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On integrability of 2-dimensional σ-models of Poisson-Lie type

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Abstract: We describe a simple procedure for constructing a Lax pair for suitable 2-dimensional σ-models appearing in Poisson-Lie T-duality

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1 Introduction

There is a class of 2-dimensional σ-models, introduced in the context of Poisson-Lie T-duality [5], whose solutions are naturally described in terms of certain flat connections. The target space of such a σ-model is D/H, where D is a Lie group and $H \subset D$ a subgroup. The σ-model is defined by the following data: an invariant symmetric non-degenerate pairing $\langle,\rangle$ on the Lie algebra $\mathfrak{d}$ such that the Lie subalgebra $\mathfrak{h} \subset \mathfrak{d}$ is Lagrangian, i.e. $\mathfrak{h}^\perp = \mathfrak{h}$, and a subspace $V_+ \subset \mathfrak{d}$ such that $\dim V_+ = (\dim \mathfrak{d})/2$ and such that $\langle,\rangle|_{V_+}$ is positive definite. The construction and properties of these σ-models are recalled in section 2 (including the Poisson-Lie T-duality, which says that the σ-model, seen as a Hamiltonian system, is essentially independent of H). Let us call them *σ-models of Poisson-Lie type.*

The solutions $\Sigma \to D/H$ of equations of motion of such a σ-model can be encoded in terms of $\mathfrak{d}$-valued 1-forms $A \in \Omega^1(\Sigma, \mathfrak{d})$ satisfying

$$dA + [A, A]/2 = 0 \tag{1.1a}$$

$$A \in \Omega^{1,0}(\Sigma, V_+) \oplus \Omega^{0,1}(\Sigma, V_-), \tag{1.1b}$$

where $V_- := (V_+)^\perp \subset \mathfrak{d}$. Namely, the flatness (1.1a) of A implies that there is a map $\ell : \tilde{\Sigma} \to D$ (where $\tilde{\Sigma}$ is the universal cover of Σ) such that $A = -d\ell\,\ell^{-1}$. If the holonomy of A is in H then ℓ gives us a well-defined map $\Sigma \to D/H$. The maps $\Sigma \to D/H$ obtained in this way are exactly the solutions of equations of motion.

As first observed by Klimčík [3], and later by Sfetsos [12], and Delduc, Magro, and Vicedo [2], some σ-models of Poisson-Lie type are integrable. Their integrability is proven by finding a Lax pair, i.e. a 1-parameter family of flat connections (with parameter λ)

$$A_\lambda \in \Omega^1(\Sigma, \mathfrak{g}) \qquad dA_\lambda + [A_\lambda, A_\lambda]/2 = 0$$

where $\mathfrak{g}$ is a suitable semisimple Lie algebra. Such a family is constructed for every element of the phase space, i.e. for every $A \in \Omega^1(\Sigma, \mathfrak{d})$ satisfying (1.1).

The aim of this note is to make the construction of A_λ transparent. We simply observe that if $A \in \Omega^1(\Sigma, \mathfrak{d})$ satisfies (1.1) and if $p : \mathfrak{d} \to \mathfrak{g}$ is a linear map such that

$$[p(X), p(Y)] = p([X, Y]) \quad \forall X \in V_+,\ Y \in V_-$$

then

$$d\, p(A) + [p(A), p(A)]/2 = 0.$$

A suitable family $p_\lambda : \mathfrak{d} \to \mathfrak{g}$ will then give us a family of flat connections

$$A_\lambda = p_\lambda(A).$$

As an example, we provide a very simple construction of such families p_λ in the case when $\mathfrak{d} = \mathfrak{g} \otimes W$, where W is a 2-dimensional commutative algebra. These families recover the deformations of the principal chiral model from [2, 3, 12]. Our purpose is thus modest — it is simply to clarify previously constructed integrable σ-models. There is possibly a less naive construction of families p_λ that might produce new integrable models, but we leave this question open.

2 σ-models of Poisson-Lie type and Poisson-Lie T-duality

In this section we review the properties of the "2-dimensional σ-models of Poisson-Lie type" introduced in [5] (together with their Hamiltonian picture from [6] and using the target spaces of the form D/H, as introduced in [7]).

Let $\mathfrak{d}$ be a Lie algebra with an invariant non-degenerate symmetric bilinear form $\langle , \rangle$ of symmetric signature and let $V_+ \subset \mathfrak{d}$ be a linear subspace with $\dim V_+ = (\dim \mathfrak{d})/2$, such that $\langle , \rangle|_{V_+}$ is positive-definite.

Let $M = D/H$ where D is a connected Lie group integrating $\mathfrak{d}$ and $H \subset D$ is a closed connected subgroup such that its Lie algebra $\mathfrak{h} \subset \mathfrak{d}$ is Lagrangian in $\mathfrak{d}$.

This data defines a Riemannian metric g and a closed 3-form η on M. They are given by

$$g(\rho(X), \rho(Y)) = \frac{1}{2}\langle X, Y\rangle \qquad \forall X, Y \in V_+$$
$$p^*\eta = -\frac{1}{2}\eta_D + \frac{1}{2} d\langle \mathcal{A}, \theta_L\rangle$$

Here ρ is the action of $\mathfrak{d}$ on $M = D/H$, $p : D \to D/H$ is the projection, $\eta_D \in \Omega^3(D)$ is the Cartan 3-form (given by $\eta_D(X^L, Y^L, Z^L) = \langle [X, Y], Z\rangle$ $(\forall X, Y, Z \in \mathfrak{d})$), $\theta_L \in \Omega^1(D, \mathfrak{d})$ is the left-invariant Maurer-Cartan form on D (i.e. $\theta_L(X^L) = X$), and $\mathcal{A} \in \Omega^1(D, \mathfrak{h})$ is the connection on the principal H-bundle $p : D \to D/H$ whose horizontal spaces are the right-translates of V_+.[1]

[1]The conceptual definition of g and η is as follows: the trivial vector bundle $\mathfrak{d} \times M \to M$ is naturally an exact Courant algebroid, with the anchor given by ρ and the Courant bracket of its constant sections being the Lie bracket on $\mathfrak{d}$. Then $V_+ \times M \subset \mathfrak{d} \times M$ is a generalized metric, which is equivalent to the metric g and the closed 3-form η. We shall not use this language in this paper, in order to keep it short.

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The metric g and the 3-form η then define a σ-model with the standard action functional

$$S(f) = \int_\Sigma g(\partial_+ f, \partial_- f) + \int_Y f^*\eta$$

where Σ is (say) the cylinder with the usual metric $d\sigma^2 - d\tau^2$ and $f : \Sigma \to M$ is a map extended to the solid cylinder Y with boundary Σ.

For our purposes, the main properties of these σ-models are the following:

- The solutions of the equations of motion are in (almost) 1-1 correspondence with 1-forms $A \in \Omega^1(\Sigma, \mathfrak{d})$ satisfying (1.1). More precisely, a map $f : \Sigma \to M$ is a solution iff it admits a lift $\ell : \tilde{\Sigma} \to D$ such that $A := -d\ell\, \ell^{-1}$ satisfies (1.1). Notice that A is uniquely specified by f (the lift ℓ is not unique — it can be multiplied by an element of H on the right).

- When we restrict A to $S^1 \subset \Sigma = S^1 \times \mathbb{R}$, we get a 1-form $j(\sigma)d\sigma \in \Omega^1(S^1, \mathfrak{d})$. The $\mathfrak{d}$-valued functions $j(\sigma)$ on the phase space of the sigma model satisfy the current algebra Poisson bracket

$$\{j_a(\sigma), j_b(\sigma')\} = f^c_{ab}\, j_c(\sigma)\, \delta(\sigma - \sigma') + t_{ab}\, \delta'(\sigma - \sigma') \tag{2.1}$$

  (written using a basis e^a of $\mathfrak{d}$, with f^c_{ab} being the structure constants of $\mathfrak{d}$ and t_{ab} the inverse of the matrix of $\langle e^a, e^b\rangle$). The Hamiltonian of the σ-model is

$$\mathcal{H} = \frac{1}{2}\int_{S^1} \langle j(\sigma), Rj(\sigma)\rangle\, d\sigma \tag{2.2}$$

  where $R : \mathfrak{d} \to \mathfrak{d}$ is the reflection w.r.t. V_+.

Finally, let us observe that the phase space of the σ-model depends on the choice of $H \subset D$ only mildly; when we impose the constraint that A has unit holonomy, the reduced Hamiltonian system is independent of H. This statement is the Poisson-Lie T-duality (in the case of no spectators). (In more detail, the phase space of the σ-model is the space of maps $\ell : \mathbb{R} \to D$ which are quasi-periodic in the sense that for some $h \in H$ we have $\ell(\sigma + 2\pi) = \ell(\sigma)h$, modulo the action of H by right multiplication. The reduced phase space is $(LD)/D$ (i.e. periodic maps modulo the action of D); it is the subspace of $\Omega^1(S^1, \mathfrak{d})$ given by the unit holonomy constraint.)

3 Constructing new flat connections

As we have seen, the solutions of our σ-model give rise to flat connections $A \in \Omega^1(\Sigma, \mathfrak{d})$ satisfying (1.1). We can obtain new flat connections out of A using the following simple observation, which is also the main idea of this paper.

Proposition 1. *Let $\mathfrak{g}$ be a Lie algebra and let $p : \mathfrak{d} \to \mathfrak{g}$ be a linear map such that*

$$[p(X), p(Y)] = p([X, Y]) \quad \forall X \in V_+,\ Y \in V_-. \tag{3.1}$$

If $A \in \Omega^1(\Sigma, \mathfrak{d})$ satisfies (1.1) then $p(A) \in \Omega^1(\Sigma, \mathfrak{g})$ is flat, i.e.

$$d\, p(A) + [p(A), p(A)]/2 = 0.$$

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Proof. Let us use the following notation: for $\alpha \in \Omega^1(\Sigma)$ let $\alpha^+ \in \Omega^{1,0}(\Sigma)$ and $\alpha^- \in \Omega^{0,1}(\Sigma)$ denote the components of α, i.e. $\alpha = \alpha^+ + \alpha^-$. In particular, $A^+ \in \Omega^{1,0}(\Sigma, V_+)$ and $A^- \in \Omega^{0,1}(\Sigma, V_-)$. We then have

$$\begin{aligned} d\,p(A) + [p(A), p(A)]/2 = d\,p(A) + [p(A^+), p(A^-)] &= p(dA + [A^+, A^-]) \\ &= p(dA + [A, A]/2) = 0. \end{aligned}$$

□

Given a 1-parameter family of maps $p_\lambda : \mathfrak{d} \to \mathfrak{g}$ satisfying (3.1) we would thus get a 1-parameter family of flat connections $A_\lambda = p_\lambda(A)$ on Σ, which may then be used to show integrability of the model. Let us observe that the Poisson brackets of the "Lax operators" $L(\sigma, \lambda) := p_\lambda(j(\sigma))$ are automatically of the form considered in [10, 11] (i.e. containing a $\delta(\sigma - \sigma')$ and a $\delta'(\sigma - \sigma')$ term), and so one can in principle extract an infinite family of Poisson-commuting integrals of motion out of the holonomy of A_λ.

Remark 1. The procedure of finding integrable deformations of integrable σ-models, due to Delduc, Magro, and Vicedo [1], can be rephrased in our formalism as follows. Suppose that for some particular pair $V_+ \subset \mathfrak{d}$ we find a family $p_\lambda : \mathfrak{d} \to \mathfrak{g}$ showing integrability of the model. Let us deform the Lie bracket on $\mathfrak{d}$, and possibly the pairing $\langle , \rangle$, in such a way that the restriction of the Lie bracket to $V_+ \times V_- \to \mathfrak{d}$ is undeformed. Then the same family p_λ will satisfy (3.1) also for the deformed structure on $\mathfrak{d}$ and show integrability of the deformed model. These deformations of $\mathfrak{d}$ do not change the system (1.1) (and if $\langle , \rangle$ is not deformed then they don't change the Hamiltonian (2.2) either), but they do change the Poisson structure (2.1) on the phase space.

Remark 2. There is a version of σ-models of Poisson-Lie type, introduced in [8], with the target space $F\backslash D/H$, where $\mathfrak{f} \subset \mathfrak{d}$ is an isotropic Lie algebra (and one needs to suppose that F acts freely on D/H). In this case $V_+ \subset \mathfrak{d}$ is required to be such that $\langle , \rangle|_{V_+}$ is semi-definite positive with kernel $\mathfrak{f}$ (in particular, $\mathfrak{f} \subset V_+$), and such that $[\mathfrak{f}, V_+] \subset V_+$ (we still have $\dim V_+ = (\dim \mathfrak{d})/2$). The phase space is the Marsden-Weinstein reduction of $\Omega^1(S^1, \mathfrak{d})$ by LF, i.e. $\Omega^1(S^1, \mathfrak{f}^\perp)/LF$. The solutions of equations of motion are still given by the solutions of (1.1), though this time A is defined only up to F-gauge transformations. In this case we can still use Proposition 1 without any changes. This setup should cover, in particular, the discussion of symmetric spaces in [1].

4 Getting a Lax pair in a simple case

In this section we give a simple example of pairs $V_+ \subset \mathfrak{d}$ with natural 1-parameter families p_λ satisfying (3.1).

Let $\mathfrak{g}$ be a Lie algebra with an invariant inner product $\langle , \rangle_\mathfrak{g}$ and let W be a 2-dimensional commutative associative algebra with unit. (W is isomorphic to one of $\mathbb{C}$, $\mathbb{R} \oplus \mathbb{R}$, $\mathbb{R}[\epsilon]/(\epsilon^2)$.) Let

$$\mathfrak{d} := \mathfrak{g} \otimes W$$

with the Lie bracket $[X_1 \otimes w_1, X_2 \otimes w_2]_\mathfrak{d} = [X_1, X_2]_\mathfrak{g} \otimes w_1 w_2$.

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We choose the following additional data in W to produce a pairing $\langle , \rangle$ on $\mathfrak{d}$ and a subspace $V_+ \subset \mathfrak{d}$:

To get the pairing, let $\theta : W \to \mathbb{R}$ be a linear form such that the pairing on W given by $\langle w_1, w_2 \rangle_W := \theta(w_1 w_2)$ is non-degenerate (i.e. such that it makes W to a Frobenius algebra) and indefinite. The pairing on $\mathfrak{d}$ is then defined via

$$\langle X_1 \otimes w_1, X_2 \otimes w_2 \rangle := \langle X_1, X_2 \rangle_{\mathfrak{g}}\, \theta(w_1 w_2).$$

To get $V_+ \subset \mathfrak{d}$, let $V_+^0 \subset W$ be a 1-dimensional subspace such that $\langle , \rangle_W$ is positive-definite on V_+^0. Let

$$V_+ = \mathfrak{g} \otimes V_+^0.$$

Then $V_- = \mathfrak{g} \otimes V_-^0$ where $V_-^0 = (V_+^0)^\perp$.

We can now describe the construction of a family $p_\lambda : \mathfrak{d} \to \mathfrak{g}$ satisfying (3.1). Let us choose non-zero elements $e_+ \in V_+^0$ and $e_- \in V_-^0$ (this choice is inessential).

Proposition 2. *If a linear form $q : W \to \mathbb{R}$ satisfies*

$$q(e_+)q(e_-) = q(e_+e_-) \tag{4.1}$$

then the map $p = \mathrm{id}_{\mathfrak{g}} \otimes q : \mathfrak{d} \to \mathfrak{g}$ satisfies (3.1).

Proof. If $X = x \otimes e_+ \in V_+$ and $Y = y \otimes e_- \in V_-$ then

$$p([X,Y]_{\mathfrak{d}}) = p\big([x,y]_{\mathfrak{g}} \otimes e_+e_-\big) = q(e_+e_-)[x,y]_{\mathfrak{g}} = [q(e_+)x, q(e_-)y]_{\mathfrak{g}} = [p(X), p(Y)]_{\mathfrak{g}}.$$

□

The solutions $q \in W^*$ of (4.1) form a curve in W^*, which is either a hyperbola or a union of two lines. If

$$e_+e_- = ae_+ + be_- \quad a, b \in \mathbb{R},$$

we rewrite (4.1) as

$$\big(q(e_+) - b\big)\big(q(e_-) - a\big) = ab.$$

We thus have a hyperbola if $ab \neq 0$ and a union of two straight lines if $ab = 0$.

One can easily check that $ab = 0$ iff one of $V_\pm^0$ is of the form $\mathbb{R}e$ where $e \in W$ satisfies $e^2 = e$. This means that one of $V_\pm = \mathfrak{g} \otimes V_\pm^0 \subset \mathfrak{d}$ is a Lie subalgebra isomorphic to $\mathfrak{g}$ and thus, according to [9], for any Lagrangian $\mathfrak{h} \subset \mathfrak{d}$, the corresponding σ-model is simply the WZW model given by G.

Let us now choose a rational parametrization $\lambda \mapsto q_\lambda$ of the hyperbola (4.1). The standard parametrization in this context seems to be the one sending $\lambda = \pm 1$ to the two points at the infinity of the hyperbola, and $\lambda = \infty$ to 0 (though any other parametrization would do). This gives

$$q_\lambda = \frac{q_+}{1+\lambda} + \frac{q_-}{1-\lambda}$$

where $q_+, q_- \in W^*$ are given by

$$q_+(e_-) = q_-(e_+) = 0, \quad q_+(e_+) = 2b, \quad q_-(e_-) = 2a.$$

(If the curve a union of two lines then this pametrizes only one of the lines, or possibly just a single point.)

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Corollary. *If $A \in \Omega^1(\Sigma, \mathfrak{d})$ satisfies (1.1), i.e. if $A = A_+ \otimes e_+ + A_- \otimes e_-$ with $A_+ \in \Omega^{1,0}(\Sigma, \mathfrak{g})$ and $A_- \in \Omega^{0,1}(\Sigma, \mathfrak{g})$ and if A is flat, then the $\mathfrak{g}$-connections*

$$A_\lambda = (1 \otimes q_\lambda)(A) = \frac{2b}{1+\lambda} A_+ + \frac{2a}{1-\lambda} A_-$$

are flat.

The $\mathfrak{g}$-valued Lax operator obtained in this way is thus

$$L(\sigma, \lambda) = \frac{2b}{1+\lambda} j_+(\sigma) + \frac{2a}{1-\lambda} j_-(\sigma) \tag{4.2}$$

where we decomposed $j(\sigma)$ as $j(\sigma) = j_+(\sigma) \otimes e_+ + j_-(\sigma) \otimes e_-$. For completeness, the Hamiltonian (2.2) is

$$\mathcal{H} = \frac{1}{2} \int_{S^1} \Big(\theta(e_+^2) \langle j_+(\sigma), j_+(\sigma) \rangle_{\mathfrak{g}} - \theta(e_-^2) \langle j_-(\sigma), j_-(\sigma) \rangle_{\mathfrak{g}} \Big) d\sigma.$$

5 Examples of the example

In this section $\mathfrak{g}$ is a compact Lie algebra and G the corresponding compact 1-connected Lie group.

Let start with the case of $W = \mathbb{R} \oplus \mathbb{R}$, i.e. $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$. The only admissible $\theta \in W^*$, up to rescaling (which can be absorbed to $\langle , \rangle_{\mathfrak{g}}$) and exchange of the two components of W, is $\theta(x, y) = x - y$. (Here the main limiting factor is existence of a lagrangian Lie subalgebra $\mathfrak{h} \subset \mathfrak{d}$: if $\theta(x, y) = cx + dy$ with $cd \neq 0$ (the non-degeneracy condition), it forces $c = -d$.) The pairing $\langle , \rangle$ on $\mathfrak{d}$ is $\langle (X_1, X_2), (Y_1, Y_2) \rangle = \langle X_1, Y_1 \rangle_{\mathfrak{g}} - \langle X_2, Y_2 \rangle_{\mathfrak{g}}$.

We have

$$e_+ = (1, t) \qquad e_- = (t, 1)$$

for some $-1 < t < 1$. The Lax operator (4.2), written in terms of $j = (j_1, j_2)$, is

$$L(\sigma, \lambda) = \frac{2t}{(1+t)(1-t)^2} \left(\frac{1}{1+\lambda} \big(j_1(\sigma) - t j_2(\sigma) \big) + \frac{1}{1-\lambda} \big(j_2(\sigma) - t j_1(\sigma) \big) \right)$$

The Poisson brackets (2.1) of $j_{1,2}$ are

$$\begin{aligned}
\{ j_{1a}(\sigma), j_{1b}(\sigma') \} &= f^c_{ab}\, j_{1c}(\sigma)\, \delta(\sigma - \sigma') + \delta_{ab}\, \delta'(\sigma - \sigma') \\
\{ j_{2a}(\sigma), j_{2b}(\sigma') \} &= f^c_{ab}\, j_{2c}(\sigma)\, \delta(\sigma - \sigma') - \delta_{ab}\, \delta'(\sigma - \sigma') \\
\{ j_{1a}(\sigma), j_{2b}(\sigma') \} &= 0
\end{aligned}$$

and the Hamiltonian (2.2) is

$$\mathcal{H} = \frac{1}{2(1-t^2)} \int_{S^1} \Big((1+t^2) \big(\langle j_1(\sigma), j_1(\sigma) \rangle_{\mathfrak{g}} + \langle j_2(\sigma), j_2(\sigma) \rangle_{\mathfrak{g}} \big) - 4t \langle j_1(\sigma), j_2(\sigma) \rangle_{\mathfrak{g}} \Big) d\sigma.$$

The degenerate case $t = 0$ (when $a = b = 0$) corresponds to the WZW-model on $\mathfrak{g}$.

The natural choice for a Lagrangian Lie subalgebra $\mathfrak{h} \subset \mathfrak{d}$ is the diagonal $\mathfrak{g} \subset \mathfrak{d}$. The target space of the σ-model is $D/G \cong G$. It is the so-called "λ-deformed σ-model" introduced by Sfetsos in [12] (Sfetsos's λ is our t).

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Let us now consider the case $W = \mathbb{C}$, which is the richest one. In this case any non-zero $\theta \in W^*$ is suitable. Let $\theta(z) = \mathrm{Im}(e^{2i\alpha} z)$ for some $\alpha \in \mathbb{R}$. We thus have $\mathfrak{d} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_\mathbb{C}$ (seen as a real Lie algebra) with the pairing $\langle X, Y \rangle = \mathrm{Im}(e^{2i\alpha} \langle X, Y \rangle_{\mathfrak{g}_\mathbb{C}})$ where $\langle , \rangle_{\mathfrak{g}_\mathbb{C}}$ is the $\mathbb{C}$-bilinear extension of $\langle , \rangle_\mathfrak{g}$.

In this case

$$e_+ = e^{-i\alpha + i\phi} \qquad e_- = e^{-i\alpha - i\phi}$$

for some $\phi \in (0, \pi/2)$. If $e^{2i\alpha} = e^{\pm 2i\phi}$ then the resulting σ-model (regardless of the choice of $\mathfrak{h} \subset \mathfrak{d}$) is the WZW-model on G.

The Lax operator (4.2) is

$$L(\sigma, \lambda) = \frac{1}{2 \sin^2 2\phi} \left(\frac{e^{2i\alpha} - e^{-2i\phi}}{1 + \lambda} + \frac{e^{2i\alpha} - e^{2i\phi}}{1 - \lambda} \right) J(\sigma) + \text{c.c.}$$

(where c.c. stands for "complex conjugate"). Here $J = j_{re} + i j_{im}$ and $\bar{J} = j_{re} - i j_{im}$ where j_{re} and j_{im} are given by $j = j_{re} \otimes 1 + j_{im} \otimes i$, their Poisson brackets (2.1) (written in an orthonormal basis of $\mathfrak{g}$) are

$$\begin{aligned}
\{J_a(\sigma), J_b(\sigma')\} &= f_{ab}^c \, J_c(\sigma) \, \delta(\sigma - \sigma') + 2i e^{-2i\alpha} \, \delta_{ab} \, \delta'(\sigma - \sigma') \\
\{\bar{J}_a(\sigma), \bar{J}_b(\sigma')\} &= f_{ab}^c \, \bar{J}_c(\sigma) \, \delta(\sigma - \sigma') - 2i e^{2i\alpha} \, \delta_{ab} \, \delta'(\sigma - \sigma') \\
\{J_a(\sigma), \bar{J}_b(\sigma')\} &= 0.
\end{aligned}$$

The Hamiltonian (2.2) is

$$\mathcal{H} = \frac{1}{2} \int_{S^1} \left(\frac{\sin 2\phi}{2} \langle J(\sigma), \bar{J}(\sigma) \rangle_\mathfrak{g} - \frac{\sin 4\phi}{4} \left(e^{2i\alpha} \langle J(\sigma), J(\sigma) \rangle_\mathfrak{g} + e^{-2i\alpha} \langle \bar{J}(\sigma), \bar{J}(\sigma) \rangle_\mathfrak{g} \right) \right) d\sigma$$

A suitable Lagrangian Lie subalgebra $\mathfrak{h} \subset \mathfrak{d}$ can be found as follows. Let $\mathfrak{n} \subset \mathfrak{g}_\mathbb{C} = \mathfrak{d}$ be the complex nilpotent Lie subalgebra spanned by the positive root spaces and let $\mathfrak{t} \subset \mathfrak{g}$ be the Cartan Lie subalgebra. Let $0 \neq z \in \mathbb{C}$ be such that $\theta(z^2) = 0$; up to a real multiple we have $z = e^{-i\alpha}$ or $z = i e^{-i\alpha}$. Then

$$\mathfrak{h} = z\mathfrak{t} + \mathfrak{n} \subset \mathfrak{g}_\mathbb{C} = \mathfrak{d}$$

is a real Lie subalgebra of $\mathfrak{d}$ which is clearly Lagrangian. If $z \notin \mathbb{R}$ then $\mathfrak{h}$ is transverse to $\mathfrak{g} \subset \mathfrak{d}$ and we have an identification $D/H \cong G$ for the target space of the σ-model.

The case of $\alpha = 0$ corresponds to Klimčík's Yang-Baxter σ-model [3]. The general case is the Yang-Baxter σ-model with WZW term introduced in [2] and reinterpreted as a σ-model of Poisson-Lie type in [4].

The final case is $W = \mathbb{R}[\epsilon]/(\epsilon^2)$. After rescaling ϵ and $\langle , \rangle_\mathfrak{g}$ we can suppose that $\theta(x + y\epsilon) = 2tx + y$ for some $t \in \mathbb{R}$ and that

$$e_+ = 1 + (1 - t)\epsilon \qquad e_- = 1 - (1 + t)\epsilon.$$

Using the notation $j = j_1 \otimes 1 + j_\epsilon \otimes \epsilon$, we get

$$L(\sigma, \lambda) = \frac{1 + t}{2(1 + \lambda)} \big((1 + t) j_1 + j_\epsilon \big) + \frac{1 - t}{2(1 - \lambda)} \big((1 - t) j_1 - j_\epsilon \big).$$

The Poisson brackets are

$$\begin{aligned}\{j_{1a}(\sigma), j_{1b}(\sigma')\} &= f^c_{ab}\, j_{1c}(\sigma)\, \delta(\sigma - \sigma')\\ \{j_{1a}(\sigma), j_{2b}(\sigma')\} &= f^c_{ab}\, j_{2c}(\sigma)\, \delta(\sigma - \sigma') + \delta_{ab}\, \delta'(\sigma - \sigma')\\ \{j_{2a}(\sigma), j_{2b}(\sigma')\} &= -2t\, \delta_{ab}\, \delta'(\sigma - \sigma')\end{aligned}$$

and the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \int_{S^1} (1 + t^2)\langle j_0, j_0\rangle_{\mathfrak{g}} + t\langle j_0, j_\epsilon\rangle_{\mathfrak{g}} + \langle j_\epsilon, j_\epsilon\rangle_{\mathfrak{g}}$$

In this case the natural $\mathfrak{h} \subset \mathfrak{d} = \mathfrak{g}[\epsilon]/(\epsilon^2)$ is $\mathfrak{h} = \epsilon\mathfrak{g}$, which gives $D/H = G$. When $t = 0$ the σ-model is the principal chiral model on G, when $t = \pm 1$ we get the WZW model, and for other values of t we get models given by the invariant metric on G and by a multiple of the Cartan 3-form.

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