Constraints on parity violating conformal field theories in $d = 3$

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ABSTRACT: We derive constraints on three-point functions involving the stress tensor, T, and a conserved U(1) current, j, in 2+1 dimensional conformal field theories that violate parity, using conformal collider bounds introduced by Hofman and Maldacena. Conformal invariance allows parity-odd tensor-structures for the $\langle TTT \rangle$ and $\langle jjT \rangle$ correlation functions which are unique to three space-time dimensions. Let the parameters which determine the $\langle TTT \rangle$ correlation function be t_4 and α_T, where α_T is the parity-violating contribution. Similarly let the parameters which determine $\langle jjT \rangle$ correlation function be a_2, and α_J, where α_J is the parity-violating contribution. We show that the parameters (t_4, α_T) and (a_2, α_J) are bounded to lie inside a disc at the origin of the t_4 - α_T plane and the a_2 - α_J plane respectively. We then show that large N Chern-Simons theories coupled to a fundamental fermion/boson lie on the circle which bounds these discs. The 't Hooft coupling determines the location of these theories on the boundary circles.

Contents

1   Introduction

Conformal field theories in $d = 3$ are of interest both in the context of holography as well as in condensed matter physics. In the context of holography there are several well studied CFTs in $d = 3$ which are known to admit a holographic dual. Just to name a few, the M2-brane theory [1], the free and critical $O(N)$ model [2–4], the ABJ(M) theories [5, 6] and more recently the Chern-Simons theories with fundamental matter [7, 8]. The critical $O(N)$ model as well as the Chern-Simons theories with fundamental matter are proposed to be dual to higher spin theories in AdS_4. Several super conformal field theories have been constructed in [9] which exhibit a rich structure of dualities [10]. Holographic theories have been used study strongly coupled phenomenon relevant to condensed matter physics [11]. Conformal field theories in $d = 3$ exhibit a rich variety of physical phenomena, and a detailed study of them can provide an understanding of quantum gravity in four dimensions.

A particularly important feature of conformal field theories in three dimensions is that physically relevant examples need not preserve parity, thanks to the possibility of a Chern-Simons term [12–14]. The study of Chern-Simons theories with matter in the past several years has been very fruitful, and led to the discovery of several new non-supersymmetric dualities: the higher spin/vector model duality, see [15] for a review and the Bose-Fermi duality [16], which has been well tested at large N [17–22], and is also believed to hold at finite N [23]. Quantum Hall fluids [24–26] are a particularly important physical example where such theories are relevant.

Consider a conformal field theory in $d = 3$ with a U(1) conserved current j. Associated with the conformal field theory, is its stress tensor T and let the theory be parity violating. In such a theory, conformal invariance constrains the three point functions $\langle jjT \rangle$ and $\langle TTT \rangle$ to be of the form [27, 28],

$$\langle jjT \rangle = n_s^j \langle jjT \rangle_{\text{free boson}} + n_f^j \langle jjT \rangle_{\text{free fermion}} + p_j \langle jjT \rangle_{\text{parity odd}}, \tag{1.1}$$
$$\langle TTT \rangle = n_s^T \langle TTT \rangle_{\text{free boson}} + n_f^T \langle TTT \rangle_{\text{free fermion}} + p_T \langle TTT \rangle_{\text{parity odd}},$$

where $\langle \ldots \rangle_{\text{free boson}}$, $\langle \ldots \rangle_{\text{free fermion}}$ denote the correlator a real free boson and a real free fermion respectively. The parity even tensor structures are written down in [29], while the parity odd structure was first discovered in [27]. The numerical coefficients $n_s^{j,T}, n_f^{j,T}$ are theory dependent, and once the normalization of the parity odd term is fixed, the parity violating coefficient $p_{j,T}$ can be determined from a given theory [7, 8].[1] At this point, it is important to mention, that there is a parity odd contact term even in the two point function as found first in [33] and discussed in detail in [34, 35]. As we will see subsequently, we will need to normalize the three point functions by dividing with the two point function. However we will see that in the required normalizations, the parity odd contact term in the two point functions do not play a role in our analysis.[2]

For parity even theories and for theories in $d = 4$ and higher dimensions the conformal collider bounds, found by [36], impose constraints on the parity even coefficients that occur in such correlators [37–39]. However the role of the parity odd coefficients $p_{j,T}$ in conformal collider bounds in $d = 3$ has not been investigated in detail. In the appendix C of [28], using general symmetry arguments, the contribution of a parity odd term to the energy flux in the conformal collider for excitations created by the stress tensor was written down. However the precise relation with the coefficient p_T was not provided.

In this paper we start from the correlators given in 1.1 and obtain the energy flux in the conformal collider as a function of the parameters in the 3 point function and impose the positive energy flux condition of Hofman and Maldacena. On normalising the 2-point function of the stress tensor and the U(1) current, the energy flux for excitations created by the stress tensor or the U(1) current is determined by 2 parameters. We show that the positive energy flux condition constrains these parameters to the region of a disc at the origin. We also show that large N Chern-Simons theories lie on the bounding circle of this disc. Their position on the circle is determined by the 't Hooft coupling.

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[1] Similar calculations of three point functions of conserved currents in supersymmetric Chern-Simons theories with matter has been done in [30–32]. Such theories do not have the parity violating contribution.

[2] We thank the referee for pointing this out to us.

To state our results, let us briefly describe the set up of [36]. A gedanken collider physics experiment was carried out, where one studies the effect of localized perturbations at the origin. The integrated energy flux per unit angle, over the states created by such perturbations, was measured at a large sphere of radius r.

$$\langle E_{\hat{n}} \rangle = \frac{\langle 0|\mathcal{O}^\dagger E_{\hat{n}} \mathcal{O}|0 \rangle}{\langle 0|\mathcal{O}^\dagger \mathcal{O}|0 \rangle},$$

$$E_{\hat{n}} = \lim_{r \to \infty} r^2 \int_{-\infty}^{\infty} dt n^i T_i^t(t, r\hat{n}),$$

$$\mathcal{O} \sim \frac{\epsilon^{ij} T_{ij}}{\sqrt{\langle \epsilon^{ij} T_{ij}|T_{ij}\epsilon^{ij} \rangle}}, \qquad \frac{\epsilon^i j_i}{\sqrt{\langle \epsilon^j j_j|j_i \epsilon^i \rangle}}, \tag{1.2}$$

where, $\hat{n}$ is a unit vector in R^3, which specifies the direction of the calorimeter and $\mathcal{O}$ is the operator creating the localised perturbation. Under a suitable transformation of coordinates it can be shown that positivity of energy flux measured in such a way is equivalent to demanding that the averaged null energy taken over the states be positive, The fact that averaged null energy is positive in any unitary interacting conformal field theory was shown in [40–42]. Thus, by demanding the positivity of energy flux, one obtains various constraints on the parameters of the three point function of the CFT, depending on the operators used to create the states $\mathcal{O}$. Similar constraints for correlators involving higher spins were obtained using unitarity in [43]. There has been a systematic study of such constraints both from the context of holography and CFT. Recently such constraints were used to place bounds on spectral sum rules of CFTs in arbitrary dimensions [44].

In this paper we perform this analysis for general CFTs in $d = 3$ including the parity odd terms in (1.1).

For states created by insertion of currents in two orthogonal directions say x, y and the calorimeter placed in the y direction, the energy observed in the conformal collider takes the form of a matrix which is given by

$$\hat{E}(j) = \begin{pmatrix} \frac{E}{4\pi}(1 - \frac{a_2}{2}) & \frac{\alpha_j E}{8\pi} \\ \\ \frac{\alpha_j E}{8\pi} & \frac{E}{4\pi}(1 + \frac{a_2}{2}) \end{pmatrix}, \tag{1.3}$$

where a_2 is the dimensionless parameter introduced by Hofman and Maldacena [36] and α_j is the contribution of the parity odd part of the three point function to the energy functional. They are related to the three parameters of the three point functions as follows

$$a_2 = -\frac{2(n_f^j - n_s^j)}{(n_f^j + n_s^j)}, \qquad \alpha_j = \frac{4\pi^4 p_j}{(n_f^j + n_s^j)}. \tag{1.4}$$

The diagonal elements of the energy matrix are due to the parity even part of the three point correlators while, parity odd contributions to the three point functions is responsible for the off diagonal elements. A similar matrix is obtained when one considers the localized

perturbations created by stress tensor insertions. This is given by

$$\hat{E}(T) = \begin{pmatrix} \frac{E}{4\pi}(1 - \frac{t_4}{4}) & \frac{\alpha_T E}{16\pi} \\ \\ \frac{\alpha_T E}{16\pi} & \frac{E}{4\pi}(1 + \frac{t_4}{4}) \end{pmatrix}, \tag{1.5}$$

where t_4 is the dimensionless parameter first introduced by [36] and α_T is the parity odd contribution. They are related to the coefficients in (1.1) by

$$t_4 = -\frac{4(n_f^T - n_s^T)}{n_f^T + n_s^T}, \qquad \alpha_T = \frac{8\pi^4 p_T}{3(n_f^T + n_s^T)}. \tag{1.6}$$

Positivity of energy requires that the eigenvalues of these symmetric matrices be positive. Since the trace of the diagonal elements is always positive, this condition boils down to the fact that the determinant must be positive. This leads us to the following constraints

$$a_2^2 + \alpha_j^2 \le 4, \qquad t_4^2 + \alpha_T^2 \le 16.$$

Thus the 2 parameters determining each of the 3 point functions are constrained to lie on a disc at the origin.

As discussed, large N Chern Simons theories at level κ coupled to fundamental matter are conformal field theories which are known to be parity violating. Let us consider the case of fundamental fermions. Using softly broken higher spin symmetry, it is known that the theory dependent parameters of the three point functions in (1.1) are given by [45]

$$n_s^T(f) = n_s^j(f) = 2N \frac{\sin\theta}{\theta} \sin^2 \frac{\theta}{2}, \qquad n_f^T(f) = n_s^j(f) = 2N \frac{\sin\theta}{\theta} \cos^2 \frac{\theta}{2},$$

$$p_T(f) = \alpha N \frac{\sin^2\theta}{\theta}, \qquad p_j(f) = \alpha' N \frac{\sin^2\theta}{\theta}, \tag{1.7}$$

where θ is the 't Hooft coupling given by

$$\theta = \pi \frac{N_f}{\kappa}. \tag{1.8}$$

Here the (f) in the brackets refer to the fact that we are dealing with the theory of fermions in the fundamental representation. The dependence of the parameters on the t 'Hooft coupling can also be found by summing over the planar diagrams as done in [16, 17]. Once the normalization of the parity odd tensor structures $\langle jjT \rangle_{\text{parity odd}}$, $\langle TTT \rangle_{\text{parity odd}}$ is agreed up on, the numerical constants α, α' can be determined by a perturbative one loop calculation. We fix the normalization of the parity odd tensor structures as given in [7].[3] We then carefully redo the perturbative one loop analysis of [7] in section 5.1 and find

$$\alpha = \frac{3}{\pi^4}, \qquad \alpha' = \frac{1}{\pi^4}. \tag{1.9}$$

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[3]These are written down in equations (3.1) and (4.1). Of course, one could instead choose to define the normalisation for the parity odd forms of three-point functions by the convention that $\alpha = \alpha' = 1$. This would be the (possibly more natural) choice of normalisation of parity-odd forms in which the results of [45] are implicitly stated.
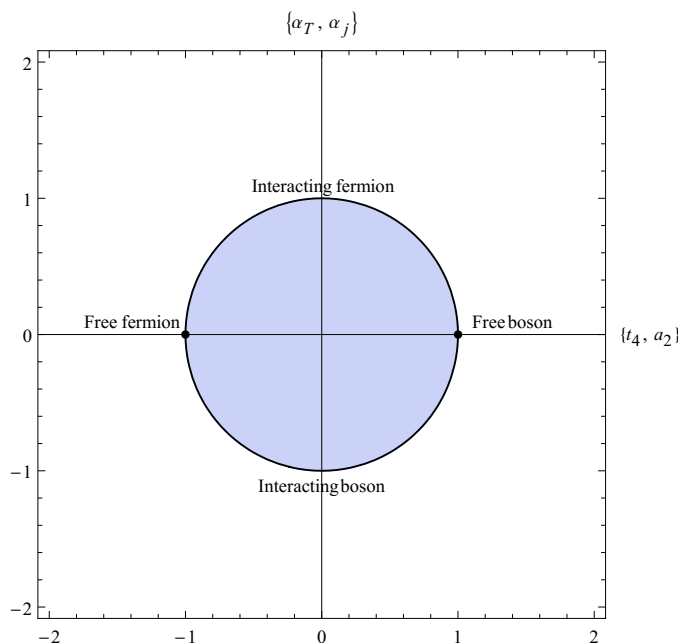
Figure 1. The space of conformal field theories in $d = 3$ obeying the conformal collider bounds is shaded. The y-axis denotes the parity odd coefficient of either the 3-pt functions $\langle TTT \rangle$ or $\langle jjT \rangle$, the x-axis denotes the parity even coefficient. Large N Chern-Simons theories lie on the boundary of the disc. The position on the disc corresponds to the 't Hooft coupling of these theories. If we choose the convention that λ is positive, the top-half circle corresponds to fundamental fermions, and the bottom half corresponds to fundamental bosons.

Now substituting these values in (1.4) and (1.6) we obtain

$$a_2 = -2\cos\theta, \qquad\qquad \alpha_j = 2\sin\theta, \qquad\qquad (1.10)$$
$$t_4 = -4\cos\theta, \qquad\qquad \alpha_T = 4\sin\theta.$$

Thus Chern-Simons theory with fundamental matter lie on the circle bounding the disc (1). Their location on the bounding circle is parametrized by the 't Hooft coupling $\theta = \frac{\pi N}{\kappa}$. Here $\theta = 0$ is the theory with free fermions, while $\theta = \pi$ is the theory with critical bosons. The range $0 < \theta < \pi$ can be thought of a theory with interacting fermions[4] in the fundamental representation with positive κ (or interacting bosons in the fundamental representation with negative κ). The range $\pi < \theta < 2\pi$ corresponds to the theory of interacting bosons with positive κ (or interacting fermions in the fundamental representation with negative κ). This is because using the bosonization map of [16, 17], the 't Hooft coupling of the interacting bosonic theory is related to the fermionic one by $\theta_b = \pi + \theta_f$. We summarise the space of conformal field theories in 3 dimensions in figure 1

The organization of the paper is as follows. In section 2, we set up the kinematics for the energy matrix observed at the collider. In section 3 and 4 we evaluate the energy matrix for excitations created by current and stress tensor respectively. We show that demanding

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[4]Note that in this convention, θ is measured from the negative $[t_4, a_2]$ axis. However if we flip the sign of t_4, a_2, then θ would have the conventional defintion.

the positivity of the eigen values of the energy matrix, the parameters a_2, α_j and t_4, α_T are constrained to lie on a disc at the origin in their respective 2-planes. In section 5 we apply these results to large N Chern-Simons theories coupled to fundamental matter and demonstrate that these theories lies on the circles bounding the discs. The position on the circle is determined by the 't Hooft coupling of the theory. Appendix A contains the detail evaluation of one of the terms contributing to the energy matrix corresponding to stress tensor excitation as an example. Appendix B contains the details of the perturbative evaluation of the parity odd three point functions of interest in large N Chern-Simons theories coupled to fundamental fermions. Finally appendix C contains the results of some integrals which arise in evaluation of the energy matrix.

2 Energy matrix and positivity of energy

In this section we set up the thought experiment in a bit more detail. Following [36], we consider localised perturbations of the CFT in minkowski space.

$$ds^2 = -dt^2 + dx^2 + dy^2. \tag{2.1}$$

These perturbations evolve in time and spread out. In order to measure the energy flux, we consider concentric circles (concentric spheres for higher dimensions) at which the detector is placed. The energy measured in a direction $\hat{n}$ is then defined as,

$$E_{\hat{n}} = \lim_{r \to \infty} r \int_{-\infty}^{\infty} dt n^i T_i^t(t, r\hat{n}), \tag{2.2}$$

where r is the radius of the circle on which the detector is placed and $\hat{n}$ is a unit vector which determines the point on the circle where the detector is placed. Alternatively one can place the detector at the future null infinity from the very beginning and integrate over the null time. These two definitions are equivalent [46]. For our convenience, we place the detector along the y direction i.e $\hat{n} = (0, 1)$. In order to calculate the contributions to the energy (eqn 2.2) at future null infinity, we introduce the light cone coordinates $x^{\pm} = t \pm y$. Let x be the other spatial co-ordinate. In the limit $r \to \infty$, eqn 2.2 becomes,

$$\begin{aligned} E &= \lim_{x^+ \to \infty} \left(\frac{x^+ - x^-}{2} \right) \int_{-\infty}^{\infty} \frac{dx^-}{2} (-T_{+y} - T_{-y}) n^y, \\ &= \lim_{x^+ \to \infty} \left(\frac{x^+ - x^-}{2} \right) \int_{-\infty}^{\infty} \frac{dx^-}{2} (-T_{++} + T_{--}), \end{aligned} \tag{2.3}$$

where we have dropped the subscript $\hat{n}$. To see that the contributions form T_{++} vanish at future null infinity $x^+ \to \infty$, we follow [36] and perform the following co-ordinate transformation

$$y^+ = -\frac{1}{x^+}, \qquad y^- = x^- - \frac{x^2}{x^+}, \qquad y^1 = \frac{x}{x^+}. \tag{2.4}$$

The stress tensor changes as

$$T_{\mu\nu}^x = \frac{\partial y^c}{\partial x^{\mu}} \frac{\partial y^d}{\partial x^{\nu}} T_{cd}^y, \tag{2.5}$$

and using the transformation in (2.4) we obtain

$$\lim_{x^+ \to \infty} T^x_{++} = (y^1)^4 T^y_{--}, \qquad T^x_{--} = T^y_{--}. \tag{2.6}$$

Here superscripts refer to the stress tensor in the respective co-ordinate system. Thus at future null infinity we see that its only the component T^y_{--} which determines all the components of the energy in the x co-ordinate system. Further more the component T^x_{++} is suppressed. Therefore the energy detected at the calorimeter is given by

$$E = \lim_{x^+ \to \infty} \left(\frac{x^+ - x^-}{2} \right) \int_{-\infty}^{\infty} \frac{dx^-}{2} T_{--}. \tag{2.7}$$

We are interested in the expectation value of the energy operator on states created by stress tensor and current insertions. The normalized states are defined as

$$\mathcal{O}_E |0\rangle = \frac{\int dt dx dy e^{iEt} \mathcal{O}(t, x, y)|0\rangle}{\sqrt{\langle \mathcal{O}_E | \mathcal{O}_E | \rangle}}, \tag{2.8}$$

where $\mathcal{O}$ are operators constructed from the current or stress tensor with definite polarizations. They are given by

$$\mathcal{O}(\epsilon; T) = \epsilon_{ij} T^{ij}, \qquad \text{or} \qquad \mathcal{O}(\epsilon, j) = \epsilon_i j^i. \tag{2.9}$$

The norm in (2.8) is defined by

$$\langle \mathcal{O}_E | \mathcal{O}_E \rangle = \int d^3 x e^{iEt} \langle \mathcal{O}(t, x, y) \mathcal{O}(0) \rangle, \tag{2.10}$$

where we have used translation invariance to factor out one integral and place one of the operators at the origin. Thus the norms are obtained by evaluating the two point function of the operator and integrating the space time point corresponding to one of the operator.

Let us first look at states created by the stress tensor excitations. Since the stress tensor is traceless, the allowed polarisations for stress tensors satisfy

$$\epsilon^\mu_\mu = 0. \tag{2.11}$$

Therefore we choose 2 independent polarisations given by

$$\epsilon^{xy} = \epsilon^{yx} = 1,$$
$$\epsilon'^{xx} = -\epsilon'^{yy} = 1. \tag{2.12}$$

Let us label the states created by these polarizations as

$$\mathcal{O}_E(\epsilon; T)|0\rangle, \qquad \mathcal{O}_E(\epsilon'; T)|0\rangle, \tag{2.13}$$

where we use the definition of the state given in (2.9) and (2.10) with the polarizations ϵ and ϵ'. We can then define the energy matrix between these states as

$$\hat{E}(T) = \begin{pmatrix} \langle 0|\mathcal{O}^\dagger_E(\epsilon; T)\mathcal{E}\mathcal{O}_E(\epsilon; T)|0\rangle & \langle 0|\mathcal{O}^\dagger_E(\epsilon; T)\mathcal{E}\mathcal{O}_E(\epsilon'; T)|0\rangle \\ \langle 0|\mathcal{O}^\dagger_E(\epsilon'; T)\mathcal{E}\mathcal{O}_E(\epsilon; T)|0\rangle & \langle 0|\mathcal{O}^\dagger_E(\epsilon'; T)\mathcal{E}\mathcal{O}_E(\epsilon'; T)|0\rangle \end{pmatrix}, \tag{2.14}$$

where $\mathcal{E}$ is defined in (2.3). To be explicit let us write out the elements in the row of the energy matrix.

$$\langle 0|\mathcal{O}_E^\dagger(\epsilon;T)\mathcal{E}\mathcal{O}_E(\epsilon;T)|0\rangle = \frac{1}{\langle\mathcal{O}_E(\epsilon;T)|\mathcal{O}_E(\epsilon,T)\rangle} \times \tag{2.15}$$

$$\times \int d^3x e^{iEt} \lim_{x_1^+\to\infty} \frac{x_1^+ - x_1^-}{4} \int dx_1^- \langle\epsilon\cdot T(x)T_{--}(x_1)\epsilon\cdot T(0)\rangle,$$

$$\langle 0|\mathcal{O}_E^\dagger(\epsilon;T)\mathcal{E}\mathcal{O}_E(\epsilon';T)|0\rangle = (\langle\mathcal{O}_E(\epsilon';T)|\mathcal{O}_E(\epsilon,'T)\rangle\langle\mathcal{O}_E(\epsilon;T)|\mathcal{O}_E(\epsilon,T)\rangle)^{-\frac{1}{2}}$$

$$\times \int d^3x e^{iEt} \lim_{x_1^+\to\infty} \frac{x_1^+ - x_1^-}{4} \int dx_1^- \langle\epsilon\cdot T(x)T_{--}(x_1)\epsilon'\cdot T(0)\rangle.$$

The other entries of the matrix in (2.14) are defined similarly. Essentially we need to evaluate ratios of 3 point functions of the stress tensor, take the $x_1^+ \to \infty$ and then perform the integral over the space time point of the last insertion of the stress tensor. Note that the normalisation involves the two point function of the stress tensor with the same polarization. It can be seen from [34] that the parity odd contact term in the two point functions vanishes when the two stress tensors have the same polarizations.

Let us examine the energy matrix corresponding to the charge excitations. We choose the two independent polarisations to be

$$\epsilon^x = 1, \qquad \epsilon'^y = 1. \tag{2.16}$$

The corresponding states are

$$\mathcal{O}_E(\epsilon;j)|0\rangle, \qquad \mathcal{O}_E(\epsilon';j)|0\rangle. \tag{2.17}$$

The energy matrix which results from these two states are given by

$$\hat{E}(j) = \begin{pmatrix} \langle 0|\mathcal{O}_E^\dagger(\epsilon;j)\mathcal{E}\mathcal{O}_E(\epsilon;j)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon;j)\mathcal{E}\mathcal{O}_E(\epsilon';j)|0\rangle \\ \\ \langle 0|\mathcal{O}_E^\dagger(\epsilon';j)\mathcal{E}\mathcal{O}_E(\epsilon;j)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon';j)\mathcal{E}\mathcal{O}_E(\epsilon';j)|0\rangle \end{pmatrix}, \tag{2.18}$$

Again to be explicit, we write down the entries corresponding to the first row

$$\langle 0|\mathcal{O}_E^\dagger(\epsilon;j)\mathcal{E}\mathcal{O}_E(\epsilon;j)|0\rangle = \frac{1}{\langle\mathcal{O}_E(\epsilon;j)|\mathcal{O}_E(\epsilon,j)\rangle} \times \tag{2.19}$$

$$\times \int d^3x e^{iEt} \lim_{x_1^+\to\infty} \frac{x_1^+ - x_1^-}{4} \int dx_1^- \langle\epsilon\cdot j(x)T_{--}(x_1)\epsilon\cdot T(0)\rangle,$$

$$\langle 0|\mathcal{O}_E^\dagger(\epsilon;j)\mathcal{E}\mathcal{O}_E(\epsilon';j)|0\rangle = (\langle\mathcal{O}_E(\epsilon';j)|\mathcal{O}_E(\epsilon,'j)\rangle\langle\mathcal{O}_E(\epsilon;j)|\mathcal{O}_E(\epsilon,j)\rangle)^{-\frac{1}{2}}$$

$$\times \int d^3x e^{iEt} \lim_{x_1^+\to\infty} \frac{x_1^+ - x_1^-}{4} \int dx_1^- \langle\epsilon\cdot j(x)T_{--}(x_1)\epsilon'\cdot j(0)\rangle.$$

Again we see that the normalisation involves the two point function of the currents with the same polarization. It can be seen from [34] that the parity odd contact term in the two point functions vanishes when the two currents have the same polarizations.

The condition of positivity of the energy observed by the calorimeter [36] translates to demanding that the eigen values of the energy matrix (2.14), (2.18) be positive. We will see that only the parity even terms in (1.1) contribute to the diagonal and the parity odd term which contributes to the off diagonal entries of these matrices.

3 Energy matrix for charge excitations

The basic ingredient to evaluate the energy matrix for charge excitations is the three point function of two U(1) currents with a single insertion of the stress tensor. Including the parity odd tensor structures, this is given by [27, 29],

$$
\langle j(x)T(x_1)j(0) \rangle = \frac{1}{|x_1 - x|^3 |x_1|^3 |x|} \epsilon_2^\sigma I_\sigma^\alpha(x - x_1) \epsilon_3^\rho I_\rho^\beta(-x_1) \epsilon_1^{\mu\nu} t_{\mu\nu\alpha\beta}(X)
$$
$$
+ p_j \frac{Q_1^2 S_1 + 2P_2^2 S_3 + 2P_3^2 S_2}{|x_1 - x||x|| - x_1|}, \tag{3.1}
$$

where the first line is the usual parity even contribution while the second line is the parity odd contribution. We use the conventions of [7] for the normalisation of the parity odd tensor structure.

These tensor structures are listed below.

$$
t_{\mu\nu\alpha\beta}(X) = \left(-\frac{2c}{3} + 2e \right) h_{\mu\nu}^1(\hat{X}) \eta_{\alpha\beta} + (3e) h_{\mu\nu}^1(\hat{X}) h_{\alpha\beta}^1 + c h_{\mu\nu\alpha\beta}^2(\hat{X}) + e h_{\mu\nu\alpha\beta}^3,
$$
$$
Q_1^2 = \epsilon_1^\mu \epsilon_1^\nu \left(\frac{x_{1\mu}}{x_1^2} - \frac{x_{1\mu} - x_\mu}{(x_1 - x)^2} \right) \left(\frac{x_{1\nu}}{x_1^2} - \frac{x_{1\nu} - x_\nu}{(x_1 - x)^2} \right),
$$
$$
P_2^2 = -\frac{\epsilon_1^\mu \epsilon_1^\nu I_{\mu\nu}(x_1)}{2x_1^2},
$$
$$
P_3^2 = -\frac{\epsilon_1^\mu \epsilon_2^\nu I_{\mu\nu}(x_1 - x)}{2(x_1 - x)^2},
$$
$$
S_1 = \frac{1}{4|x_1 - x||x|^3| - x_1|} \left(\varepsilon^{\mu\nu}{}_\rho x_\mu (x_1 - x)_\nu \epsilon_2^\rho \epsilon_3^\alpha x_\alpha \right.
$$
$$
\left. - \frac{\varepsilon^\mu{}_{\nu\rho}}{2} \left(|x_1 - x|^2 x_\mu + |x|^2 (x_1 - x)_\mu \right) \epsilon_2^\nu \epsilon_3^\rho \right),
$$
$$
S_2 = \frac{1}{4|x_1 - x||x|| - x_1|^3} \left(\varepsilon^{\mu\nu}{}_\rho (x_{1\mu}) x_\nu \epsilon_3^\rho \epsilon_1^\alpha x_{1\alpha} - \frac{\varepsilon^\mu{}_{\nu\rho}}{2} \left(-|x|^2 x_{1\mu} + |x_1|^2 x_\mu \right) \epsilon_3^\nu \epsilon_1^\rho \right),
$$
$$
S_3 = \frac{1}{4|x_1 - x|^3 |x|| - x_1|} \left(\varepsilon^{\mu\nu}{}_\rho (x_1 - x)_\mu (-x_{1\nu}) \epsilon_1^\rho \epsilon_2^\alpha (x_1 - x_\alpha) \right.
$$
$$
\left. - \frac{\varepsilon^\mu{}_{\nu\rho}}{2} \left(|x|^2 (x - x_1)_\mu + |x - x_1|^2 (-x_\mu) \right) \epsilon_1^\nu \epsilon_2^\rho \right), \tag{3.2}
$$

where,

$$
\hat{X} = \frac{x - x_1}{|x - x_1|^2} + \frac{x_1}{|x_1|^2},
$$
$$
I_{\alpha\beta}(x) = \eta_{\alpha\beta} - \frac{2x_\alpha x_\beta}{x^2},
$$
$$
h_{\mu\nu}^1(\hat{x}) = \frac{x_\mu x\nu}{x^2} - \frac{1}{3}\eta_{\mu\nu},
$$
$$
h_{\mu\nu\sigma\rho}^2(\hat{x}) = \frac{x_\mu x_\sigma}{x^2} \eta_{\nu\rho} + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma) - \frac{4}{3}\frac{x_\mu x_\nu}{x^2}\eta_{\sigma\rho} - \frac{4}{3}\frac{x_\sigma x_\rho}{x^2}\eta_{\mu\nu} + \frac{3}{16}\eta_{\mu\nu}\eta_{\sigma\rho},
$$

$$h^3_{\mu\nu\sigma\rho} = \eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \frac{2}{3}\eta_{\mu\nu}\eta_{\sigma\rho},$$

$$c = \frac{3(2n^j_f + n^j_s)}{256\pi^3}, \qquad e = \frac{3n^j_s}{256\pi^3}. \tag{3.3}$$

For normalising the excited states, we also need the two point function of currents which is given by

$$\langle j_\mu(x)j_\nu\rangle = \frac{C_V}{x^4}I_{\mu\nu}(x), \tag{3.4}$$

with

$$C_V = \frac{8}{3}\pi(c + e). \tag{3.5}$$

3.1 Parity even contribution

That contribution to the energy deposited in the conformal collider due to charge excitations from the parity even part has been obtained before for arbitrary dimensions in [47]. Here we repeat this analysis as a cross check as well as to fix our conventions. Its easy to see that diagonal terms in the energy matrix (3.1) result only from the parity even terms in the three point function (2.18). Let us first choose the polarisations to be given by

$$\epsilon^x_2 = \epsilon^x_3 = \epsilon^{--}_1 = 1. \tag{3.6}$$

This corresponds to the first entry of the energy matrix and is given by the ratio

$$\hat{E}(j)_{11} = \frac{g^1_j(E)}{g^2_j(E)}, \tag{3.7}$$

$$g^1_j(E) = \int d^3x e^{iEt} \lim_{x^+_1 \to \infty} \frac{x^+_1 - x^-_1}{4} \int_{-\infty}^{\infty} dx^-_1 \langle j_x(x)T_{--}(x_1)j_x(0)\rangle, \tag{3.8}$$

and

$$g^2_j(E) = \int d^3x e^{-iEt}\langle j_x(x)j_x(0)\rangle. \tag{3.9}$$

We look at the individual contributions to $g^1_j(E)$. On substituting the expression for the three point function in (3.1) we obtain four terms

$$g^1_j(E) = I'_1 + I'_2 + I'_3 + I'_4, \tag{3.10}$$

$$I'_1 = \int d^3x e^{iEt} \lim_{x^+_1 \to \infty} \frac{x^+_1 - x^-_1}{4} \int_{-\infty}^{\infty} dx^-_1 \left(-\frac{2c}{3} + 2e\right)$$
$$\times I^\alpha_x(x - x_1)I^\beta_x(-x_1)h^1_{--}(\hat{X})\eta_{\alpha\beta}. \tag{3.11}$$

Let us describe the steps involved in evaluating the limit and the integral. First the limit $x^+_1 \to \infty$ is taken which results in

$$I'_1 = \left(-\frac{2c}{3} + 2e\right)\int d^3x e^{iEt}\int_{-\infty}^{\infty} dx^-_1 \frac{(x^-)^2}{16(x^- - x^-_1 - i\epsilon)^{5/2}(-x^-_1 + i\epsilon)^{5/2}(x^2 - x^-x^+)^{3/2}}. \tag{3.12}$$

We follow the $i\epsilon$ prescription introduced by [36, 38] to evaluate the integral. Operators are assigned a negative imaginary part to time depending on their position in the correlation function. Operators to the left are assigned a larger negative imaginary part than the operators to the right i.e, $t_1 \to t_1 - i\epsilon$, $t \to t - 2i\epsilon$. Hence the light-cone coordinates change $x_1^\pm \to x_1^\pm - i\epsilon$, $x^\pm \to x^\pm - 2i\epsilon$. The integral over x_1^- is then performed by using the integral in (C.3). The integrals over the other directions are then best performed by integrating over x direction, followed by integrating the light cone directions.

$$
\begin{aligned}
I_1' &= \frac{1}{4}\left(-\frac{2c}{3} + 2e\right) \int dx^+ dx^- e^{\frac{iEx^+}{2}} e^{\frac{iEx^-}{2}} \frac{16}{3(x^- - 2i\epsilon)^2 x^- x^+}, \\
&= \frac{2}{3}\left(-\frac{2c}{3} + 2e\right) E^2 \pi^2.
\end{aligned}
\tag{3.13}
$$

where we have used C.3 and C.4. Following the same method we obtain

$$
\begin{aligned}
I_2' &= \int d^3x\, e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{2} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} (3e) I_x^\alpha(x - x_1) I_x^\beta(-x_1) h_{--}^1(\hat{X}) h_{\alpha\beta}^1, \\
&= (3e) \int d^3x\, e^{iEt} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} \frac{(x^-)^2 \left(2x^2 + x^- x^+\right)}{24(x^- - x_1^-)^{5/2}(-x_1^-)^{5/2} \left(x^2 - x^- x^+\right)^{5/2}}, \\
&= 0.
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
I_3' &= \int d^3x\, e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{4} \int_{-\infty}^{\infty} dx_1^-\, c I_x^\alpha(x - x_1) I_x^\beta(-x_1) h_{--\alpha\beta}^2(\hat{X}), \\
&= -c \int d^3x\, e^{iEt} \int_{-\infty}^{\infty} \frac{dx_1^-}{12} - \frac{(x^-)^2}{(x^- - x_1^-)^{5/2}(-x_1^-)^{5/2} \left(x^2 - x^- x^+\right)^{3/2}}, \\
&= -\frac{c}{4} \int dx^+ dx^- e^{\frac{iEx^+}{2}} e^{\frac{iEx^-}{2}} \left(\frac{64}{9(x^- - 2i\epsilon)^2 \sqrt{(x^- x^+)^2 + i\epsilon x^- x^+}}\right), \\
&= \frac{c}{2}\left(\frac{-16}{9} E^2 \pi^2\right).
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
I_4' &= \int d^3x\, e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{2} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} e I_x^\alpha(x - x_1) I_x^\beta(-x_1) h_{--\alpha\beta}^3, \\
&= 0.
\end{aligned}
\tag{3.16}
$$

Putting all this together the result for the numerator in (3.7) is given by

$$
g_j^1(E) = \frac{4}{3} E^2 \pi^2 (e - c).
\tag{3.17}
$$

Now the denominator in (3.7) is defined by

$$
\begin{aligned}
g_j^2(E) &= \int d^3x\, e^{-iEt} \langle j_x(x) j_x(0) \rangle, \\
&= -\frac{8}{3}\pi(c + e) \int d^3x\, e^{-iEt} \frac{x^2 + x^- x^+}{(x^2 - x^- x^+)^3}.
\end{aligned}
\tag{3.18}
$$

The calculation proceeds similarly as before, with the spatial integrals being performed first, followed by the light cone directions using the $i\epsilon$ prescription.

$$
\begin{aligned}
g_j^2(E) &= -\frac{8}{3}\pi(c+e)\left(\frac{1}{2}\int dx^+ dx^- e^{\frac{iEx^+}{2}} e^{\frac{iEx^-}{2}} \frac{i\pi}{4\left(x^- - 2i\epsilon\right)^{3/2}\left(x^+ - 2i\epsilon\right)^{3/2}}\right), \\
&= -\frac{8}{3}E\pi^3(c+e).
\end{aligned}
\tag{3.19}
$$

Therefore we obtain

$$
\begin{aligned}
\hat{E}(j)_{11} &= \frac{(c-e)E}{2(c+e)\pi}, \\
&= \frac{E}{4\pi}\left(1 - \frac{a_2}{2}\right),
\end{aligned}
\tag{3.20}
$$

where,

$$
a_2 = \frac{2(3e-c)}{(e+c)} = -\frac{2(n_f^j - n_s^j)}{(n_f^j + n_s^j)}.
\tag{3.21}
$$

Let us examine the second diagonal element of the charge matrix. For this we choose the polarisations to be given by

$$
\epsilon_2^y = \epsilon_3^y = \epsilon_1^{--} = 1.
\tag{3.22}
$$

Proceeding identically we obtain

$$
\begin{aligned}
\hat{E}(j)_{22} &= \frac{e}{c+e}\frac{E}{\pi}, \\
&= \frac{E}{4\pi}\left(1 + \frac{a_2}{2}\right).
\end{aligned}
\tag{3.23}
$$

If we restrict the class of theories to be parity preserving, then from (3.20) and (3.23) we obtain the constraint

$$
|a_2| \leq 2.
\tag{3.24}
$$

This agrees with the results of [47].

3.2 Parity odd contribution

It can be seen that the off diagonal contribution is entirely due to the parity odd terms in the three point function (3.1). Let us first examine the (12) element of the energy matrix. For this we choose the polarizations to be given by

$$
\epsilon_2^x = \epsilon_3^y = 1.
\tag{3.25}
$$

Again this can be written as a ratio

$$
\hat{E}(j)_{12} = \frac{f_j^1(E)}{f_j^2(E)}.
\tag{3.26}
$$

From the structure of the parity odd term, the numerator naturally breaks up into 3 parts which are defined as follows.

$$f_j^1(E) = \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{2} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} j_x(x) T_{--}(x_1) j_y(0),$$

$$= \mathcal{I}_1^p + \mathcal{I}_2^p + \mathcal{I}_3^p, \tag{3.27}$$

where,

$$\mathcal{I}_n^p = \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{2} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} I_n^p. \tag{3.28}$$

Now let us evaluate each of the integrals following the same methods introduced earlier. The first integrand is given by

$$I_1^p = p_j \frac{Q_1^2 S_1}{|x_1 - x||x||-x_1|} \bigg|_{\epsilon_1^{--} = \epsilon_2^x = \epsilon_3^y = 1},$$

$$= \frac{16 p_j}{64|x_1-x|^2|x|^4|-x_1|^2} \left(\frac{x_{1-}}{x_1^2} - \frac{(x_1-x)_-}{(x_1-x)^2} \right)^2 \tag{3.29}$$

$$\times \left(\varepsilon^{\mu\nu}{}_x x_\mu (x_1-x)_\nu (x_+ - x_-) - \frac{\varepsilon^\mu{}_{x+} - \varepsilon^\mu{}_{x-}}{2} \left(|x_1-x|^2 x_\mu + |x|^2 (x_1-x)_\mu \right) \right),$$

where the ε tensor is given by

$$\varepsilon_{+-x} = \frac{1}{2}. \tag{3.30}$$

Taking the $x_1^+ \to \infty$ limit and then performing the integral we obtain

$$\mathcal{I}_1^p = \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{2} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} I_1^p,$$

$$= p_j \int d^3x e^{iEt} \int_{-\infty}^{\infty} \frac{dx_1^-}{4} \frac{(x^-)^2 \left(x^2 + x^- (x^+ - x_1^-) - x_1^- x^+ \right)}{64 (x_1^-)^3 (x^- - x_1^-)^3 (x^2 - x^- x^+)^2},$$

$$= \frac{p_j}{128} \int d^3x e^{iEt} \left[\frac{6i\pi \left(x^2 + x^- x^+ \right)}{(x^-)^3} + \frac{3i\pi(-x^- - x^+)}{(x^-)^2} \right] \left[\frac{1}{(x^2 - x^- x^p)^2} \right],$$

$$= -\frac{p_j}{128} \frac{1}{4} \int dx^+ dx^- e^{\frac{iEx^+}{2}} e^{\frac{iEx^-}{2}} \left(\frac{3\pi^2}{(x^- - 2i\epsilon)^{5/2}(x^+ - 2i\epsilon)^{3/2}} + \frac{3\pi^2}{(x^- - 2i\epsilon)^{7/2}\sqrt{x^+ - 2i\epsilon}} \right),$$

$$= -\frac{3p_j}{160} E^2 \pi^3. \tag{3.31}$$

The integrand for the second term in (3.27) is given by

$$I_2^p = p_j \frac{2 P_2^2 S_3}{|x_1 - x||x||-x_1|} \bigg|_{\epsilon_1^{--} = \epsilon_2^x = \epsilon_3^y = 1},$$

$$= \frac{-p_j}{4|x_1-x|^4|x|^2|-x_1|^4} I_{y-}(x_1) \tag{3.32}$$

$$\times \left(\varepsilon^{\mu\nu}{}_-(x_1-x)_\mu(-x_1)_\nu(x_1-x)_x - \frac{\varepsilon^\mu{}_{-x}}{2} \left(|-x_1|^2 (x_1-x)_\mu + |x_1-x|^2 (-x_1)_\mu \right) \right).$$

Taking the limit and performing the integrals we obtain

$$
\begin{aligned}
\mathcal{I}_2^p &= \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{4} \int_{-\infty}^{\infty} dx_1^- I_2^p, \\
&= \frac{p_j}{128} \int d^3x e^{iEt} \int_{-\infty}^{\infty} dx_1^- \frac{x^-}{(x_1^-)^3 (x^- - x_1^-)^2 (x^2 - x^- x^+)}, \\
&= \frac{p_j}{256} \int d^3x e^{iEt} \frac{12i\pi}{(x^-)^3 (x^2 - x^- x^+)}, \\
&= -\frac{p_j}{80} E^2 \pi^3.
\end{aligned}
\tag{3.33}
$$

Finally the integrand for the last term in (3.27) is given by

$$
\begin{aligned}
I_3^p &= p_j \frac{2P_3^2 S_2}{|x_1 - x||x|| - x_1|} \Big|_{\epsilon_1^{--} = \epsilon_2^x = \epsilon_3^y = 1}, \\
&= \frac{-16p_j}{64|x_1 - x|^4 |x|^2| - x_1|^4} I_{-x}(x_1 - x) \\
&\quad \times \left((\varepsilon^{\mu\nu}{}_+ - \varepsilon^{\mu\nu}{}_-)(x_1)_\mu x_\nu (x_1)_- - \frac{\varepsilon^\mu{}_{+-}}{2} \left(-|x|^2 x_{1\mu} + |x_1|^2 x_\mu \right) \right),
\end{aligned}
\tag{3.34}
$$

Evaluating the limit and the integrals we obtain

$$
\begin{aligned}
\mathcal{I}_3^p &= \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{2} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} I_3^p, \\
&= 0.
\end{aligned}
\tag{3.35}
$$

Putting all the terms for the numerator together we get

$$
\begin{aligned}
f_1(E) &= \mathcal{I}_1^p + \mathcal{I}_2^p + \mathcal{I}_3^p, \\
&= -\frac{E^2 \pi^3 p_j}{32}.
\end{aligned}
$$

The denominator in (3.26) is given by

$$
f_2(E) = -\frac{8}{3} E \pi^3 (c + e).
\tag{3.36}
$$

Therefore the (12) element of the energy matrix is given by

$$
\begin{aligned}
\hat{E}(j)_{12} &= \frac{3p_j E}{256(c + e)}, \\
&= \frac{E}{8\pi} \alpha_j,
\end{aligned}
\tag{3.37}
$$

where α_j is defined as

$$
\alpha_j \equiv \frac{3p_j \pi}{32(c + e)} = \frac{4\pi^4 p_j}{(n_f^j + n_s^j)}.
\tag{3.38}
$$

Let us now examine the second off diagonal element in the energy matrix. To extract this component we choose the polarisations to be given by

$$\epsilon_2^y = \epsilon_3^x = 1. \tag{3.39}$$

Again this element is given by the ratio

$$\hat{E}(j)_{21} = \frac{f_j'^1(E)}{f_j^2(E)}. \tag{3.40}$$

By evaluating the numerator explicitly using the same methods it can be seen that

$$f_j'^1(E) = f_j^1(E). \tag{3.41}$$

Therefore the two off diagonal elements are identical. This is consistent with the fact that the energy matrix must be symmetric and is in fact a cross check for our calculations.

To summarise, the energy matrix for charge excitations is given by

$$\hat{E}(j) = \begin{pmatrix} \frac{E}{4\pi}(1 - \frac{a_2}{2}) & \frac{E}{8\pi}\alpha_j \\ \\ \frac{E}{8\pi}\alpha_j & \frac{E}{4\pi}(1 + \frac{a_2}{2}) \end{pmatrix}, \tag{3.42}$$

where

$$a_2 = \frac{2(3e - c)}{(e + c)} = -\frac{2(n_f^j - n_s^j)}{(n_f^j + n_s^j)},$$

$$\alpha_j = \frac{3p_j\pi}{32(c + e)} = \frac{4\pi^4 p_j}{(n_f^j + n_s^j)}. \tag{3.43}$$

The condition that the energy observed at the calorimeter is positive leads to the fact that the eigen values of the energy matrix are positive. The trace of the matrix in (3.42) is positive, this implies that the determinant is positive which leads to the condition

$$a_2^2 + \alpha_j^2 \le 4. \tag{3.44}$$

This region is a disc of radius 2 centered at the origin in the a_2, α_j plane.

4 Energy matrix for stress tensor excitations

In this section we evaluate the energy matrix corresponding to stress tensor excitations $\hat{E}(T)$ defined in (2.14). Here the basic ingredient is the three point function of the stress energy tensor including the parity odd term. The parity even tensor structures in the three point function was found earlier by [29, 48] while the parity odd contribution was written down by [27]. Combining both these contributions the three point function of the stress tensor for a conformal field theory in $d = 3$ is given by

$$\langle T(x)T(x_1)T(0)\rangle = \frac{\epsilon_1^{\mu\nu}\mathcal{I}_{\mu\nu,\mu'\nu'}^T(x)\epsilon_2^{\sigma\rho}\mathcal{I}_{\sigma\rho,\sigma'\rho'}^T(x_1)\epsilon_3^{\alpha\beta}t^{\mu'\nu'\sigma'\rho'}{}_{\alpha\beta}}{x^6 x_1^6} + \tag{4.1}$$

$$+ p_T \frac{(P_1^2 Q_1^2 + 5P_2^2 P_3^2)S_1 + (P_2^2 Q_2^2 + 5P_3^2 P_1^2)S_2 + (P_3^2 Q_3^2 + 5P_3^2 P_1^2)S_3}{|x - x_1||x_1|| - x|}.$$

Here the first line denotes the parity even contribution to the three point function and the parity odd contribution is given in the second line. We will choose the normalisation of the parity odd tensor structure as given in [7]. The parity even tensor structures are given by

$$t_{\mu\nu\sigma\rho\alpha\beta} = \mathcal{A}\mathcal{E}^T_{\mu\nu,\epsilon\eta}\mathcal{E}^{T\ \eta}_{\ \sigma\rho,\ \lambda}\mathcal{E}^{T\ \lambda\epsilon}_{\ \alpha\beta,}\frac{1}{(Z^2)^{\frac{3}{2}}} \tag{4.2}$$

$$+(\mathcal{B}-2\mathcal{A})\mathcal{E}^T_{\alpha\beta,\epsilon\eta}\mathcal{E}^{T\ \eta}_{\ \sigma\rho,\ \kappa}\mathcal{E}^{T\ \epsilon}_{\ \mu\nu,\lambda}\frac{Z^\kappa Z^\lambda}{(Z^2)^{\frac{5}{2}}}$$

$$-\mathcal{B}\left(\mathcal{E}^T_{\mu\nu,\epsilon\eta}\mathcal{E}^{T\ \eta}_{\ \sigma\rho,\ \kappa}\mathcal{E}^{T\ \epsilon}_{\ \alpha\beta,\lambda} + (\mu\nu)\leftrightarrow(\sigma\rho)\right)\frac{Z^\kappa Z^\lambda}{(Z^2)^{\frac{5}{2}}}$$

$$+\mathcal{C}\left(\mathcal{E}^T_{\mu\nu,\sigma\rho}\left(\frac{Z_\alpha Z_\beta}{Z^2}-\frac{1}{3}\eta_{\alpha\beta}\right) + \mathcal{E}^T_{\sigma\rho,\alpha\beta}\left(\frac{Z_\mu Z_\nu}{Z^2}-\frac{1}{3}\eta_{\mu\nu}\right)\right.$$

$$\left.\times\mathcal{E}^T_{\alpha\beta,\mu\nu}\left(\frac{Z_\sigma Z_\rho}{Z^2}-\frac{1}{3}\eta_{\sigma\rho}\right)\right)\frac{1}{(Z^2)^{\frac{3}{2}}}$$

$$+(\mathcal{D}-4\mathcal{C})\left(\mathcal{E}^T_{\mu\nu,\epsilon\kappa}\mathcal{E}^{T\ \epsilon}_{\ \sigma\rho,\ \lambda}\right)\left(\frac{Z_\alpha Z_\beta}{Z^2}-\frac{1}{3}\eta_{\alpha\beta}\right)\frac{Z^\kappa Z^\lambda}{(Z^2)^{\frac{5}{2}}}$$

$$-(\mathcal{D}-2\mathcal{B})\left(\mathcal{E}^T_{\sigma\rho,\epsilon\kappa}\mathcal{E}^{T\ \epsilon}_{\ \alpha\beta,\ \lambda}\left(\frac{Z_\mu Z_\nu}{Z^2}-\frac{1}{3}\eta_{\mu\nu}\right) + (\mu\nu)\leftrightarrow(\sigma\rho)\right)\frac{Z^\kappa Z^\lambda}{(Z^2)^{\frac{5}{2}}}$$

$$+(\mathcal{E}+4\mathcal{C}-2\mathcal{D})\left(\frac{Z_\mu Z_\nu}{Z^2}-\frac{1}{3}\eta_{\mu\nu}\right)\left(\frac{Z_\alpha Z_\beta}{Z^2}-\frac{1}{3}\eta_{\alpha\beta}\right)\left(\frac{Z_\sigma Z_\rho}{Z^2}-\frac{1}{3}\eta_{\sigma\rho}\right)\frac{1}{Z^{\frac{3}{2}}},$$

where tensors involved and the theory dependent parameters given by

$$\mathcal{E}^T_{\mu\nu,\alpha\beta} = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta}+\eta_{\mu\beta}\eta_{\nu\alpha})-\frac{1}{3}\eta_{\alpha\beta}\eta_{\mu\nu}, \tag{4.3}$$

$$I_{\alpha\beta}(x) = \eta_{\alpha\beta}-\frac{2x_\alpha x_\beta}{x^2},$$

$$\mathcal{I}^T_{\mu\nu,\alpha\beta}(x) = I_{\mu\mu'}(x)I_{\nu\nu'}(x)\mathcal{E}^{T\mu'\nu'}_{\ \ \ \ \alpha\beta},$$

$$\mathcal{D} = \frac{5\mathcal{A}}{2}+\frac{5\mathcal{B}}{2}-6\mathcal{C}, \qquad\qquad \mathcal{E} = 5\mathcal{A}+\frac{3}{4}(9\mathcal{B}-26\mathcal{C}),$$

$$\mathcal{A} = \frac{27n_s^T}{512\pi^3}, \qquad\qquad \mathcal{B} = -\frac{9(4n_f^T+3n_s^T)}{512\pi^3}, \quad \mathcal{C} = -\frac{9(8n_f^T+n_s^T)}{2048\pi^3}.$$

The parity odd tensor structures are given by

$$P_1^2 = \frac{-\epsilon_2^\mu\epsilon_3^\nu I_{\mu\nu}(x_1)}{2x_1^2}, \qquad\qquad P_2^2 = \frac{-\epsilon_3^\mu\epsilon_1^\nu I_{\mu\nu}(x)}{2x^2}, \tag{4.4}$$

$$P_3^2 = \frac{-\epsilon_1^\mu\epsilon_2^\nu I_{\mu\nu}(x-x_1)}{2(x-x_1)^2},$$

$$Q'_{1\alpha} = \left(\frac{x_\alpha}{x^2}-\frac{(x-x_1)_\alpha}{(x-x_1)^2}\right), \qquad\qquad Q'_{2\alpha} = \left(-\frac{x_{1\alpha}}{x_1^2}+\frac{(x_1-x)_\alpha}{(x_1-x)^2}\right),$$

$$Q'_{3\alpha} = \left(\frac{x_{1\alpha}}{x_1^2}-\frac{x_\alpha}{x^2}\right),$$

$$Q_1^2 = \epsilon_1^\alpha\epsilon_1^\beta Q'_{1\alpha}Q'_{1\beta}, \qquad\qquad Q_2^2 = \epsilon_2^\alpha\epsilon_2^\beta Q'_{2\alpha}Q'_{2\beta},$$

$$Q_3^2 = \epsilon_3^\alpha\epsilon_3^\beta Q'_{3\alpha}Q'_{3\beta},$$

and

$$S_1 = \frac{1}{4|x-x_1||x_1|^3|-x|} \left(\varepsilon^{\mu\nu}{}_\rho x_{1\mu}(x-x_1)_\nu \epsilon_2^\rho \epsilon_3^\alpha x_{1\alpha} \right.$$
$$\left. - \frac{\varepsilon^\mu{}_{\nu\rho}}{2} \left(|x-x_1|^2 x_{1\mu} + |x_1|^2 (x-x_1)_\mu \right) \epsilon_2^\nu \epsilon_3^\rho \right),$$

$$S_2 = \frac{1}{4|x-x_1||x_1||-x|^3} \left(\varepsilon^{\mu\nu}{}_\rho(-x_\mu)x_{1\nu}\epsilon_3^\rho \epsilon_1^\alpha(-x_\alpha) - \frac{\varepsilon^\mu{}_{\nu\rho}}{2} \left(|x_1|^2(-x_\mu) + |x|^2 x_{1\mu} \right) \epsilon_3^\nu \epsilon_1^\rho \right),$$

$$S_3 = \frac{1}{4|x-x_1|^3|x_1||-x|} \left(\varepsilon^{\mu\nu}{}_\rho(x-x_1)_\mu(-x_\nu)\epsilon_1^\rho \epsilon_2^\alpha(-x_\alpha) \right.$$
$$\left. - \frac{\varepsilon^\mu{}_{\nu\rho}}{2} \left(|x|^2(x-x_1)_\mu + |x-x_1|^2(-x_\mu) \right) \epsilon_1^\nu \epsilon_2^\rho \right). \tag{4.5}$$

For obtaining the normalized states created by the stress tensor we also need its two point function which is given by

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(0) = \frac{C_T}{x^6} \mathcal{I}^T_{\mu\nu,\sigma\rho}(x), \tag{4.6}$$

with

$$C_T = \frac{2}{15}\pi(10\mathcal{A} - 2\mathcal{B} - 16\mathcal{C}). \tag{4.7}$$

4.1 Parity even contribution

Though the derivation of the parity even contribution to the energy at the conformal collider in $d = 3$ can be obtained from the results of [38] we repeat the analysis as a check. It can be seen that it is only the parity even terms in (4.1) that contribute to the diagonal terms of the energy matrix $\hat{E}(T)$. From (4.2) we write the parity even part of the three point function as a contribution from 7 terms.

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(x_1)T_{\alpha\beta}(0)\rangle_{\text{even}}, = I_7 + I_6 + I_5 + I_4 + I_3 + I_2 + I_1. \tag{4.8}$$

Let us proceed to evaluate the (11) element of $\hat{E}(T)$ given in (2.14). For this we choose the polarization to be given by,

$$\epsilon_1^{xy} = \epsilon_1^{yx} = \frac{1}{2}, \qquad \epsilon_3^{xy} = \epsilon_3^{yx} = \frac{1}{2}, \tag{4.9}$$
$$\epsilon_2^{--} = 1.$$

The matrix element can be written as the ratio

$$\hat{E}(T)_{11} = \frac{g_T^1(E)}{g_T^2(E)}. \tag{4.10}$$

The denominator in (4.10) is the norm which depends on the two point function of the stress tensor. We will first examine the numerator. From the break up the three point function in (4.8) we have,

$$g_T^1(E) = \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{4} \int_{-\infty}^{\infty} dx_1^- \langle T_{xy}(x)T_{--}(x_1)T_{xy}(0)\rangle,$$
$$= \mathcal{I}_7 + \mathcal{I}_6 + \mathcal{I}_5 + \mathcal{I}_4 + \mathcal{I}_3 + \mathcal{I}_2 + \mathcal{I}_1. \tag{4.11}$$

Performing the limit and the integrals in each of the 7 terms is tedious but straight forward. The procedure is identical to that carried out for the charge excitations in the previous section. The details for evaluating the term $\mathcal{I}_1$ are provided in appendix A. Here we list out the contribution of each of the 7 terms.

$$\mathcal{I}_1 = \frac{(\mathcal{E} + 4\mathcal{C} - 2\mathcal{D})}{2}\left(-\frac{149\pi^2 E^4}{9450}\right), \tag{4.12}$$

$$\mathcal{I}_2 = \frac{-(\mathcal{D} - 2\mathcal{B})}{2}\left(-\frac{29\pi^2 E^4}{5400} + \frac{19\pi^2 E^4}{37800}\right),$$

$$\mathcal{I}_3 = \frac{(\mathcal{D} - 4\mathcal{C})}{2}\left(-\frac{989\pi^2 E^4}{37800}\right),$$

$$\mathcal{I}_4 = \frac{\mathcal{C}}{2}\left(\frac{-11}{525}\pi^2 E^4\right),$$

$$\mathcal{I}_5 = \frac{\mathcal{B}}{2}\left(\frac{127 E^4 \pi^2}{75600} + \frac{91 E^4 \pi^2}{10800}\right),$$

$$\mathcal{I}_6 = \frac{(\mathcal{B} - 2\mathcal{A})}{2}\frac{29 E^4 \pi^2}{10800},$$

$$\mathcal{I}_7 = \frac{\mathcal{A}}{2}\frac{11 E^4 \pi^2}{3600}.$$

Summing all these terms together we get for the numerator in (4.10)

$$g_T^1(E) = -\frac{1}{180}(5\mathcal{A} + 7\mathcal{B} - 24\mathcal{C})\pi^2 E^4. \tag{4.13}$$

Evaluating the denominator we obtain

$$g_T^2(E) = -\frac{1}{180}(10\mathcal{A} - 2\mathcal{B} - 16\mathcal{C})\pi^3 E^3. \tag{4.14}$$

Therefore the (11) matrix element is given by

$$\hat{E}(T)_{11} = \frac{E}{\pi}\frac{(5\mathcal{A} + 7\mathcal{B} - 24\mathcal{C})}{10\mathcal{A} - 2\mathcal{B} - 16\mathcal{C}},$$
$$= \frac{E}{4\pi}\left(1 - \frac{t_4}{4}\right), \tag{4.15}$$

where t_4 is defined by

$$t_4 \equiv -\frac{4(30\mathcal{A} + 90\mathcal{B} - 240\mathcal{C})}{3(10\mathcal{A} - 2\mathcal{B} - 16\mathcal{C})} = -\frac{4(n_f^T - n_s^T)}{n_f^T + n_s^T}. \tag{4.16}$$

Let us now examine the second diagonal element of the energy matrix $\hat{E}(T)$. For this we choose the polarisations to be

$$\epsilon_1^{xx} = -\epsilon_1^{yy} = 1, \qquad \epsilon_2^{--} = 1,$$
$$\epsilon_3^{xx} = -\epsilon_3^{yy} = 1. \tag{4.17}$$

The matrix element is given by

$$\hat{E}(T)_{22} = \frac{h_1(E)}{h_2(E)} \tag{4.18}$$

where

$$h_1(E) = \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{4} \int_{-\infty}^{\infty} dx_1^- \times \tag{4.19}$$
$$\times \langle (T_{xx}(x) - T_{yy}(x)) \, T_{--}(x_1) \, (T_{xx}(0) - T_{yy}(0)) \rangle,$$
$$h_2(E) = \int d^3x e^{iEt} \, \langle (T_{xx}(x) - T_{yy}(x)) \, (T_{xx}(0) - T_{yy}(0)) \rangle.$$

Proceeding identically we obtain

$$h_1(E) = \frac{1}{2}\frac{16}{45}(\mathcal{B} - 2\mathcal{C})\pi^2 E^4,$$
$$h_2(E) = -\frac{4}{180}(10\mathcal{A} - 2\mathcal{B} - 16\mathcal{C})\pi^3 E^3. \tag{4.20}$$

Evaluating the ratio we get

$$\hat{E}(T)_{22} = \frac{E}{4\pi}\left(-\frac{16(\mathcal{B} - 2\mathcal{C})}{5\mathcal{A} - \mathcal{B} - 8\mathcal{C}}\right),$$
$$= \frac{E}{4\pi}\left(1 + \frac{t_4}{4}\right). \tag{4.21}$$

Note that if one restricts the theories to be only parity even and require that the energy is positive results. Then from (4.15) and (4.21) we obtain the constraint

$$|t_4| \le 4, \tag{4.22}$$

which agrees with the results of [38].

4.2 Parity odd contribution

The off diagonal elements in the energy matrix receive contributions only from the parity odd terms in the three point function (4.1). The parity odd term consists of 6 terms which are defined as

$$\langle T(x)T(x_1)T(0)\rangle_{\text{odd}} = p_T \frac{(P_1^2 Q_1^2 + 5P_2^2 P_3^2)S_1 + (P_2^2 Q_2^2 + 5P_3^2 P_1^2)S_2 + (P_3^2 Q_3^2 + 5P_3^2 P_1^2)S_3}{|x - x_1||x_1|| - x|},$$
$$\equiv p_T(I_1^p + I_2^p + I_3^p + I_4^p + I_5^p + I_6^p). \tag{4.23}$$

To obtain the (12) element of the energy matrix we choose the polarisations to be given by

$$\epsilon_1^{xy} = \epsilon_1^{yx} = \frac{1}{2}, \qquad \epsilon_3^{xx} = -\epsilon_3^{yy} = 1,$$
$$\epsilon_2^{--} = 1. \tag{4.24}$$

We symmetrise the (x, y) component of the polarisation tensor ϵ_1^{xy}

$$\epsilon_1^{\mu\nu} T_{\mu\nu} = \frac{1}{2}(T_{xy} + T_{yx}). \tag{4.25}$$

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Then the required matrix element can be written as the ratio

$$\hat{E}(T)_{12} = \frac{f_1(E)}{f_2(E)}, \tag{4.26}$$

where,

$$f_1(E) = \int d^3 x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{8} \int_{-\infty}^{\infty} dx_1^- (T_{xy} + T_{yx})(x) T_{--}(x_1)(T_{xx}(0) - T_{yy}(0)),$$
$$= \frac{p_T}{2}(\mathcal{I}_1^p + \mathcal{I}_2^p + \mathcal{I}_3^p + \mathcal{I}_4^p + \mathcal{I}_5^p + \mathcal{I}_6^p), \tag{4.27}$$

and

$$\mathcal{I}_n^p = \int d^3 x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{4} \int_{-\infty}^{\infty} dx_1^- I_n^p. \tag{4.28}$$

To evaluate the denominator we use the two point function of the stress tensor (4.6) and it results in

$$f_2(E) = -\frac{1}{90}(10\mathcal{A} - 2\mathcal{B} - 16\mathcal{C})\pi^3 E^3. \tag{4.29}$$

As an example the contribution $\mathcal{I}_1^p$ to the numerator is evaluated in the appendix A. Here we write down the result for each of the contributions

$$\mathcal{I}_1^p = -\frac{1}{256}\frac{8}{315}\pi^3 E^4, \tag{4.30}$$
$$\mathcal{I}_2^p = \frac{1}{256}\left(\frac{1}{7} - \frac{4}{63}\right)\pi^3 E^4,$$
$$\mathcal{I}_3^p = \frac{1}{256}\left(\frac{4}{105} - \frac{22}{315} - \frac{4}{105} + \frac{2}{315}\right)\pi^3 E^4,$$
$$\mathcal{I}_4^p = \frac{1}{256}\left(-\frac{4}{63} - \frac{2}{21}\right)\pi^3 E^4,$$
$$\mathcal{I}_5^p = \frac{1}{256}\left(-\frac{1}{105} - \frac{29}{315}\right)\pi^3 E^4,$$
$$\mathcal{I}_6^p = -\frac{1}{256}\frac{25}{63}\pi^3 E^4.$$

Summing all the contributions to the numerator we obtain

$$f_T^1(E) = \frac{p_T}{2}(\mathcal{I}_1^p + \mathcal{I}_2^p + \mathcal{I}_3^p + \mathcal{I}_4^p + \mathcal{I}_5^p + \mathcal{I}_6^p),$$
$$= -\frac{p_T}{3 \times 256}\pi^3 E^4. \tag{4.31}$$

Using (4.29) and (4.31) we see that the (12) element of the energy matrix is given by

$$\hat{E}(T)_{12} = \frac{f_T^1(E)}{f_T^2(E)},$$
$$= \frac{p_T}{256}\frac{15E}{5\mathcal{A} - \mathcal{B} - 8\mathcal{C}},$$
$$= \frac{E}{16\pi}\alpha_T, \tag{4.32}$$

where we define α_T by

$$\alpha_T \equiv \frac{p_T}{256} \frac{240\pi}{5\mathcal{A} - \mathcal{B} - 8\mathcal{C}} = \frac{8\pi^4 p_T}{3(n_f^T + n_s^T)}. \tag{4.33}$$

Finally let us examine the second off diagonal element of the energy matrix. We choose the polarisations to be given by

$$\epsilon_3^{xy} = \epsilon_3^{yx} = \frac{1}{2}, \qquad \epsilon_1^{xx} = -\epsilon_1^{yy} = 1,$$
$$\epsilon_2^{--} = 1. \tag{4.34}$$

Again this matrix element can be written as the ratio

$$\hat{E}(T)_{21} = \frac{f_T'^1(E)}{f_T^2(E)}, \tag{4.35}$$

where the numerator is given by

$$f_T'^1(E) = \int d^3x\, e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{8} \int_{-\infty}^{\infty} dx_1^- (T_{xx}(x) - T_{yy}(x)) T_{--}(x_1)(T_{xy}(0) + T_{yx}(0)). \tag{4.36}$$

The denominator in (4.35) is the same as given in (4.29). Evaluating the numerator using the same methods we have shown that the numerator is given by

$$f_T'^1(E) = f_T^1(E) = -\frac{\alpha}{256} \frac{1}{3} \pi^3 E^4. \tag{4.37}$$

This result coincides with the numerator of the (12) element given in (4.31). This implies that

$$\hat{E}(T)_{12} = \hat{E}(T)_{21}. \tag{4.38}$$

As mentioned earlier for the energy matrix corresponding to the charge excitations, the fact that the off diagonal entries of the matrix coincide serve a simple check of our calculations.

Using all the results we can write the energy matrix for the excitations created by the stress tensor to be given by

$$\hat{E}(T) = \begin{pmatrix} \frac{E}{4\pi}(1 - \frac{t_4}{4}) & \frac{E}{16\pi}\alpha_T \\ \\ \frac{E}{16\pi}\alpha_T & \frac{E}{4\pi}(1 + \frac{t_4}{4}) \end{pmatrix}, \tag{4.39}$$

where,

$$t_4 = -\frac{4(30\mathcal{A} + 90\mathcal{B} - 240\mathcal{C})}{3(10\mathcal{A} - 2\mathcal{B} - 16\mathcal{C})} = -\frac{4(n_f^T - n_s^T)}{n_f^T + n_s^T},$$
$$\alpha_T = \frac{p_T}{256} \frac{240\pi}{5\mathcal{A} - \mathcal{B} - 8\mathcal{C}} = \frac{8\pi^4 p_T}{3(n_f^T + n_s^T)}. \tag{4.40}$$

The condition that the energy observed at the calorimeter of the conformal collider is positive leads to the fact that the eigen values of the matrix $\hat{E}(T)$ are positive. The

trace is positive, which implies that the determinant has to be positive. This leads to the constraint

$$t_4^2 + \alpha_T^2 \leq 16. \tag{4.41}$$

Thus the theory dependent parameters t_4, α_T of the three point function are constrained to lie on a disc of radius 4 at the origin. All three-dimensional conformal field theories which satisfy the conformal collider bounds of [36] lie in this disc. The constraint (4.41) was also obtained in appendix C of [28] by writing down the energy observed at the conformal collider purely from symmetry arguments. However the precise relation (4.40) between α_T and the coefficient p_T of the parity odd term in the three point function of the stress tensor was not given.

5 Large N Chern Simons theories

U(N) Chern Simons theory at level κ coupled to either fermions or bosons in the fundamental representation [7, 16] are examples of conformal field theories which violate parity. In the large N limit, these can be solved to all orders in the 't Hooft coupling[5]

$$\lambda = \frac{N}{\kappa}. \tag{5.1}$$

Lets write the three point functions of interest again

$$\langle jjT \rangle = n_s^j \langle jjT \rangle_{\text{free boson}} + n_f^j \langle jjT \rangle_{\text{free fermion}} + p_j \langle jjT \rangle_{\text{parity odd}}, \tag{5.2}$$
$$\langle TTT \rangle = n_s^T \langle TTT \rangle_{\text{free boson}} + n_f^T \langle TTT \rangle_{\text{free fermion}} + p_T \langle TTT \rangle_{\text{parity odd}}.$$

where $\langle \ldots \rangle_{\text{free boson}}$, $\langle \ldots \rangle_{\text{free fermion}}$ denote the correlator of a single real free boson and a single real free fermion respectively. The theory dependent coefficients $n_s^{j,T}, n_f^{j,T}$ and $p_{j,T}$ in the three point functions (5.2) are functions of the 't Hooft coupling which can be determined using softly broken higher spin symmetry [45] or direct calculation [7, 16, 17]. These coefficients for U(N) Chern-Simons theory coupled to fermions in the fundamental representation are given by

$$n_s^T(f) = n_s^j(f) = 2N\frac{\sin\theta}{\theta}\sin^2\frac{\theta}{2}, \qquad n_f^T(f) = n_s^j(f) = 2N\frac{\sin\theta}{\theta}\cos^2\frac{\theta}{2},$$
$$p_j(f) = \alpha'N\frac{\sin^2\theta}{\theta}, \qquad\qquad p_T(f) = \alpha N\frac{\sin^2\theta}{\theta}, \tag{5.3}$$

where the t 'Hooft coupling is related to θ by

$$\theta = \frac{\pi N}{\kappa}. \tag{5.4}$$

Given the normalisation of the parity odd tensor structure in (3.1) and (4.1), the numerical coefficients α, α' can be determined by a one loop computation either in the theory with fundamental fermions or in theory with fundamental bosons. In this section we present the

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[5]Note that κ is the level defined using dimensional regularization, and differs from the level k defined using Yang-Mills regularization by $\kappa = k + N$. $|\kappa| > N$ hence $|\lambda| \leq 1$.

results of this perturbative calculation. This was done earlier in [7], we repeat the analysis to precisely determine the factors α, α'.

Before we proceed, lets examine the coefficients for the parity even terms in (5.3). Note that we have chosen the normalisation of the stress tensor T and the current j in (5.2) to agree with that given in [29]. This is evident from the tensor structures we have used for the parity even part in (3.1) and (4.1) and the normalisations of the two point functions in (3.4) and (4.6). Also observe that taking the free limit $\theta \to 0$, we see that the three point function must reduce to that of $2N$ decoupled real fermions. This is clear from the limit $n_f^{j,T} \to 2N$ as $\theta \to 0$. To fix the factor in front the parity even bosonic contribution, we use the fact that we can obtain the three point functions of the (critical) bosonic theory by the duality transformation [16, 17]

$$n_s^{j,T}(f) \to n_s^{j,T}(b) = 2N_b \frac{\sin\theta_b}{\theta_b} \cos^2 \frac{\theta_b}{2}. \tag{5.5}$$

Now taking the limit $\theta_b \to 0$ we see that $n_s^{j,T}(b) \to 2N_b$.

5.1 Parity odd three point functions

The parity odd coefficients α, α' can be determined by performing a one loop computation of $\langle jjT \rangle$ and $\langle TTT \rangle$ in the U(N) Chern-Simons theory coupled to fundamental fermions. At one-loop these correlators are necessarily parity odd. While these calculations were previously done in [7],[6] we explicitly redo the calculation here for completeness, and to ensure the precise numerical factors are correct.[7] The calculation closely follows appendix G of [7].

The action for the theory can be written in Euclidean space as

$$S = \frac{i\kappa}{4\pi} \int \text{Tr}\left(AdA + \frac{2}{3}A^3\right) + \int d^3x \bar{\psi}\gamma^\mu D_\mu\psi, \tag{5.6}$$

where

$$D_\mu\psi = \partial_\mu\psi - iA^a T^a \psi,$$

and the generators T_a are normalized as: $\text{Tr}\, T_a^2 = \frac{1}{2}$. It is convenient to work in the gauge $A_3 = 0$. We have the following propagators:

$$\langle A_i^a(x) A_j^b(0) \rangle = \frac{2\pi i}{\kappa} \epsilon_{ij} \text{sign}(x_3) \delta^2(\vec{x}) \delta^{ab}, \tag{5.7}$$

$$\langle \psi^n(x) \bar{\psi}_m(0) \rangle = \frac{1}{4\pi} \frac{x_\mu \gamma^\mu}{|x|^3} \delta_m^n. \tag{5.8}$$

Here the gauge field $A_\mu = A_\mu^a T^a$, where T^a are generators normalised by

$$\sum_a (T^a)_m^n (T^a)_p^q = \frac{1}{2} \delta_m^q \delta_p^n. \tag{5.9}$$

Indices i and j can take on the values 1 and 2. It is also convenient to define Euclidean light cone coordinates as: $x_\mp = x^\pm = \frac{1}{\sqrt{2}}(x^1 \pm ix^2)$.

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[6]A one-loop calculation of $\langle jjT \rangle$ in Chern-Simons coupled to fundamental bosons also appears in [8].
[7]The numerical factors α, α' differ slightly from the one-loop calculations in [7, 8].
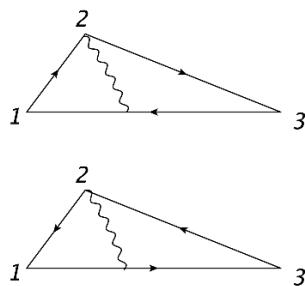
Figure 2. Diagram IA and IB.

We will calculate correlation functions with all free indices in the x_- direction. We define the normalisations of $\tilde{T}_{--}$ and j_- as

$$j_- = \bar{\psi}\gamma_-\psi, \tag{5.10}$$

$$\tilde{T}_{--} = \bar{\psi}\gamma_-(\overrightarrow{D}_- - \overleftarrow{D}_-)\psi = \bar{\psi}\gamma_-\overleftrightarrow{D}_-\psi. \tag{5.11}$$

Note that here the normalisation of the stress tensor is twice that of [29] which is what we have been using in the rest of the paper.

$$\tilde{T} = 2T. \tag{5.12}$$

We will incorporate this change in normalisation towards the end of our calculations. With these normalizations, we have, in free theory,

$$\langle J_-(x)J_-(0)\rangle = -N\frac{1}{4\pi^2}\frac{x_-^2}{x^6} = -N\frac{1}{4\pi^2}\frac{P_3^2}{x^2}, \tag{5.13}$$

$$\langle \tilde{T}_{--}(x)\tilde{T}_{--}(0)\rangle = N\frac{3}{\pi^2}\frac{x_-^4}{x^{10}} = N\frac{3}{\pi^2}\frac{P_3^4}{x^2}. \tag{5.14}$$

These are also valid at one-loop, since the two-point functions must be parity even. We will calculate all our correlation functions at the following points:

$$x_1 = (x_1^+, x_1^-, x_1^3) = (0, 0, 0), \tag{5.15}$$

$$x_2 = -(\delta^+, \delta^-, 0), \tag{5.16}$$

$$x_3 = (0, 0, t), \tag{5.17}$$

and we will assume that

$$|\delta| \ll t, \qquad \text{and} \quad t > 0. \tag{5.18}$$

With this particular choice of points, temporal gauge and all free indices on operators in the same null direction, many diagrams vanish. The only diagrams which are non-vanishing are given in figures 2, 3 and 4. After carrying out the calculation, we find that, the dominant contributions in our limit are from diagram IIIA and IIIB given in figure 4. Other diagrams are sub-leading in our limit. (There are also some artefacts of temporal gauge produced by each diagram, which cancel amongst each other. See appendix G of [7] for more details.) In the appendix B we present the calculations only of diagram III.
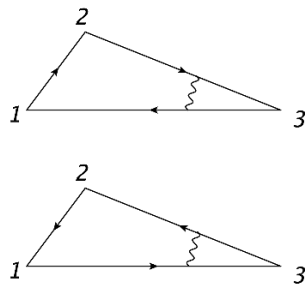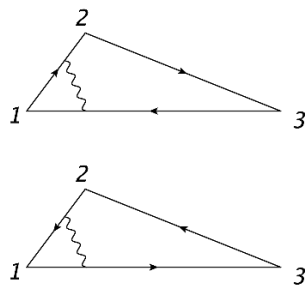
Figure 3. Diagram IIA and IIB.

Figure 4. Diagram IIIA and IIIB.

Summary of the results. Note that, using definitions[8] in [7], in the limit given by (5.15) and (5.18)

$$P_3^2 = \frac{(x_{12}^+)^2}{x_{12}^4} = \frac{(\delta^+)^2}{\delta^4},$$ (5.19)

$$P_1^2 = \frac{(x_{23}^+)^2}{x_{23}^4} = \frac{(\delta^+)^2}{(t^2 + \delta^2)^2} \sim \frac{(\delta^+)^2}{t^4},$$

$$P_2^2 = \frac{(x_{31}^+)^2}{x_{31}^4} = 0,$$

$$Q_1 = -\frac{\delta^+}{\delta^2}, \qquad\qquad Q_2 = 0, \qquad Q_3 = \frac{\delta^+}{t^2},$$

$$S_1 = i\frac{(\delta^+)^2}{4\delta t^3}, \qquad\qquad S_2 = 0, \qquad S_3 = -i\frac{(\delta^+)^2 t}{4\delta^3 t^2}.$$

Using the limits (5.19), the odd tensor structure for $\langle jj\tilde{T}\rangle$ reduces to

$$\frac{1}{|x_{12}||x_{23}||x_{31}|}\left(Q_3^2 S_3 + 2P_1^2 S_2 + 2P_2^2 S_1\right)\Big|_{|\delta/t\to|0} = -i\frac{\delta_-^4}{4\delta^4 t^7}.$$ (5.20)

In appendix B we have evaluated the diagrams in figure 4 and we find

$$\lim_{|\delta/t|\to 0}\langle jj\tilde{T}\rangle_{\mathrm{odd}} = N\lambda\frac{1}{2\pi^3}\frac{\delta_-^4}{\delta^4 t^7}.$$ (5.21)

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[8]Note that the definitions of P_i^2, Q_i^2 and S_i given in [7] and [27] differ by a factor of 4.

Therefore using (5.20), the parity odd part of the three point function is given by

$$\langle jj\tilde{T}\rangle_{\text{odd}} = N\lambda\frac{2i}{\pi^3}\ \frac{\left(Q_3^2 S_3 + 2P_1^2 S_2 + 2P_2^2 S_1\right)}{|x_{12}||x_{23}||x_{31}|}. \tag{5.22}$$

Now going back to Minkowski signature and taking into account of the normalisation (5.12) we obtain

$$\langle jjT\rangle_{\text{odd}} = N\lambda\frac{1}{\pi^3}\ \frac{\left(Q_3^2 S_3 + 2P_1^2 S_2 + 2P_2^2 S_1\right)}{|x_{12}||x_{23}||x_{31}|}. \tag{5.23}$$

Finally let us compare the above equation with the normalisation for p_j written in (5.3). For this we need to take the $\theta \to 0$ limit. We see that

$$\alpha' = \frac{1}{\pi^4}. \tag{5.24}$$

Let us perform the same analysis for the parity odd part of the three point function of the stress tensor. In the limit (5.19) the parity odd tensor structure reduces to

$$\frac{1}{|x_{12}||x_{23}||x_{31}|}\left[(P_1^2 Q_1^2 + 5P_2^2 P_3^2)S_1 + (P_2^2 Q_2^2 + 5P_3^2 P_1^2)S_2 + (P_3^2 Q_3^2 + 5P_1^2 P_2^2)S_3\right]\Big|_{|\delta/t\to|0}$$

$$= -i\frac{\delta_-^6}{4\delta^8 t^7}. \tag{5.25}$$

Evaluating the corresponding diagram in figure 4 in appendix we obtain

$$\langle\tilde{T}\tilde{T}\tilde{T}\rangle = N\lambda\frac{6}{\pi^3}\frac{(\delta^+)^6}{\delta^8 t^7}. \tag{5.26}$$

Therefore using (5.25) we can write the parity odd part of the three point function of the stress tensor as

$$\langle\tilde{T}\tilde{T}\tilde{T}\rangle_{\text{odd}} = N\lambda\frac{24i}{\pi^3}\times \tag{5.27}$$

$$\times\frac{1}{|x_{12}||x_{23}||x_{31}|}\left[(P_1^2 Q_1^2 + 5P_2^2 P_3^2)S_1 + (P_2^2 Q_2^2 + 5P_3^2 P_1^2)S_2 + (P_3^2 Q_3^2 + 5P_1^2 P_2^2)S_3\right].$$

In minkowski signature and using the normalisation of the stress tensor in (5.12) we obtain

$$\langle TTT\rangle_{\text{odd}} = N\lambda\frac{3}{\pi^3}\times \tag{5.28}$$

$$\times\frac{1}{|x_{12}||x_{23}||x_{31}|}\left[(P_1^2 Q_1^2 + 5P_2^2 P_3^2)S_1 + (P_2^2 Q_2^2 + 5P_3^2 P_1^2)S_2 + (P_3^2 Q_3^2 + 5P_1^2 P_2^2)S_3\right].$$

Finally we fix the normalisation of the coefficient p_T in (5.3) by taking the $\theta \to 0$ limit. We obtain

$$\alpha = \frac{3}{\pi^4}. \tag{5.29}$$

5.2 Saturation of the conformal collider bounds

Given values for the coefficients $n_s^{j,T}, n_f^{j,T}$ and $p_{j,T}$ given in (5.3) with α, α' as evaluated in (5.29) and (5.24) we can evaluate the entries of the energy matrix corresponding to the charge and stress tensor excitations. Before we present the final result, for reference we list here the intermediate parameters $\mathcal{A}, \mathcal{B}, \mathcal{C}$ which appear in the three point function of the stress tensor. We use the relations [29]

$$\mathcal{A} = \frac{27 n_s^T}{512\pi^3}, \qquad \mathcal{B} = -\frac{9(4n_f^T + 3n_s^T)}{512\pi^3}, \qquad \mathcal{C} = -\frac{9(8n_f^T + n_s^T)}{2048\pi^3}, \qquad (5.30)$$

to obtain

$$\mathcal{A} = \frac{27N \sin\frac{\theta}{2}^2 \sin\theta}{256\pi^3\theta},$$
$$\mathcal{B} = -\frac{9N \sin\theta(\cos\theta + 7)}{512\pi^3\theta},$$
$$\mathcal{C} = -\frac{9N \sin\theta(7\cos\theta + 9)}{2048\pi^3\theta}. \qquad (5.31)$$

Similarly the intermediate parameters c, e, that occur in the 3 point function of the stress tensor with 2 insertions of the current is related to the coefficients in (5.3) by

$$c = \frac{3(2n_f^j + n_s^j)}{256\pi^3}, \qquad e = \frac{3n_s^j}{256\pi^3}. \qquad (5.32)$$

This leads to

$$c = \frac{3N(3 + \cos\theta)\sin\theta}{256\pi^3\theta}, \qquad (5.33)$$
$$e = \frac{3N \sin\frac{\theta}{2}^2 \sin\theta}{128\pi^3\theta}.$$

Now using (5.3) with α, α' as evaluated in (5.29) and (5.24) and the relations (3.43) and (4.40) we obtain

$$a_2 = -2\cos\theta, \qquad \alpha_j = 2\sin\theta, \qquad t_4 = -4\cos\theta, \qquad \alpha_T = 4\sin\theta. \qquad (5.34)$$

Thus the Chern-Simons theories with a single fundamental boson or fermion saturate the conformal collider bounds and they lie on the circles

$$a_2^2 + \alpha_j^2 = 4, \qquad t_4^2 + \alpha_T^2 = 16. \qquad (5.35)$$

The location of the theory on the circle is given by θ the t 'Hooft coupling.

We note that, in the large N limit, Chern-Simons theories coupled to fundamental bosons or fermions are very similar to free theories, because they also contain an infinite tower of higher spin operators, whose scaling dimensions ($\Delta_s = s+1$) saturate the unitarity bound ($\Delta_s \geq s + 1$), hence it is perhaps not surprising that they saturate the conformal collider bounds as well. The fact that these theories saturate the bounds implies that

there exist a polarisation of the energy matrix for which the eigen value vanishes. It will be interesting to show this directly from the conservation of higher spin currents in these theories. Such a proof will demonstrate that conservation of higher spin currents ensures that the conformal collider bounds will be saturated. Large N Chern-Simons theory coupled to more general types of fundamental matter, such as theories with both a fundamental boson and fermion [49], as well as supersymmetric theories, or ABJ in the limit $M \ll N$ [50] also contain a tower of higher-spin operators at the unitarity bound. Let us consider the case of large N Chern-Simons theory with both fundamental boson and fermion, in such a theory there exists two sets of U(1) as well as spin 2 currents. Therefore the polarisation at which the energy vanishes are different for each of currents. The combined currents will therefore not have a polarisation at which the energy vanishes.[9]

On the other hand, we might expect that strongly-interacting large N Chern-Simons theories coupled to matter in other representations, such as adjoint or bi-fundamental matter [5, 51, 52], where the higher-spin operators are not at the unitarity bound, must lie inside the disc, away from the boundary. Correlation functions in pure Einstein gravity, which are of course, parity-preserving, lie at the centre of the disk. However, perhaps that one could find a counter example to this using parity-violating gravity theories with an axion [53, 54].

6 Conclusions

We have obtained constraints on the three-point functions $\langle jjT \rangle, \langle TTT \rangle$ that apply to all (both parity-even and parity-odd) conformal field theories in $d = 3$. These constraints were obtained by imposing the condition that energy observed at the conformal collider be positive. The constraints we obtain imply that the space of all allowed $\langle jjT \rangle, \langle TTT \rangle$ correlation functions in conformal field theories in $d = 3$ lie on a two-dimensional disc. These constraints are particularly relevant for Chern-Simons theories with matter, and we explicitly showed that the $\langle jjT \rangle, \langle TTT \rangle$ correlation functions of large N, U(N) Chern-Simons theories with a single fundamental fermion or a fundamental boson lie on the bounding circles of these disc.

In this paper we have restricted our analysis to excitations created by the U(1) current and the stress tensor. However we expect similar results for excitations carrying arbitrary spin. It will be interesting to generalise the observations of this paper to these correlation functions.

Recent evaluation of parity violating three point functions in certain higher spin theories Vasiliev theories in AdS_4 show that these theories are dual to large N matter Chern-Simons theories [55, 56]. Our results then imply that these Vasiliev like theories also saturate the conformal collider bounds and lie on the circles that bound the disc in the parameter space of the three point functions. It will be interesting to show this directly from a shock wave type analysis as was done in AdS_5 by [41].

Another direction to explore is to obtain a proof of the conformal collider bounds for parity odd theories in $d = 3$ from unitarity and causality of the CFT along the lines

---

[9]We thank Ofer Aharony for raising these points.

of [41, 42, 57–59]. For this we need to understand the structure of parity odd conformal blocks in $d = 3$ which in itself is an useful enterprise.

Acknowledgments

A   Details for evaluation of the energy matrix $\hat{E}(T)$

In this appendix we will illustrate the procedure involved in evaluating the contributions to the energy matrix $\hat{E}(T)$.

We first detail the steps for the term $\mathcal{I}_1$ that occurs in the parity even element (11) of the energy matrix $\hat{E}(T)$ defined in (4.11). All the other terms are evaluated similarly. We need to first take the $x_1^+ \to \infty$ limit, then the integral over the null time x_1^- and finally perform the integral over the 3 remaining spatial directions. The steps are outlined in the following equations.

$$
\mathcal{I}_1 = \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{2} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} I_1, \tag{A.1}
$$

$$
= (\mathcal{E}+4\mathcal{C}-2\mathcal{D}) \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{2} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} \frac{\mathcal{I}_{xy,\mu'\nu'}^T(x)\mathcal{I}_{--,\sigma'\rho'}^T(x_1)}{x^6 x_1^6}
$$

$$
\times \left(\frac{Z^{\mu'}Z^{\nu'}}{Z^2} - \frac{1}{3}\eta^{\mu'\nu'} \right) \left(\frac{Z_x Z_y}{Z^2} \right) \left(\frac{Z^{\sigma'}Z^{\rho'}}{Z^2} - \frac{1}{3}\eta^{\sigma'\rho'} \right) \frac{1}{Z^{\frac{3}{2}}},
$$

$$
= (\mathcal{E}+4\mathcal{C}-2\mathcal{D}) \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{2} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} \frac{(\mathcal{I}_{x+,\mu'\nu'}^T(x) - \mathcal{I}_{x-,\mu'\nu'}^T(x))\mathcal{I}_{--,\sigma'\rho'}^T(x_1)}{x^6 x_1^6}
$$

$$
\times \left(\frac{Z^{\mu'}Z^{\nu'}}{Z^2} - \frac{1}{3}\eta^{\mu'\nu'} \right) \left(\frac{Z_x(Z_+ - Z_-)}{Z^2} \right) \left(\frac{Z^{\sigma'}Z^{\rho'}}{Z^2} - \frac{1}{3}\eta^{\sigma'\rho'} \right) \frac{1}{Z^{\frac{3}{2}}},
$$

$$
= (\mathcal{E}+4\mathcal{C}-2\mathcal{D}) \int d^3x e^{iEt} \int_{-\infty}^{\infty} \frac{dx_1^-}{2}
$$

$$
\times \frac{x^2(x^-)^2 \sqrt{\frac{-x^- + x_1^-}{(x^2 - x^- x^+)x_1^-}} \left(x^2 - (x^-)^2 + (x^- - x^+)x_1^- \right) \left(x^2 + x^+ x_1^- - x^-(x^+ + x_1^-) \right)}{32(x^2 - (x^- - 2i\epsilon)(x^+ - 2i\epsilon))^4 (x^- - x_1^- + i\epsilon)^4 (x_1^- - i\epsilon)^3},
$$

$$
= (\mathcal{E}+4\mathcal{C}-2\mathcal{D}) \int d^3x e^{iEt}
$$

$$\times \int_{-\infty}^{\infty} \frac{dx_1^-}{2} \frac{(x^2(x^-)^2)}{32(x^2-x^-x^+)^{\frac{9}{2}}} \left(\frac{x^4-x^2(x^-)^2-x^2x^-x^++(x^-)^3x^+}{(x_1^--i\epsilon)^{\frac{7}{2}}(x_1^--x^-+i\epsilon)^{\frac{7}{2}}} \right.$$

$$\left. + \frac{-(x^-)^2+2x^-x^+-(x^+)^2}{(x_1^--i\epsilon)^{\frac{3}{2}}(x_1^--x^-+i\epsilon)^{\frac{7}{2}}} + \frac{(x^-)^3+x^-x^+-2(x^-)^2x^+}{(x_1^--i\epsilon)^{\frac{5}{2}}(x_1^--x^-+i\epsilon)^{\frac{7}{2}}} \right),$$

$$= \frac{(\mathcal{E}+4\mathcal{C}-2\mathcal{D})}{2}$$

$$\times \int d^3x e^{iEt} \left(\frac{2x^2\left(16x^4-16x^2x^-(x^-+x^+)+(x^-)^2\left(5(x^-)^2+6x^-x^++5(x^+)^2\right)\right)}{15(x^--2i\epsilon)^4(x^2-(x^--2i\epsilon)(x^+-2i\epsilon))^{9/2}} \right).$$

In the last step we have used integrals (C.3), (C.6) , (C.1). To perform the final integrations we introduce the light cone coordinates $x^\pm = t \pm y$. The integrations are done by first integrating over the direction orthogonal to the light cone directions (x) first and then the light cone directions. The steps are indicated below

$$\mathcal{I}_1 = \frac{(\mathcal{E}+4\mathcal{C}-2\mathcal{D})}{2}\frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{iEx^+}{2}} \int_{-\infty}^{\infty} dx^- e^{\frac{iEx^-}{2}}$$

$$\times \int_{-\infty}^{\infty} dx \left(\frac{2x^2\left(16x^4-16x^2x^-(x^-+x^+)+(x^-)^2\left(5(x^-)^2+6x^-x^++5(x^+)^2\right)\right)}{15(x^--2i\epsilon)^4\left(x^2-(x^--2i\epsilon)(x^+-2i\epsilon)\right)^{9/2}} \right),$$

$$= -\frac{(\mathcal{E}+4\mathcal{C}-2\mathcal{D})}{2}\frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{iEx^+}{2}} \int_{-\infty}^{\infty} dx^- e^{\frac{iEx^-}{2}} \frac{1504\sqrt{-x^-x^+}\sqrt{-\frac{1}{x^-x^+}}}{1575\left(x^--2i\epsilon\right)^5\left(x^+-2i\epsilon\right)}$$

$$+\frac{64\sqrt{-x^-x^+}\sqrt{-\frac{1}{x^-x^+}}}{175\left(x^--2i\epsilon\right)^4\left(x^+-2i\epsilon\right)^2} + \frac{32\sqrt{-x^-x^+}\sqrt{-\frac{1}{x^-x^+}}}{315\left(x^--2i\epsilon\right)^3\left(x^+-2i\epsilon\right)^3},$$

$$= \frac{(\mathcal{E}+4\mathcal{C}-2\mathcal{D})}{2}\left(-\frac{149\pi^2 E^4}{9450} \right). \tag{A.2}$$

We have used (C.4), for integrating over the light cone directions.

As an illustration of the steps involved in obtaining the parity odd element (12) of the energy matrix $\hat{E}(T)$ we examine the term corresponding to I_1^p in (4.23).

$$I_1^p = \frac{P_1^2 Q_1^2 S_1}{|x-x_1||x_1||-x|}\big|_{\epsilon_1^{xy}=\epsilon_1^{yx}=\epsilon_2^{--}=\epsilon_3^{xx}=1} + \frac{P_1^2 Q_1^2 S_1^2}{|x-x_1||x_1||-x|}\big|_{\epsilon_1^{xy}=\epsilon_1^{yx}\epsilon_2^{--}=-\epsilon_3^{yy}=1},$$

$$= \frac{2}{256}\frac{-32Q_{1x}Q_{1y}I_{-x}(x_1)}{x_1^6(x-x_1)^2x^2}\left(\epsilon^{\gamma'\delta}_{-}x_{1\gamma'}(x-x_1)_\delta x_{1x} - \frac{\epsilon^{\gamma'}_{-x}}{2}\left(|x-x_1|^2x_{1\gamma'}+|x_1|^2(x-x_1)_{\gamma'}\right) \right)$$

$$+ \frac{2}{256}\frac{32Q_{1x}Q_{1y}I_{-y}(x_1)}{x_1^6(x-x_1)^2x^2}\left(\epsilon^{\gamma'\delta}_{-}x_{1\gamma'}(x-x_1)_\delta x_{1y} - \frac{\epsilon^{\gamma'}_{-y}}{2}\left(|x-x_1|^2x_{1\gamma'}+|x_1|^2(x-x_1)_{\gamma'}\right) \right),$$

$$\equiv I_{1,1}^p + I_{1,2}^p \tag{A.3}$$

The term $I_{1,1}^p$ does not contribute to the final result since in the limit $x_1^+ \to \infty$, it is

sub-leading.

$$
\begin{aligned}
\mathcal{I}_{1,1}^p &= \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{2} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} I_{1,1}^p, \\
&= \int d^3x e^{iEt} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{2} \frac{2}{256} \frac{-32Q_{1x}Q_{1y}I_{-x}(x_1)}{x_1^6(x-x_1)^2x^2} \\
&\quad \times \left(\epsilon^{\gamma'\delta}{}_{-}x_{1\gamma'}(x-x_1)_\delta x_{1x} - \frac{\epsilon^{\gamma'}{}_{-x}}{2} \left(|x-x_1|^2 x_{1\gamma'} + |x_1|^2 (x-x_1)_{\gamma'} \right) \right), \\
&= \int d^3x e^{iEt} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} O\left(\frac{1}{x_1^+} \right), \\
&= O\left(\frac{1}{x^+} \right).
\end{aligned}
\tag{A.4}
$$

The term $I_{1,2}^p$ is handled as before in the following steps.

$$
\begin{aligned}
\mathcal{I}_{1,2}^p &= \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{2} \int_{-\infty}^{\infty} \frac{dx_1^-}{2} I_{1,2}^p, \\
&= \frac{1}{256} \int d^3x e^{iEt} \int_{-\infty}^{\infty} dx_1^- \frac{2x^2 \left(x^2 - (x^-)^2 + x^- x_1^- - x^+ x_1^- \right)}{2(-x^2 + (x^- - 2i\epsilon)(x^+ - 2i\epsilon))^3 (x^- - x_1^- + i\epsilon)^2 (x_1^- - i\epsilon)^4}, \\
&= \frac{1}{256} \int d^3x e^{iEt} \int_{-\infty}^{\infty} dx_1^- \frac{-2}{2(x^2 - (x^- - 2i\epsilon)(x^+ - 2i\epsilon))^3} \left(\frac{x^2 \left(x^2 - (x^-)^2 \right)}{(x_1^- - x^- + i\epsilon)^2 (x_1^- - i\epsilon)^4} \right. \\
&\quad \left. + \frac{x^2 (x^- - x^+)}{(x_1^- - x^- + i\epsilon)^2 (x_1^- - i\epsilon)^3} \right), \\
&= \frac{1}{256} \int d^3x e^{iEt} \frac{2i\pi x^2 \left(4x^2 - x^- (x^- + 3x^+) \right)}{(x^- - 2i\epsilon)^5 (-x^2 + (x^- - 2i\epsilon)(x^+ - 2i\epsilon))^3}, \\
&= \frac{1}{256} \frac{1}{2} \int_{-\infty}^{\infty} dx^+ e^{\frac{iEx^+}{2}} \int_{-\infty}^{\infty} dx^- e^{\frac{iEx^-}{2}} \frac{2\pi^2}{8(x^- - 2i\epsilon)^{9/2}(x^+ - 2i\epsilon)^{3/2}} \\
&\quad + \frac{30\pi^2}{8(x^- - 2i\epsilon)^{11/2}\sqrt{x^+ - 2i\epsilon}}, \\
&= \frac{1}{256} \left(-\frac{8}{315} \pi^3 E^4 \right).
\end{aligned}
\tag{A.5}
$$

We have used the integrals (C.5), (C.2) and (C.4)

$$
\mathcal{I}_1^p = -\frac{1}{32} \frac{1}{315} \pi^3 E^4
\tag{A.6}
$$

B Parity odd three-point functions at one loop

In this appendix we detail the steps involved in evaluating diagrams $IIIA, IIIB$ in figure 4 for both the $\langle jj\tilde{T} \rangle$ and the $\langle \tilde{T}\tilde{T}\tilde{T} \rangle$ correlator.

B.1 Perturbative calculation of $\langle jj\tilde{T}\rangle$

The integral corresponding to diagram IIIA is

$$(\text{III}_A) = \int d^3y_1 d^3y_2 \langle A_i(y_1)A_j(y_2)\rangle \text{Tr}\left[\gamma^i\langle\psi(y_1)\bar\psi(x_1)\rangle\gamma_-\langle\psi(x_1)\bar\psi(y_2)\rangle\gamma^j\langle\psi(y_2)\bar\psi(x_2)\rangle\right.$$
$$\left.\times\,\gamma_-\langle\psi(x_2)\bar\psi(x_3)\rangle\gamma_-(\overset{\leftrightarrow}{\partial}_{x_3^-})\langle\psi(x_3)\bar\psi(y_1)\rangle\right]. \quad \text{(B.1)}$$

The overall minus sign comes from the fermion loop. Processing this we have:

$$\int d^3y_1 d^3y_2\,\text{sign}(y_{12}^0)\delta^2(\vec{y}_{12})\epsilon_{ij}\text{Tr}\left[\gamma^i\slashed{y}_1\gamma^+(-\slashed{y}_2)\gamma^j(\slashed{y}_2+\slashed{\delta})\gamma^+(-\slashed{x}-\slashed{\delta})\gamma^+(\slashed{x}-\slashed{y}_1)\right] \quad \text{(B.2)}$$

$$\times\frac{1}{|y_1|^3|y_2|^3|\delta+y_2|^3}\left(\frac{1}{|x+\delta|^3}\overset{\leftrightarrow}{\partial}_{x^-}\frac{1}{|x-y_1|^3}\right)\left(\frac{1}{4\pi}\right)^5\frac{2\pi i}{k}\left(\frac{N^2}{2}\right)$$

$$=\frac{N^2}{64\pi^4 k}\delta^+\int d^2z dt_1 dt_2\,\text{sign}(t_{12})[(z^+)^2 t_2^2 + z^+(z+\delta)^+ t_1(t-t_1)]$$

$$\times\frac{1}{(z^2+t_1^2)^{\frac{3}{2}}(z^2+t_2^2)^{\frac{3}{2}}[(z+\delta)^2+t_2^2]^{\frac{3}{2}}}\left\{\frac{1}{(t^2+\delta^2)^{\frac{3}{2}}}(\overset{\leftarrow}{\partial}_{\delta^-}+\overset{\rightarrow}{\partial}_{z^-})\frac{1}{[(t-t_1)^2+z^2]^{\frac{3}{2}}}\right\}.$$

The trace over gamma matrices is carried out explicitly in the section below. The factor of $N^2/2$ is a group theory factor associated, with the factor of $1/2$ arising from the normalisation of the generators T^a.

Diagram IIIB is the same as diagram IIIA with $x_2 \leftrightarrow x_3$:

$$(\text{III}_B) = \int d^3y_1 d^3y_2 \langle A_i(y_1)A_j(y_2)\rangle \text{Tr}\left[\gamma^i\langle\psi(y_1)\bar\psi(x_1)\rangle\gamma_-\langle\psi(x_1)\bar\psi(y_2)\rangle\gamma^j\langle\psi(y_2)\bar\psi(x_3)\rangle\right.$$
$$\left.\times\,\gamma_-(\overset{\leftrightarrow}{\partial}_{x_3^-})\langle\psi(x_3)\bar\psi(x_2)\rangle\gamma_-\langle\psi(x_2)\bar\psi(y_1)\rangle\right]. \quad \text{(B.3)}$$

Processing this diagram, we see that, after evaluating the trace, it is identical to Diagram IIIA (with t_1 and t_2 interchanged):

$$=-\frac{N^2}{64\pi^4 k}\delta^+\int d^2z dt_1 dt_2\,\text{sign}(t_{12})[(z^+)^2 t_1^2 + z^+(z+\delta)^+ t_2(t-t_2)] \quad \text{(B.4)}$$

$$\times\frac{1}{(z^2+t_1^2)^{\frac{3}{2}}(z^2+t_2^2)^{\frac{3}{2}}(t_1^2+(\delta+z)^2)^{\frac{3}{2}}}\left\{\frac{1}{[(t_2-t)^2+z^2]^{\frac{3}{2}}}(\overset{\leftarrow}{\partial}_{z^-}+\overset{\rightarrow}{\partial}_{\delta^-})\frac{1}{(t^2+\delta^2)^{\frac{3}{2}}}\right\}.$$

Now summing both the diagrams we obtain

$$\text{III}_A + \text{III}_B = \frac{N^2}{32\pi^4 k}\delta^+\int d^2z dt_1 dt_2\,\text{sign}(t_{12})[(z^+)^2 t_2^2 + z^+(z+\delta)^+ t_1(t-t_1)] \quad \text{(B.5)}$$

$$\times\frac{1}{(z^2+t_1^2)^{\frac{3}{2}}(z^2+t_2^2)^{\frac{3}{2}}[(z+\delta)^2+t_2^2]^{\frac{3}{2}}}\left\{\frac{1}{(t^2+\delta^2)^{\frac{3}{2}}}(\overset{\leftarrow}{\partial}_{\delta^-}+\overset{\rightarrow}{\partial}_{z^-})\frac{1}{[(t-t_1)^2+z^2]^{\frac{3}{2}}}\right\}.$$

This includes all group theory factors and both diagrams.

The leading contribution to these integrals comes from the regions D1: $z \sim O(\delta)$, $t_1, t_2 \sim O(\delta)$, and D2:$z \sim O(\delta)$, and $t - t_1 \sim O(\delta)$ $t_2 \sim O(\delta)$. Evaluating the integral in the region $D2$ gives rise to an un-physical artefact of temporal gauge, which cancels with

another diagram. See [7] for details. For the final answer, therefore, we only require the contribution from region D1, which is:

$$\left.(\text{III}_A + \text{III}_B)\right|_{D1} = -\frac{3N^2}{32\pi^4 k}\frac{\delta^+}{t^7}\int d^2 z \, dt_1 dt_2 \frac{\text{sign}(t_{12})z^+(z^+ + \delta^+)^2 t_1}{(z^2 + t_1^2)^{\frac{3}{2}}(z^2 + t_2^2)^{\frac{3}{2}}[(z + \delta)^2 + t_2^2]^{\frac{3}{2}}}. \quad (\text{B.6})$$

Using

$$\int dt_1 \, \text{sign}(t_1 - t_2)\frac{t_1}{(z^2 + t_1^2)^{3/2}} = \frac{2}{(z^2 + t_2^2)^{1/2}}, \quad (\text{B.7})$$

we are left with an integral over $d^2 z$ and t_2. Defining $z_3 = t_2$, we find it can be written as:

$$\left. I \right|_{D1} = -\frac{3N^2}{16\pi^4 k}\frac{\delta^+}{t^7}\int d^3 z \frac{z^+(z^+ + \delta^+)^2}{z^4(z + \delta)^3}. \quad (\text{B.8})$$

Evaluating this remaining integral over z by Feynman parameters, we have:

$$\int d^3 z \frac{z^+(z^+ + \delta^+)^2}{z^4(z + \delta)^3} = -\frac{8\pi}{3}\frac{\delta^+}{\delta^4}. \quad (\text{B.9})$$

Therefore we obtain

$$\lim_{|\delta/t|\to 0}\langle jjT\rangle = \frac{N^2}{2\pi^3 k}\frac{(\delta^+)^4}{\delta^4 t^7}. \quad (\text{B.10})$$

B.2 Perturbative calculation of $\langle \tilde{T}\tilde{T}\tilde{T}\rangle$

Again, we focus on Diagram III (corrections to the x_1 vertex). As before there are two permutations that contribute:

$$\text{III}_A = \int d^3 y_1 d^3 y_2 \langle A_i(y_1)A_j(y_2)\rangle \text{Tr}\left[\gamma^j\langle\psi(y_1)\bar\psi(x_1)\rangle\gamma_-(\overleftrightarrow{\partial}_{x_1^-})\langle\psi(x_1)\bar\psi(y_2)\rangle\gamma^i\langle\psi(y_2)\bar\psi(x_2)\rangle\right.$$
$$\left.\times\;\gamma_-(\overleftrightarrow{\partial}_{x_2^-})\langle\psi(x_2)\bar\psi(x_3)\rangle\gamma_-(\overleftrightarrow{\partial}_{x_3^-})\langle\psi(x_3)\bar\psi(y_1)\rangle\right]. \quad (\text{B.11})$$

and

$$\text{III}_B = \int d^3 y_1 d^3 y_2 \langle A_i(y_1)A_j(y_2)\rangle \text{Tr}\left[\gamma^j\langle\psi(y_1)\bar\psi(x_1)\rangle\gamma_-(\overleftrightarrow{\partial}_{x_1^-})\langle\psi(x_1)\bar\psi(y_2)\rangle\gamma^i\langle\psi(y_2)\bar\psi(x_3)\rangle\right.$$
$$\left.\times\;\gamma_-(\overleftrightarrow{\partial}_{x_3^-})\langle\psi(x_3)\bar\psi(x_2)\rangle\gamma_-(\overleftrightarrow{\partial}_{x_2^-})\langle\psi(x_2)\bar\psi(y_1)\rangle\right]. \quad (\text{B.12})$$

As before, we see that they turn out to be identical.

Diagram $IIIA$ can be written as:

$$C\int d^3 y_1 d^3 y_2 \, \text{sign}(t_{12})\delta^2(\vec{y}_{12})\epsilon_{ij}\text{Tr}\left[\gamma^i\slashed{y}_1\gamma^+(-\slashed{y}_2)\gamma^j(\slashed{y}_2+\slashed{\delta})\gamma^+(-\slashed{x}-\slashed{\delta})\gamma^+(\slashed{x}-\slashed{y}_1)\right] \quad (\text{B.13})$$

$$\times\left(\frac{1}{|y_1|^3}(\overleftarrow{\partial}_{y_1^-}-\overrightarrow{\partial}_{y_2^-})\frac{1}{|y_2|^3}\right)\left[\frac{1}{|\delta+y_2|^3}(\overleftarrow{\partial}_{\delta-}-\overrightarrow{\partial}_{\delta-})\frac{1}{|x+\delta|^3}(-\overleftarrow{\partial}_{x-}+\overrightarrow{\partial}_{x-})\frac{1}{|x-y_1|^3}\right]$$

$$=-16iC\delta^+\int d^2 z \, dt_1 dt_2 \, \text{sign}(t_{12})[(z^+)^2 t_2^2 + z^+(z+\delta)^+ t_1(t-t_1)]\left[\frac{1}{(t_1^2+z^2)^{\frac{3}{2}}}\overleftrightarrow{\partial}_{z^-}\frac{1}{(t_2^2+z^2)^{\frac{3}{2}}}\right]$$

$$\times\left\{\frac{1}{[t_2^2+(\delta'+z)^2]^{\frac{3}{2}}}(\overleftarrow{\partial}_{z-}-\overrightarrow{\partial}_{\delta-})\frac{1}{(t^2+\delta^2)^{\frac{3}{2}}}(-\overleftarrow{\partial}_{\delta-}-\overrightarrow{\partial}_{z-})\frac{1}{[(t-t_1)^2+z^2]^{\frac{3}{2}}}\right\}_{\delta'=\delta},$$

where $C = \left(\frac{1}{4\pi}\right)^5 \frac{2\pi i}{k} \left(\frac{N^2}{2}\right)$. This receives contributions from the two regions D1 and D2. Evaluating the region D2 gives rise to an un-physical artefact of temporal gauge which cancels with another diagram. The final answer for the correlation function in our limit thus only comes from region D1.

$$
= -16iC\delta^+ t \int d^2z dt_1 dt_2 \, \text{sign}(t_{12})[z^+(z+\delta)^+ t_1] \left[\frac{1}{(t_1^2+z^2)^{\frac{3}{2}}} \overset{\leftrightarrow}{\partial}_{z^-} \frac{1}{(t_2^2+z^2)^{\frac{3}{2}}} \right] \tag{B.14}
$$

$$
\times \left\{ \frac{1}{[t_2^2+(\delta'+z)^2]^{\frac{3}{2}}} (\overset{\leftarrow}{\partial}_{z^-}) \frac{1}{(t^2+\delta^2)^{\frac{3}{2}}} (-\overset{\leftarrow}{\partial}_{\delta^-} - \overset{\rightarrow}{\partial}_{z^-}) \frac{1}{[(t-t_1)^2+z^2]^{\frac{3}{2}}} \right\},
$$

$$
= -16iC\frac{\delta^+}{t^7} \int d^2z dt_1 dt_2 \, \text{sign}(t_{12})[z^+(z+\delta)^+ t_1] \left[\frac{1}{(t_1^2+z^2)^{\frac{3}{2}}} \overset{\leftrightarrow}{\partial}_{z^-} \frac{1}{(t_2^2+z^2)^{\frac{3}{2}}} \right]
$$

$$
\times \left\{ \frac{-9(\delta^+ + z^+)^2}{[t_2^2+(\delta+z)^2]^{\frac{5}{2}}} \right\},
$$

$$
= -16iC\frac{\delta^+}{t^7} \int d^2z dt_1 dt_2 \, \text{sign}(t_{12})[z^+(z+\delta)^+ t_1] \left[\frac{-3z^+(t_1^2-t_2^2)}{(t_1^2+z^2)^{\frac{5}{2}}(t_2^2+z^2)^{\frac{5}{2}}} \right] \left\{ \frac{-9(\delta^+ + z^+)^2}{[t_2^2+(\delta+z)^2]^{\frac{5}{2}}} \right\}.
$$

Now we use the integral

$$
\int dt_1 \, \text{sign}(t_1 - t_2) t_1 (t_1^2 - t_2^2)(t_1^2 + z^2)^{-5/2} = \frac{4}{3} \frac{1}{(t^2 + z^2)^{1/2}}, \tag{B.15}
$$

and let $z_3 = t$, to obtain

$$
\text{III}_A = -16iC\frac{\delta^+}{t^7} \int d^3z \frac{36(\delta^+ + z^+)^3(z^+)^2}{z^6(\delta + z)^5}, \tag{B.16}
$$

$$
= 12 \cdot 256\pi iC \frac{(\delta^+)^6}{\delta^8 t^7}.
$$

which was evaluated using Feynman parameters.

Diagram IIIB is

$$
C \int d^3y_1 d^3y_2 \, \text{sign}(t_{12}) \delta^2(\vec{y}_{12}) \epsilon_{ij} \text{Tr} \left[\gamma^i \slashed{y}_1 \gamma^+ (-\slashed{y}_2) \gamma^j (\slashed{y}_2 - \slashed{x}) \gamma^+ (\slashed{x} + \slashed{\delta}) \gamma^+ (-\slashed{\delta} - \slashed{y}_1) \right] \tag{B.17}
$$

$$
\times \left(\frac{1}{|y_1|^3} (\overset{\leftarrow}{\partial}_{y_1^-} - \overset{\rightarrow}{\partial}_{y_2^-}) \frac{1}{|y_2|^3} \right) \left[\frac{1}{|y_2 - x|^3} (-\overset{\leftarrow}{\partial}_{x^-} + \overset{\rightarrow}{\partial}_{x^-}) \frac{1}{|x + \delta|^3} (\overset{\leftarrow}{\partial}_{\delta^-} - \overset{\rightarrow}{\partial}_{\delta^-}) \frac{1}{|\delta + y_1|^3} \right]
$$

$$
= -16iC\delta^+ \int d^2z dt_1 dt_2 \, \text{sign}(t_{12})[(z^+)^2 t_1^2 + z^+(z+\delta)^+ t_2(t-t_2)] \left[-\frac{1}{(t_2^2+z^2)^{\frac{3}{2}}} \overset{\leftrightarrow}{\partial}_{z^-} \frac{1}{(t_1^2+z^2)^{\frac{3}{2}}} \right]
$$

$$
\times \left\{ \frac{1}{[t_1^2+(\delta'+z)^2]^{\frac{3}{2}}} (\overset{\leftarrow}{\partial}_{z^-} - \overset{\rightarrow}{\partial}_{\delta^-}) \frac{1}{(t^1+\delta^2)^{\frac{3}{2}}} (-\overset{\leftarrow}{\partial}_{\delta^-} - \overset{\rightarrow}{\partial}_{z^-}) \frac{1}{[(t-t_2)^2+z^2]^{\frac{3}{2}}} \right\}_{\delta'=\delta}.
$$

Using $\text{sign}(t_{12}) = -\text{sign}(t_{12})$ we see that this is equal to diagram IIIA (with t_1 and t_2 interchanged).

To summarize, the result is then:

$$
\lim_{|\delta/t| \to 0} \langle \tilde{T}\tilde{T}\tilde{T} \rangle = \text{III}_A + \text{III}_B = 24 \cdot 256\pi iC \frac{(\delta^+)^6}{\delta^8 t^7} = \frac{6N^2}{\pi^3 k} \frac{(\delta^+)^6}{\delta^8 t^7}. \tag{B.18}
$$

Traces over gamma matrices. To evaluate the traces over gamma matrices, we use $\epsilon_{+-} = i$, $(\gamma^+)^2 = 0$ and relations such as

$$\text{Tr}(\gamma^{\alpha_1}\gamma^{\alpha_2}\gamma^{\alpha_3}\gamma^{\alpha_4}) = 2\delta^{\alpha_1\alpha_2}\delta^{\alpha_3\alpha_4} + 2\delta^{\alpha_1\alpha_4}\delta^{\alpha_2\alpha_3} - 2\delta^{\alpha_1\alpha_3}\delta^{\alpha_2\alpha_4}, \tag{B.19}$$

from which we find:

$$i\text{Tr}\ \left(\gamma^+\gamma^\mu\gamma^+\gamma^\nu\gamma^-\gamma^\rho\gamma^+\gamma^\sigma\gamma^+\gamma^\eta\right), \tag{B.20}$$

$$= 8i\eta^{+\eta}\eta^{+\mu}\eta^{+\sigma}\text{Tr}\ \left(\gamma^+\gamma^\nu\gamma^-\gamma^\rho\right), \tag{B.21}$$

$$= 16i\eta^{+\eta}\eta^{+\mu}\eta^{+\sigma}\left(\eta^{+\nu}\eta^{-\rho} + \eta^{+\rho}\eta^{-\nu} - \eta^{\nu\rho}\right). \tag{B.22}$$

and

$$-i\text{Tr}\ \left(\gamma^-\gamma^\mu\gamma^+\gamma^\nu\gamma^+\gamma^\rho\gamma^+\gamma^\sigma\gamma^+\gamma^\eta\right), \tag{B.23}$$

$$= -8i\eta^{+\sigma}\eta^{+\nu}\eta^{+\rho}\text{Tr}\ \left(\gamma^-\gamma^\mu\gamma^+\gamma^\eta\right), \tag{B.24}$$

$$= -16i\eta^{+\sigma}\eta^{+\nu}\eta^{+\rho}\left(\eta^{-\mu}\eta^{+\eta} + \eta^{-\eta}\eta^{+\mu} - \eta^{\mu\eta}\right). \tag{B.25}$$

These imply that

$$\epsilon_{ij}\text{Tr}\left[\gamma^i \not{y}_1 \gamma^+(-\not{y}_2)\gamma^j(\not{y}_2 + \not{\delta})\gamma^+(-\not{z} - \not{\delta})\gamma^+(\not{z} - \not{y}_1)\right] = \tag{B.26}$$

$$= 16i\delta^+\left((z^+)^2 t_2^2 + z^+(z^+ + \delta^+)t_1(t - t_1)\right).$$

where, from the delta function in the propagator we can set $\vec{y}_1 = \vec{y}_2 = \vec{z}$, $x^3 = t$, $y_1^3 = t_1$ and $y_2^3 = t_2$.

C   Table of integrals

In this appendix we list out the various integrals used to evaluate the Energy matrix. We also outline the steps in performing some of them.

$$\int_{-\infty}^{\infty} dz \frac{1}{(z-i\epsilon)^d(z-x+i\epsilon)^{d+1}} = \int_{-\infty}^{\infty} dz \int_0^\infty ds \frac{(z-i\epsilon)s^d e^{(-s(z-i\epsilon)(z-x+i\epsilon))}}{\Gamma[d+1]},$$

$$= \int_0^\infty ds \frac{e^{-\frac{1}{4}s(2\epsilon+ix)^2}\sqrt{\pi}(-2i\epsilon+x)}{2\sqrt{s}}\frac{s^d}{\Gamma[d+1]},$$

$$= \frac{4^d\sqrt{\pi}\left((2\epsilon+ix)^2\right)^{-\frac{1}{2}-d}(-2i\epsilon+x)\Gamma\left[\frac{1}{2}+d\right]}{\Gamma[1+d]}. \tag{C.1}$$

$$\int_{-\infty}^{\infty} dz \frac{1}{(z-i\epsilon)^{d+1}(z-x+i\epsilon)^d} = \int_{-\infty}^{\infty} dz \int_0^\infty ds \frac{(z-x+i\epsilon)s^d e^{(-s(z-i\epsilon)(z-x+i\epsilon))}}{\Gamma[d+1]},$$

$$= \frac{4^d\sqrt{\pi}(2i\epsilon-x)\left((2\epsilon+ix)^2\right)^{-\frac{1}{2}-d}\Gamma\left[\frac{1}{2}+d\right]}{\Gamma[1+d]}. \tag{C.2}$$

$$\int_{-\infty}^{\infty} dz \frac{1}{(z-i\epsilon)^d(z-x+i\epsilon)^d} = \frac{\sqrt{\pi}(-1)^{\frac{1}{2}-d}(2)^{-1+2d}\Gamma\left[-\frac{1}{2}+d\right]}{(x-2i\epsilon)^{2d}\Gamma[d]}. \tag{C.3}$$

$$\int_{-\infty}^{\infty} dz \frac{e^{\frac{iEz}{2}}}{(z-i\epsilon)^d} = \int_0^\infty dz \frac{e^{\frac{iEz}{2}}}{(z-i\epsilon)^d} + \int_{-\infty}^0 dz \frac{e^{\frac{iEz}{2}}}{(z-i\epsilon)^d},$$

$$= \int_0^\infty dz \frac{e^{\frac{iEz}{2}}}{(z-i\epsilon)^d} + e^{i\pi d}\int_0^\infty dz \frac{e^{\frac{-iEz}{2}}}{(z+i\epsilon)^d},$$

$$= \int_0^\infty dz \int_0^\infty ds \frac{s^{d-1}}{\Gamma[d]} e^{-s(z-i\epsilon)+\frac{iEz}{2}}$$

$$+ e^{i\pi d}\int_0^\infty dz \int_0^\infty ds \frac{s^{d-1}}{\Gamma[d]} e^{-s(z+i\epsilon)-\frac{iEz}{2}},$$

$$= \frac{1}{\Gamma[d]}\int_0^\infty ds s^{d-1} e^{is\epsilon} \frac{1}{s-i\epsilon}$$

$$+ e^{i\pi d}\frac{1}{\Gamma[d]}\int_0^\infty ds s^{d-1} e^{-is\epsilon} \frac{1}{-s+i\epsilon},$$

$$= \frac{1}{\Gamma[d]}\int_{-\infty}^\infty ds s^{d-1} e^{is\epsilon} \frac{1}{s-i\epsilon},$$

$$= \frac{2\pi i}{\Gamma(d)}\left(\frac{iE}{2}\right)^{d-1}. \tag{C.4}$$

$$\int_{-\infty}^\infty dz \frac{1}{(z-i\epsilon)^{d+2}(z-x+i\epsilon)^d} = -\frac{2^{1+2d}d\sqrt{\pi}\left((2\epsilon+ix)^2\right)^{-\frac{1}{2}-d}\Gamma\left[\frac{1}{2}+d\right]}{\Gamma[2+d]}. \tag{C.5}$$

$$\int_{-\infty}^\infty dz \frac{1}{(z-i\epsilon)^d(z-x+i\epsilon)^{d+2}} = -\frac{2^{1+2d}d\sqrt{\pi}\left((2\epsilon+ix)^2\right)^{-\frac{1}{2}-d}\Gamma\left[\frac{1}{2}+d\right]}{\Gamma[2+d]}. \tag{C.6}$$

$$\int_{-\infty}^\infty dz \frac{1}{(z-i\epsilon)^d(z-x+i\epsilon)^{d+3}} = \frac{(2\pi i)(-1)^{-d-3}\Gamma[2d+2]}{\Gamma[d]\Gamma[d+3](x-2i\epsilon)^{2d+2}}. \tag{C.7}$$

$$\int_{-\infty}^\infty dz \frac{1}{(z-i\epsilon)^{d+3}(z-x+i\epsilon)^d} = \frac{(2\pi i)(-1)^{-d}\Gamma[2d+2]}{\Gamma[d]\Gamma[d+3](x-2i\epsilon)^{2d+2}}. \tag{C.8}$$

$$\int_{-\infty}^\infty dz \frac{1}{(z-i\epsilon)^{d+4}(z-x+i\epsilon)^d} = \frac{2^{3+2d}d(1+d)\sqrt{\pi}\left((2\epsilon+ix)^2\right)^{-\frac{3}{2}-d}\Gamma\left[\frac{3}{2}+d\right]}{\Gamma[4+d]}. \tag{C.9}$$

$$\int_{-\infty}^\infty dz \frac{1}{(z-i\epsilon)^d(z-x+i\epsilon)^{d+4}} = \frac{2^{3+2d}d(1+d)\sqrt{\pi}\left((2\epsilon+ix)^2\right)^{-\frac{3}{2}-d}\Gamma\left[\frac{3}{2}+d\right]}{\Gamma[4+d]}. \tag{C.10}$$

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