

Spinorial flux tubes in $SO(N)$ gauge theories in 2+1 dimensions

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ABSTRACT: We investigate whether one can observe in $SO(3)$ and $SO(4)$ (lattice) gauge theories the presence of spinorial flux tubes, i.e. ones that correspond to the fundamental representation of $SU(2)$; and similarly for $SO(6)$ and $SU(4)$. We do so by calculating the finite volume dependence of the $J^p = 2^+$ glueball in 2 + 1 dimensions, using lattice simulations. We show how this provides strong evidence that these $SO(N)$ gauge theories contain states that are composed of (conjugate) pairs of winding spinorial flux tubes, i.e. ones that are in the (anti)fundamental of the corresponding $SU(N')$ gauge theories. Moreover, these two flux tubes can be arbitrarily far apart. This is so despite the fact that the fields that are available in the $SO(N)$ lattice field theories do not appear to allow us to construct operators that project onto single spinorial flux tubes.

KEYWORDS: Gauge Symmetry, Lattice Quantum Field Theory, Wilson, 't Hooft and Polyakov loops

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Contents

1	Introduction	1
2	Lattice preliminaries	3
3	Strategy: glueballs in small volumes	4
3.1	SU(2)	5
3.2	SU(4)	8
4	Glueballs in small volumes: SO(3), SO(4) and SO(6)	10
4.1	SO(3) and SU(2)	10
4.2	SO(4) and SU(2)	11
4.3	SO(6) and SU(4)	13
5	SO(6) flux tube as a bound state	15
6	Conclusions	16

1 Introduction

There are pairs of $SU(N)$ and $SO(N')$ gauge theories that share the same Lie algebra. These pairs are $SO(3)$ and $SU(2)$, $SO(4)$ and $SU(2) \times SU(2)$, and $SO(6)$ and $SU(4)$. It is interesting to ask how, if at all, the differing global properties of the groups in each pair affect the dynamics. This question has been addressed using lattice methods in [1–3] for gauge theories in 2+1 dimensions. It turns out [1, 2] that those glueball masses that have been reliably calculated, in particular the $J^P = 0^+$ and $J^P = 2^+$ ground states, are indeed consistent within each pair, as are string tensions and couplings. So also is the deconfining temperature, T_c , for $SO(4)$ compared to $SU(2) \times SU(2)$ and for $SO(6)$ compared to $SU(4)$ [3]. (There is no calculation of T_c for $SO(3)$.) The lightest excited glueball states, although not so reliably calculated, also appear to be consistent [2]. All this suggests quite strongly that the low-lying non-perturbative physics is the same within each of these pairs of gauge theories.

This raises some interesting questions. The comparison between the pairs of theories can only be made in their common sector of states. So consider for example a periodic space-time and a fundamental flux tube that winds around one of the spatial tori. In, say, $SO(6)$ such a flux tube with flux in the fundamental representation corresponds to a flux tube in $SU(4)$ with flux in the $k = 2A$ representation (the antisymmetric piece of the product of two fundamentals). And indeed one finds that the corresponding string tensions are equal when expressed in units of the coupling or the mass gap [1, 2]. On the other hand the fundamental flux tube of $SU(4)$ corresponds to the spinorial of $SO(6)$ and this cannot

be constructed from tensor products of the $SO(6)$ fundamental representation. That is to say, we cannot construct an operator using our $SO(6)$ lattice gauge fields that projects onto such a spinorial flux tube and therefore one might expect such a flux tube not to exist in the spectrum of the $SO(6)$ theory. However a state consisting of two winding flux tubes, one spinorial and one conjugate-spinorial, can have vacuum quantum numbers and should be accessible to our operators. If these winding flux tubes are sufficiently far apart from each other then they will not interact and this appears paradoxical if the theory does not possess states consisting of single spinorial flux tubes.

In this paper this is the issue we address. Since the $SO(N)$ gauge theory does not contain operators that project onto a single spinorial flux tube, our strategy will have to be indirect. We use the fact that in an $SU(N)$ gauge theory on sufficiently small volumes the lightest glueballs are composed of a pair of (conjugate) winding flux tubes in the fundamental representation. This can occur because a state consisting of a winding flux tube and its conjugate will have a non-zero overlap onto the usual contractible loops that form a basis for glueball operators. (In contrast to a single winding flux tube that has zero overlap by the standard centre symmetry argument.) The reason it actually does occur is that the energy of a winding flux tube decreases with its length (roughly linearly) and so as the lattice volume decreases a conjugate pair of such flux tubes should eventually become lighter than the ‘normal’ glueball state, and will then become the ground state glueball. The signal for this happening is thus the sudden onset of a rapid decrease in the glueball ground state energy as the volume decreases. If we find exactly the same finite volume behaviour in, say, $SO(6)$ as in $SU(4)$, then we can infer that the $SO(6)$ theory possesses states composed of a pair of (conjugate) flux tubes in the spinorial representation of $SO(6)$, i.e. ones that are in the fundamental of $SU(4)$. And similarly for $SO(3)$ and $SU(2)$ and also for $SO(4)$ and $SU(2) \times SU(2)$. This is indeed what we shall find. This is not such a surprise if one believes that the finite volume spectra of such pairs of $SO(N)$ and $SU(N')$ theories should be identical — but of course this need not be the case, especially since the effect of the differing global properties of the groups might well be enhanced on small volumes.

There is a corresponding puzzle in the context of the glueball spectrum which arises because the $SO(N)$ group elements are real so naively the fields available in the theory do not allow us to write an operator with $C = -$. Thus, for example, the $C = -$ states of $SU(4)$ would appear to have no counterpart in $SO(6)$, just like the fundamental flux tubes of $SU(4)$ discussed above. (These are not unrelated: after all contractible closed loops of fundamental flux can provide a basis for $C = -$ glueballs in $SU(4)$.) However a pair of $C = -$ glueballs will have $C = +$ and so one can ask whether such ‘scattering’ states appear in the $SO(N)$ theory. Indeed in $SU(4)$ an excited $J^{PC} = 0^{++}$ glueball that has a mass larger than about 3 times the mass gap has enough energy to decay into two 0^{--} glueballs [4] and will do so with some non-zero decay width. So if the low-lying $SU(4)$ and $SO(6)$ $J^{PC} = 0^{++}$ spectra are identical, the decay widths will be the same and such decay products of two 0^{--} glueballs will be present in the $SO(6)$ gauge theory. This includes states where the two $C = -$ glueballs are arbitrarily far apart. It is interesting to understand how this can be consistent with the apparent absence of single-particle $C = -$ states. This is a question that turns out to have a simple solution, as we have shown very

recently in [5], although it is not clear at this stage whether the methods developed in that paper will be useful in helping to resolve the puzzle that is the topic of the present paper.

In the next section we briefly describe our lattice setup. In section 3 we explain in more detail how pairs of (anti)fundamental flux tubes affect the small volume glueball spectra of $SU(N)$ gauge theories, and we provide some explicit calculations, in both $SU(2)$ and $SU(4)$, that provide evidence for this. We then move on, in section 4, to provide evidence for exactly the same behaviour in $SO(3)$, $SO(4)$ and $SO(6)$ gauge theories. Section 5 is something of an aside: we provide some evidence that the fundamental flux tube in $SO(6)$ is a bound state, presumably of two spinorial flux tubes. We then finish with some conclusions.

2 Lattice preliminaries

Our lattice calculations are standard and we refer to [4] for a detailed description of the $SU(N)$ calculations, and to [1] for details of our $SO(N)$ calculations. We simply remark here that for $SO(N)$ our degrees of freedom are $N \times N$ real orthogonal matrices U_l with unit determinant, assigned to the links l of the cubic periodic space-time lattice, while for $SU(N)$ the matrices are $N \times N$ complex unitary matrices with unit determinant. The partition function is

$$Z = \int \mathcal{D}U \exp\{-\beta S[U]\} \tag{2.1}$$

where $\mathcal{D}U$ is the Haar measure and we use the standard plaquette action for $S[U]$, and $\beta = 2N/ag^2$ where ag^2 is the dimensionless running coupling on the length scale of the lattice spacing a . (We recall that g^2 has dimensions $[m]^1$ in $D = 2 + 1$.)

Ground state masses M are calculated from the asymptotic time dependence of correlators, i.e.

$$\langle \phi^\dagger(t)\phi(0) \rangle = \sum_n |\langle n|\phi|\text{vac} \rangle|^2 e^{-E_n t} \xrightarrow{t \rightarrow \infty} e^{-Mt} \tag{2.2}$$

where M is the mass of the lightest state with the quantum numbers of the operator ϕ . The operator ϕ will be the product of link matrices around some closed path, with the trace then taken. To calculate the excited states E_n in eqn (2.2), one calculates (cross)correlators of several operators and uses these as a basis for a systematic variational calculation in e^{-Ht_0} where H is the Hamiltonian (corresponding to our lattice transfer matrix) and t_0 is some convenient distance. (Typically we choose $t_0 = a$.) To have good overlaps onto the desired states, so that one can evaluate masses at values of t where the signal has not yet disappeared into the statistical noise, one uses blocked and smeared operators. (For more details see e.g. [1, 4].) For $SO(N)$ at the smallest values of N , these operators, as used in [1], do not have very good overlaps onto the desired states. In [2] we have improved the algorithm for both $SO(3)$ and $SO(4)$ and are confident that our mass estimates in the present paper are reliable.

We label glueball states by spin J , parity P , and charge conjugation C . Of course on our square lattice this is misleading since many continuum states fall into the same representation, e.g. $J = 0, 4, 8, \dots$ are all in the same representation of the lattice rotation group. We shall take the convention of labelling the lightest state by the smallest spin J

in that representation. This will be correct for the lightest 0^+ and 2^+ states, the states of interest in this paper, but it would not be correct, for example, for the lightest ‘ 0^- ’ which is known to be in fact a 4^- [6].

Since we are interested in the qualitative behaviour of glueball masses as the spatial volume decreases, rather than in some precision calculation of mass ratios, we do not attempt an extrapolation to the continuum limit, but rather perform calculations at a single bare coupling (i.e at a single lattice spacing) that is chosen to be small enough that lattice corrections should be insignificant. Our specific choices are $\beta = 12$ for SU(2), $\beta = 13.7$ for SO(4), $\beta = 7$ for SO(3), $\beta = 46$ for SO(6), and $\beta = 63$ for SU(4). For SU(2) and SU(4) the lattice corrections to $m_G/\sqrt{\sigma}$ are much less than 1% (see tables 8,10 of [4]) For SO(4) and SO(6) the lattice corrections are also smaller than 1% (see tables 12,14,21,23, 28 of [1]). This also appears to be the case for SO(3) (see tables 11,20,28 of [1]) but the instability of the flux tube means that the errors are relatively large. In addition $\beta = 7$ is on the weak edge of the weak-strong coupling transition but this appears to have no effect on the states relevant to our calculations. In summary, we believe that our lattice calculations provide an accurate picture of what occurs in the corresponding continuum limits.

At various points in this paper we will make a statement that some mass μ in an SO(N) gauge theory equals a corresponding mass in the SU(N') gauge theory, where the groups share the same Lie algebra. What we mean by this is that when one calculates the continuum limits of μ/g^2 in SO(N) and in SU(N') then they are equal within errors once one relates the two values of g^2 by the theoretically predicted factors [1].

3 Strategy: glueballs in small volumes

As remarked in the Introduction, on small enough spatial volumes the energy of a conjugate pair of winding flux tubes may become small enough for this state to become the lightest glueball state. In particular this is most likely to occur for glueball quantum numbers that are accessible to a pair of flux tubes that are in their ground state, i.e. for $J^P = 0^+, 2^+$. For other quantum numbers at least one flux tube has to be in an excited state, which means its energy is considerably higher and so it may be that the flux tube state never becomes lighter than the ‘normal’ glueball. And clearly all this is only relevant if the volume on which the glueball ground state changes character from a normal glueball to a pair of winding flux tubes is large enough to still support a confining vacuum.

Of course on a very small volume a conjugate pair of flux tubes will strongly overlap and interact, and this may increase the energy of the flux tube state sufficiently that it never becomes light enough to take over as the lightest glueball. So we can be confident that a crossing will occur if the lightest glueball is sufficiently heavy that the level crossing takes place on a large enough volume for the flux tube interaction to be modest. Since the $J^P = 2^+$ glueball is heavier than the 0^+ , most of our calculations will focus on that state.

Much of the past discussion of this particular small volume physics has been in the context of $D = 3 + 1$ SU(N) gauge theories. Some early references are [7, 8] and a more recent reference is [9]. We shall now examine how this works in $D = 2 + 1$ for our SU(2) and SU(4) gauge theories.

SU(2) $\beta = 12.0$				
$l_x \times l_y \times l_t$	am_{2^+}	am_{2^-}	am_{0^+}	$2aE_f(l_x)$
$50 \times 50 \times 40$	0.9132(25)	0.9207(28)	0.5594(14)	1.360(12)
$42 \times 42 \times 40$	0.9146(20)	0.9155(24)	0.5589(10)	1.1470(34)
$34 \times 34 \times 40$	0.9142(23)	0.9165(25)	0.5578(9)	0.9206(28)
$30 \times 30 \times 48$	0.8776(59)	0.9215(30)	0.5585(21)	0.8062(32)
$26 \times 26 \times 56$	0.7876(65)	0.9222(36)	0.5565(17)	0.6852(28)
$20 \times 20 \times 56$	0.616(11)	0.9122(61)	0.5288(17)	0.5040(18)
$18 \times 18 \times 68$	0.5852(24)	0.9177(71)	0.5161(39)	0.4430(12)
$14 \times 14 \times 80$	0.4690(32)	0.8893(85)	0.4986(65)	0.3290(12)

Table 1. Lightest $J^P = 2^+, 2^-, 0^+$ glueball masses on various $l_x \times l_y$ spatial lattice volumes. Also the energy $E_f(l_x)$ of the lightest fundamental flux tube that winds around the x -torus. In SU(2) at an inverse bare coupling $\beta = 4/ag^2 = 12.0$.

3.1 SU(2)

We shall illustrate how this works in the $D = 2 + 1$ SU(2) gauge theory working at $\beta = 4/ag^2 = 12.0$. We calculate the lightest 2^+ , 2^- and 0^+ glueball masses on a range of lattice space-time volumes, $l_x \times l_y \times l_t$, as listed in table 1. Here we use $l_x = l_y = l$ so as to maintain the square lattice rotation symmetry that allows us to identify $J = 0$ and $J = 2$ states. We also calculate the energy $aE_f(l_x)$ of the lightest flux tube that winds around the periodic x -direction. We list twice that value in table 1 as that provides us an estimate of the energy of the state containing a flux tube and its conjugate, although, since it neglects the interaction energy between the two flux tubes, it can only serve as a rough estimate, especially on smaller volumes.

We display how the lightest SU(2) glueball mass varies with the spatial volume $V = l^2$ in figure 1. We have chosen to express everything in units of the $V = \infty$ mass gap, i.e. the mass of the lightest 0^+ glueball on large volumes. In the continuum limit and on an infinite volume the 2^+ and 2^- states should be degenerate. (Parity doubling for $J \neq 0$ in 2 space dimensions.) So as we decrease the volume, the breaking of the near-degeneracy is a useful indication of the onset of important finite volume effects. As we see in figure 1 the 2^+ and 2^- glueballs are indeed (nearly) degenerate on our largest volumes. As we decrease l we observe a sudden breaking of the degeneracy close to the value of l where the energy $2aE_f(l)$ of two flux tubes drops below the large- V mass of the lightest 2^+ glueball. As we decrease l further the lightest 2^+ mass decreases approximately in step with the value of $2aE_f(l)$. Moreover on these smaller volumes the first excited 2^+ state, which on larger volumes was nearly degenerate with the first excited 2^- , becomes nearly degenerate with the lightest 2^- . All this provides strong evidence for the scenario of level crossing, where the 2^+ ground state becomes, at small l , a state of two winding flux tubes, while the first excited 2^+ state is now the ‘normal’ glueball state that satisfies approximate parity doubling.

We also see from figure 1 that as we decrease l there is a slowly growing gap between the 2^+ ground state and the energy of two non-interacting flux tubes, $2aE_f(l)$. This is presumably due to the interaction energy of the two flux tubes which will increasingly

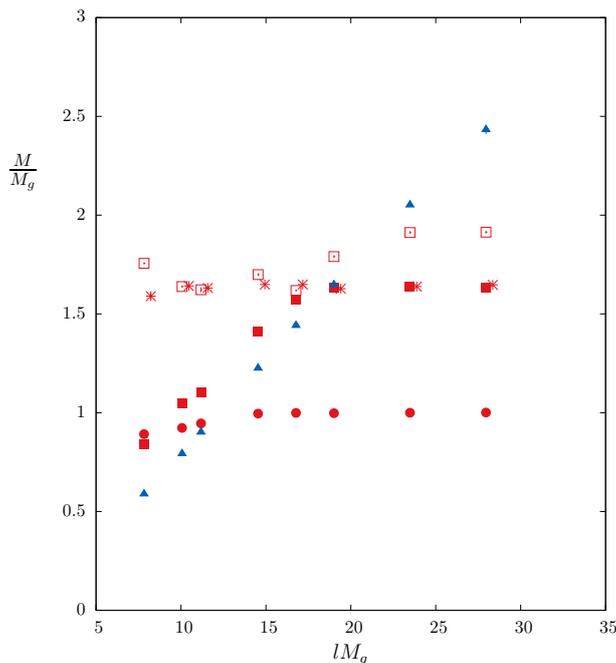


Figure 1. Lightest $J^P = 0^+$ (\bullet), 2^+ (\blacksquare) and 2^- (\star) glueball masses in SU(2) at $\beta = 12.0$ on various spatial volumes, l^2 , all in units of the infinite volume mass gap M_g . The 2^- has been shifted for clarity. Also shown is the first excited 2^+ (\square) glueball mass and twice the fundamental flux loop energy (\blacktriangle).

overlap as l decreases. (We also see that because the 0^+ is much lighter, any level crossing occurs at a much smaller value of l so its interpretation as a pair of flux tubes is much less compelling.) We would expect that if we increase the transverse size of the volume, i.e. $V = l \times l \rightarrow l \times l_\perp$ with $l_\perp \uparrow$, then the interaction energy between the flux tubes will decrease because the physical overlap between the flux tubes decreases. Of course once $l \neq l_\perp$ we lose the $\pi/2$ rotation symmetry and the would-be 0^+ and 2^+ operators will mix, so it only makes sense to talk of the mass gap. This is what we plot in figure 2 for our smallest value $l = 14$, using the values listed in table 2 (which have been obtained using ‘rotationally symmetric’ operators). We also list there the values of the lightest 0^+ and 2^+ glueballs on the symmetric l^2 volume. We see how the mass gap appears to approach the energy of two non-interacting flux tubes as l_\perp/l increases, further confirming the above scenario.

We have not attempted a more systematic study because this interpretation of the small volume behaviour of the 2^+ and 0^+ is not controversial. It would of course be useful to show how the energy of the two interacting flux tubes varies with the spatial volume but this would be a delicate calculation given the mixing of such a state with the normal lightest glueballs and would go beyond the scope of this paper. We did however calculate the correlators of a few operators of the form $l_f(p=0) \times l_f^\dagger(p=0)$ where $l_f(p=0)$ is (the trace of) a winding flux tube operator that is summed over all spatial sites, and hence has $p=0$. This gives us an operator that, neglecting interactions, projects onto a state with two conjugate $p=0$ winding flux tubes, i.e. with zero total and relative momentum. Adding

SU(2) $\beta = 12.0$				
$l \times l_{\perp} \times l_t$	am_{2^+}	am_{0^+}	am_g	$2aE_f(l)$
$14 \times 14 \times 80$	0.4690(32)	0.4986(65)	—	0.3290(12)
$14 \times 20 \times 80$	—	—	0.4328(56)	0.3090(32)
$14 \times 32 \times 80$	—	—	0.3565(65)	0.3106(14)
$14 \times 48 \times 80$	—	—	0.3333(33)	0.3098(16)

Table 2. Lightest $J^P = 2^+, 0^+$ glueball masses and the mass gap m_g on the spatial lattice volumes shown. Also the energy $E_f(l_x)$ of the lightest fundamental flux tube that winds around the x -torus. In SU(2) at an inverse bare coupling $\beta = 4/ag^2 = 12.0$.

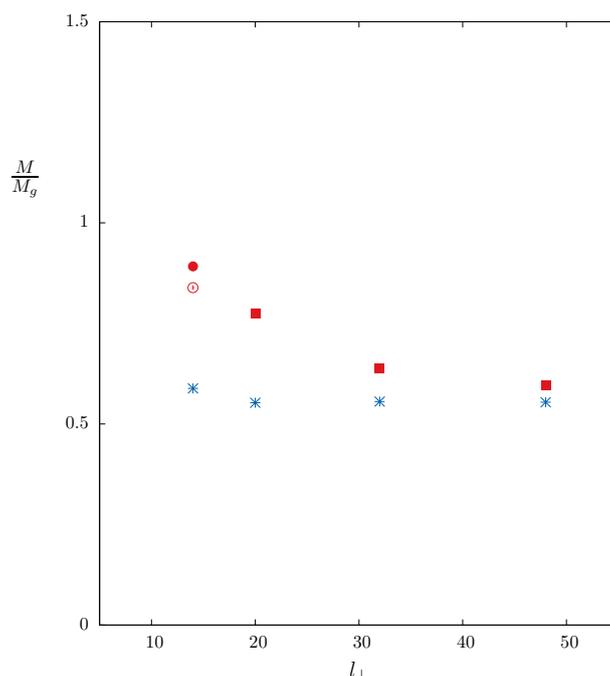


Figure 2. At $\beta = 12.0$ in SU(2) for our shortest flux tube, $l = 14$: variation of lightest masses as a function of the transverse spatial size, l_{\perp} , in units of the infinite volume mass gap M_g . For $l_{\perp} = 14$ the lightest $J^P = 0^+$ (\bullet) and $J^P = 2^+$ (\circ) masses are shown, and for $l_{\perp} > 14$ we show the mass gap (\blacksquare). Also displayed is twice the energy of the winding flux loop (\star).

or subtracting such product operators winding around the x and y tori gives us $J = 0$ and $J = 2$ operators. Of course what we are interested in are the states of two flux tube states that interact and once one has significant interactions (as we certainly will have in SU(2)) our operator will also project onto pairs of flux tube states with $p \neq 0$ (although the total momentum of the state must remain zero) as well as mixing with ‘normal’ glueballs. As the volume increases the allowed $p \neq 0$ values become smaller so that the energy spectrum becomes denser and it becomes increasingly difficult to extract, accurately, the lightest flux tube state. Nonetheless we find that on our smaller volumes we are more-or-less able to do so and we list in table 3 our best estimates of the energies of the lightest $J = 0$ and $J = 2$ states composed of a pair of conjugate flux tubes. (For $J = 0$ we perform a vacuum

SU(2): $l_f(p=0)l_f^\dagger(p=0)$			
$l_x \times l_y$	am_{2^+}	am_{0^+}	$2aE_f(l)$
14×14	0.4670(30)	0.4945(13)	0.3290(12)
18×18	0.5822(25)	0.5432(43)	0.4430(12)
20×20	0.618(11)	0.5661(57)	0.5040(18)
26×26	0.626(105)	0.668(11)	0.6852(28)

Table 3. Energies of lightest $J^P = 2^+, 0^+$ states formed from the product of a pair of (conjugate) winding flux tube operators on various $l_x \times l_y$ spatial lattice volumes. Also twice the energy $E_f(l_x)$ of the lightest fundamental flux tube that winds around the x -torus. In SU(2) at an inverse bare coupling $\beta = 4/ag^2 = 12.0$.

subtraction, since otherwise this would be the lightest state.) We see that as the volume increases these energies are consistent with approaching those of two non-interacting flux tubes and, indeed, a comparison with the 0^+ and 2^+ glueball masses in table 1 reinforces that interpretation.

3.2 SU(4)

We recall that the operators that project onto the lightest glueball in a large volume are formed by taking single traces of the path ordered product of link matrices around contractible closed loops and these are the operators we use in our glueball correlators on all our volumes. The states composed of two (conjugate) winding flux tubes correspond to double trace operators (up to the effects of mixing). Thus their appearance in glueball correlators will be increasingly suppressed as N increases, both for SU(N) and for SO(N), by the usual large- N counting rules [10–13]. It is thus necessary to check whether the small volume glueball dynamics described above manifests itself as clearly in our SU(4) glueball correlators as it did in our above SU(2) calculations.

Our SU(4) calculations are performed at a bare coupling $\beta = 8/ag^2 = 63.0$. We list in table 4 the lightest $2^+, 2^-$ and 0^+ masses that we obtain from our glueball correlators, on the various spatial volumes listed. We also list twice the energy, $aE_f(l)$, of the ground state of the winding flux tube of length l that carries fundamental flux. Since it will be useful later on, we also list the corresponding energy $aE_{2A}(l)$ of a flux tube carrying flux in the $k = 2$ antisymmetric representation.

We display in figure 3 the SU(4) analogue of our SU(2) plot in figure 1. The main features are similar in both plots: in SU(4) we have clear evidence, just as in SU(2), for a level crossing as the spatial volume decreases below the point where a pair of flux tubes becomes lighter than the large-volume normal glueball mass. So as the volume decreases beyond this value, one expects the corresponding glueball ground state to be composed of a pair of conjugate winding flux tubes.

At a more detailed level there are some differences between figure 3 and figure 1. In particular on the smallest volumes we see in figure 3 that there is little difference between the value of $2aE_f(l)$ and either the 0^+ or the 2^+ glueball mass. This is in contrast to what we see in figure 1. The natural interpretation is that what we are seeing is the suppression

SU(4) $\beta = 63.0$					
$l_x \times l_y \times l_t$	am_{2^+}	am_{2^-}	am_{0^+}	$2aE_f(l_x)$	$2aE_{2A}(l_x)$
$50 \times 50 \times 56$	0.7203(33)	0.7219(41)	0.4350(15)	1.0302(26)	1.3996(52)
$36 \times 36 \times 56$	0.7167(60)	0.7258(20)	0.4326(18)	0.7236(14)	0.9882(28)
$30 \times 30 \times 60$	0.672(15)	0.7293(22)	0.4316(12)	0.5904(18)	0.8096(20)
$26 \times 26 \times 70$	0.549(20)	0.695(15)	0.4262(12)	0.5000(16)	0.6792(30)
$22 \times 22 \times 80$	0.423(12)	0.680(16)	0.3972(47)	0.4034(12)	0.5446(18)
$18 \times 18 \times 90$	0.3244(42)	0.525(13)	0.3524(67)	0.3032(8)	0.4030(14)
$14 \times 14 \times 100$	0.202(11)	0.381(14)	0.233(10)	0.2044(12)	0.2700(26)

Table 4. Lightest $J^P = 2^+, 2^-, 0^+$ glueball masses on various $l_x \times l_y$ spatial lattice volumes. Also twice the energy of the lightest fundamental, $E_f(l_x)$, and $k = 2$, $E_{2A}(l_x)$, flux tubes that wind around the spatial torus. In SU(4) at an inverse bare coupling $\beta = 8/ag^2 = 63.0$.

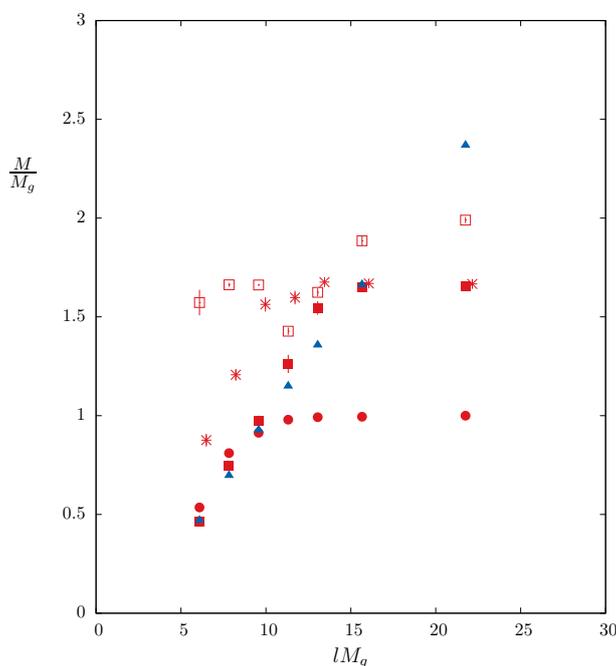


Figure 3. Lightest $J^P = 0^+$ (\bullet), 2^+ (\blacksquare) and 2^- (\star) glueball masses in SU(4) at $\beta = 63.0$ on various spatial volumes, l^2 , all in units of the infinite volume mass gap M_g . The 2^- has been shifted for clarity. Also shown is the first excited 2^+ (\square) glueball mass and twice the fundamental flux loop energy (\blacktriangle).

of the interaction between the two flux tubes as N increases, since standard counting tells us that the interaction energy between two colour singlets vanishes as $N \rightarrow \infty$. Of course the same counting also tells us that the overlap of the state consisting of two flux tubes onto the single trace glueball operators should decrease as N increases and so the worry is that we could lose the corresponding signal in the statistical noise of our glueball correlators. Although the overlaps do indeed appear to decrease as we move from SU(2) to SU(4) we find, fortunately, that this does not provide a significant obstacle to seeing this physics in

SO(3) $\beta = 7.0$			
$l_x \times l_y \times l_t$	am_{2^+}	am_{2^-}	am_{0^+}
$62 \times 62 \times 48$	0.6541(94)	0.661(5)	0.4004(21)
$46 \times 46 \times 48$	0.6582(83)	0.661(8)	0.4003(17)
$38 \times 38 \times 48$	0.587(15)	0.665(5)	0.3961(20)
$34 \times 34 \times 54$	0.513(11)	0.651(8)	0.3866(40)
$30 \times 30 \times 60$	0.4704(51)	0.648(7)	0.3793(23)
$26 \times 26 \times 72$	0.4160(33)	0.652(9)	0.3699(15)
$22 \times 22 \times 80$	0.3653(23)	0.638(14)	0.3502(74)
$18 \times 18 \times 100$	0.3051(39)	0.582(3)	0.3374(20)

Table 5. Lightest $J^P = 2^+, 2^-, 0^+$ glueball masses on various $l_x \times l_y$ spatial lattice volumes. In SO(3) at an inverse bare coupling $\beta = 6/ag^2 = 7.0$.

SU(4). This provides us with some confidence that we can perform a useful comparison between SU(4) and SO(6).

4 Glueballs in small volumes: SO(3), SO(4) and SO(6)

4.1 SO(3) and SU(2)

Since SO(3) has the same Lie algebra as SU(2) it is natural to ask if the light glueball spectra are the same [1]. In particular, if the ground states are the same in a finite volume then we can infer that on small volumes the state is composed of a pair of conjugate winding flux tubes that carry a flux which corresponds to the fundamental of SU(2), i.e. the spinorial of SO(3).

So in figure 4 we plot the lightest SO(3) 0^+ and 2^+ glueball masses (listed in table 5) as a function of the spatial volume $V = l^2$. The calculation is at an inverse bare coupling $\beta = 6/ag^2 = 7.0$ and we express all the quantities in units of the large volume mass gap, M_g , at this coupling. We also plot the corresponding quantities obtained in SU(2) at $\beta = 8/ag^2 = 12.0$, together with twice the energy of the lightest flux tube, $2E_f(l)$, in SU(2). The SU(2) quantities are also expressed in units of the large volume mass gap, but this time that of SU(2) rather than SO(3). Since one finds that in the continuum limit the large volume SU(2) and SO(3) mass gaps are the same within small errors, we can regard the SU(2) and SO(3) quantities in figure 4 as being expressed with a common scale.

What we see in figure 4 is that within quite small errors the volume dependences of the 2^+ and 0^+ glueball masses are identical in SO(3) and SU(2). So we can infer that on the smallest volumes the 2^+ glueball ground state in SO(3) is in fact a state composed of a conjugate pair of spinorial flux tubes, corresponding to a conjugate pair of fundamental flux tubes in SU(2).

As in SU(2) the SO(3) glueball masses on the smallest volumes are significantly heavier than $2E_f(l)$, the energy of two non-interacting flux tubes, and just as for SU(2) we interpret this gap as being due to the interaction energy arising from the fact that on these small volumes the flux tubes are necessarily overlapping. So we would expect that if we perform

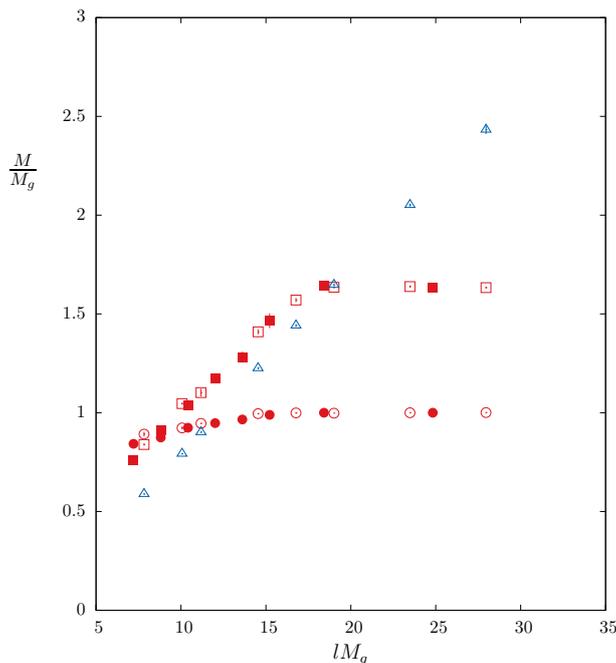


Figure 4. Lightest scalar (●) and tensor (■) glueballs in SO(3) on various spatial volumes, l^2 , all in units of the infinite volume mass gap M_g . And same for SU(2) (○, □ respectively). Also shown is twice the fundamental flux loop mass in SU(2) (△).

SO(3) $\beta = 7.0$			
$l \times l_{\perp} \times l_t$	am_{0^+}	am_{2^+}	am_g
$18 \times 18 \times 100$	0.3374(20)	0.3051(39)	—
$18 \times 30 \times 100$	—	—	0.2539(69)
$18 \times 60 \times 100$	—	—	0.2165(31)

Table 6. Lightest $J^P = 2^+, 0^+$ glueball masses and the mass gap m_g on various $l_x \times l_y$ spatial lattice volumes. In SO(3) at an inverse bare coupling $\beta = 6/ag^2 = 7.0$.

calculations on an asymmetric lattice volume, $V = ll_{\perp}$, and increase l_{\perp} then the flux tubes will overlap less and this gap will decrease, just as we observed for SU(2) in figure 2. We have performed such a study for SO(3) leading to the values listed in table 6, and we observe in figure 5 the same behaviour as in SU(2). All this helps to corroborate our interpretation of the nature of the finite volume glueball ground states in the SO(3) gauge theory.

4.2 SO(4) and SU(2)

The groups SO(4) and $SU(2) \times SU(2)$ share the same Lie algebra so the naive expectation is that the spectrum of SO(4) will be that of two mutually non-interacting SU(2) groups. If this is so, then we expect the lightest 0^+ and 2^+ glueballs to have the same masses in SU(2) and in SO(4), and on large volumes that indeed appears to be the case [1]. Since in SO(4) the confining flux tube in the fundamental representation has a string tension that is twice the fundamental SU(2) string tension [1] it will be approximately twice as massive

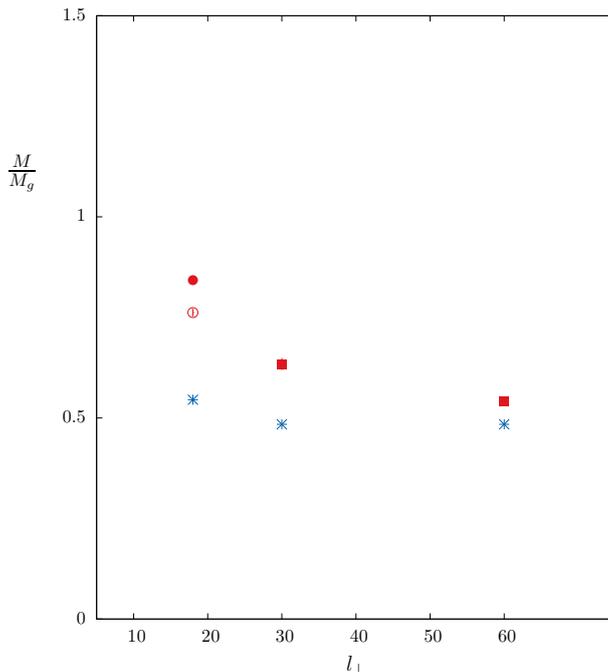


Figure 5. At $\beta = 7.0$ in $\text{SO}(3)$ for our shortest flux tube, $l = 18$: variation of lightest masses as a function of the transverse spatial size, l_{\perp} , in units of the infinite volume mass gap M_g . For $l_{\perp} = 18$ the lightest $J^P = 0^+$ (●) and $J^P = 2^+$ (○) masses are shown, and for $l_{\perp} > 18$ we show the mass gap (■). Also shown are estimates of twice the energy of the fundamental $\text{SU}(2)$ flux loop energy (*).

for a given length l of the flux tube. So as we decrease l the location of the level crossing where the lightest glueball becomes a (conjugate) pair of these flux tubes, and where its mass begins to decrease strongly with decreasing l , should occur at a much smaller value of l than in the case of $\text{SU}(2)$. If on the other hand we find that this decrease in the glueball mass begins at the same value of l as in $\text{SU}(2)$, which would of course be consistent with the very naive expectation that the $\text{SO}(4)$ and $\text{SU}(2)$ masses are the same at any volume, then this is evidence for a level crossing with a (conjugate) pair of spinorial flux tubes (corresponding to a pair of fundamentals in $\text{SU}(2)$).

In Figure 6 we display how the masses of the lightest 0^+ and 2^+ glueballs (listed in table 7) vary with the spatial volume in $\text{SO}(4)$ at $\beta = 13.7$ and in $\text{SU}(2)$ at $\beta = 12.0$ using the corresponding $V = \infty$ mass gaps to set the scale. (These mass gaps are equal in the continuum limit within small uncertainties [1].) We also show the energy of a pair of non-interacting fundamental flux tubes that wind around the torus, for both $\text{SU}(2)$ and $\text{SO}(4)$. We see that the latter energy is about twice the former at a given l , reflecting the factor of two difference in the string tensions. We also see that the 2^+ masses are essentially identical for the two groups, and that they start decreasing rapidly at the same value of l . At this value of l the energy of two $\text{SO}(4)$ fundamental flux tubes is much too large for it to be driving this decrease. So just as for $\text{SO}(3)$ we conclude that the $\text{SO}(4)$ gauge theory contains states that are composed of a pair flux tubes that carry a flux that corresponds to the fundamental of $\text{SU}(2)$. This despite the fact that we cannot construct

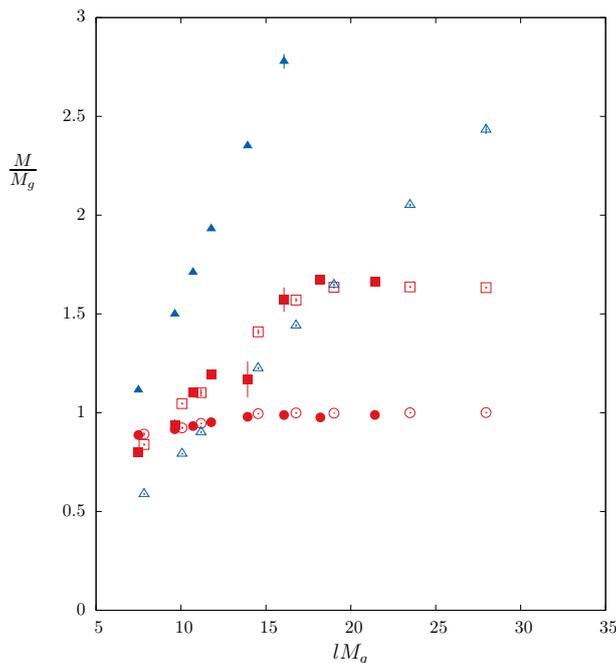


Figure 6. Lightest scalar (●) and tensor (■) glueballs in SO(4) on various spatial volumes, l^2 , all in units of the infinite volume mass gap M_g . And the same for SU(2) (○, □ respectively). Also shown is twice the fundamental flux loop mass in SU(2) (△) and SO(4) (▲).

SO(4) $\beta = 13.7$				
$l_x \times l_y \times l_t$	am_{2^+}	am_{2^-}	am_{0^+}	$2aE_f(l_x)$
$40 \times 40 \times 44$	0.8908(82)	0.862(17)	0.5294(89)	2.016(28)
$34 \times 34 \times 44$	0.8975(81)	0.9145(36)	0.5227(86)	1.745(15)
$30 \times 30 \times 48$	0.842(33)	0.878(16)	0.5292(67)	1.488(20)
$26 \times 26 \times 56$	0.626(49)	0.887(10)	0.5246(61)	1.259(5)
$22 \times 22 \times 56$	0.640(12)	0.8889(50)	0.5096(27)	1.034(3)
$20 \times 20 \times 56$	0.590(5)	0.8955(54)	0.4994(38)	0.9162(46)
$18 \times 18 \times 56$	0.501(17)	0.8884(61)	0.4906(38)	0.8028(28)
$14 \times 14 \times 68$	0.4288(89)	0.834(11)	0.4751(75)	0.5972(14)

Table 7. Lightest $J^P = 2^+, 2^-, 0^+$ glueball masses on various $l_x \times l_y$ spatial lattice volumes. Also twice the energy $E_f(l_x)$ of the lightest fundamental flux tube that winds around the x -torus. In SO(4) at an inverse bare coupling $\beta = 8/ag^2 = 13.7$.

a wave-functional for a single such flux tube, using our SO(4) gauge fields, and so might naively expect that a single flux tube of this type would not appear in the spectrum of the theory.

4.3 SO(6) and SU(4)

We now turn to SU(4) and SO(6) which also share a common Lie algebra. The fundamental of SO(6) corresponds to the $k = 2A$ of SU(4), i.e the totally antisymmetric piece of $f \otimes f$

SO(6) $\beta = 46.0$				
$l_x \times l_y \times l_t$	am_{2^+}	am_{2^-}	am_{0^+}	$2aE_f(l_x)$
$46 \times 46 \times 48$	0.7718(51)	0.7729(63)	0.4633(24)	1.4628(60)
$42 \times 42 \times 44$	0.7753(52)	0.7675(68)	0.4614(15)	1.3204(48)
$36 \times 36 \times 44$	0.7832(23)	0.7881(27)	0.4643(18)	1.1204(34)
$30 \times 30 \times 50$	0.730(21)	0.760(12)	0.4577(25)	0.9276(36)
$26 \times 26 \times 56$	0.671(13)	0.780(5)	0.4569(24)	0.7822(28)
$22 \times 22 \times 60$	0.529(21)	0.768(6)	0.4455(15)	0.6344(22)
$18 \times 18 \times 90$	0.390(10)	0.706(12)	0.4014(43)	0.4758(22)
$14 \times 14 \times 100$	0.226(13)	0.470(11)	0.3137(49)	0.3196(20)

Table 8. Lightest $J^P = 2^+, 2^-, 0^+$ glueball masses on various $l_x \times l_y$ spatial lattice volumes. Also twice the energy $E_f(l_x)$ of the lightest fundamental flux tube that winds around the x -torus. In SO(6) at an inverse bare coupling $\beta = 12/ag^2 = 46.0$.

of SU(4), while the fundamental f of SU(4) corresponds to the spinorial of SO(6). Thus the relation between the string tensions will be [4]

$$\frac{\sigma_{f,so6}}{\sigma_{sp,so6}} = \frac{\sigma_{2A,su4}}{\sigma_{f,su4}} \simeq 1.357(3). \tag{4.1}$$

using an obvious notation. So the difference in energy between a pair of flux tubes in the fundamental and spinorial representations will be much less in SO(6) than in SO(4). This would suggest that our evidence for spinorials might be less impressive for SO(6) than for SO(3) or for SO(4). In addition the larger value of N will reduce the overlap of our single trace operators onto the double trace operators that are natural for two flux tubes and might make it hard to see any level crossing at all. However in practice, as we have already seen for SU(4) in figure 3, the same large N effects appear to reduce the interaction energy in a way that makes the level crossing more stark, and this appears to largely compensate for the negative factors described above.

Turning now to our results, listed in table 8, we show in figure 7 the finite volume dependence of our lightest 2^+ and 0^+ glueball masses calculated in SO(6) at $\beta = 46$ and compare to that of SU(4) calculated at $\beta = 63$. We also show the energy of two non-interacting winding flux tubes for the case where both are in the fundamental of SO(6) and also the case where both are in the fundamental of SU(4) (corresponding to the spinorial of SO(6)) and in the $k = 2A$ of SU(4) (corresponding to the fundamental of SO(6)). As before we use the infinite volume mass gaps as our scale, noting that we know these to be equal in the continuum limit within small uncertainties [1]. We observe in figure 7 that the energy of two free flux tubes in the fundamental of SO(6) coincides with that of flux tubes in the $k = 2A$ of SU(4) over the whole range of volumes, indicating that the differing global properties of the SO(6) and SU(4) groups are unimportant for this physics. We also observe that the variation with volume of the lightest 2^+ and 0^+ glueball masses appears to be identical for SO(6) and SU(4). In particular the onset of the rapid decrease in the masses with decreasing l appears to occur at the same value of l for SO(6) and SU(4), indicating an identical level crossing dynamics. So it is very plausible that the small volume SO(6)

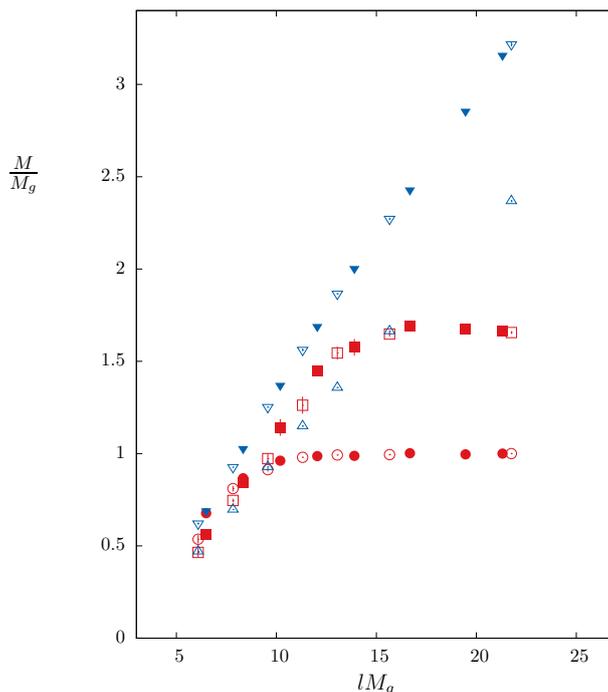


Figure 7. Lightest scalar (\circ) and tensor (\square) glueballs in SU(4) on various spatial volumes, l^2 , all in units of the infinite volume mass gap M_g . And same for SO(6) (\bullet , \blacksquare respectively). Also shown is twice the fundamental flux loop energy in SU(4) (\triangle) and in SO(6) (\blacktriangledown), and the $k = 2$ flux loop energy in SU(4) (∇).

glueballs are composed of a pair of winding (conjugate) spinorial flux tubes. Note that the comparison with SU(4) is crucial here: if we were relying only on the SO(6) results, then it would be difficult to argue from figure 7 that the level crossing is associated with pairs of spinorial rather than fundamental flux tubes, because these do not differ very much in energy.

5 SO(6) flux tube as a bound state

In SU(4) one finds [14] that the gap between the ground and first excited states of the $k = 2A$ winding flux tube is roughly constant with l as long as l is small enough that the higher excited states are significantly heavier (to avoid ambiguities due to level crossings). This is in marked contrast to the fundamental flux tube where the first excited state closely tracks the ‘Nambu-Goto’ string formula even down to very small values of l [14]. A natural explanation of this difference is that the $k = 2A$ spectrum is more complicated because the $k = 2A$ flux tube should be regarded as a (loosely) bound state of two fundamental flux tubes, with the binding being encoded in the world sheet action by a massive particle which will then appear in the flux tube spectrum [14]. (For a relevant discussion of the world sheet action see [15–18] and references therein.) Then one interprets this first excited state as being composed of the ground state flux tube with this massive excitation bound to it [14]. Conversely the existence of this ‘anomalous’ excitation encourages the interpretation of

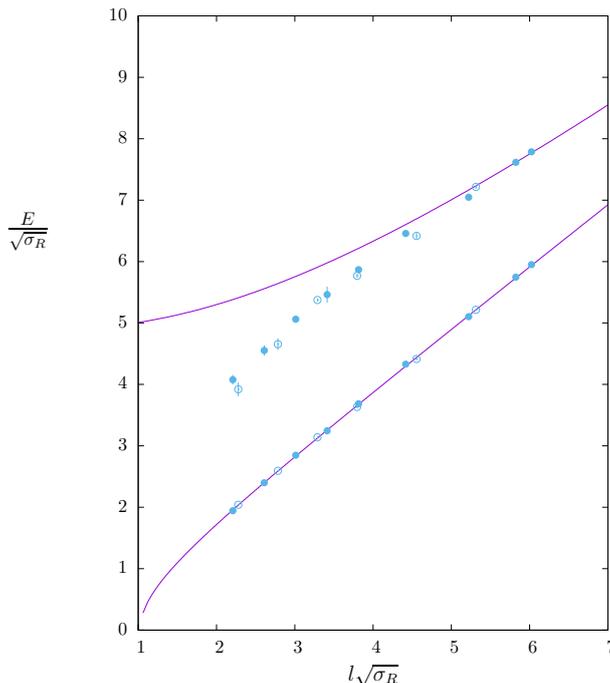


Figure 8. Energies of the lightest two winding flux tubes of length l on spatial volumes $l \times l_{\perp}$ with $l_{\perp} \gg l$ for small l . For SU(4) (\bullet) and SO(6) (\circ), scaled by $\sigma_R = \sigma_{2A}$ and $\sigma_R = \sigma_f$ respectively. Lines are ‘Nambu-Goto’ predictions.

the $k = 2A$ flux tube as a (loosely) bound state of two fundamental flux tubes. It is thus interesting to see if the SO(6) fundamental flux tube spectrum also shows such an ‘anomalous’ excited state. If it does then we would be motivated to interpret the SO(6) fundamental flux tube as a loosely bound state of two spinorial flux tubes.

So in figure 8 we plot the energies of the ground and first excited winding flux tubes in SO(6). For comparison we also plot the corresponding energies of the $k = 2A$ flux tubes in SU(4). The energies and lengths are expressed in units of the corresponding string tensions, which are known to be equal (within small uncertainties) in the continuum limit [1]. We see that the SU(4) and SO(6) spectra are identical within small errors. That is to say, we have some evidence that the fundamental flux tube of SO(6) is in fact a ‘loosely’ bound state of two spinorial flux tubes.

6 Conclusions

By calculating in SO(3), SO(4) and SO(6) gauge theories the finite volume behaviour of the lightest 2^+ glueball and, to a lesser extent, that of the 0^+ , and by comparing to the corresponding SU(N) gauge theories with the same Lie algebra, we obtained strong evidence that on small volumes the lightest glueball becomes a state composed of a pair of loosely bound spinorial flux tubes that wind around the spatial volume.

This is interesting because in our SO(N) gauge theories we cannot construct from the SO(N) lattice fields operators that project onto single spinorial flux tubes and so one

might naively expect that these would not exist in the spectrum of the theory. Of course our small volume glueball states are typically composed of a pair of winding (conjugate) spinorial flux tubes that are physically overlapping and which are interacting rather than free. However, as we have seen, if we work on asymmetric $l \times l_{\perp}$ spatial volumes and increase l_{\perp} then this interaction energy appears to decrease towards zero. Thus the theory does seem to include states that are composed of a pair of winding spinorial flux tubes that are arbitrarily far apart. Having established this brings into sharp focus the interesting question of how to square it with the naive expectation that the theory does not include states composed of a single spinorial flux tube.

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