

$$\tilde{k}_{000} = k_{000}m^3 - 3k_{001}nm^2 + 3k_{011}n^2m - k_{111}n^3 \tag{3.43}$$

$$\begin{aligned} &= \frac{m^3}{120}(H - V - T - 3) \\ &\quad + \frac{1}{4}\sum_{\mathcal{R}} H(\mathcal{R}) \sum_{\mathbf{w}\in\mathcal{R}} \Big(- m^3 l_{\mathbf{w}}^2 (l_{\mathbf{w}}+1)^2 \\ &\qquad + 2nm^2 q_1(\mathbf{w}) l_{\mathbf{w}}(l_{\mathbf{w}}+1)(2l_{\mathbf{w}}+1)\,\text{sign}(\mathbf{w}) \\ &\qquad - n^2 m q_1(\mathbf{w})^2\,(1 + 6 l_{\mathbf{w}}(l_{\mathbf{w}}+1)) \\ &\qquad + 2n^3 q_1(\mathbf{w})^3 (2l_{\mathbf{w}}+1)\,\text{sign}(\mathbf{w})\Big)\,. \end{aligned} \tag{3.44}$$

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Furthermore, one finds the one-loop contribution to k_I to be

$$\begin{aligned}\tilde{k}_0 &= k_0 m - k_1 n \\ &= \frac{m}{6}\,(H - V + 5T + 15) \\ &\quad + \sum_{\mathcal{R}} H(\mathcal{R}) \sum_{\mathbf{w}\in\mathcal{R}} \Big(m l_{\mathbf{w}}(l_{\mathbf{w}}+1) - n q_1(\mathbf{w})(2l_{\mathbf{w}}+1)\,\mathrm{sign}(\mathbf{w}) \Big)\,. \end{aligned} \tag{3.45}$$

Having presented the field theory result for the Chern-Simons terms obtained by integrating out all massive modes, we are now in the position to compare this with the reduction on $\mathcal{X}$. To try and understand the above discussion from a different angle, let us consider the fiber geometry of a bi-section for a moment. By definition, a bi-section cuts out two different points over a generic point in the base manifold. Let us call these points P and Q. *Locally*, the bi-section is therefore indistinguishable from the sum of two separate sections cutting out P and Q, respectively. In a given patch, one could therefore try and define divisors $V(P)$ and $V(Q)$ and follow the usual procedure of applying the Shioda map [46, 47] to obtain a suitable set of massless gauge fields. Choosing $V(P)$ as the zero section, one would thus obtain the two "local divisors"

$$D_0 = V(P)\,, \qquad D_1 = \lambda\,(V(Q) - V(P)) \tag{3.46}$$

up to some irrelevant vertical parts, where λ is an arbitrary normalization constant. However, since we have a bi-section, *globally* the two points P and Q undergo monodromies and the only well-defined quantity is the divisor $V(P) + V(Q)$. Consequently, as the massless U(1) gauge field corresponds to the bi-section, its associated divisor must satisfy

$$\widetilde{D}_0 \sim 2\lambda D_0 + D_1\,, \tag{3.47}$$

where the proportionality constant is just another normalization factor that we can choose arbitrarily. Comparing (3.47) to $\widetilde{A}^0$, one hence finds

$$m = 2\lambda\,, \qquad n = -1\,. \tag{3.48}$$

This geometric argument therefore implies that the fluxes present in the circle reduction are in fact fixed uniquely up to physically irrelevant rescalings of the massless U(1) gauge field.

4 Examples: transitions removing the section

The discussion so far has been general. We now illustrate how the physics works in a particularly transparent set of examples. These are given by pairs of Calabi-Yau threefolds $(\mathbb{X}, \mathcal{X})$ related by a conifold transition, where $\mathbb{X}$ has two independent sections and $\mathcal{X}$ has no section, but rather a multi-section. Our discussion begins in subsection 4.1 by keeping the treatment of the $(\mathbb{X}, \mathcal{X})$ pairs independent of the base manifolds. In subsection 4.2 we review some well-known facts about the physics of conifold transition, before we proceed in subsection 4.3 by constructing explicit Calabi-Yau manifolds with base manifold $\mathbb{P}^2$. Finally, we evaluate the Chern-Simons terms of some of the specific examples in subsection 4.4 and give a general argument explaining why they have to match. In figure 5 we give a pictorial description of the essential physical process studied in the following subsections.

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M-theory on $\mathbb{X}$

Massless sector:

2	gauge fields A^a
H_{neutral}	neutral hypers

$\xrightarrow{\text{Conifold transition}}$

M-theory on $\mathcal{X}$

Massless sector:

1	gauge field $\widetilde{A}^0$
$H_{\text{neutral}} + \delta - 1$	neutral hypers

Figure 5. The two theories obtained by compactifying M-theory on $\mathbb{X}$ and $\mathcal{X}$, respectively, are connected by a conifold transition.

4.1 Constructing $(\mathbb{X}, \mathcal{X})$ pairs with general base manifold

The basic observation allowing us to construct large numbers of such pairs is that there is a natural conifold transition implicit in most recent constructions of spaces with two sections. As described in [18], for example, the generic model with two sections is obtained by taking a Calabi-Yau hypersurface in $\widehat{\mathbb{P}^{1,1,2}}$. Let us parametrize $\widehat{\mathbb{P}^{1,1,2}}$ by the coordinates

	y_1	y_2	w	t
$\mathbb{C}^*_1$	1	1	2	0
$\mathbb{C}^*_2$	0	0	1	1

(4.1)

We blow-up the $\mathbb{Z}_2$ singularity in the fiber to have a nicer ambient space, and to be able to realize torically the Cartan divisor in some of the examples below. The Stanley-Reisner ideal (*SRI* in what follows) is generated by $\langle y_1 y_2, wt\rangle$. The generic Calabi-Yau hypersurface is a degree $(4,2)$ hypersurface in these coordinates, which we parametrize as

$$gw^2 + wtP(y_1, y_2) + t^2 Q(y_1, y_2) = 0\,, \tag{4.2}$$

with $P(y_1, y_2)$ a quadratic function in y_i

$$P(y_1, y_2) = \alpha y_1^2 + \beta y_1 y_2 + f y_2^2 \tag{4.3}$$

and $Q(y_1, y_2)$ a quartic

$$Q = y_1(by_1^3 + cy_1^2 y_2 + dy_1 y_2^2 + ey_2^3) + ay_2^4 \equiv y_1 Q'(y_1, y_2) + ay_2^4\,. \tag{4.4}$$

Since the elliptic fiber will be fibered over a base, g and the coefficients of P, Q will be sections of appropriate degree in the coordinates of the base (we will study some explicit examples below).[4] In order to have two sections, we set $a = 0$, so Q takes the form

$$Q = y_1(by_1^3 + cy_1^2 y_2 + dy_1 y_2^2 + ey_2^3) = y_1 Q'(y_1, y_2)\,. \tag{4.5}$$

The restricted Calabi-Yau equation becomes

$$\phi \equiv gw^2 + wtP(y_1, y_2) + t^2 y_1 Q'(y_1, y_2) = 0\,. \tag{4.6}$$

[4]The models constructed in [18] correspond to taking $g = 1$, which imposes some restrictions on the allowed fibrations. We do not impose such restriction.

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When the coefficients are chosen in this way, there are two sections of (4.6) that can easily be found. Take $y_1 = 0$. Since $y_1 y_2$ belongs to the SRI of $\widehat{\mathbb{P}^{1,1,2}}$, we can set $y_2 = 1$. We end up with

$$w(gw + tf) = 0 \tag{4.7}$$

where f is the coefficient of y_2^2 in σ_0 (again a section of some line bundle on the base, in general). We thus find a first section at $w = 0$ (we can then set $t = 1$ using $\mathbb{C}_2^*$), and a second section at $gw = -tf$. For generic choices of g, f and at generic points of the base, this equation has a unique solution, giving a second section, but at the zeroes of g, f it will behave in interesting ways.

Singularities. The hypersurface (4.6) will be singular when $\phi = d\phi = 0$. It is easy to check that solutions of this set of equations exist for $w = y_1 = e = f = 0$. For two-dimensional bases of the fibration, $e = f = 0$ generically has a set of solutions given by points. Close to one such zero, for generic values of the coefficients, equation (4.6) becomes

$$\lambda_1 w^2 + \lambda_2 wf + \lambda_3 wy_1 + \lambda_4 y_1^2 + \lambda_4 y_1 e = 0 \tag{4.8}$$

where λ_i are constants,[5] and one should see w, y_1, f, e as local variables for a $\mathbb{C}^4$ neighborhood of the singularity in the ambient space. Generically this is a non-degenerate quadratic form on the ambient space variables, defining locally a conifold singularity. For later reference, note that the number of such singularities is given by the number of points in $e = f = 0$, or slightly more formally by the intersection of the homology classes of the divisors $[e] \cdot [f]$ on the base. Associated with these singularities there will be massless hypermultiplets coming from wrapped M2 branes, which will be the essential states in our discussion.

Deformation. Since the singularities are conifolds, we expect that there are two ways of smoothing out the singularities. The first is by deformation, i.e. changing the Calabi-Yau equation (4.6). Our only option is to consider deformations away from $a = 0$. This indeed modifies the analysis above in that a singularity would require $a = f = e = 0$, but for non-vanishing a and a two-dimensional base there is generically no solution to this system (by simple dimension counting), so there is no singularity anymore. An important observation for our purposes below is that under this deformation the two sections no longer exist independently, but they rather recombine into a unique global object. Setting $y_1 = 0$ in (4.2) gives

$$gw^2 + wtf + at^2 = 0\,, \tag{4.9}$$

which no longer factorizes globally. The two sections above still exist locally and can be found by solving for w, but there is a $\mathbb{Z}_2$ monodromy coming from going around zeros of the discriminant $t^2(f^2 - 4ag)$, which exchanges the two roots. This is thus a case with a bi-section, but no section. In the examples below the non-existence of a section can also

[5]These constants can be easily read from (4.6), but we only need that they are non-vanishing constants.

be easily verified using Oguiso's criteria [48, 49], we collect some of the relevant details in appendix B. All in all, this gives the first element of our pair, the deformed Calabi-Yau threefold $\mathcal{X}$.

Resolution. On the other hand, one can do a blow-up of the conifold in order to desingularize the geometry. A simple toric way of achieving this is by blowing up the $y_1 = w = 0$ point, which is the point of intersection of the conifolds with the fiber, as done in [18]. More concretely, we replace the fiber by the following GLSM:

$$\begin{array}{|c|ccccc|} \hline & y_1 & y_2 & w & t & s \\ \hline \mathbb{C}^*_1 & 1 & 1 & 2 & 0 & 0 \\ \mathbb{C}^*_2 & 0 & 0 & 1 & 1 & 0 \\ \mathbb{C}^*_3 & 1 & 0 & 1 & 0 & -1 \\ \hline \end{array} \tag{4.10}$$

The new Stanley-Reisner ideal is given by $\langle wy_1, wt, st, sy_2, y_1y_2\rangle$. Notice in particular that $w = y_1 = 0$ does not belong to the ambient space anymore. The Calabi-Yau hypersurface in this space is of degree $(4, 2, 1)$ and can be parametrized, matching with the proper transform of (4.6), by

$$\widetilde{\phi} \equiv gw^2s + wtP(sy_1, y_2) + t^2y_1Q'(sy_1, y_2) = 0\,. \tag{4.11}$$

The sections transform naturally under the blow-up. In particular, the $w = y_1 = 0$ section transforms to $s = 0$. Setting $s = 0$ in (4.11), and setting $t = y_2 = 1$ since they cannot vanish when $s = 0$, one gets

$$wf + y_1e = 0 \tag{4.12}$$

so this section maps to $(y_1, y_2, w, t, s) = (-f, 1, e, 1, 0)$. Let us denote this section by σ_0. We will take it to be our zero section, parametrizing the F-theory limit.

The other section is given by $y_1 = 0$. Plugging this into (4.11), and setting $w = y_2 = 1$, one gets

$$gs + tf = 0\,. \tag{4.13}$$

We thus find a second section at $(y_1, y_2, w, t, s) = (0, 1, 1, -g, f)$, which we denote by σ. We think of this section as generating a U(1) symmetry in the six-dimensional theory obtained by putting F-theory on $\mathbb{X}$, choosing σ_0 as the zero section.

So, as expected, deformation does not recombine the sections, but rather we stay with two independent sections of the fibration.

It is also not hard to see that the resulting space is generically non-singular, as one may have expected from the fact that we are considering the most general equation over the blown-up fiber. We denote the resulting space by $\mathbb{X}$.

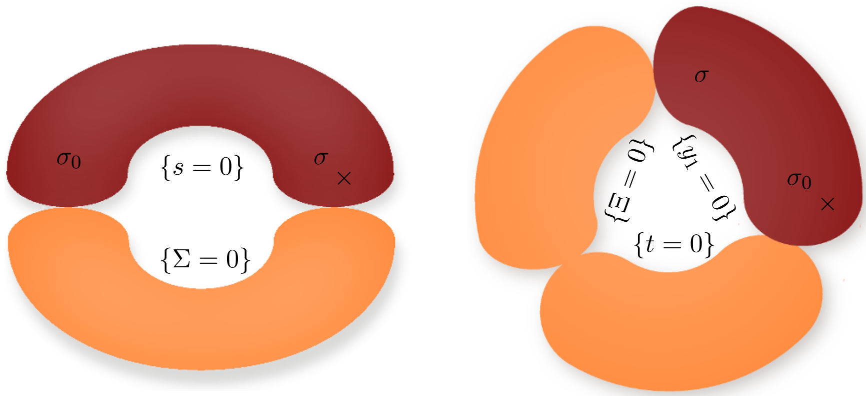

Figure 6. Schematic behavior of the fiber geometry over the two non-holomorphic loci. On the left, the locus $\{e = f = 0\}$ is depicted. σ_0 wraps the entire fiber component, while σ cuts out a single point. On the right, the locus $\{f = g = 0\}$ is shown, where σ becomes non-holomorphic and σ_0 cuts out a point in the same fiber component. Fiber components wrapped by a section are colored dark red.

Holomorphy of the sections. Looking at the sections we just found, we see that they are ill-defined over some points in the base. In particular, σ_0 is ill-defined over $f = e = 0$, since over these points σ_0 would be $(0, 1, 0, 1, 0)$, but $y_1 w$ is in the Stanley-Reisner ideal. Similarly, σ becomes ill-defined over $g = f = 0$, since st is in the Stanley-Reisner ideal. This is a hallmark of rationality of the sections (as opposed to holomorphy): the sections are not given by a single point in the fiber everywhere, but over some subspaces (where σ_0 and σ becomes ill-defined in our examples) they wrap components of the fiber.

It is not hard to be more explicit about the behavior of these sections on the problematic points. Setting $f = e = 0$, and $s = 0$, the Calabi-Yau equation (4.11) becomes identically satisfied, so the section at this point jumps in dimension. Similarly for σ, since at $y_1 = f = g = 0$ (4.11) is identically satisfied, so σ again jumps in dimension at these points.

Let us study the behavior of the elliptic fiber at these points more carefully. For $f = e = 0$, the Calabi-Yau equation becomes

$$s(gw^2 + wty_1P'(sy_1, y_2) + t^2y_1^2Q''(sy_1, y_2)) \equiv s\Sigma = 0 \tag{4.14}$$

where $P' = P/(sy_1)$, and $Q'' = Q'/(sy_1)$, which are homogeneous polynomials when f and e vanish, of degrees 1 and 2 respectively in the y_i. We see that at this locus the elliptic fiber degenerates into two components, given by $s = 0$ and $\Sigma = 0$. When $s = 0$ we can gauge fix $\mathbb{C}^*_1$ and $\mathbb{C}^*_2$ in (4.10) by setting $t = y_2 = 0$, so we end up with the y_1, w coordinates, with relative SRI $\langle wy_1\rangle$, and identified by the $\mathbb{C}^*$ action $(y_1, w) = (\lambda y_1, \lambda w)$. This is the usual description of $\mathbb{P}^1$, as one could have expected from the fact that $s = 0$ was the blow-up divisor. The curve Σ defines a degree $(4, 2, 2)$ divisor on the ambient space, and a simple adjunction computation gives then that Σ has genus 0, i.e. it is also a $\mathbb{P}^1$. More explicitly

$$\chi(\Sigma) = \int_\Sigma c_1(T\Sigma) = \int_A (c_1(TA) - \Sigma)\Sigma = -\int_A [0, 0, 1] \wedge [4, 2, 2] = 2\int_A [w] \wedge [s] = 2 \tag{4.15}$$

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where A denotes the ambient toric space (4.10), and on the second line we have denoted the divisor classes by their toric weights.

These two spheres intersect over a point: setting $s = 0$ (and thus $y_2 = t = 1$) in the equation for Σ we get:

$$gw^2 + wy_1P'(0, y_2) + y_1^2Q''(0, y_2) = 0\,. \tag{4.16}$$

This is a quadratic on the exceptional $\mathbb{P}^1$, which has exactly two solutions. So we recover the usual picture of the T^2 fiber degenerating into two spheres, touching at two points. The rational section σ_0 wraps one of the two sphere components, namely $s = 0$.

A similar analysis holds for σ. Setting $g = f = 0$ on (4.11) the Calabi-Yau equation factorizes as

$$y_1t(wsP'(sy_1, y_2) + tQ'(sy_1, y_2)) \equiv y_1t\,\Xi = 0\,. \tag{4.17}$$

We find that there are three components in the fiber. By the same kind of analysis as above we find that they are $\mathbb{P}^1$s: for $y_1 = 0$ and $t = 0$ this is immediate by looking to (4.10). One also has that $\Xi = 0$ is an equation of degree $(3, 1, 0)$, and an adjunction computation gives that it has genus 0.

The intersections between the three spheres can be computed easily, with the result that any two of the three spheres intersect at exactly one point. Our section σ wraps the $y_1 = 0$ component. A summary of the fiber geometry is contained in figure 6.

4.2 Physics of the conifold transition

The low energy description of the conifold transition is well understood, starting with the seminal paper by Strominger [50] (see also [51, 52], and [53] for a treatment specialized to M-theory on Calabi-Yau threefolds), so we will be brief here.

The basic physics mechanism in effective field theory language is simply a Coulomb/Higgs branch transition: at the conifold point there are a number of massless hypermultiplets, coming from M2 branes wrapped on the collapsed S^2 cycles. We can smooth the conifold points in two ways: deformation or resolution. On the resolved side the 2-spheres take finite size, and this corresponds to making the M2 states massive. In field theoretic terms, this mass terms are associated with the introduction of (geometry dependent) mass terms for the hypermultiplets. More in detail, in M-theory compactified on a smooth Calabi-Yau threefold $\mathbb{X}$, there are $n_H = h^{2,1}(\mathbb{X}) + 1$ hypermultiplets, and $h^{1,1}(\mathbb{X})$ U(1) gauge fields. A particular combination of these belongs to the gravity multiplet, and the other $n_V = h^{1,1}(\mathbb{X}) - 1$ U(1) fields belong to vector multiplets. These vector multiplets have a real bosonic scalar component. The size of the resolved 2-spheres (keeping the overall size of the Calabi-Yau threefold fixed) is precisely encoded in the values of these scalars, so resolving the conifold singularities corresponds to going into a Coulomb branch of the field theory.

On the other hand, there is a Higgs branch obtained by giving VEVs to the massless hypermultiplets. This corresponds to smoothing out the conifold singularities by complex deformations. Since the massless hypermultiplets are naturally charged under the U(1) symmetries (M2 branes couple electrically to C_3), giving a VEV will make some of the U(1) vector multiplets massive.

There is a simple relation between the counting of massless fields in the five-dimensional theory and the Hodge numbers of the spaces related by the conifold transition. Assume that there are P 2-spheres degenerating at P conifold points. Typically not all of these 2-spheres are linearly independent, but there are R homology relations between them (so $P - R$ independent classes vanish). Writing down the low energy effective field theory for the hypermultiplets at the conifold point, one can easily see [51, 52] that there are precisely R flat directions of the hypermultiplets, along which one can Higgs them. A generic such Higgsing will then give mass to $P - R$ vectors. All in all, M-theory on the resolved Calabi-Yau threefold $\mathbb{X}$ gives rise to a massless spectrum with $(n_H(\mathbb{X}), n_V(\mathbb{X})) = (h^{2,1}(\mathbb{X}) + 1, h^{1,1}(\mathbb{X}) - 1)$. At the conifold point, P extra hypers become massless: $(n_H^0, n_V^0) = (h^{2,1}(\mathbb{X}) + 1 + P, h^{1,1}(\mathbb{X}) - 1)$. Higgsing then removes $P - R$ hyper-vector pairs: $(n_H(\mathcal{X}), n_V(\mathcal{X})) = (h^{2,1}(\mathbb{X})+1+R, h^{1,1}(\mathbb{X})-1-P+R)$. On the other hand, these numbers are just $h^{2,1}(\mathcal{X}) + 1$ and $h^{1,1}(\mathcal{X}) - 1$, respectively, so we learn that the conifold transition acts on the Hodge numbers as

$$(h^{2,1}(\mathcal{X}), h^{1,1}(\mathcal{X})) = (h^{2,1}(\mathbb{X}) + R, h^{1,1}(\mathbb{X}) - P + R)\,. \tag{4.18}$$

This formula will provide a nice consistency check that we are identifying the geometry properly in our forthcoming examples (in our examples, $P - R = 1$, so $h^{1,1}(\mathbb{X}) - h^{1,1}(\mathcal{X}) = 1$). A simple quantity to check, in particular, is the difference in Euler numbers

$$\begin{aligned}\chi(\mathbb{X}) - \chi(\mathcal{X}) &= 2(h^{2,1}(\mathcal{X}) - h^{2,1}(\mathbb{X})) - 2(h^{1,1}(\mathcal{X}) - h^{1,1}(\mathbb{X})) \\ &= 2P\end{aligned} \tag{4.19}$$

giving the number of conifold points involved in the transition.

4.3 Explicit examples with base $\mathbb{P}^2$

Having described the general setup for our main class of examples, we are now ready to construct a number of examples of conifold transitions removing the section. For simplicity, we will stay with a $\mathbb{P}^2$ base.

Let us start on the deformed side $\mathcal{X}$. The set of Calabi-Yau threefolds T^2-fibered over $\mathbb{P}^2$ can be described as hypersurfaces on the toric ambient space described by the GLSM

	x_1	x_2	x_3	y_1	y_2	w	t
$\mathbb{C}_1^*$	1	1	1	0	a	b	0
$\mathbb{C}_2^*$	0	0	0	1	1	2	0
$\mathbb{C}_3^*$	0	0	0	0	0	1	1

(4.20)

The last four coordinates parametrize the fiber $\widehat{\mathbb{P}^{1,1,2}}$, while the first three coordinates parametrize the base $\mathbb{P}^2$. The fibration map $\pi\colon X \to \mathbb{P}^2$ simply "forgets" about the last four coordinates of any point in X. In principle the last four entries in the first row (the charges of y_1, y_2, w, t under $\mathbb{C}_1^*$) can be arbitrary integers, but it is easy to convince oneself

(a,b)	$h^{1,1}(\mathcal{X})$	$h^{2,1}(\mathcal{X})$	$\deg(a)$	$\deg(e)$	$\deg(f)$	$\deg(g)$
$(0,3)$	2	128	6	6	3	0
$(1,4)$	2	132	4	5	2	0
$(2,5)$	2	144	2	4	1	0
$(0,-2)$	3	59	1	1	3	5
$(0,-1)$	3	65	2	2	3	4
$(0,0)$	3	75	3	3	3	3
$(0,1)$	3	89	4	4	3	2
$(0,2)$	3	107	5	5	3	1
$(1,0)$	3	69	0	1	2	4
$(1,1)$	3	79	1	2	2	3
$(1,2)$	3	93	2	3	2	2
$(1,3)$	3	111	3	4	2	1
$(2,3)$	3	105	0	2	1	2
$(2,4)$	3	123	1	3	1	1
$(3,6)$	3	165	0	3	0	0
$(0,-3)$	6	60	0	0	3	6

Table 2. Hodge numbers and polynomials degrees for various fibrations over $\mathbb{P}^2$.

that by redefining (if necessary) the y_i and the $\mathbb{C}^*_i$, any such fibration can be brought to the canonical form (4.20), with $a \geq 0$.

The generic equation in these variables is given by (4.2). In order to have a Calabi-Yau threefold, (4.2) must be a homogeneous polynomial of degree $(3+a+b, 4, 2)$. Tracing the definitions above, this implies that the interesting coefficients of (4.2) are homogeneous functions on the x_i of degrees

$$\deg(a) = 3 - 3a + b \tag{4.21}$$
$$\deg(e) = 3 - 2a + b \tag{4.22}$$
$$\deg(f) = 3 - a \tag{4.23}$$
$$\deg(g) = 3 + a - b\,. \tag{4.24}$$

There are a finite number of allowed values for (a,b), obtained by imposing that all the coefficients of (4.2) be holomorphic functions on the x_i (in particular, there should be no poles). These conditions define a polygon in the (a,b) plane, as pointed out in [23, 24], and the different cases, given in table 2, correspond to integral points of this auxiliary polygon.

There are some interesting features in this table. Notice that the first three entries have $\deg(g) = 0$. Taking g a generic non-zero constant, we find that Q becomes a holomorphic section, since the $f = g = 0$ locus does not exist anymore. Similarly, for the $(0,-3)$ example the σ_0 section is holomorphic, and for the $(3,6)$ example both sections are holomorphic. In the rest of the cases both sections are rational.

The resolved side $\mathbb{X}$ is given by hypersurfaces on toric ambient spaces described by GLSMs of the form

	x_1	x_2	x_3	y_1	y_2	w	t	s
$\mathbb{C}^*_1$	1	1	1	0	a	b	0	0
$\mathbb{C}^*_2$	0	0	0	1	1	2	0	0
$\mathbb{C}^*_3$	0	0	0	0	0	1	1	0
$\mathbb{C}^*_4$	0	0	0	1	0	1	0	-1

(4.25)

As before, in principle we could have given a charge to s under $\mathbb{C}^*_1$, but there is always a way of redefining the fields and $\mathbb{C}^*$ symmetries in order to set this charge to 0. Imposing that the coefficients of (4.11) are sections of line bundles of non-negative degree on the $\mathbb{P}^2$ base, one finds 31 different possible values for (a,b). All those in table 2 are included, and in addition there are a few models which are only possible on the resolved side, since the blow-up fixes the coefficient of the y_2^4 term in Q to vanish, so there is one less constraint. We will only be interested in the ones coming from conifold transitions on $\mathcal{X}$.

Identifying the models in the canonical way, we can immediately compute the Hodge numbers of the resolved spaces using PALP, for instance, the results are given in table 3. Computing from here the expected number of conifold points, with the results shown in the last column of table 3, one sees easily by comparing with the values in table 2 that in all cases the expected number of conifold points precisely agrees with the expectation from the discussion given above:

$$\frac{1}{2}(\chi(\mathbb{X}) - \chi(\mathcal{X})) = \deg(e) \cdot \deg(f)\,. \tag{4.26}$$

In table 3 we summarize information about the models obtained by resolving the manifolds from table 2, including the chiral spectrum in six dimensions, obtained via the techniques described in [22, 38]. Here $H(\mathcal{R})$ denotes the net amount of chiral matter (six-dimensional hypers) in the representation $\mathcal{R}$. We denote the representation by $\mathbf{N}_m$, where $\mathbf{N}$ is the representation under the gauge group SU(2) (to be explained below), and m the U(1) charge. We define the divisor class generating the U(1) charge by [22]

$$D_{\mathrm{U}(1)} = 2\sigma - 2\sigma_0 - 4\pi^* c_1(TB) + E\,. \tag{4.27}$$

We have denoted $\pi\colon \mathbb{X} \to \mathbb{P}^2$ the fibration map, π^* its pullback to cohomology on X, σ, σ_0 denote the extra section and the zero section described above, and E is the divisor associated with the Cartan of SU(2). The single manifold with $h^{1,1}(\mathbb{X}) = 6$ has three divisors that do not descend from the ambient space and it is unclear what the full gauge group and matter spectrum are, so we will not analyze it here. Lastly, let us remark that we find that

$$H(\mathbf{1}_4) = \frac{1}{2}(\chi(\mathbb{X}) - \chi(\mathcal{X})) = [e] \cdot [f] \tag{4.28}$$

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(a,b)	$h^{1,1}(\mathbb{X})$	$h^{2,1}(\mathbb{X})$	P	$H(\mathbf{1}_2)$	$H(\mathbf{1}_4)$	$H(\mathbf{2}_1)$	$H(\mathbf{2}_3)$	$H(\mathbf{3}_0)$
$(0,3)$	3	111	18	144	18	0	0	0
$(1,4)$	3	123	10	140	10	0	0	0
$(2,5)$	3	141	4	128	4	0	0	0
$(0,-2)$	4	57	3	64	3	55	15	6
$(0,-1)$	4	60	6	76	6	52	12	3
$(0,0)$	4	67	9	90	9	45	9	1
$(0,1)$	4	78	12	106	12	34	6	0
$(0,2)$	4	93	15	124	15	19	3	0
$(1,0)$	4	68	2	72	2	56	8	3
$(1,1)$	4	76	4	86	4	48	6	1
$(1,2)$	4	88	6	102	6	36	4	0
$(1,3)$	4	104	8	120	8	20	2	0
$(2,3)$	4	104	2	90	2	38	2	0
$(2,4)$	4	121	3	108	3	21	1	0
$(3,6)$	3	165	0	108	0	0	0	0
$(0,-3)$	6	60	0	–	–	–	–	–

Table 3. Hodge numbers and chiral spectra for the resolved versions of the manifolds in table 2. All U(1) charges have been rescaled by 2. P denotes the expected number of conifold points, obtained from (4.19). The last entry in the table corresponds to a space with many non-torically realized divisors, so we will not analyze it here.

which strongly suggests that it is precisely the $\mathbf{1}_4$ multiplets that are involved in the conifold transition.

The existence of an SU(2) symmetry in the cases with $h^{1,1}(\mathbb{X}) > 3$ can be argued for as follows. Consider the $g = 0$ locus on the base (this is only possible if $\deg(g) > 0$). Over this divisor, the Calabi-Yau equation becomes

$$\widetilde{\phi}|_{g=0} = t(wP + ty_1Q') \equiv t\Lambda = 0\,. \tag{4.29}$$

We see that over this divisor on the base the T^2 factorizes. The $t = 0$ piece defines a $\mathbb{P}^1$, and it is not hard to prove that $\Lambda = 0$ is also a $\mathbb{P}^1$, intersecting $t = 0$ at two points. This is the familiar affine SU(2) structure over a zero of the discriminant, so we expect a SU(2) enhancement over $g = 0$. A short computation shows, in addition, that the section σ_0 intersects Λ at a point, and σ intersects $t = 0$ at a point. Since we chose σ_0 as our zero section, we interpret the component not intersecting it, namely $t = 0$, as the one associated with the W bosons enhancing the gauge symmetry to SU(2). All in all, we learn that E in (4.27) is just $\{t = 0\} \cap \{\widetilde{\phi} = 0\}$, or $[t]$ in brief (abusing notation slightly).

We are in fact in a position to compute the charges of some of the multiplets in table 3 from first principles. We start by discussing the $\mathbf{1}_4$ multiplets, which are the main actors

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in the conifold transition. The other representations can be obtained analogously, with some extra effort. Since these representations are less directly relevant for the conifold transition, we demote their discussion to appendix A.

We claim that the $\mathbf{1}_4$ multiplets comes from $f = e = 0$. We have explained above that when $f = e = 0$ the fiber becomes split into two components, given by $\{s = 0\} \cup \{\Sigma = 0\}$. Since st belongs in the Stanley-Reisner ideal, the hyper wrapping $s = 0$ has no charge under the SU(2) symmetry. Its charge under the U(1) is given by

$$Q_{\mathrm{U}(1)} = \mathcal{C}_s \cdot (2\sigma - 2\sigma_0 - 12[x_1] + [t])\,. \tag{4.30}$$

We have denoted by $\mathcal{C}_s$ the component of the fiber over $f = e = 0$ given by $s = 0$, and we used the fact that $[x_1]$ is the pullback of the hyperplane on $\mathbb{P}^2$. Since $x_1 = 0$ will generically not intersect $f = e = 0$, we have $\mathcal{C}_s \cdot [x_1] = 0$. Similarly, since st is in the Stanley-Reisner ideal, $\mathcal{C}_s \cdot [t] = 0$. We already determined above that σ intersects $\mathcal{C}_s$ at a point, so $\mathcal{C}_s \cdot \sigma = 1$. On the other hand, σ_0 becomes rational at $f = e = 0$, so the calculation is less straightforward. Consider the total class of the (factorized) T^2 fiber, given by $\mathcal{C}_s + \mathcal{C}_\Sigma$, with the last component being the $\Sigma = 0$ locus. Since the total fiber can move as a holomorphic divisor into a smooth T^2, which intersects σ_0 at a point, it must be the case that $(\mathcal{C}_s + \mathcal{C}_\Sigma) \cdot \sigma_0 = 1$. On the other hand, on the factorized locus it is clear that $\mathcal{C}_\Sigma \cdot \sigma_0 = 2$ (the two points where the $\mathbb{P}^1$ components touch). So we conclude $\mathcal{C}_s \cdot \sigma_0 = -1$. Substituting all this into (4.30) we obtain $Q_{\mathrm{U}(1)} = 4$, as claimed.

4.4 Chern-Simons terms

In this final subsection, we confirm geometrically that the Chern-Simons terms of the theory obtained by compactifying M-theory on $\mathcal{X}$ are in fact related to the Chern-Simons terms of M-theory on $\mathbb{X}$ as described in equation (3.40). Instead of delving into concrete examples right away and showing explicitly that this prescription is correct on a case by case basis, let us make a general geometric argument first. As the Chern-Simons terms of the five-dimensional models are given in terms of intersection numbers, we need to understand how the intersection form on $\mathcal{X}$ is obtained from the intersection form of $\mathbb{X}$. Fortunately for us, this was studied long ago, see for example [53]. Denoting by $\mathcal{K}_i$, $i = 1, \ldots, h^{1,1}(\mathbb{X})$ a basis of the Kähler cone on $\mathbb{X}$ and by $\widetilde{\mathcal{K}}_i$, $i = 1, \ldots, h^{1,1}(\mathcal{X})$ the corresponding Kähler cone basis on $\mathcal{X}$, we choose the $\mathcal{K}_i$ such that under the conifold transition they are mapped to divisors on $\mathcal{X}$ according to

$$\mathcal{K}_i \mapsto \begin{cases} \widetilde{\mathcal{K}}_i & \text{if } i \leq h^{1,1}(\mathcal{X}) \\ 0 & \text{otherwise.} \end{cases} \tag{4.31}$$

Then the intersection numbers of the $\widetilde{\mathcal{K}}_i$ on $\mathcal{X}$ are the same as of the $\mathcal{K}_i$ on $\mathbb{X}$, i.e.

$$\widetilde{\mathcal{K}}_i \cdot \widetilde{\mathcal{K}}_j \cdot \widetilde{\mathcal{K}}_k = \mathcal{K}_i \cdot \mathcal{K}_j \cdot \mathcal{K}_k\,. \tag{4.32}$$

Put differently, the intersection form on $\mathcal{X}$ is obtained by restricting the intersection form on $\mathbb{X}$. That is, given expressions for the volumes $\mathcal{V}$ and $\widetilde{\mathcal{V}}$ of $\mathbb{X}$ and $\mathcal{X}$ in terms of the

Kähler parameters v^i and $\tilde{v}^i$, one has that

$$\widetilde{\mathcal{V}} = \mathcal{V}(v^1 = \tilde{v}^1, \dots, v^{h^{1,1}(\mathcal{X})} = \tilde{v}^{h^{1,1}(\mathcal{X})}, 0, \dots) . \tag{4.33}$$

Presented with this simple relation between triple intersections on $\mathbb{X}$ and $\mathcal{X}$, let us now return to the discussion of the Chern-Simons terms of M-theory on $\mathcal{X}$. Given two independent sections on $\mathbb{X}$ we know that only a certain linear combination $D_{\mathrm{U}(1)}$ is left untouched by the conifold transition — the other U(1)-divisor is eliminated as the corresponding gauge field gains a mass term. Identifying the surviving U(1) amounts to making the same clever choice of basis as for the $\mathcal{K}_i$ above. Then, equation (4.32) tells us that the intersection numbers of the surviving U(1)-divisor are *precisely the same* as on the resolved side. Therefore, we are left with two questions to examine in our specific examples, namely:

1. Which divisor $D_{\mathrm{U}(1)}$ survives the conifold transition?
2. Why is $D_{\mathrm{U}(1)} \cdot c_2(\mathbb{X}) = \tilde{D}_{\mathrm{U}(1)} \cdot c_2(\mathcal{X})$?

In subsection 3.3 we gave a general argument for how to identify $D_{\mathrm{U}(1)}$ and, in fact, we will show explicitly that this prescription does in fact select the correct divisor for the examples below. The second point is more difficult to answer generally, but we can confirm it on a case by case basis.

Put in a nutshell, we have explained generally that after a clever change of basis the Chern-Simons terms of the theories corresponding to $\mathbb{X}$ and $\mathcal{X}$ are simply obtained by "dropping" the massive U(1). Of course, one can also confirm this statement explicitly through the calculation of intersection numbers and in the remainder of this section we will perform an example calculation.

4.4.1 A close look at the model with $(a, b) = (0, 3)$

For concreteness, let us study the manifold with $(a, b) = (0, 3)$, beginning on the resolved side. We find that the Mori cone is generated by the three curves

	x_1	x_2	x_3	y_1	y_2	w	s
$\mathcal{C}_1$	1	1	1	−3	0	0	3
$\mathcal{C}_2$	0	0	0	−1	1	0	2
$\mathcal{C}_3$	0	0	0	1	0	1	−1

(4.34)

and we can hence choose

$$\mathcal{K}_1 = x_1 \,, \quad \mathcal{K}_2 = y_2 \,, \quad \mathcal{K}_3 = w \tag{4.35}$$

as a basis of the Kähler cone satisfying $\mathcal{K}_i \cdot \mathcal{C}^j = \delta_i^j$. Expressing the Kähler form $J = \sum_{i=1}^3 v^i [\mathcal{K}_i]$ in terms of two-forms dual to these divisors, one finds that the overall volume of the Calabi-Yau can be written as

$$\mathcal{V} = (v^1)^2 v^2 + \frac{3}{2}(v1)^2 v^3 + 6 v^1 v^2 v^3 + \frac{15}{2} v^1 (v^3)^2 + 9 v^2 (v^3)^2 + \frac{21}{2}(v^3)^3 . \tag{4.36}$$

Let us turn to the two divisors generating the U(1) symmetries in five dimensions. One is obtained by appropriately shifting the zero section [18, 22], while the other can be computed by applying the Shioda map to the other section. Naturally, a different choice of zero section will lead to interchanged results for the divisor expansions. Since the resulting physics remain unaffected, we choose the divisor $s = 0$, or σ_0 in the notation of subsection 4.1, as the zero section during the rest of this discussion. Note that in this particular basis the divisors generating the two U(1)s have the expansion

$$D_0 = \frac{9}{2}\mathcal{K}_1 + 2\mathcal{K}_2 - \mathcal{K}_3\,, \quad D_1 = -24\mathcal{K}_1 - 6\mathcal{K}_2 + 4\mathcal{K}_3\,. \tag{4.37}$$

Now we discuss the deformed manifold $\mathcal{X}$. Its Mori cone is spanned by

	x_1	x_2	x_3	y_1	y_2	w
$\tilde{\mathcal{C}}_1$	1	1	1	0	0	3
$\tilde{\mathcal{C}}_2$	0	0	0	1	1	2

(4.38)

and a good choice of Kähler basis is for example given by

$$\tilde{\mathcal{K}}_1 = x_1\,, \quad \tilde{\mathcal{K}}_2 = y_2\,. \tag{4.39}$$

Then the volume of the deformed manifold is

$$\tilde{\mathcal{V}} = (\tilde{v}^1)^2\tilde{v}^2\,. \tag{4.40}$$

Obviously, the intersection rings of $\mathbb{X}$ and $\mathcal{X}$ are related as in equation (4.33), with $\mathcal{K}_3$ the divisor eliminated during the conifold transition. Up to an overall rescaling, there is hence a unique combination of D_0 and D_1 that is left invariant under the conifold map, namely the one not containing $\mathcal{K}_3$. It is[6]

$$D_{\mathrm{U}(1)} \sim 4D_0 + D_1\,. \tag{4.41}$$

Since we rescaled the six-dimensional U(1) divisor on $\mathbb{X}$ by $\lambda = 2$, this is precisely the expression that we expect from equation (3.47). Lastly, we can check by explicit computation that $D_{\mathrm{U}(1)} \cdot c_2(\mathbb{X}) = \widetilde{D}_{\mathrm{U}(1)} \cdot c_2(\mathcal{X})$.

4.4.2 A close look at the model with $(a, b) = (0, -2)$

As a second example, we repeat the analysis for one of the models that contain an additional SU(2) factor to show that the above discussion is independent of the existence of additional gauge group factors. Again, we begin with the resolved manifold $\mathbb{X}$, whose Mori cone is this time spanned by the curves

	x_1	x_2	x_3	y_1	y_2	w	s	t
$\mathcal{C}_1$	1	1	1	0	0	−2	0	0
$\mathcal{C}_2$	0	0	0	1	1	0	0	−2
$\mathcal{C}_3$	0	0	0	1	0	1	−1	0
$\mathcal{C}_4$	0	0	0	−1	0	0	1	1

(4.42)

[6]Note that in subsection 3.3 we denoted the U(1)-divisor remaining massless by $\widetilde{D}_0$. Here we call it $D_{\mathrm{U}(1)}$ to emphasize that it not necessarily a divisor on $\mathcal{X}$.

and we pick

$$\mathcal{K}_1 = x_1\,, \quad \mathcal{K}_2 = y_2\,, \quad \mathcal{K}_3 = t + 2y_2\,, \quad \mathcal{K}_4 = w + 2x_1 \tag{4.43}$$

as the basis of the Kähler cone. The volume of the resolved manifold is then

$$\begin{aligned}\mathcal{V} = (v^1)^2v^2 + 2(v^1)^2v^3 + 5v^1v^2v^3 + 5v^1(v^3)^2 + 5v^2(v^3)^2 + \frac{10}{3}(v^3)^3 + \frac{3}{2}(v^1)^2v^4 \\ + 5v^1v^2v^4 + 10v^1v^3v^4 + 10v^2v^3v^4 + 10(v^3)^2v^4 + \frac{7}{2}v^1(v^4)^2 \\ + 5v^2(v^4)^2 + 10v^3(v^4)^2 + \frac{7}{3}(v^4)^3\,. \end{aligned} \tag{4.44}$$

Choosing $\sigma_0 = \{s = 0\}$ as zero section and expanding the U(1) divisors of the five-dimensional theory in a basis of $\mathcal{K}_i$ one finds

$$D_0 = \frac{3}{2}\mathcal{K}_1 + \mathcal{K}_3 - \mathcal{K}_4\,, \quad D_1 = -12\mathcal{K}_1 - 3\mathcal{K}_3 + 4\mathcal{K}_4\,. \tag{4.45}$$

Additionally, there is a third U(1) which is enhanced to the non-Abelian SU(2) factor in the F-theory limit. We denote it by E and its expansion reads

$$E = -2\mathcal{K}_1 + \mathcal{K}_2\,. \tag{4.46}$$

Changing to the deformed manifold $\mathcal{X}$ corresponding to F-theory with a massive U(1), we find that its Mori cone is generated by

	x_1	x_2	x_3	y_1	y_2	w	t
$\tilde{\mathcal{C}}_1$	1	1	1	0	0	-2	0
$\tilde{\mathcal{C}}_2$	0	0	0	1	1	0	-2
$\tilde{\mathcal{C}}_3$	0	0	0	0	0	1	1

(4.47)

and we parametrize the Kähler form in terms of two-forms Poincaré-dual to

$$\tilde{\mathcal{K}}_1 = x_1\,, \quad \tilde{\mathcal{K}}_2 = y_2\,, \quad \tilde{\mathcal{K}}_3 = t + 2y_2\,. \tag{4.48}$$

The volume of $\mathcal{X}$ is given by

$$\tilde{\mathcal{V}} = (\tilde{v}^1)^2\tilde{v}^2 + 2(\tilde{v}^1)^2\tilde{v}^3 + 5\tilde{v}^1\tilde{v}^2\tilde{v}^3 + 5\tilde{v}^1(\tilde{v}^3)^2 + 5\tilde{v}^2(\tilde{v}^3)^2 + \frac{10}{3}(\tilde{v}^3)^3 \tag{4.49}$$

and one can see that it is obtained by restricting the volume of the resolved phase according to

$$\tilde{\mathcal{V}} = \mathcal{V}|_{v^4=0, v^i=\tilde{v}^i}\,. \tag{4.50}$$

Consequently, we see that the above choice of $\mathcal{K}_i$ is again a good one in the sense of equations (4.31) and (4.33) and one transitions from $\mathbb{X}$ to $\mathcal{X}$ by dropping $\mathcal{K}_4$. Since equation (4.46) does not contain $\mathcal{K}_4$, we observe that it is left untouched by the conifold transition and does not take part in the mixing involving the remaining two U(1)s.

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(a,b)	V	H_{neutral}	$H(\mathbf{1}_2)$	$H(\mathbf{1}_4)$	$\tilde{k}_0$	$\tilde{k}_{000}$
$(0,3)$	1	112	144	18	-168	432
$(1,4)$	1	124	140	10	-128	304
$(2,5)$	1	142	128	4	-80	208
$(3,6)$	1	166	108	0	-24	144

Table 4. Spectra and Chern-Simons coefficients of $\widetilde{A}^0$ for the models with two sections and $h^{1,1} = 3$. Here, the Chern-Simons terms are obtained from the geometry and can be shown to match the field theory computation. All U(1) charges have been rescaled by 2.

Requiring again that the surviving U(1) must not contain $\mathcal{K}_4$, one finds that, up to an overall rescaling, it is given by

$$D_{\mathrm{U}(1)} = 4D_0 + D_1 \,, \tag{4.51}$$

which, as before, matches the prescription of (3.47) with $\lambda = 2$. In summary, we find that the discussion of the case with additional SU(2) gauge symmetry is almost identical to the one of the simpler models with only Abelian gauge groups. As before, we identify a curve shrinking to zero volume in the conifold limit. The intersection form of the deformed model is then obtained by dropping the divisor dual to that curve from the intersection form of the resolved phase. As the SU(2) Cartan divisor does not contain the divisor that is eliminated in the conifold transition, it does not mix with any of the other U(1)s during the conifold transition. Finally, one can again confirm that $D_{\mathrm{U}(1)} \cdot c_2(\mathbb{X}) = \widetilde{D}_{\mathrm{U}(1)} \cdot c_2(\mathcal{X})$, thereby showing that the Chern-Simons terms corresponding to the higher curvature terms are matched as well.

4.4.3 Explicit formulas for the Chern-Simons terms

Technically, the previous discussion already ensures the matching of the Chern-Simons terms as discussed in section 3.3. Nevertheless, it may be illuminating to consider the discussion from a different angle. Let us therefore evaluate formulas (3.44) and (3.45) for the examples at hand and show that they predict the correct intersection numbers. Turning the discussion around, one can also use these relations to *compute the spectrum of* $\mathcal{X}$ without making use to the resolved manifold $\mathbb{X}$.

This time, we restrict ourselves to models with purely Abelian gauge group, where we know the spectrum to consist of $\mathbf{1}_2$ and $\mathbf{1}_4$ states. Assuming furthermore that

$$l_{\mathbf{1}_2} = 0 \,, \qquad l_{\mathbf{1}_4} = 1 \tag{4.52}$$

as is the case when $\mathbb{X}$ has a non-holomorphic zero section (corresponding to σ_0 as above), the formulas for $\tilde{k}_{000}$ and $\tilde{k}_0$ simplify to

$$\begin{aligned} \tilde{k}_{000} = {} & \frac{m^3}{120}\,(H - V - T - 3) \\ & + \frac{1}{4} H(\mathbf{1}_2)\left(-4n^2 m + 16 n^3\,\mathrm{sign}(\mathbf{1}_2)\right) \\ & + \frac{1}{4} H(\mathbf{1}_4)\left(-4m^3 - 208 n^2 m + (384 n^3 + 48 n m^2)\,\mathrm{sign}(\mathbf{1}_4)\right) \,. \end{aligned} \tag{4.53}$$

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and

$$\begin{aligned}\tilde{k}_0 = &\frac{m}{6}\,(H - V + 5T + 15) \\ &+ H(\mathbf{1}_2)(-2n\,\mathrm{sign}(\mathbf{1}_2)) + H(\mathbf{1}_4)\,(2m - 12n\,\mathrm{sign}(\mathbf{1}_4))\ .\end{aligned} \tag{4.54}$$

To be as concrete as possible, we plug in $n = -1$ and $m = 4$ as we found above and use that for these manifolds $\mathrm{sign}(\mathbf{1}_2) = \mathrm{sign}(\mathbf{1}_4) = -1$ and $T = 0$ to find

$$\tilde{k}_{000} = \frac{8}{15}\,(H - V - 3) + 16H(\mathbf{1}_4) \tag{4.55}$$

$$\tilde{k}_0 = \frac{2}{3}(H - V + 15) - 2H(\mathbf{1}_2) - 4H(\mathbf{1}_4)\,. \tag{4.56}$$

Evaluating the formulas, one easily confirms that they indeed match the intersection numbers given in table 4. Note that table 4 contains the spectra of the F-theory models on the resolved manifolds $\mathbb{X}$. However, they can easily be translated to the case of a massive U(1) corresponding to F-theory on $\mathcal{X}$. F-theory on $\mathcal{X}$ has $H_{\mathrm{neutral}} - 1$ neutral hypermultiplets and $V = 0$ massless vectors as shown in figure 7. In six dimensions, the charged spectrum is the same on $\mathbb{X}$ and $\mathcal{X}$ with the difference that the U(1) field in F-theory on $\mathcal{X}$ is massive. However, upon doing the fluxed circle reduction to five dimensions, the $\mathbf{1}_4$ states with KK-level $\hat{n} = -1$ are neutral under the mixed massless U(1) gauge field $\widetilde{A}^0$ and must therefore be counted as additional neutral states not counted by $h^{2,1}(\mathcal{X})$.

We remark that these are the same results as one would get by starting with the conjectured six-dimensional F-theory set-up with a massive U(1). In fact, by computing the Mori cones of $\mathbb{X}$ and $\widetilde{X}$ one can show that the sign functions for the states $\mathbf{1}_2$ and $\mathbf{1}_4$ agree in the deformed and the resolved phases.

Finally, let us comment on directly computing spectra of F-theory models $\mathcal{X}$ without section. In the examples studied, we gained an computational advantage by finding models $\mathbb{X}$ with section that are related to $\mathcal{X}$ by conifold transitions. Ideally, however, one would like to compute the spectra of F-theory on $\mathcal{X}$ without making this detour. In general, this is going to be more difficult due to the fact that there are less divisors on $\mathcal{X}$ and therefore less intersection numbers to extract information from even though the spectra are equally complicated. As it turns out, for cases with a single U(1) there are generally more unknown variables than equations obtained from matching the Chern-Simons terms. However, if one also requires all anomalies to be canceled, it *is possible* to compute the spectra directly from $\mathcal{X}$ for the cases presented here. Incorporating these methods into a general approach by extending the variety of models studied here seems to be a promising direction of study.

5 Conclusions

In this paper we studied the effective physics of F-theory compactifications on elliptically fibered Calabi-Yau threefolds that do not have sections, but instead admit a bi-section. Applying M- to F-theory duality, we found that an ordinary circle reduction of a six-dimensional theory without U(1) gauge symmetries is not sufficient to match M-theory compactified to five dimensions. Instead, we claimed that there exists a six-dimensional

U(1) symmetry made massive by a geometric Stückelberg mechanism. Nevertheless, a simple circle reduction of this putative effective theory in six dimensions is not yet sufficient to achieve a match with M-theory either. Due to the absence of a section, non-trivial NS-NS and R-R fluxes appear along the circle direction used in the compactification to five-dimensional and an axionic degree of freedom is shift-gauged by the respective Kaluza-Klein vector. Caused by this additional gauging, the Kaluza-Klein vector and the massive U(1) vector mix in the fluxed circle compactification and a linear combination of both vectors remains massless in the five-dimensional effective theory. Geometrically, this massless U(1) vector is identified with the bi-section of the genus-one fibration.

Having treated such set-ups generally, we presented a class of example geometries in section 4. In this class of examples, we found that one can employ a conifold transition to pass from an F-theory model without section to one that does admit a section. Geometrically, the manifolds with bi-section correspond to deformations of the singular conifold points, while one obtains the manifolds with sections by resolving the conifold singularities. As expected from the vast physics literature on this subject, we confirm that physically, a certain set of states becomes massless during the conifold transition and Higgs one of the two massless five-dimensional U(1) gauge fields.

For completeness we have combined the figures of section 2, section 3.2, section 3.3, and section 4 into figure 7, which summarizes the relations between all the theories discussed in this paper.

Throughout the entire paper, we made extensive use of the information contained in the Chern-Simons terms of the different five-dimensional theories that are generated by integrating out charged massive matter fields. In order to perform validity checks of the proposed F-theory models, we computed the matter spectra of the five-dimensional theories corresponding to manifolds with section and tracked the Chern-Simons terms through the transition to the manifolds that possess solely multi-sections. Eventually, however, we were able to propose how to use the Chern-Simons terms to *directly compute* the matter spectra with respect to the massive U(1)s. The absence of an additional divisor caused by the massiveness of the U(1) appears to imply that one has somewhat less control over F-theory models without section. Fortunately, though, it seems that one can use anomaly cancelation to nevertheless compute the spectra without needing a conifold transition to a model with section.

5.1 Open questions and future directions of study

Since the study of F-theory compactifications without sections is still a fairly unexplored topic, there exists a plethora of ways to extend the results of [13, 14] and this work. Given that the focus of these papers has been on the study of models with bi-sections, it would certainly be desirable to have comparable control or at least access to a similar number of example geometries with multi-sections of higher degree.

As discussed above, all of our example geometries in this paper are connected to Calabi-Yau manifold with two sections by straightforward conifold transition. That this could be a general feature of such genus-one fibrations is an enticing prospect. In particular, one can imagine that F-theory on genus-one fibrations with 3- or 4-sections could be linked to

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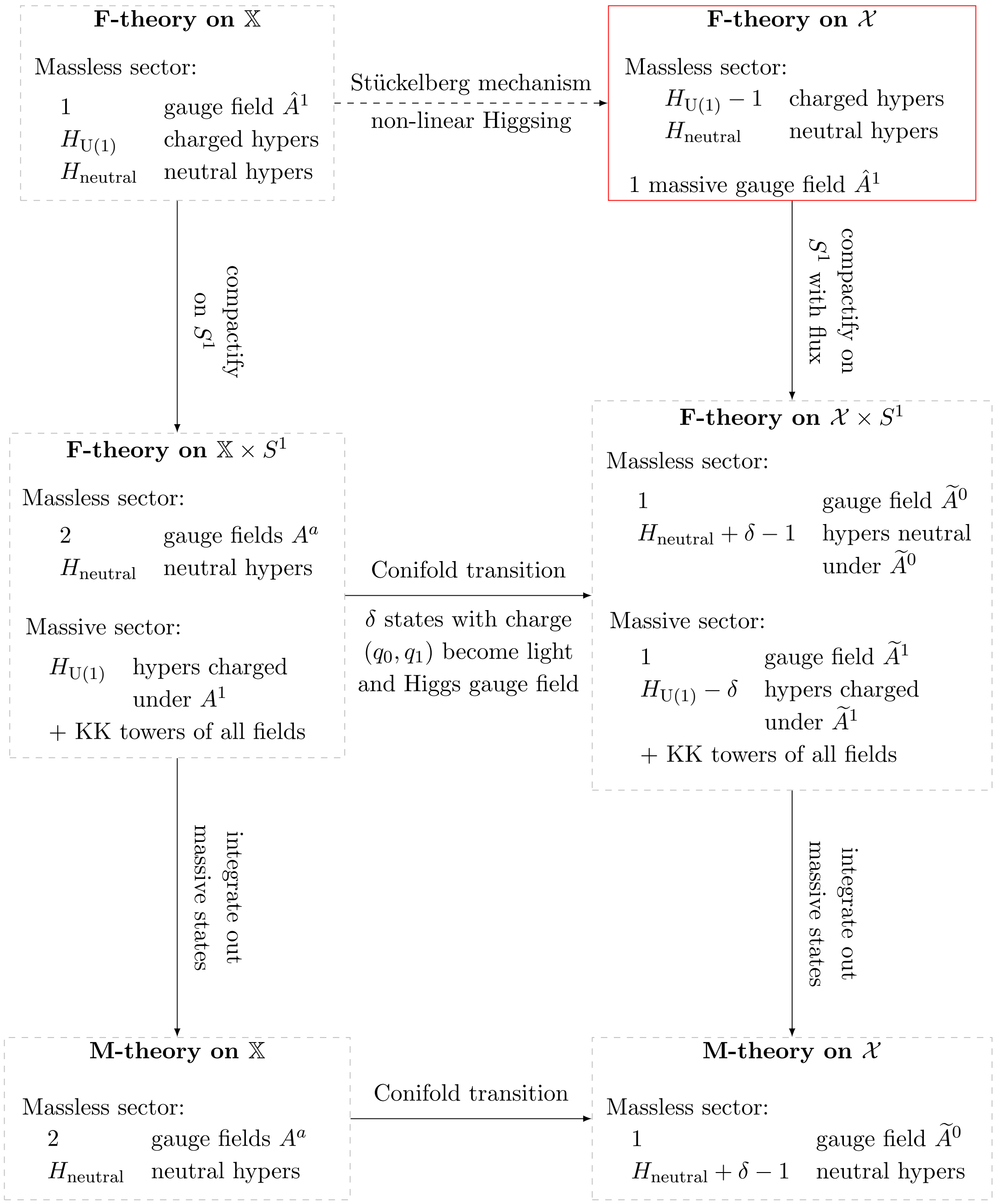

Figure 7. A comprehensive summary of relations between the different theories and their spectra.

F-theory models with multiple independent 1-sections by performing not one, but several such conifold transitions, passing through models with, say, a 1-section and a 2-section in intermediate steps.

Naturally, as just mentioned above, it would nevertheless be most convenient to access physical observables of F-theory on manifolds without section directly — that is without using additional related manifolds such as the Jacobians or the manifolds obtained here by conifold transition. While we have demonstrated for the explicit models studied above that this can be achieved, it still needs to be shown that such an approach can be employed also for arbitrary gauge groups. Developing a general framework for computing matter spectra under the massive U(1)s or determining other physical observables would therefore certainly be a promising direction of research.

Ultimately, to make contact with realistic F-theory models, one should extend the models studied here and in [13, 14] to Calabi-Yau fourfolds without section. In principle, it is completely straightforward to take the class of models studied in section 4 and fiber the genus-one curves presented there over a 3-complex-dimensional base instead. However, it would be interesting to examine whether there exist additional features in F-theory compactifications to 4d and understand, for example, whether the (non-)existence of certain Yukawa points has an impact on the states taking part in the conifold transitions. In this context it may also prove useful to compute not only the chiral indices of the 4d matter states, but rather their exact multiplicities using the formalism recently developed in [54].

Lastly, we note that over the past years considerable effort (see for example [55–59]) has been made to systematically investigate and classify six-dimensional supergravity models obtained from F-theory. In this approach, one usually considers maximally Higgsed gauge groups and focuses on the remaining unbroken gauge group that a given base manifold requires the overall fibration to have. At first sight, one might therefore expect not to detect the presence of the sort of massive U(1)s treated in this paper. It would be interesting to see whether there exists a way of nevertheless extracting such information.

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A Geometric description of the matter multiplets in $\mathbb{X}$

For the purposes of understanding the conifold transition, it was sufficient to understand the $\mathbf{1}_4$ states in table 3. It is nevertheless interesting and somewhat illuminating to describe the geometric origin of the rest of the matter multiplets in the six-dimensional theory arising from F-theory on $\mathbb{X}$.

We start with the $\mathbf{1}_2$ multiplets. In fact, the relevant curves have already been described in the $h^{1,1} = 3$ cases explicitly in [18] (under the names $\mathcal{T}_n$, $0 \leq n \leq 3$). We now review the discussion in that paper (using a slightly different approach). Let us assume $f \neq 0$. We want to understand under which conditions (4.11) factorizes into two $\mathbb{P}^1$s. This happens whenever the Calabi-Yau equation factorizes as

$$\widetilde{\phi} = (w + B)(ws + C) = 0 \tag{A.1}$$

for B, C to be determined. For simplicity we restrict ourselves to the case with $\deg(g) = 0$, and set $g = 1$. In this case, an easy argument shows that a holomorphic redefinition of w allows one to set $\alpha = \beta = 0$ in (4.3). In what follows we will implicitly perform such a redefinition.

Expanding (A.1), and comparing with (4.11), we immediately conclude that

$$\begin{aligned} BC &= y_1 Q' \\ C + sB &= f y_2^2 \,. \end{aligned} \tag{A.2}$$

By homogeneity and holomorphy, the most general form for B is given by

$$B = F y_1^2 s + G y_1 y_2 \tag{A.3}$$

with F, G polynomials in the x_i variables of the appropriate degree. (A term linear in w is also possible, but this can be reabsorbed in a redefinition of w.) Expanding the equations, and comparing order by order, we arrive at the equations

$$b = -F^2 \tag{A.4}$$

$$c = -2FG \tag{A.5}$$

$$d = Ff - G^2 \tag{A.6}$$

$$e = fG \tag{A.7}$$

which can be solved by

$$\begin{aligned} G &= \frac{e}{f} \\ F &= \frac{1}{f^3}(df^2 + e^2) \end{aligned} \tag{A.8}$$

as long as

$$\begin{aligned} b &= -\frac{1}{f^6}(d^2 f^4 + 2df^2e^2 + e^4) \\ c &= -\frac{2}{f^4}(df^2 e + e^3) \,. \end{aligned} \tag{A.9}$$

The $\mathbf{1}_2$ multiplets live at the points in the base where this equation is satisfied. In order to count these points, we multiply the whole equation by appropriate powers of f (recall that $f \neq 0$ by assumption), obtaining the equations

$$\begin{aligned} P_1 &\equiv bf^6 + d^2 f^4 + 2df^2e^2 + e^4 = 0 \\ P_2 &\equiv cf^4 + 2df^2 e + 2e^3 = 0 \,. \end{aligned} \tag{A.10}$$

This set of equations has $(3\deg(e))(4\deg(e)) = 12\deg(e)^2$ solutions. Not all of these solutions correspond to $\mathbf{1}_2$ states, though, some solutions come from $f = e = 0$, which as discussed in section 4 correspond to $\mathbf{1}_4$ multiplets instead. Each one of the solutions of $f = e = 0$ contributes $\deg_e(\mathrm{Res}_f(P_1, P_2)) = 16$ spurious solutions to (A.10) (see [21]), so the final count for $\mathbf{1}_2$ multiplets is given by

$$H(\mathbf{1}_2) = 12\deg(e)^2 - 16\deg(f)\cdot\deg(e)\,. \tag{A.11}$$

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It is easy to check that this formula gives the right values for the entries with $\deg(g) = 0$ in table 3.

Over the solutions of (A.10) with $f \neq 0$ in the base, the elliptic fiber factorizes into the curves

$$\begin{aligned} c_B &= \{w + F y_1^2 s + G y_1 y_2 = 0\} \\ c_C &= \{ws + f y_2^2 - F(sy_1)^2 - G(sy_1)y_2 = 0\}\,. \end{aligned} \tag{A.12}$$

The claim is that the hypermultiplets coming from wrapping $M2$ branes on these curves have charge 2 under (4.27). Notice first that, since we are assuming $f \neq 0$, both sections are holomorphic, and in particular $(c_B + c_C) \cdot \sigma_0 = (c_B + c_C) \cdot \sigma = 1$, since the two components of the fiber, taken together, span the class of the elliptic fiber. By the same token, the intersection is transversal, so necessarily one of the intersections vanishes, and the other is equal to 1. More explicitly, an easy calculation gives

$$c_B \cdot \sigma_0 = c_C \cdot \sigma = 1\,, \tag{A.13}$$

$$c_B \cdot \sigma = c_C \cdot \sigma_0 = 0\,. \tag{A.14}$$

In addition, it is clear that $c_B \cdot [x_1] = c_C \cdot [x_1] = 0$, since the curves are localized over points in the base $\mathbb{P}^2$, and for the $g \neq 0$ case that we are considering there is no intersection with the non-abelian divisor. All in all, we obtain that $Q_{\mathrm{U}(1)} = 2$.

We now consider $H(\mathbf{2}_3)$. We claim that these hypers come from the contracting spheres at $f = g = 0$. As discussed above, over this locus the T^2 fiber decomposes into three $\mathbb{P}^1$ components. We denote these components by $\mathcal{C}_t$, $\mathcal{C}_{y_1}$ and $\mathcal{C}_\Xi$, and claim that the $\mathbf{2}_3$ hypers come from $\mathcal{C}_{y_1}$ and $\mathcal{C}_\Xi$ (the $M2$ states wrapping $\mathcal{C}_t$ are rather associated with W bosons of SU(2)).

Consider first $\mathcal{C}_\Xi$. From the discussion above, we know that $\mathcal{C}_\Xi \cdot \sigma = 1$, $\mathcal{C}_\Xi \cdot \sigma_0 = 0$ (since σ_0 intersects the σ rational component), $\mathcal{C}_\Xi \cdot [x_1] = 0$ (by genericity) and $\mathcal{C}_\Xi \cdot [t] = 1$. Plugging into the charge formula, we conclude that $Q_{\mathrm{U}(1)} = 3$. In addition, the SU(2) Cartan is associated with $[t]$, so this is a charged state in the fundamental, with charge one under the Cartan.

Similarly, for $\mathcal{C}_{y_1}$ we have that $\mathcal{C}_{y_1} \cdot \sigma_0 = 1$, $\mathcal{C}_{y_1} \cdot [x_1] = 0$ and $\mathcal{C}_{y_1} \cdot [t] = 1$. The intersection with σ is again somewhat subtle, since σ is rational, wrapping the whole $\mathcal{C}_{y_1}$. By the moving fiber argument, $(\mathcal{C}_{y_1} + \mathcal{C}_\Xi + \mathcal{C}_t) \cdot \sigma = 1$, and from $(\mathcal{C}_\Xi + \mathcal{C}_t) \cdot \sigma = 2$ we conclude that $Q_{\mathrm{U}(1)} = -1$. Plugging these values into the charge formula, we obtain $Q_{\mathrm{U}(1)} = -3$. This state is also charged under the SU(2) Cartan with charge one. Taking the conjugate state, we can complete the $\mathbf{2}_3$ multiplet, as advertised.

Let us now consider the $\mathbf{2}_1$ states. We consider factorizations of the form

$$\widetilde{\phi} = t(b_0 y_1 s + b_1 y_2)(b_2 y_1^3 + b_3 y_1^2 y_2 st + b_4 y_1 y_2^2 t + b_5 y_1 ws + b_6 y_2 w)\,. \tag{A.15}$$

Here the b_i are coefficients to be determined, and will depend on the coefficients $b, c, \ldots$ of the Calabi-Yau equation. Such a splitting exists whenever

$$g(x_i) = I_1(x_i) = 0\,, \tag{A.16}$$

with $I_1(x_i) = b^2 f^3 + \dots$ a certain polynomial of the $\mathbb{P}^2$ coordinates x_i.[7] This will hold at

$$\begin{aligned} \deg(g) \cdot \deg(I_1) &= \deg(g) \cdot (2\deg(b) + 3\deg(f)) \\ &= -a^2 + 3ab - 2b^2 + 12a - 9b + 45 \end{aligned} \tag{A.17}$$

points in the base. Comparing with table 3 one easily sees that this expression reproduces the $H(\mathbf{2}_1)$ multiplicities, so we expect that these hypermultiplets come from M2 branes wrapping these degenerations. Let us check this claim explicitly.

Over a point satisfying (A.16) we have that the fiber degenerates, and in addition, generically $b_1 \neq 0$ in (A.15), since otherwise we would have three polynomials intersecting over a point in $\mathbb{P}^2$, which is non-generic. We can thus locally redefine y_2 in such a way that (A.15) becomes

$$\widetilde{\phi} = ty_2(b_2 s^2 y_1^3 + b_3 y_1^2 y_2 st + b_4 y_1 y_2^2 t + b_5 y_1 ws + b_6 y_2 w)\,. \tag{A.18}$$

(This redefinition of y_2 is not necessary, but it simplifies the presentation of the analysis.) Furthermore, comparing with the generic form (4.6) we can immediately identify $b_4 = e$, $b_6 = f$, and similarly for the other coefficients. We see that the fiber degenerates into three components: $\mathcal{C}_t = \{t = 0\}$, $\mathcal{C}_{y_2} = \{y_2 = 0\}$ and $\mathcal{C}_{\Xi'} = \{b_2 y_1^3 + \dots\}$. Computing the intersections amongst the components, and between the components and the sections, is a completely straightforward exercise. The resulting non-vanishing intersections are

$$\mathcal{C}_t \cdot \mathcal{C}_{y_2} = \mathcal{C}_{y_2} \cdot \mathcal{C}_{\Xi'} = \mathcal{C}_{\Xi'} \cdot \mathcal{C}_t = 1 \tag{A.19}$$

$$\mathcal{C}_{\Xi'} \cdot P = \mathcal{C}_t \cdot Q = 1\,. \tag{A.20}$$

Plugging into the charge formula (4.30), we obtain that the M2 branes wrapped on $\mathcal{C}_{y_2}, \mathcal{C}_{\Xi'}$ form a doublet under SU(2) (since they are charged under the Cartan) with U(1) charge 1, as expected from the counting above.

The last remaining set of states is $\mathbf{3}_0$. These have a somewhat different origin. Notice that they are adjoints of the SU(2) group, this suggests that their origin comes from Wilson lines on the SU(2) divisor, which we will call G. Recall that this divisor is given by $\{g = 0\} \subset \mathbb{P}^2$, so its Euler character is, by adjunction:

$$\begin{aligned} \chi(\mathsf{G}) &= \int_{\mathsf{G}} c_1(T\mathsf{G}) = \int_{\mathbb{P}^2} [g] \wedge (3[x_1] - [g]) \\ &= \deg(g)(3 - \deg(g)) \end{aligned} \tag{A.21}$$

or, equivalently, in terms of the genus g_{G} of G

$$g_{\mathsf{G}} = 1 - \frac{\deg(g)}{2}(3 - \deg(g))\,. \tag{A.22}$$

The SU(2) Wilson lines on the (two) one-cycles associated with each element of g_{G}, together with scalars coming from reduction of C_3 on the same set of one-cycles (plus the contracting

[7] We computed (A.16) by computing the elimination ideal associated to solving for the b_i variables in (A.15) in terms of the Calabi-Yau coefficients, using SAGE [60].

Cartan divisor), one obtains exactly g_{G} five-dimensional hypers in the adjoint representation, which lift to g_{G} six-dimensional hypers in F-theory. This reproduces precisely the count displayed in table 3.

As an aside, let us highlight a small subtlety in checking six-dimensional anomaly cancellation. If one naively plugs the matter content in table 3 into the six-dimensional anomaly cancellation conditions, one will see that the examples with $\mathbf{3}_0$ multiplets do not satisfy gravitational anomaly cancellation. The explanation is simple: deformations of G can be described by complex structure moduli variation of the total Calabi-Yau, i.e. elements of $h^{2,1}(\mathcal{X})$, but they are also encoded in the values of the Wilson lines over G. In particular, since the gauge group is SU(2), there is a single Casimir invariant, and each Wilson line degree of freedom encodes one deformation modulus. We can see this a bit more precisely: as emphasized in [3], for instance, deformations of the G locus are counted by sections the anticanonical bundle K_{G} of G, and using Serre duality

$$\dim H^0(K_{\mathsf{G}}) = \dim H^1(\mathcal{O}_{\mathsf{G}}) = h^{0,1}(\mathsf{G}) \tag{A.23}$$

which is precisely equal to g_{G} for a connected Riemann surface, such as G. All in all, in order to avoid overcounting one should subtract g_{G} neutral hypers from the contribution of $h^{2,1}(\mathcal{X})$ to the gravitational anomaly, or alternatively count the $\mathbf{3}_0$ multiplets with a multiplicity of 2, instead of 3.

B Non-existence of a section for $\mathcal{X}$

In this appendix we would like to show that the deformed spaces $\mathcal{X}$ considered in section 4 do not admit a section, but rather a bi-section. I.e. there is no rational embedding of the base $\mathbb{P}^2$ into the total space such that the fiber is generically intersected at a single point. The best that we can do is finding divisors of the total space that project down to the base, but generically intersect the fiber twice, i.e. a bi-section. The basic idea was described in [48, 49].

In order to prove this, we need to identify the fiber curve first. This is easy, it is simply given by $\mathcal{T} = [x_1]^2 \cap \mathcal{X}$. This is intuitively easy to understand: the fiber is obtained by taking the preimage of a point (with class $[x_1]^2$) in the base $\mathbb{P}^2$.

Now we need to prove that there is no section S. In all of our examples, the Kähler cone of the Calabi-Yau $\mathcal{X}$ can be generated by the restrictions of the toric divisors $[x_1]$, $[y_1]$, and in the cases with $h^{1,1}(\mathcal{X}) = 3$, also $[w]$. We thus parametrize

$$\mathsf{S} = a[x_1] + b[y_1] + c[w] \tag{B.1}$$

with coefficients (a priori not necessarily integral) to be determined. The generic intersection between the T^2 fiber and the section is given by

$$\mathcal{T} \cdot \mathsf{S} = 2b + 4c\,. \tag{B.2}$$

Showing that this can never be equal to one would follow if $b, c \in \mathbb{Z}$. This is indeed the case, as we now show. Consider first the case with $h^{1,1}(\mathcal{X}) = 3$, since it is somewhat simpler.

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Over a locus in the base given by

$$g(x_i) = I_2(x_i) = 0 \tag{B.3}$$

with[8]

$$\begin{aligned} I_2(x_i) = {} & f^4b^2 - \beta f^3bc + \alpha f^3c^2 + \beta^2 f^2bd - 2\alpha f^3bd - \alpha\beta f^2cd + \alpha^2 f^2d^2 - \beta^3 fbe \\ & + 3\alpha\beta f^2be + \alpha\beta^2 fce - 2\alpha^2 f^2ce - \alpha^2\beta fde + \alpha^3 fe^2 + \beta^4 ba - 4\alpha\beta^2 fba \\ & + 2\alpha^2 f^2ba - \alpha\beta^3 ca + 3\alpha^2\beta fca + \alpha^2\beta^2 da - 2\alpha^3 fda - \alpha^3\beta ea + \alpha^4 a^2 \,, \end{aligned} \tag{B.4}$$

the Calabi-Yau equation (4.2) factorizes into three factors

$$\phi = t(b_0y_1 + b_1y_2)(b_2y_1^3 + b_3y_1^2y_2t + b_4y_1y_2^2t + b_5y_2^3t + b_6y_1w + b_7y_2w)\,. \tag{B.5}$$

The important part for our analysis is that this defines three holomorphic curves in the Calabi-Yau: $\mathcal{C}_t = \{t = 0\}$, $\mathcal{C}_y = \{b_0y_1 + b_1y_2 = 0\}$ and $\mathcal{C}_\Xi$ for the other component. (The notation is intended to be reminiscent of that used in appendix A. Indeed, the matter we just found is precisely the $\mathbf{2}_1$ and $\mathbf{2}_3$ multiplets on the resolved side taken together, since after the Higgsing of the U(1) they cannot be separated anymore.) Computing the intersection numbers with the generators of the Kähler cone chosen in (B.1) is an easy exercise, we get

$$\begin{aligned} \mathcal{C}_t \cdot [y_1] &= 1 \\ \mathcal{C}_y \cdot [w] &= 1 \end{aligned} \tag{B.6}$$

with all other intersections vanishing. Since the intersection between a divisor and a curve in a smooth space has to be integral, by intersecting S with these curves we conclude that $b, c \in \mathbb{Z}$, and thus $\mathcal{T} \cdot \mathsf{S} \in 2\mathbb{Z}$. I.e. there is no section, but rather a bi-section.

This argument fails for the cases with $h^{1,1}(\mathcal{X}) = 2$, since $g = 0$ has no solutions. From the previous discussion it is nevertheless clear what to do, though: the $\mathbf{1}_2$ states on the resolved side $\mathbb{X}$ that we described in appendix A will survive the conifold transition, and appear on the deformed side $\mathcal{X}$ as loci on the $\mathbb{P}^2$ base where the fiber degenerates as

$$\phi = (w + B)(w + D)\,. \tag{B.7}$$

Computing the intersection numbers one gets

$$\begin{aligned} \mathcal{C}_B \cdot [y_1] = \mathcal{C}_C \cdot [y_1] &= 1 \\ \mathcal{C}_B \cdot [x_1] = \mathcal{C}_C \cdot [x_1] &= 0 \end{aligned} \tag{B.8}$$

and since a putative section $\mathsf{S} = a[x_1] + b[y_1]$ has intersection $\mathsf{S} \cdot \mathcal{T} = 2b$ with the fiber $\mathcal{T}$, this shows that indeed we have no section, but rather a bi-section.

[8] As in appendix A this is obtained using SAGE [60].

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