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$c - a$ from the $\mathcal{N} = 1$ superconformal index

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ABSTRACT: We present a prescription for obtaining the difference of the central charges, $c - a$, of a four dimensional superconformal quantum field theory from its single-trace index. The formula is derived from a one-loop holographic computation, but is expected to be valid independent of holography. We demonstrate the prescription with several holographic and non-holographic examples. As an application of our formula, we show the AdS/CFT matching of $c - a$ for arbitrary toric quiver CFTs without adjoint matter that are dual to smooth Sasaki-Einstein 5-manifolds.

KEYWORDS: Supersymmetric gauge theory, Anomalies in Field and String Theories, AdS-CFT Correspondence

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1 Introduction

Over the last few decades, there has been enormous progress in the understanding of supersymmetric field theories, both from the field theory side and from AdS/CFT. Of particular interest are superconformal gauge theories admitting AdS duals. As is often the case when exploring strong/weak coupling dualities, AdS/CFT is most powerful when applied to BPS sectors that are protected from quantum corrections and therefore can be understood at all couplings. In this context, the $\mathcal{N} = 1$ superconformal index [1, 2] has received much attention as a quantity which efficiently encodes the protected information in superconformal quantum field theories.

The superconformal index is defined as a refined Witten index for the theory in radial quantization. In four dimensions it is given by

$$\mathcal{I}_{\text{s.t.}}^R(t, y; a_i) = \text{Tr}_{\text{s.t.}}(-1)^F e^{-\beta\delta} t^{-2(E+j_2)/3} y^{2j_1} \prod a_i^{2s_i}, \quad (1.1)$$

where $\delta = E - \frac{3}{2}r - 2j_2$, and $\{E, j_1, j_2, r; s_i\}$ are the quantum numbers of the superconformal group $SU(2,2|1)$ and the global flavor symmetries, and $\{t, y; a_i\}$ are the corresponding fugacities.¹ As defined this is the right-handed index. One can also define a left-handed

¹As we discuss below, we have chosen the normalization of the exponent of t such that a chiral primary operator would contribute with an exponent given by the negative of its R-charge. Hence our parameter t is the inverse of the one in [3–6], and the inverse cubed of the one in [2, 7].

index $\mathcal{I}_{\text{s.t.}}^L$ in which one replaces r with $-r$ and swaps j_1 and j_2 in both the definition of the index and of δ . Only states with $\delta = 0$ contribute to the index; thus it is independent of β . This condition means that only states which lie within shortened representations of the superconformal algebra will contribute to the index. The index is hence a protected quantity and is independent of the coupling, and therefore can be employed to test various supersymmetric weak/strong dualities [2–7].

In this article we demonstrate that it is possible to obtain the difference of central charges, $c - a$, from the large- N single-trace superconformal index. This difference is related to the mixed $U(1)_R$ chiral anomaly [8] via $c - a = -\frac{1}{16}\text{Tr}R$, where the trace is over the fundamental fermions that can run in the loop of the associated triangle diagram. Thus there is a direct connection between the superconformal index and $\text{Tr}R$. The combination $c - a$ is known to partially determine the subleading logarithmic contribution of quantum fields to the entropy of various extremal and non-extremal black-hole backgrounds [9–11]. It also appears in the subleading correction to the shear viscosity-to-entropy ratio in gauge theory plasmas admitting a holographic dual [12].

The expression we have obtained for extracting $c - a$ from the large- N single-trace index is

$$c - a = \lim_{t \rightarrow 1} -\frac{1}{32} (t\partial_t + 1) (6(y\partial_y)^2 - 1) \times [(1 - t^{-1}y)(1 - t^{-1}y^{-1})\mathcal{I}_{\text{s.t.}}^+(t, y; a_i)] \Big|_{y=1, a_i=1}^{\text{finite}}, \quad (1.2)$$

where $\mathcal{I}_{\text{s.t.}}^+ \equiv \frac{1}{2}(\mathcal{I}_{\text{s.t.}}^R + \mathcal{I}_{\text{s.t.}}^L)$, and the fugacities are set to one after acting with the differential operator on the index. Note that the factor $(1 - t^{-1}y)(1 - t^{-1}y^{-1})$ multiplying the single-trace index removes the contribution from descendant states. The result obtained is often divergent, as we are working in the large- N limit, so the prescription is that the finite term in an expansion about $t = 1$ yields the value of $c - a$.

Some care must be taken in removing the divergent terms in (1.2), as different regularization procedures will affect the finite term and thus the result for $c - a$. For example, a double pole of the form $1/(t-1)^2$ will be converted into $t^2/(t-1)^2 = 1/(t-1)^2 + 2/(t-1) + 1$ under the replacement $t \rightarrow t^{-1}$. What we find is that the choice of t in (1.1) is what yields $c - a$ after simply dropping the pole terms in the Laurent expansion around $t = 1$. Moreover, as we will see in later sections, this choice is unique in that it yields only a double pole (and no simple pole) in t for all of the large- N examples we consider. Of course, if the divergent terms were absent, then the result for $c - a$ would be manifestly independent of the normalization of the exponent of t . At finite- N this is in fact the case, i.e. there is no divergence and $c - a$ is independent of the normalization. However, in the infinite- N limit this divergence arises due to the infinite sums encountered in this limit.

For a holographic CFT, eq. (1.2) can be thought of as an expression that allows one to obtain $\text{Tr}R$ (or $c - a$) from the protected mesonic single-trace states of the CFT at large 't Hooft coupling. In fact, we came upon this expression starting from a one-loop holographic Weyl anomaly computation which amounts to [13, 14]

$$c - a = -\frac{1}{360} \sum (-1)^F (E_0 - 2) d(j_1, j_2) (1 + f(j_1) + f(j_2)). \quad (1.3)$$

Shortening	Condition	Representation	$(1 - t^{-1}y)(1 - t^{-1}y^{-1})\mathcal{I}^+$
conserved	$E_0 = 2 + j_1 + j_2, \frac{3}{2}r = j_1 - j_2$	$\mathcal{D}(E_0, j_1, j_2, r)$	$\frac{1}{2}(-1)^{2(j_1+j_2)+1}t^{-(2E_0+2j_2+2)/3}\chi_{j_1}(y) + (j_1 \leftrightarrow j_2)$
chiral	$E_0 = \frac{3}{2}r$	$\mathcal{D}(E_0, j_1, 0, r)$	$\frac{1}{2}(-1)^{2j_1}t^{-2E_0/3}\chi_{j_1}(y)$
anti-chiral	$E_0 = -\frac{3}{2}r$	$\mathcal{D}(E_0, 0, j_2, r)$	$\frac{1}{2}(-1)^{2j_2}t^{-2E_0/3}\chi_{j_2}(y)$
SLI	$E_0 = 2 + 2j_1 - \frac{3}{2}r$	$\mathcal{D}(E_0, j_1, j_2, r)$	$\frac{1}{2}(-1)^{2(j_1+j_2)+1}t^{-(2E_0+2j_1+2)/3}\chi_{j_2}(y)$
SLII	$E_0 = 2 + 2j_2 + \frac{3}{2}r$	$\mathcal{D}(E_0, j_1, j_2, r)$	$\frac{1}{2}(-1)^{2(j_1+j_2)+1}t^{-(2E_0+2j_2+2)/3}\chi_{j_1}(y)$

Table 1. Contributions to the superconformal index from the various shortened multiplets.

Here the sum is over all dual supergravity fields with AdS_5 quantum numbers $(E_0, j_1, j_2; r)$, and where $d(j_1, j_2) = (2j_1 + 1)(2j_2 + 1)$ and $f(j) = j(j + 1)[6j(j + 1) - 7]$. It is easy to see that only shortened multiplets of $\text{SU}(2, 2|1)$ contribute to this quantity [14], and that it has the characteristics of a combination of left and right indices.

The shortened multiplets of $\text{SU}(2, 2|1)$ are listed in table 1, along with their contributions to the single-trace index. The chiral and SLII multiplets contribute to the right-handed index, while the CP-conjugate multiplets, namely the anti-chiral and SLI multiplets, contribute to the left-handed index. Conserved multiplets, which are CP self-conjugate, contribute to both.

In order to relate $c - a$ to the index, we first consider the chiral and SLII multiplets. The contribution to $c - a$ from a generic chiral multiplet $\mathcal{D}(E_0, j_1, 0; r)$ is given by summing (1.3) over all the fields of the multiplet. This yields

$$(c - a)|_{\text{chiral}} = -\frac{1}{192}(-1)^{2j_1}(2E_0 - 3)(2j_1 + 1)(1 - 8j_1(j_1 + 1)). \quad (1.4)$$

Similarly, a generic SLII multiplet $\mathcal{D}(E_0, j_1, j_2; r)$ contributes

$$(c - a)|_{\text{SLII}} = \frac{1}{192}(-1)^{2j_1+2j_2}(2E_0 + 2j_2 - 1)(2j_1 + 1)(1 - 8j_1(j_1 + 1)). \quad (1.5)$$

It is now possible to see how these expressions may be obtained from the contributions to the right-handed index given in table 1. Since the $\text{SU}(2)$ character $\chi_j(y)$ is given by

$$\chi_j(y) = \frac{y^{2j+1} - y^{-(2j+1)}}{y - y^{-1}}, \quad (1.6)$$

the differential operator $(6(y\partial_y)^2 - 1)$ acting on the contributions to the index gives $(2j + 1)[8j(j + 1) - 1]$ when y is set to one. The operator $(t\partial_t + 1)$ then produces the E_0 -dependent factors in (1.4) and (1.5).

The CP conjugate multiplets (anti-chiral and SLI) contribute similarly to (1.4) and (1.5) with the appropriate replacement of quantum numbers, and are accounted for in the left-handed index. Finally, since conserved multiplets contribute as the sum of one SLI and one SLII multiplet, they are implicitly included in both the left- and right-handed indices. Our key observation is that the contribution to $c - a$ has an uniform expression for every single bulk multiplet. Hence a single differential operator acting on the index can yield the appropriate contribution to $c - a$ regardless of the shortening condition. Summing

over all multiplets, one finally arrives at (1.2), where the index is now the single-particle supergravity index which is equal to the single-trace index of the SCFT. As a side comment, note that the index thus provides a natural regulator for the Kaluza-Klein sums encountered in the holographic $c - a$ calculations of [14–16].

Although eq. (1.2) was derived via a holographic calculation, it refers only to field theoretic objects, and so we conjecture it to be true in general for all superconformal field theories, regardless of whether they admit a dual holographic description or not. This is most straightforward in the large- N limit, where the single-trace index is well-defined. However, eq. (1.2) can be extended away from the large- N limit by simply replacing the single-trace index by the plethystic log of the full index [17, 18]. (Another connection between the central charges and the full index is suggested by the results of [19], which demonstrated that c and a are independently encoded in the prefactor relating the supersymmetric partition function on $S^3 \times S^1$ to the index.) In the remainder of this article we discuss the application of (1.2) to a variety of examples.

2 Examples

Since all the theories we consider are CP invariant, we have $\mathcal{I}_{\text{s.t.}}^+ = \mathcal{I}_{\text{s.t.}}^R = \mathcal{I}_{\text{s.t.}}^L$, and therefore we dispense with the $+$ superscript of $\mathcal{I}_{\text{s.t.}}$ in the following.

2.1 Holographic theories

We begin with an examination of eq. (1.2) in the context of four-dimensional superconformal field theories admitting an AdS_5 dual at large N . For such theories it is well-known that even though c and a are both of order N^2 , their difference $c - a$ is $\mathcal{O}(1)$, and therefore it vanishes at leading order [20, 21]. Our prescription thus gives an $\mathcal{O}(1/N^2)$ correction to the leading order quantities at large N .

2.1.1 The $\mathcal{N} = 4$ SYM theory and the $\mathbb{Z}_2$ orbifold

We first consider $\mathcal{N} = 4$ SYM theory dual to IIB string theory on $\text{AdS}_5 \times S^5$. The $\mathcal{N} = 1$ large- N single-trace index for $\mathcal{N} = 4$ SYM with gauge group $\text{SU}(N)$ is [2]

$$\mathcal{I}_{\text{s.t.}} = \frac{3}{t^{2/3} - 1} - \frac{3t^{-2/3}(1 - t^{-2/3})}{(1 - t^{-1}y)(1 - t^{-1}/y)}. \quad (2.1)$$

Inserting this into (1.2) gives

$$c - a = \left[-\frac{27}{16(t-1)^2} + \mathcal{O}(t-1) \right]_{t \rightarrow 1}^{\text{finite}} = 0, \quad (2.2)$$

as expected, since $c = a$ exactly for the $\mathcal{N} = 4$ theory. It is also possible to verify that the decoupled $\text{U}(1)$ has vanishing contribution to $c - a$, so the result (2.2) holds for the $\text{U}(N)$ case as well. While this example is somewhat trivial, it nevertheless demonstrates that a divergent term needs to be removed when extracting $c - a$ from the single-trace index. Moreover, a possible simple pole divergence is absent, but would have been present had

we not made the choice for t given in (1.1).² The double pole divergence appears to be an artifact of the large- N limit, and we will comment further on this below.

We now consider the $S^5/\mathbb{Z}_2$ orbifold, which has $\mathcal{N} = 2$ supersymmetry. The index was computed in [22, 23] and is given by

$$\mathcal{I}_{\text{s.t.}} = \frac{2}{t^{2/3} - 1} + \frac{2}{t^{4/3} - 1} - \frac{2t^{-2/3}(1 - t^{-2/3})}{(1 - t^{-1}y)(1 - t^{-1}/y)}. \quad (2.3)$$

Using this expression in (1.2) gives

$$c - a = \left[-\frac{27}{16(t-1)^2} + \frac{1}{12} + \mathcal{O}(t-1) \right]_{t \rightarrow 1}^{\text{finite}} = \frac{1}{12}, \quad (2.4)$$

which matches the field theory result.

The computation can be extended to more general abelian $\mathcal{N} = 2$ orbifolds of S^5 , although we have not explicitly done so here. Instead we turn next to toric quivers without adjoint matter and with smooth Sasaki-Einstein dual geometries, these encompass $\mathcal{N} = 1$ orbifolds of S^5 as a special case. To be precise, this is true of the $S^5/\mathbb{Z}_n$ orbifold theories only for odd n . The even n orbifolds are not special cases of the toric theories considered below because their dual geometry is singular. However, their index and value of $c - a$ are obtained correctly if we treat them as limiting cases of the toric $Y^{p,q}$ theories [15].

2.1.2 Toric quivers without adjoint matter

We now apply the relation (1.2) to the class of toric quiver gauge theories without adjoint matter dual to smooth Sasaki-Einstein five-manifolds. These include the theories dual to $\mathcal{N} = 1$ orbifolds of S^5 as well as the $Y^{p,q}$ manifolds which were discussed in [14–16]. Here we reproduce the $c - a$ results of those papers utilizing the relation to the superconformal index.

The index for such a toric gauge theory was reported in [24]. It is given by

$$\mathcal{I}_{\text{s.t.}} = \sum_i \frac{1}{t^{r_i/3} - 1}, \quad (2.5)$$

where r_i are the R -charges of extremal BPS mesons, to be determined by a -maximization [25]. In fact, $\sum r_i$ can be seen from comparing equations (3.13) and (3.17) in [24] to be given by

$$\sum r_i = 6(\# \text{ nodes in the quiver}). \quad (2.6)$$

Note that in obtaining (2.6) one must assume $0 \leq R_i \leq 2$, where R_i are the R -charges of chiral fields in the quiver; the lower bound comes from unitarity while the upper bound follows from the fact that in toric quiver theories all chiral fields participate in superpotential terms [26] that have R -charge equal to 2.

²The vanishing of the simple pole holds for all the examples we have investigated. This appears to be a universal property of the Laurent expansion of (1.2), although it is sensitive to the normalization of the exponent of t in (1.1). It would be interesting to understand this further.

Applying (1.2) to (2.5) gives

$$c - a = \left[\frac{1}{96} \sum_i \left(3(1 - t^{-2}) \frac{1}{t^{r_i/3} - 1} - (1 + 10t^{-1} + t^{-2}) \frac{r_i t^{-r_i/3}}{(1 - t^{-r_i/3})^2} \right) \right]_{t \rightarrow 1}^{\text{finite}}, \quad (2.7)$$

which upon expanding around $t = 1$ yields

$$c - a = \left[-\frac{9}{8(t-1)^2} \sum_{i=1}^{n_v} \frac{1}{r_i} + \frac{1}{96} \sum_{i=1}^{n_v} r_i \right]_{t \rightarrow 1}^{\text{finite}} = \frac{1}{96} \sum_{i=1}^{n_v} r_i. \quad (2.8)$$

Comparing (2.8) with (2.6) then yields

$$c - a = \frac{1}{16} (\# \text{ nodes in the quiver}). \quad (2.9)$$

This matches the expected result for $c - a$ based on the decoupling of a $U(1)$ at each node in the quiver. (Since there are no adjoints in the quiver, there are no additional $\mathcal{O}(1)$ contributions to $c - a$ in the field theoretical computation through $c - a = -\frac{1}{16} \text{Tr} R$.)

Note that since eq. (1.2) is derived from a one-loop holographic computation, and since the AdS/CFT matching of the index has been demonstrated in [24] for the theories described above, the success of (1.2) for the theories considered so far may be interpreted as a successful AdS/CFT matching of $c - a$. This significantly generalizes our earlier results in [14–16].

2.1.3 Suspended Pinch Point

Next, we will illustrate the prescription (1.2) for a theory with a singular holographic dual geometry, namely, the Suspended Pinch Point (SPP) quiver CFT [27]. This is a toric gauge theory with three $SU(N)$ gauge factors, one chiral multiplet with R -charge $2 - 2/\sqrt{3}$ in the adjoint of one of the gauge factors and a number of chiral bifundamental fields [26]. Since, in the absence of fundamental matter, $\text{Tr} R$ for any holographic CFT is order one (contrary to a or c which are of order N^2), we may count only the order one contributions to it and be sure that order N^2 contributions cancel. $\text{Tr} R$ in this case is $-3 - (1 - 2/\sqrt{3})$, where the first term comes from the three gauginos, and the second term from the fermion in the adjoint chiral. Thus $c - a = -\frac{1}{16} \text{Tr} R = (6 - \sqrt{3})/24$.

We now demonstrate that this result can be reproduced via eq. (1.2). We need the large- N index of the SPP quiver, which can be computed following the approach of [7]. The essential ingredient in the computation is the single-letter index matrix i_{SPP} for the quiver, which we find to be

$$i_{\text{SPP}} = \begin{pmatrix} i_V + (a_1 a_2)^{-1} i_{\chi(2-2/\sqrt{3})} + (a_1 a_2) i_{\bar{\chi}(2-2/\sqrt{3})} & a_2 i_{\chi(1/\sqrt{3})} + a_1^{-1} i_{\bar{\chi}(1/\sqrt{3})} & a_1 i_{\chi(1/\sqrt{3})} + a_2^{-1} i_{\bar{\chi}(1/\sqrt{3})} \\ a_1 i_{\chi(1/\sqrt{3})} + a_2^{-1} i_{\bar{\chi}(1/\sqrt{3})} & i_V & a_2^{-1} i_{\chi(1-1/\sqrt{3})} + a_1 i_{\bar{\chi}(1-1/\sqrt{3})} \\ a_2 i_{\chi(1/\sqrt{3})} + a_1^{-1} i_{\bar{\chi}(1/\sqrt{3})} & a_1^{-1} i_{\chi(1-1/\sqrt{3})} + a_2 i_{\bar{\chi}(1-1/\sqrt{3})} & i_V \end{pmatrix}. \quad (2.10)$$

Following [7], we are denoting by i_V , $i_{\chi(r)}$, and $i_{\bar{\chi}(r)}$, respectively the single-letter index of a vector multiplet, a chiral multiplet, and an anti-chiral multiplet. We have introduced fugacities a_1 and a_2 for the global $U(1) \times U(1)$ flavor symmetry of the theory. From the

single-letter index matrix the large- N single-trace index of the theory can be computed as explained in [7]. The result is

$$\mathcal{I}_{\text{s.t.}}(t, y; a_1, a_2) = -3 + \frac{1}{1 - (a_1 a_2)^{-1} t^{-2/\sqrt{3}}} + \frac{1}{1 - a_1^{-1} t^{-1-1/\sqrt{3}}} + \frac{1}{1 - a_2^{-1} t^{-1-1/\sqrt{3}}} - \frac{2}{1 - (a_1 a_2)^{-1} t^{2-2/\sqrt{3}}} + \frac{(a_1 a_2)^{-1} t^{-2/\sqrt{3}} - a_1 a_2 t^{-2+2/\sqrt{3}}}{(1 - t^{-1} y)(1 - t^{-1} y^{-1})}. \quad (2.11)$$

This could also be obtained from the results of [28].

Setting a_1 and a_2 equal to one, and plugging the formula for the index of the SPP quiver into (1.2) we obtain

$$c - a = \left[-\frac{27}{16(t-1)^2} + \frac{6 - \sqrt{3}}{24} + \mathcal{O}(t-1) \right]_{t \rightarrow 1}^{\text{finite}} = \frac{6 - \sqrt{3}}{24}, \quad (2.12)$$

which matches the field theory result. Note that the index (2.11) was obtained by field theoretic means. Thus, in particular, it is not necessary to work with a holographic dual when applying the expression (1.2) for $c - a$.

2.1.4 Del Pezzo theories

For $k > 3$, the quiver theories dual to dP_k surfaces are not toric, so they can not be considered as special cases of the theories studied above. However, our prescription works very simply. The index is [24]

$$\mathcal{I}_{\text{s.t.}} = (k+3) \frac{1}{t^2 - 1}, \quad (2.13)$$

which when combined with eq. (1.2) gives

$$c - a = \left[-\frac{9(k+3)}{16(t-1)^2} + \frac{k+3}{16} + \mathcal{O}(t-1) \right]_{y=1, t \rightarrow 1}^{\text{finite}} = \frac{k+3}{16}. \quad (2.14)$$

This matches the field theory result since the quivers have $k+3$ $\text{SU}(N)$ nodes and no adjoints.

2.2 Non-holographic theories

We now consider several examples where the field theory does not admit an AdS dual, and show that the expression (1.2) continues to hold. Note, however, that we continue to work in a large- N limit where the single-trace index is well defined.

2.2.1 $\text{U}(1)^N$ gauge theory with matter

Consider a $\text{U}(1)^N$ gauge theory with N_χ chiral multiplets (along with their conjugates) having R -charges R_i , and neutral under the gauge group. We are not claiming such a theory exists as an SCFT for generic values of N, N_χ , and R_i , however as described below there exist specific values for which these theories do describe particular SCFTs. With this example we only want to show that our prescription is able to extract $-\frac{1}{16} \text{Tr} R$ from

the single-trace index even for very simple non-holographic systems. Note that $(N, N_\chi) = (0, 1)$, and $(1, 0)$ correspond to a single chiral multiplet and a single $U(1)$ vector multiplet, respectively. Also, for the case $(N, N_\chi) = (0, (N_c + 1)^2 + 2(N_c + 1))$, with $2(N_c + 1)$ of the chiral multiplets having R -charge $\frac{N_c}{N_c+1}$ and the rest having R -charge $\frac{2}{N_c+1}$ corresponds to the IR s -confining phase of $SU(N_c)$ SQCD with $N_c + 1$ flavors, where the chiral matter described above are the confined mesons and baryons. In particular, for $N_c = 2$ the mesons and baryons attain the expected superconformal R -charges for a free theory.

The index is

$$\begin{aligned}\mathcal{I}_{\text{s.t.}} &= N\mathcal{I}_{\text{s.t.}}(V) + \sum_{i=1}^{N_\chi} \mathcal{I}_{\text{s.t.}}(\chi, \bar{\chi}, R_i) \\ &= N \left(1 - \frac{1 - t^{-2}}{(1 - t^{-1}y)(1 - t^{-1}y^{-1})} \right) + \sum_{i=1}^{N_\chi} \frac{t^{-R_i} - t^{R_i-2}}{(1 - t^{-1}y)(1 - t^{-1}y^{-1})},\end{aligned}\quad (2.15)$$

which when combined with eq. (1.2) gives

$$\begin{aligned}c - a &= \left[-\frac{N}{16} - \sum_{i=1}^{N_\chi} \frac{R_i - 1}{16} + \mathcal{O}(t - 1) \right]_{y=1, t \rightarrow 1}^{\text{finite}} \\ &= -\frac{N}{16} - \sum_{i=1}^{N_\chi} \frac{R_i - 1}{16},\end{aligned}\quad (2.16)$$

which is indeed $-\frac{1}{16}\text{Tr}R$. Curiously, this expression for $c - a$ is finite as $t \rightarrow 1$.

2.2.2 Large- N SQCD

We now consider standard SQCD [29]. This theory has a large- N limit which does not admit a dual gravity description. In particular, $c - a$ for this theory is [30]

$$c - a = \frac{1}{16}(N_c^2 + 1) \quad (2.17)$$

which is $\mathcal{O}(N^2)$. Nevertheless, we will see that our prescription successfully computes $c - a$ from the large- N expression of the index.

Consider SQCD in the Veneziano limit $N_c, N_f \gg 1$ with N_c/N_f held fixed. The superconformal index for this theory has been computed in [4], and the single-trace index can be obtained by taking its plethystic log. After simplification, the result is given (in our notation) by

$$\mathcal{I}_{\text{s.t.}} = \frac{1}{t^2 - 1} \left(1 + N_f^2 \frac{(t^{N_c/N_f} - t^{-N_c/N_f})^2}{(1 - t^{-1}y)(1 - t^{-1}y^{-1})} \right). \quad (2.18)$$

Applying eq. (1.2) to this we find

$$c - a = \left[-\frac{3}{16(t-1)^2} + \frac{1}{16}(N_c^2 + 1) + \mathcal{O}(t - 1) \right]_{t \rightarrow 1}^{\text{finite}} = \frac{1}{16}(N_c^2 + 1), \quad (2.19)$$

which recovers the expected value of $c - a$.

2.2.3 A_k theories

Next, consider the A_k generalizations of SQCD [30] which add an additional adjoint chiral superfield X along with a superpotential of the form

$$W = \text{tr } X^{k+1}. \quad (2.20)$$

For $k = 1$ this is just a massive deformation, and the X field can be integrated out to yield the standard SQCD of the previous subsection. For generic k the difference of central charges is given by

$$c - a = \frac{1}{8(k+1)}(N_c^2 + 1). \quad (2.21)$$

The superconformal index has been computed for these theories in the Veneziano limit [4], and we can again extract the single-trace index to obtain

$$\begin{aligned} \mathcal{I}_{\text{s.t.}} = & \frac{t^{-\frac{2}{k+1}}}{1 - t^{-\frac{2}{k+1}}} + \frac{t^{-\frac{4k}{k+1}}}{1 - t^{-\frac{4k}{k+1}}} - \frac{t^{-\frac{2k}{k+1}}}{1 - t^{-\frac{2k}{k+1}}} \\ & - \frac{\left(t^{-\frac{2}{k+1}} - t^{-\frac{2k}{k+1}}\right) - N_f^2 \frac{\left(t^{\frac{2N_c}{(k+1)N_f}} - t^{-\frac{2N_c}{(k+1)N_f}}\right)^2}{t^2 \left(1 - t^{-\frac{2}{k+1}}\right) \left(1 + t^{-\frac{2k}{k+1}}\right)}}{(1 - t^{-1}y)(1 - t^{-1}y^{-1})}. \end{aligned} \quad (2.22)$$

From this we find

$$c - a = \left[-\frac{3(2k-1)(k+1)}{32k(t-1)^2} + \frac{1}{8(k+1)}(N_c^2 + 1) + \mathcal{O}(t-1) \right]_{t \rightarrow 1}^{\text{finite}} = \frac{1}{8(k+1)}(N_c^2 + 1), \quad (2.23)$$

and we again arrive at the expected value for $c - a$.

3 A comment on $\mathcal{N} = 2$ to $\mathcal{N} = 1$ RG flows

For theories preserving $\mathcal{N} = 2$ supersymmetry the index can be further refined by introducing a fugacity for the enlarged $\text{SU}(2)_R \times \text{U}(1)_r$ R -symmetry. In this case the right-handed index can be written (in this section we are explicitly setting $a_i = 1$ for all i)

$$\mathcal{I}_{\mathcal{N}=2}^R = \text{tr} (-1)^F e^{-\beta\delta} t^{-2(E+j_2)/3} y^{2j_1} v^{-(r_{\mathcal{N}=2} + R_{\mathcal{N}=2})}, \quad (3.1)$$

where $R_{\mathcal{N}=2}$ and $r_{\mathcal{N}=2}$ are the quantum numbers under the $\text{SU}(2)_R \times \text{U}(1)_r$ R -symmetry which are related to those of the $\mathcal{N} = 1$ R -symmetry by

$$r_{\mathcal{N}=1} = \frac{2}{3}(2R_{\mathcal{N}=2} - r_{\mathcal{N}=2}). \quad (3.2)$$

In general, $\mathcal{N} = 2$ theories can be deformed by giving a mass term to the $\mathcal{N} = 1$ chiral multiplet that sits in the $\mathcal{N} = 2$ vector multiplet. In this case, one arrives at an $\mathcal{N} = 1$ theory in the infrared (IR). When all of the gauge couplings are exactly marginal and when one gives mass terms to the chiral multiplets in all of the $\mathcal{N} = 2$ vector multiplets, ref. [7] demonstrated an interesting relation between the $\mathcal{N} = 2$ index of the ultraviolet (UV)

theory and the $\mathcal{N} = 1$ index of the IR theory of such RG flows. In particular, they showed that the IR $\mathcal{N} = 1$ index is given by simply setting the fugacity $v = t^{-1/3}$ in the UV $\mathcal{N} = 2$ index, so that

$$\mathcal{I}_{\mathcal{N}=2}^{\text{UV}}(t, y, v = t^{-1/3}) = \mathcal{I}_{\mathcal{N}=1}^{\text{IR}}(t, y). \quad (3.3)$$

This implies a simple relation between the flow of $c - a$, parametrized by $\Delta_{c-a} \equiv (c - a)_{\text{UV}} - (c - a)_{\text{IR}}$, and the $\mathcal{N} = 2$ index of the theory in the UV. There are known universal relations between the UV and IR central charges of such theories [31], and it would be interesting to know if these results are related to what we discuss below.

Notice first that one can evaluate the $\mathcal{N} = 1$ index for an $\mathcal{N} = 2$ theory by simply setting $v = 1$, and so our prescription for computing $c - a$ from the index applies also to theories preserving more supersymmetry with the additional fugacities set to unity. We now apply the observation (3.3) to the expression for $c - a$ and use it to compute Δ_{c-a} . The difference is given by

$$\begin{aligned} \Delta_{c-a} = \lim_{t \rightarrow 1} & -\frac{1}{32} (t\partial_t + 1) (6(y\partial_y)^2 - 1) \\ & \times \left[(1 - t^{-1}y)(1 - t^{-1}y^{-1}) \left(\mathcal{I}_{\text{s.t.}}^{\mathcal{N}=2}(t, y, v = 1) - \mathcal{I}_{\text{s.t.}}^{\mathcal{N}=2}(t, y, v = t^{-1/3}) \right) \right]_{y=1}^{\text{finite}}. \end{aligned} \quad (3.4)$$

When acting on the second term in the parentheses, the t derivative can be re-written as

$$t\partial_t + t \frac{dv}{dt} \partial_v = t\partial_t - \frac{1}{3} v \partial_v, \quad (3.5)$$

where one should afterwards set $v = t^{-1/3}$ before taking the limit in (3.4).

If the index expressions were finite, then the only terms that survive in the difference once we take $t \rightarrow 1$ are those in which the v derivative acts explicitly on the IR index. In this case, the result can then be written purely in terms of the $\mathcal{N} = 2$ index

$$\Delta_{c-a} = \lim_{t \rightarrow 1} -\frac{1}{96} v \partial_v (6(y\partial_y)^2 - 1) \left[(1 - t^{-1}y)(1 - t^{-1}y^{-1}) \mathcal{I}_{\text{s.t.}}^{\mathcal{N}=2}(t, y, v) \right] \Big|_{y=1, v=t^{-1/3}}^{\text{finite}}. \quad (3.6)$$

In fact, this is only strictly true for functions which are finite as t approaches one, as would be the case when (3.4) is applied to the single-trace index at finite N . However we know the result to be divergent at $t = 1$ in the large- N limit from the examples studied in the previous sections. Nevertheless, since we expect (3.6) to hold for the index at finite N , we should expect it also to be true in the large- N limit. This can be seen in a simple example.

Consider the Klebanov-Witten flow between the holographic duals to the $S^5/\mathbb{Z}_2$ orbifold theory and the $T^{1,1}$ theory [32]. The single-trace $\mathcal{N} = 2$ index for the $S^5/\mathbb{Z}_2$ theory is [23]

$$\mathcal{I}_{\text{s.t.}}^{\mathcal{N}=2} = 2 \left(\frac{t^{-2/3}v}{1 - t^{-2/3}v} + \frac{t^{-4/3}/v}{1 - t^{-4/3}/v} - \frac{t^{-2/3}v - t^{-4/3}/v}{(1 - t^{-1}y)(1 - t^{-1}y^{-1})} \right). \quad (3.7)$$

Note that setting $v = 1$ reproduces the $\mathcal{N} = 1$ index given in (2.3). We can evaluate (3.6) for this theory to find

$$\Delta_{c-a}(S^5/\mathbb{Z}_2 \rightarrow T^{1,1}) = -\frac{1}{24}, \quad (3.8)$$

which is the expected result.

It is also worth noting that this result is completely finite at $t = 1$. Also, if one evaluates the remaining terms in (3.4), i.e. those that would vanish for a function that is regular at $t = 1$, we find that they correctly reproduce the $(t - 1)^{-2}$ divergence of the difference of the $c - a$ results from applying (1.2) directly. In particular, they contain no finite term. It would be interesting to understand if Δ_{c-a} as defined by (3.6) is finite generically.

4 Moving away from the large- N limit

Since the relation (1.2) was motivated by holographic considerations, it is natural for it to hold in the large- N limit. However, one can ask whether it will remain valid even at finite N (provided the single-trace index is replaced by the plethystic log of the full index). One possible obstruction in making this connection between large and finite N arises from the regularization of the $t \rightarrow 1$ divergence that is present at large N . As seen in the above examples, this divergence shows up as a second order pole in $t - 1$, which needs to be removed by hand in order to obtain a finite answer for $c - a$. (One exception to this behavior is in the $U(1)^N$ theory of section. 2.2.1, for which N is not necessarily large.)

Because of the t -derivative in (1.2), we see that the second order pole in $c - a$ arises from a first order pole in the single-trace index. Pole terms in the index arise due to geometric sums over an infinite number of states. The reason the $U(1)^N$ theory avoids this pole is that the only infinite series of states at finite- N is due to the descendent operators. In this case the single-trace index multiplied by $(1 - t^{-1}y)(1 - t^{-1}y^{-1})$ (to remove the descendent contributions) is finite at $t = 1$ and fixed $y \neq 1$ for finite N . This may be understood physically by realizing that only a finite number of protected single-trace operators may arise in the abelian theory. More generally, the number of such operators in a gauge theory with gauge group $SU(N)$ is of order N , since traces of products with more than N terms can be written in terms of products of shorter ones. So a theory at finite- N implicitly cuts off the potential geometric sums over single-trace states. What this suggests is that the pole term of $c - a$ in $t - 1$ arises as an artifact of the large- N limit.

The combination of t and y serve to regulate the index, so the $t \rightarrow 1$ and $y \rightarrow 1$ limits are somewhat delicate. When the descendant states are removed from the single-trace index by multiplication with $(1 - t^{-1}y)(1 - t^{-1}y^{-1})$, the expression at $t = 1$ essentially counts the number of protected single-trace operators in the theory. In this case it ought to remain finite at $t = 1$ when N is finite. On the other hand, the single-trace index by itself at finite N diverges when $y = 1$ and $t \rightarrow 1$ because of the contribution of the descendant states. Considering again the $U(1)^N$ example, we see that the finite- N single-trace index at $y = 1$ has a first order pole at $t = 1$ with structure given by

$$\mathcal{I}_{\text{s.t.}}^{\text{finite-}N}(t \rightarrow 1, y = 1) \simeq -2 \frac{\text{Tr} R}{t - 1} = 32 \frac{c - a}{t - 1}. \quad (4.1)$$

This behavior of the index has also been shown to be generically true [33]. In particular, we can see this by defining $t = e^\beta$ and taking the Plethystic exponential of the above single-trace index. Doing this for this example and extracting the leading $\beta \rightarrow 0$ behavior

gives

$$\begin{aligned}\mathcal{I}^{\text{finite-}N} &= \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \mathcal{I}_{\text{s.t.}}^{\text{finite-}N}(e^{n\beta}, y^n) \right) \Big|_{y=1} \\ &\simeq \exp \left(\frac{32(c-a)}{\beta} \sum \frac{1}{n^2} \right) = \exp \left(\frac{16\pi^2(c-a)}{3\beta} \right),\end{aligned}\quad (4.2)$$

which precisely agrees with the general results of [33].

For large- N single-trace indices, even when $y \neq 1$ there are pole terms around $t = 1$ because at infinite N there are infinitely many protected single-trace operators in the theory. We have not been able to recognize the coefficient of this pole in general. However, it is instructive to examine the pole structure somewhat more carefully.

The behavior of $\mathcal{I}_{\text{s.t.}}$ as $t \rightarrow 1$ and $y \rightarrow 1$ depends on how the limit is taken. By taking $t \rightarrow 1$ first, we obtain an expansion of the form

$$\mathcal{I}_{\text{s.t.}}(t, y) = \frac{a'_0}{t-1} + b'_0 + (t-1) \left(\frac{c'_{-2}}{(y-1)^2} + \frac{c'_{-1}}{y-1} + c'_0 \right) + \dots \quad (4.3)$$

Although we do not claim that this is the most general structure, it nevertheless holds for all examples we have looked at. In this expansion, there is at most a simple pole in $t-1$, while the higher order terms have increasing poles in $y-1$. In contrast, taking $y \rightarrow 1$ first yields the expansion

$$\begin{aligned}\mathcal{I}_{\text{s.t.}}(t, y) &= \frac{a_0}{t-1} + a_1 + a_2(t-1) + \dots \\ &+ (y-1)^2 \left(\frac{b_0}{(t-1)^3} + \frac{b_1}{(t-1)^2} + \frac{b_2}{t-1} + b_3 + b_4(t-1) + \dots \right) + \dots\end{aligned}\quad (4.4)$$

That there is no term proportional to $(y-1)$ in the above expansion follows from CP invariance. While we do not have an argument to demonstrate that the leading pole in the second line is third order in $t-1$, this nevertheless holds for all of the examples we have considered.

More generally, it is possible to approach $t \rightarrow 1$ and $y \rightarrow 1$ from different directions. For example, if we considered the supersymmetric partition function on the squashed three-sphere times a circle [34], then the natural limit would be to hold the squashing parameter b fixed. The approach to $t \rightarrow 1$ and $y \rightarrow 1$ then follows the curve $y = t^{(1-b^2)/(1+b^2)}$. (Note that this reduces to $y = 1$ for the round S^3 .) This choice of holding b fixed was taken in [33].

A point to be made about the two expansions in (4.3) and (4.4) is that the coefficients a'_0 and a_0 of the pole terms do not necessarily agree, as demonstrated in table 2, because the order of the limits $y \rightarrow 1$ and $t \rightarrow 1$ is important. The expansion that is relevant for computing $c-a$ using (1.2) is (4.4), where the expansion around $y = 1$ is only needed to second order. In this case, we obtain

$$c-a = -\frac{3}{8} \frac{a_0 - b_0}{(t-1)^2} + \frac{1}{32} (a_0 + 12(a_2 - b_0 + b_1 - b_2)) + \dots \quad (4.5)$$

Theory	a'_0	a_0	a_1	a_2	b_0	b_1	b_2
Toric w/o adjoint	$3 \sum \frac{1}{r_i}$	$3 \sum \frac{1}{r_i}$	$-4 + \frac{3}{2} \sum \frac{1}{r_i}$	$\frac{1}{36} \sum r_i - \frac{1}{4} \sum \frac{1}{r_i}$	0	0	0
$\mathcal{N} = 4$ theory	9/2	5/2	-1/4	19/216	-2	-3	-19/27
$\mathbb{Z}_2$ orbifold theory	9/2	19/6	-5/12	101/648	-4/3	-2	-38/81
SPP quiver	9	$\frac{13}{2} - \frac{4}{\sqrt{3}}$	$\frac{3}{4} - \frac{2}{\sqrt{3}}$	$\frac{35}{24} - \frac{20}{9\sqrt{3}}$	$2 - \frac{4}{\sqrt{3}}$	$3 - 2\sqrt{3}$	$\frac{7}{3} - \frac{38}{9\sqrt{3}}$
SQCD	1/2	$1/2 + 2N_c^2$	$-1/4 + N_c^2$	$\frac{1}{8} - \frac{2N_c^2}{3} + \frac{2N_c^4}{3N_f^2}$	$2N_c^2$	$3N_c^2$	$\frac{N_c^2}{3} + \frac{2N_c^4}{3N_f^2}$

Table 2. The coefficients of the expansions (4.3) and (4.4) of the single-trace index at large N .

Moreover, the vanishing of the simple pole term in $c - a$ follows directly from the form of the operator $t\partial_t + 1$ in (1.2) acting on (4.4). (This would not be true if we had chosen a different convention for the normalization of t .) It is the finite part in (4.5) that correctly reproduces the value expected from weak-coupling field theory computation, namely $-\frac{1}{16}\text{Tr}R$, in all examples we have considered. Note in particular that for the large- N Veneziano limit of A_k theories, $c - a = (N_c^2 + 1)/8(k + 1)$ is large (in contrast with the holographic examples), but eq. (1.2) still works perfectly fine. The coefficients relevant to $c - a$ are listed for a number of examples in table 2.

As seen in (4.5), the regulated $c - a$ receives contributions from several terms in the expansion of the single-trace index. In particular, it includes information away from $y = 1$, as encoded in the b_0 , b_1 and b_2 coefficients. This is, of course, due to the y -dependent operator in (1.2), which is needed to bring in the proper spin-dependence to reproduce the holographic expressions (1.4) and (1.5). In this sense, it appears that while the information of $c - a$ is contained in the large- N index, it is encoded in a rather non-trivial manner.

The expansion (4.5) also applies to the $U(1)^N$ theory of section 2.2.1. For this particular case it turns out that $a_0 = b_0$ so that the pole term in $c - a$ vanishes. Furthermore, while all the coefficients in the expansion (4.4) are non-vanishing, it turns out that the sum $(a_2 - b_0 + b_1 - b_2)$ also vanishes. This leaves only the finite $a_0/32$ as the result for $c - a$, which agrees with (4.1), i.e. that the coefficient of the pole term of the finite- N single-trace index is proportional to $c - a$.³

In fact, the observation of the preceding paragraph can be generalized assuming a particular form for the index. Let us suppose that the finite- N index has a pole structure determined only by the descendent contributions such that it has the form

$$\mathcal{I}_{\text{s.t.}}^{\text{finite-}N}(t, y) = \frac{F(t, y)}{(1 - t^{-1}y)(1 - t^{-1}y^{-1})}, \quad (4.6)$$

where $F(t, y)$ is a regular function with a first order zero at $t = 1$ when $y = 1$. This behavior of the function $F(t, y)$ can be seen heuristically by considering the fact that in an expansion around $t = y = 1$ one expects the bare sum (i.e. neglecting the denominators from the descendent contributions) on single trace operators to vanish since the index ought to pick up a contribution from an equal number of bosonic and fermionic states, thus leading to (at least) a first order zero. This can be seen in $\mathcal{N} \geq 2$ theories because

³We would like to thank Z. Komargodski for discussions on this point.

states contributing to the index live in representations of a supersymmetric commuting sub-group of the superconformal group. For $\mathcal{N} = 1$ the commuting sub-group no longer contains a fermionic generator. Nonetheless one can see a cancelation of this sort by grouping all chiral (or semi-long) operators into triplets which, together with the bare chiral (semi-long) operator, correspond to inserting e.g. the gaugino superfields W_α and $W^\alpha W_\alpha$ into the trace. The sum of contributions to the single-trace index over any such triplet indeed vanishes at $t = y = 1$. (Note that this argument breaks down if one must sum over an infinite tower of states, as we have seen in the large- N examples discussed previously, in which case $F(t, y)$ acquires a pole term in t for y not strictly equal to unity. This leads to the second order pole in $c - a$.) In this case, one can Taylor expand the numerator near $t = y = 1$

$$F(t, y) = f_1(t - 1) + f_2(t - 1)^2 + f_3(t - 1)^3 + \dots \\ + (y - 1)^2(g_0 + g_1(t - 1) + \dots) + \dots, \quad (4.7)$$

where we have kept only the terms relevant for the calculation of $c - a$ in (4.5). Assuming the above form and computing $c - a$ via (4.5), one finds again that the coefficients in the expansion (4.4) satisfy $a_0 = b_0$. However, the sum $(a_2 - b_0 + b_1 - b_2) = -g_0 - g_1$. While we do not have an argument for the vanishing of the linear combination of coefficients $g_0 + g_1$, if it does vanish, then we recover the result that $c - a$ is simply proportional to the coefficient a_0 of the pole term in the expansion (4.4). It would be interesting to see if a general argument exists that implies the structure assumed in (4.6) and that also enforces $g_0 + g_1 = 0$.

A more heuristic argument for the vanishing of the pole term in (4.5) for finite N theories is that in this case it appears that only a finite number of protected operators would enter into the computation of $c - a$. This would require the plethystic log of the index (which we take to be the finite- N counterpart of the single-trace index) to satisfy the condition $a_0 = b_0$ (while a_0 itself is non-zero), regardless of the regularity of $F(t, y)$ in (4.6). It is not clear to us what the significance of this relation is, and how it sees the distinction between finite N , where $a_0 = b_0$ may plausibly hold, and large N , where it is clearly violated. A better understanding of the structure of the finite- N index may be needed in order to clarify this condition and to explore the applicability of (1.2) at finite N .

As indicated above, the pole term in $c - a$ is non-vanishing in the large- N limit. This is essentially the counterpart of the divergence in the corresponding holographic computation [14–16]. In particular, the index provides a natural regulator of the sum over the shortened spectrum. For theories admitting a holographic dual, we conjecture (guided by [35] and following [16]) that the pole term may be related to geometric data given by

$$-\frac{3(a_0 - b_0)}{8} = -\frac{9}{128\pi^3} \left(19 \text{vol}(\text{SE}) + \frac{1}{8} \text{Riem}^2(\text{SE}) \right). \quad (4.8)$$

This is valid for all the holographic theories dual to smooth SE_5 manifolds that we considered in the present work.

It would be interesting to apply (1.2) to the finite- N theories where the full index is known as a means of testing the conjecture that this relation remains valid at finite N . A

possible issue is resolving how single- versus multi-trace operators contribute to $c - a$, and whether such contributions are properly handled by the plethystic log. Ideally, one would wish to replace (1.2) by some sort of operator expression acting directly on the index without having to resort to the plethystic log. Ref. [33] suggests that a simple relation exists of the form (4.1). It would be curious to see how this connects with (1.2), both at finite and large N .

Finally, at least in the large- N limit, we have demonstrated that the superconformal index contains information about the difference $c - a$. At some level, this is perhaps not all that surprising, as the holographic computation of $c - a$ depends only on the spectrum of protected operators, and the index contains the full information about such operators. One may ask whether it is possible to obtain a and c individually from the index. However, the holographic computation suggests that this would be a challenge, as long operators (which are not captured by the index) will contribute to the individual central charges. It may nevertheless be possible that the protected operators themselves retain sufficient information to allow for the extraction of a and c separately (see [19, 36–38] for interesting related discussions).⁴ In any case, it is clear that the index encodes a wide range of information that can be used to more fully characterize the SCFT, and is an important tool for their investigation, especially at strong coupling.

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⁴See in particular appendix B of [36] for relations between the central charges and the *single-letter* index.

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