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Gauged spinning models with deformed supersymmetry

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Abstract: New models of the SU(2|1) supersymmetric mechanics based on gauging the systems with dynamical $(\mathbf{1},\mathbf{4},\mathbf{3})$ and semi-dynamical $(\mathbf{4},\mathbf{4},\mathbf{0})$ supermultiplets are presented. We propose a new version of SU(2|1) harmonic superspace approach which makes it possible to construct the Wess-Zumino term for interacting $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets. A new $\mathcal{N}=4$ extension of $d=1$ Calogero-Moser multiparticle system is obtained by gauging the U(n) isometry of matrix SU(2|1) harmonic superfield model.

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Contents

1 Introduction

In recent papers [1–3] there was initiated the systematic study of the models of deformed $\mathcal{N} = 4$ supersymmetric mechanics with SU(2|1) as a substitute of the standard "flat" $\mathcal{N} = 4, d = 1$ superalgebra. Earlier examples of SU(2|1) supersymmetric $d = 1$ models have been pioneered in [4–6]. The higher-dimensional systems with curved rigid supersymmetry based on the supergroup SU(2|1) and its central extension were studied in [7–10].

The centrally-extended superalgebra $\hat{su}(2|1)$ [1–3] is spanned by the fermionic generators Q^i and $\bar{Q}_i = (Q^i)^\dagger$, $i = 1, 2$, satisfying

$$\{Q^i, \bar{Q}_k\} = 2mI^i_k + 2\delta^i_k\,(H - mF)\,, \qquad \{Q^i, Q^k\} = \{\bar{Q}_i, \bar{Q}_k\} = 0\,. \tag{1.1}$$

The generator $H = H^\dagger$ commutes with all other generators and can be interpreted as an operator central charge. The $\mathrm{SU}(2)_{\rm int}$ generators $I^i_k = (I^k_i)^\dagger$ and the $\mathrm{U}(1)_{\rm int}$ generator $F = F^\dagger$,

$$\left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,, \qquad \left[I^i_j, F\right] = 0\,, \tag{1.2}$$

possess the non-vanishing commutators with supercharges

$$\left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\,\delta^i_j Q^k\,, \qquad [I^i_j, \bar{Q}_l] = -\delta^i_l \bar{Q}_j + \frac{1}{2}\,\delta^i_j \bar{Q}_l\,, \tag{1.3}$$

$$\left[F, Q^k\right] = \frac{1}{2}\,Q^k\,, \qquad [F, \bar{Q}_l] = -\frac{1}{2}\,\bar{Q}_l\,. \tag{1.4}$$

Furthermore, the $su(2|1)$ superalgebra has the automorphism group $\mathrm{SU}(2)_{\mathrm{ext}}$ with the generators $T^i_j = (T^k_i)^\dagger$ which rotate the supercharges in the precisely same manner as the internal $\mathrm{SU}(2)_{\mathrm{int}}$ generators I^i_j do:

$$\left[T^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\,\delta^i_j Q^k\,, \qquad [T^i_j, \bar{Q}_l] = -\delta^i_l \bar{Q}_j + \frac{1}{2}\,\delta^i_j \bar{Q}_l\,. \tag{1.5}$$

The $\mathrm{SU}(2)_{\mathrm{ext}}$ generators rotate, in the same way, the indices of the $\mathrm{SU}(2)_{\mathrm{int}}$ generators I^i_j, so these two SU(2) groups form a semi-direct product

$$\left[T^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,. \tag{1.6}$$

In [1, 2] the SU(2|1) invariant one-particle $d = 1$ models were constructed, proceeding from the superfield formalism based on the superspace with the coordinates $\left(t, \theta_k, \bar{\theta}^k\right)$ $\bar{\theta}^i = \overline{(\theta_i)}$. These coordinates are related to the SU(2|1) coset representative $\exp\left\{itH + \vartheta_k Q^k + \bar{\vartheta}^k \bar{Q}_k\right\}$ via the substitutions $\vartheta_i = \left(1 + \frac{2}{3}\,m\,\theta_k\bar{\theta}^k\right)\theta_i$. $\bar{\vartheta}^i = \left(1 + \frac{2}{3}\,m\,\theta_k\bar{\theta}^k\right)\bar{\theta}^i$. The fermionic SU(2|1) transformations are realized on them as

$$\delta t = i\left(\epsilon_k\,\bar{\theta}^k + \bar{\epsilon}^k\,\theta_k\right), \qquad \delta\theta_i = \epsilon_i + 2m\,\bar{\epsilon}^k\,\theta_k\,\theta_i\,, \qquad \delta\bar{\theta}^i = \bar{\epsilon}^i - 2m\,\epsilon_k\,\bar{\theta}^k\,\bar{\theta}^i\,. \tag{1.7}$$

As a further step, in [3] there was considered the "minimal" complex harmonic coset

$$\frac{\{H, Q^{\pm}, \bar{Q}^{\pm}, F, I^{\pm\pm}, I^0, T^{\pm\pm}, T^0\}}{\{F, I^{++}, I^0, I^{--} - T^{--}, T^0\}} \sim \left(t_A, \theta^{\pm}, \bar{\theta}^{\pm}, w^{\pm}_i\right) \equiv \zeta_H\,, \tag{1.8}$$

where

$$I^{++} \equiv I^1_2\,, \qquad I^{--} \equiv I^2_1\,, \qquad I^0 \equiv I^1_1 - I^2_2 = 2I^1_1\,, \tag{1.9}$$

$$T^{++} \equiv T^1_2\,, \qquad T^{--} \equiv T^2_1\,, \qquad T^0 \equiv T^1_1 - T^2_2 = 2T^1_1\,, \tag{1.10}$$

$$Q^{+} \equiv Q^1\,, \qquad Q^{-} \equiv Q^2\,, \qquad \bar{Q}^{-} \equiv \bar{Q}_1\,, \qquad \bar{Q}^{+} \equiv -\bar{Q}_2\,. \tag{1.11}$$

This SU(2|1) harmonic approach, as a deformation of the analogous formalism in $\mathcal{N} = 4$ supersymmetric mechanics [11], have provided additional opportunities to build new SU(2|1) models, in particular those associated with the multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and its "mirror" counterpart. As was pointed out in [1–3] (see also [12]), many issues of $\mathcal{N} = 4$ supersymmetric mechanics still await their SU(2|1) generalization. The list includes the $\mathcal{N} = 4$ supersymmetric Calogero-like systems, the gauging procedure in superspace, coupling to the background gauge fields, etc. In the framework of $\mathcal{N} = 4$ supersymmetric mechanics, all these topics were found to be tightly interrelated. E.g., the Wess-Zumino (WZ) type actions describe the interaction of the proper $d = 1$ supermultiplets with external gauge fields [11].

The actions of the same type describe semi-dynamical degrees of freedom [14, 15], the use of which proved to be of pivotal importance for constructing the many-particle supersymmetric $d = 1$ systems [13] (see also the review [16]). Additional important technical ingredients of the $\mathcal{N} = 4$ model-building which essentially exploit the WZ type $d = 1$ actions are the pure gauge "topological" multiplet and the superfield gauging procedure relating diverse models [17, 18].

In this paper we construct new models of the $\mathcal{N} = 4$ deformed supersymmetric mechanics that make use of a few different types of SU(2|1) supermultiplets: dynamical, semi-dynamical and pure gauge supermultiplets. The outcome are new SU(2|1)-invariant one-particle model with spinning degrees of freedom, as well as new SU(2|1) superextension of the Calogero-Moser multi-particle system.

The harmonic superspace (1.8) is not directly applicable for tackling these tasks. The main problem roots in the algebra of the covariant constraints to be imposed on the relevant harmonic superfields Ψ for singling out various irreducible SU(2|1) multiplets. The Grassmann analyticity conditions in the harmonic superspace (1.8) (specifically, $\mathcal{D}^+\Psi = 0$, $\bar{\mathcal{D}}^+\Psi = 0$) necessarily entail the harmonic condition (specifically, $\mathcal{D}^{++}\Psi = 0$). However, such harmonic constraints turn out to be too strong if we wish to describe some supermultiplets in the harmonic approach, e.g. the "topological" gauge multiplet which is the main object of the $d = 1$ gauging [17, 18] efficiently exploited in refs. [13–16]. As we will see, the only way around is to pass to an extended SU(2|1) harmonic superspace involving two sets of harmonic variables: those associated with the group $\mathrm{SU}(2)_{\mathrm{int}}$ and those parametrizing the external automorphism group $\mathrm{SU}(2)_{\mathrm{ext}}$.

In section 2 we introduce new harmonic superspace with two sets of harmonic variables, including the standard (unitary) harmonics on $\mathrm{SU}(2)_{\mathrm{ext}}$. As a result, we gain an opportunity to perform a gauging procedure and define interacting dynamical and semi-dynamical multiplets. In section 3 we construct the system of dynamical $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplet interacting with a semi-dynamical $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet. This coupling is used to define the WZ term for the $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ multiplet, which, as was noticed in [3], is impossible in the framework of the harmonic superspace (1.8). The gauging procedure relevant to this SU(2|1) invariant system is described in section 5. In section 6 we present a matrix generalization of the SU(2|1) invariant model with dynamical, semi-dynamical and pure gauge supermultiplets. When reduced on shell, it describes SU(2|1) supersymmetrization of the Calogero-Moser multi-particle system [19–22], with the mass specified by the deformation parameter m of the $su(2|1)$ algebra. Section A contains the concluding remarks. In appendix we present the "master" SU(2|1) harmonic formalism which yields the settings developed in [3] and in section 2 of the present paper upon two different reductions with respect to the extra harmonic variables.

2 SU(2|1) harmonic superspace revisited

As opposed to the "minimal" harmonic coset (1.8), we will use now the coset

$$\hat{\zeta}_H = \left(t_A, \theta^\pm, \bar{\theta}^\pm, u^\pm_i, z^{++}\right) \sim \frac{\{H, Q^\pm, \bar{Q}^\pm, F, I^{\pm\pm}, I^0, T^{\pm\pm}, T^0\}}{\{F, I^{++}, I^0, T^0\}}, \tag{2.1}$$

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where the variables

$$u_i^{\pm}\,, \qquad u^{+i}u_i^- = 1\,, \qquad u_i^+u_k^- - u_k^+u_i^- = \varepsilon_{ik} \tag{2.2}$$

are the standard unitary harmonics on the coset $\mathrm{SU}(2)_{\mathrm{ext}}/\mathrm{U}(1)_{\mathrm{ext}} \sim S^2$ [23], while the coordinate z^{++} is associated with the generator I^{--}. The elements of this coset are defined as

$$g_H = e^{i\left(\xi T^{++}+\bar{\xi}T^{--}\right)}\exp\left\{z^{++}I^{--}\right\}\exp\left\{it_AH - \theta^+Q^- + \bar{\theta}^+\bar{Q}^-\right\}\exp\left\{\theta^-Q^+ - \bar{\theta}^-\bar{Q}^+\right\}, \tag{2.3}$$

where $e^{i\left(\xi\tau^{++}+\bar{\xi}\tau^{--}\right)} = (u_i^{\pm})$, $\tau^{\pm\pm} = \frac{1}{2}(\tau^1 \pm i\tau^2)$, τ^p, $p = 1,2,3$, are the Pauli matrices, and we use the notations (1.9), (1.10), (1.11).

The relation with the standard SU(2|1) superspace coordinates is given by

$$\begin{aligned}
& t_A = t + i\left(\theta^+\bar{\theta}^- + \theta^-\bar{\theta}^+\right), \\
& \theta^- = \theta^i w_i^-\,, \qquad\qquad \theta^+ = \theta^i w_i^+\left(1 + m\,\theta^k w_k^-\bar{\theta}^l w_l^+\right), \\
& \bar{\theta}^- = \bar{\theta}^k w_k^-\,, \qquad\qquad \bar{\theta}^+ = \bar{\theta}^k w_k^+\left(1 - m\,\theta^k w_k^+\bar{\theta}^l w_l^-\right),
\end{aligned} \tag{2.4}$$

where $w_i^{\pm}$ are the non-unitary harmonics which define the "minimal" complex harmonic coset (1.8) and are related to the harmonics (2.2) as [24, 25]

$$w_i^+ = u_i^+ + z^{++}u_i^-\,, \qquad w_i^- = u_i^-\,, \qquad w_i^+w_k^- - w_k^+w_i^- = \varepsilon_{ik}\,. \tag{2.5}$$

The relations (2.4) imply [3]

$$\begin{aligned}
& t = t_A - i\left(\theta^+\bar{\theta}^- + \theta^-\bar{\theta}^+\right), \\
& \theta^i w_i^- = \theta^-, \quad \theta^i w_i^+ = \theta^+\left(1 - m\theta^-\bar{\theta}^+\right), \quad \bar{\theta}^k w_k^- = \bar{\theta}^-, \quad \bar{\theta}^k w_k^+ = \bar{\theta}^+\left(1 + m\theta^+\bar{\theta}^-\right).
\end{aligned} \tag{2.6}$$

The fermionic SU(2|1) transformations induced by the left shifts of the coset representative (2.3) are written as

$$\begin{aligned}
& \delta t_A = 2i\left(\epsilon^-\bar{\theta}^+ - \bar{\epsilon}^-\theta^+\right), \\
& \delta\theta^+ = \epsilon^+ + \epsilon^-\left(z^{++} - m\,\theta^+\bar{\theta}^+\right), \qquad \delta\bar{\theta}^+ = \bar{\epsilon}^+ + \bar{\epsilon}^-\left(z^{++} + m\,\theta^+\bar{\theta}^+\right), \\
& \delta\theta^- = \epsilon^- + 2m\,\bar{\epsilon}^-\theta^-\theta^+, \qquad \delta\bar{\theta}^- = \bar{\epsilon}^- + 2m\,\epsilon^-\bar{\theta}^-\bar{\theta}^+, \\
& \delta z^{++} = m\left(\epsilon^+\bar{\theta}^+ + \bar{\epsilon}^+\theta^+\right) + m\,z^{++}\left(\epsilon^-\bar{\theta}^+ + \bar{\epsilon}^-\theta^+\right), \\
& \delta u_i^{\pm} = 0\,,
\end{aligned} \tag{2.7}$$

where

$$\epsilon^{\pm} = \epsilon^i u_i^{\pm}\,, \qquad \bar{\epsilon}^{\pm} = \bar{\epsilon}^k u_k^{\pm}\,. \tag{2.8}$$

It follows from the transformations (2.7) that the SU(2|1) harmonic superspace contains an analytic harmonic subspace parametrized by the reduced coordinate set,

$$\hat{\zeta}_A = \left(t_A, \bar{\theta}^+, \theta^+, u_i^{\pm}, z^{++}\right), \tag{2.9}$$

which is closed under the action of SU(2|1). It can be identified with the supercoset

$$\hat{\zeta}_A \sim \frac{\{H, Q^{\pm}, \bar{Q}^{\pm}, F, I^{\pm\pm}, I^0, T^{\pm\pm}, T^0\}}{\{Q^+, \bar{Q}^+, F, I^{++}, I^0, T^0\}}\,. \tag{2.10}$$

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The transformations (2.7) rewritten through harmonics $w_i^\pm$ defined in (2.5) take just the form given in [3]

$$\begin{aligned}
&\delta t_A = 2i\left(\eta^-\bar\theta^+ - \bar\eta^-\theta^+\right), \\
&\delta\theta^+ = \eta^+ - m\,\eta^-\theta^+\bar\theta^+\,, \qquad && \delta\bar\theta^+ = \bar\eta^+ + m\,\bar\eta^-\theta^+\bar\theta^+\,, \\
&\delta\theta^- = \eta^- + 2m\,\bar\eta^-\theta^-\theta^+, && \delta\bar\theta^- = \bar\eta^- + 2m\,\eta^-\bar\theta^-\bar\theta^+, \\
&\delta w_i^+ = m\left(\eta^+\bar\theta^+ + \bar\eta^+\theta^+\right) w_i^-\,, && \delta w_i^- = 0\,,
\end{aligned} \tag{2.11}$$

where $\eta^\pm = \epsilon^i w_i^\pm$, $\bar\eta^\pm = \bar\epsilon^i w_i^\pm$. The extra coordinate z^{++} transforms in this basis as

$$\delta z^{++} = m\left(\eta^+\bar\theta^+ + \bar\eta^+\theta^+\right). \tag{2.12}$$

Applying the routine coset techniques to the coset (2.1) (see, for example, [1]) we derive the following expressions for the covariant derivatives

$$\mathcal{D}_{t_A} = \partial_{t_A} = \frac{\partial}{\partial t_A}\,, \tag{2.13}$$

$$\begin{aligned}
\mathcal{D}^- &= -\frac{\partial}{\partial\theta^+} - 2i\,\bar\theta^-\partial_{t_A} - m\,\bar\theta^-\theta^-\frac{\partial}{\partial\theta^-} + m\,\bar\theta^+\frac{\partial}{\partial z^{++}} + m\,\bar\theta^-\left(\tilde I^0 + 2\tilde F\right), \\
\bar{\mathcal{D}}^- &= \frac{\partial}{\partial\bar\theta^+} - 2i\,\theta^-\partial_{t_A} + m\,\theta^-\bar\theta^-\frac{\partial}{\partial\bar\theta^-} - m\,\theta^+\frac{\partial}{\partial z^{++}} - m\,\theta^-\left(\tilde I^0 - 2\tilde F\right),
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
\mathcal{D}^+ &= \frac{\partial}{\partial\theta^-} - m\,\bar\theta^-\tilde I^{++}\,, \\
\bar{\mathcal{D}}^+ &= -\frac{\partial}{\partial\bar\theta^-} + m\,\theta^-\tilde I^{++}\,,
\end{aligned} \tag{2.15}$$

$$\mathcal{D}_z^{--} = \frac{\partial}{\partial z^{++}} + 2i\,\theta^-\bar\theta^-\partial_{t_A} + m\left(\theta^+\bar\theta^- - \theta^-\bar\theta^+\right)\frac{\partial}{\partial z^{++}} + \theta^-\frac{\partial}{\partial\theta^+} + \bar\theta^-\frac{\partial}{\partial\bar\theta^+} - 2m\,\theta^-\bar\theta^-\tilde F\,, \tag{2.16}$$

$$\mathcal{D}^{--} = \partial_u^{--} + 2i\,\theta^-\bar\theta^-\partial_{t_A} + m\left(\theta^+\bar\theta^- - \theta^-\bar\theta^+\right)\frac{\partial}{\partial z^{++}} + \theta^-\frac{\partial}{\partial\theta^+} + \bar\theta^-\frac{\partial}{\partial\bar\theta^+} - 2m\,\theta^-\bar\theta^-\tilde F\,, \tag{2.17}$$

$$\begin{aligned}
\mathcal{D}^{++} = \partial_u^{++} &+ 2i\,\theta^+\bar\theta^+\partial_{t_A} + \left(\theta^+ + m\,\theta^+\bar\theta^+\theta^-\right)\frac{\partial}{\partial\theta^-} + \left(\bar\theta^+ - m\,\theta^+\bar\theta^+\bar\theta^-\right)\frac{\partial}{\partial\bar\theta^-} \\
&- z^{++}\partial_u^0 - (z^{++})^2\frac{\partial}{\partial z^{++}} \\
&+ z^{++}\left(\mathcal{D}^0 + \tilde I^0\right) - 2m\,\theta^+\bar\theta^+\tilde F - m\left(\theta^-\bar\theta^+ - \theta^+\bar\theta^-\right)\tilde I^{++}\,,
\end{aligned} \tag{2.18}$$

$$\mathcal{D}^0 = \partial_u^0 + 2z^{++}\frac{\partial}{\partial z^{++}} + \left(\theta^+\frac{\partial}{\partial\theta^+} + \bar\theta^+\frac{\partial}{\partial\bar\theta^+}\right) - \left(\theta^-\frac{\partial}{\partial\theta^-} + \bar\theta^-\frac{\partial}{\partial\bar\theta^-}\right). \tag{2.19}$$

The partial harmonic derivatives in these expressions are defined as

$$\partial_u^{\pm\pm} = u_i^\pm\frac{\partial}{\partial u_i^\mp}\,, \quad \partial_u^0 = u_i^+\frac{\partial}{\partial u_i^+} - u_i^-\frac{\partial}{\partial u_i^-}\,, \qquad [\partial_u^{++},\partial_u^{--}] = \partial_u^0\,, \quad [\partial_u^0,\partial_u^{\pm\pm}] = \pm 2\partial_u^{\pm\pm}\,, \tag{2.20}$$

and $\tilde F$, $\tilde I^0$, $\tilde I^{++}$ are matrix parts of the generators F, I^0, I^{++} properly acting on the matrix indices of the superfields and the operators. In particular, note the U(1) assignments

$$\tilde I^0\mathcal{D}^\pm = \mp\mathcal{D}^\pm\,, \quad \tilde I^0\bar{\mathcal{D}}^\pm = \mp\bar{\mathcal{D}}^\pm\,, \qquad \tilde F\mathcal{D}^\pm = -\frac{1}{2}\,\mathcal{D}^\pm\,, \quad \tilde F\bar{\mathcal{D}}^\pm = \frac{1}{2}\,\bar{\mathcal{D}}^\pm\,, \tag{2.21}$$

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which will be used below. Note the non-zero commutation relation

$$[\tilde{I}^0, \tilde{I}^{++}] = 2\tilde{I}^{++} \,. \tag{2.22}$$

Also, the notable property is

$$\mathcal{D}_z^{--} - \mathcal{D}^{--} = \frac{\partial}{\partial z^{++}} - \partial_u^{--} \,. \tag{2.23}$$

The covariant derivatives act on the harmonic superfields $\Psi^{(q)}(t_A, \theta^\pm, \bar{\theta}^\pm, u^\pm, z^{++}) = \Psi^{(q)}(\hat{\zeta}_H)$ which are assumed to transform under SU(2|1) supersymmetry in accord with the general rules of the (super)coset realizations

$$\delta\Psi^{(q)} = m\left[2\left(\epsilon^-\bar{\theta}^+ - \bar{\epsilon}^-\theta^+\right)\tilde{F} - \left(\epsilon^-\bar{\theta}^+ + \bar{\epsilon}^-\theta^+\right)\tilde{I}^0 - \left(\epsilon^-\bar{\theta}^- + \bar{\epsilon}^-\theta^-\right)\tilde{I}^{++}\right]\Psi^{(q)} . \tag{2.24}$$

As usual, these superfields are eigenfunctions of the harmonic U(1) charge operator $\mathcal{D}^0$:

$$\mathcal{D}^0\Psi^{(q)} = q\Psi^{(q)} \,. \tag{2.25}$$

We treat the dependence of $\Psi^{(q)}(t_A, \theta^\pm, \bar{\theta}^\pm, u^\pm, z^{++})$ on two sorts of harmonic variables in the same way as in [25]. Namely, we assume the polynomial dependence on z^{++} and the standard harmonic expansion in $u^\pm$ [23].

It is worth pointing out that $\mathcal{D}^{++}\Psi^{(q)}$, $\mathcal{D}^+\Psi^{(q)}$ and $\bar{\mathcal{D}}^+\Psi^{(q)}$ transform according to the general superfield rule (2.24), while the SU(2|1) variations of $\mathcal{D}^{--}\Psi^{(q)}$ and $\mathcal{D}^-\Psi^{(q)}, \bar{\mathcal{D}}^-\Psi^{(q)}$ exhibit some deviations from (2.24), involving the superfield $\Psi^{(q)}$ itself. However, this subtlety is harmless for our subsequent consideration.

In what follows we will mainly limit our study to the harmonic superfields subjected to some additional covariant conditions

$$\left(\mathcal{D}^0 + \tilde{I}^0\right)\Psi^{(q)} = 0 \Rightarrow \tilde{I}^0\Psi^{(q)} = -q\Psi^{(q)} \,, \tag{2.26}$$

$$\tilde{F}\,\Psi^{(q)} = 0 \,, \tag{2.27}$$

$$\tilde{I}^{++}\Psi^{(q)} = 0 \,, \tag{2.28}$$

as well as the constraint

$$\left(\mathcal{D}_z^{--} - \mathcal{D}^{--}\right)\Psi^{(q)} = 0 \,. \tag{2.29}$$

The constraint (2.29) effectively eliminates the dependence of the harmonic superfields on the variable z^{++}

$$\Psi^{(q)}(t_A, \theta^\pm, \bar{\theta}^\pm, u^\pm, z^{++}) = e^{\,z^{++}\partial_u^{--}}\,\Phi^{(q)}(t_A, \theta^\pm, \bar{\theta}^\pm, u^\pm) \,, \tag{2.30}$$

where $\Phi^{(q)}$ satisfies the condition

$$D^0\Phi^{(q)} = q\Phi^{(q)} \,, \qquad D^0 = \partial_u^0 + \theta^+\frac{\partial}{\partial\theta^+} + \bar{\theta}^+\frac{\partial}{\partial\bar{\theta}^+} - \theta^-\frac{\partial}{\partial\theta^-} - \bar{\theta}^-\frac{\partial}{\partial\bar{\theta}^-} \,, \tag{2.31}$$

as a consequence of (2.25) and has the standard expansion in $u^\pm$. The superfield solution (2.30) can be rewritten as

$$\Psi^{(q)}(t_A, \theta^\pm, \bar{\theta}^\pm, u^\pm, z^{++}) = \Phi^{(q)}(t_A, \theta^\pm, \bar{\theta}^\pm, w^\pm) = \Phi^{(q)}(\zeta_H) \,, \tag{2.32}$$

where $w_i^\pm$ and ζ_H were defined in (2.5) and (1.8).

The constraint (2.28) is the self-consistency condition for the covariant definition of the analytic SU(2|1) superfields which live on the analytic subspace (2.9). This definition amounts to the Grassmann-analyticity constraints

$$\mathcal{D}^{+}\Psi^{(q)}=\bar{\mathcal{D}}^{+}\Psi^{(q)}=0\,, \tag{2.33}$$

which, due to the relation

$$\{\mathcal{D}^{+},\bar{\mathcal{D}}^{+}\}=2m\tilde{I}^{++} \tag{2.34}$$

following from (2.15), necessarily imply (2.28). Similar to (2.32), the analytic harmonic superfields are expressed as

$$\Psi^{(q)}(t_A,\theta^{+},\bar{\theta}^{+},u^{\pm},z^{++})=e^{\,z^{++}\partial_u^{--}}\,\Phi^{(q)}(t_A,\theta^{+},\bar{\theta}^{+},u^{\pm})=\Phi^{(q)}(t_A,\theta^{+},\bar{\theta}^{+},w^{\pm})=\Phi^{(q)}(\zeta_A)\,. \tag{2.35}$$

As opposed to the approach of ref. [3], the constraints (2.33) and (2.34) by no means require the condition $\mathcal{D}^{++}\Psi^{(q)}=0$. Of course the latter can be imposed as an *independent* additional constraint, but it is not necessitated now by the Grassmann analyticity conditions (2.33). The relationship between two alternative SU(2|1) harmonic approaches is explained in appendix.

The constraint (2.27) leads to some simplification of the expressions for other covariant derivatives. For example, on harmonic superfields obeying the constraints (2.25)–(2.29) the covariant derivative $\mathcal{D}^{++}$ (2.18) takes the form

$$\mathcal{D}^{++}\Psi^{(q)}=e^{\,z^{++}\partial_u^{--}}\,D^{++}\,\Phi^{(q)}\,, \tag{2.36}$$

where

$$D^{++}=\partial_u^{++}+2i\,\theta^{+}\bar{\theta}^{+}\partial_{t_A}+\left(\theta^{+}+m\,\theta^{+}\bar{\theta}^{+}\theta^{-}\right)\frac{\partial}{\partial\theta^{-}}+\left(\bar{\theta}^{+}-m\,\theta^{+}\bar{\theta}^{+}\bar{\theta}^{-}\right)\frac{\partial}{\partial\bar{\theta}^{-}}\,. \tag{2.37}$$

The general transformation law (2.24) for the superfields subjected to the constraints (2.25)–(2.29) is simplified to the form

$$\delta\Psi^{(q)}=qm\left(\epsilon^{-}\bar{\theta}^{+}+\bar{\epsilon}^{-}\theta^{+}\right)\Psi^{(q)}\,. \tag{2.38}$$

One more comment concerns the possibility to use, along with the harmonic basis $(u_i^{\pm},z^{++})$, the basis $(w_i^{\pm},z^{++})$ with the non-unitary harmonics. Due to the relation (2.5), these two bases are equivalent to each other, while many formulas and constraints are simplified in the second basis. The dictionary between these bases is as follows

$$\begin{aligned}
\partial_u^{++}&\Rightarrow\partial_w^{++}+z^{++}\partial_w^{0}-(z^{++})^2\partial_w^{--}\,, &\qquad \partial_u^{--}&\Rightarrow\partial_w^{--}\,,\\
\partial_u^{0}&\Rightarrow\partial_w^{0}-2z^{++}\partial_w^{--}\,, &\qquad \frac{\partial}{\partial z^{++}}&\Rightarrow\frac{\partial}{\partial z^{++}}+\partial_w^{--}\,.
\end{aligned} \tag{2.39}$$

For instance, in the $(w_i^{\pm},z^{++})$ basis the constraint (2.29) becomes just the condition of z^{++} independence

$$\frac{\partial}{\partial z^{++}}\Psi^{(q)}=0\Rightarrow\Psi^{(q)}=\Phi^{(q)}(t_A,\theta^{\pm},\bar{\theta}^{\pm},w^{\pm})\,. \tag{2.40}$$

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Its SU(2|1) covariance immediately follows from the property $\delta\frac{\partial}{\partial z^{++}} = 0$. Also, it is instructive to present the $(w_i^{\pm}, z^{++})$ form of the pure harmonic part of the covariant derivative $\mathcal{D}^{++}$ (2.18):

$$\partial_u^{++} - z^{++}\partial_u^0 - (z^{++})^2\frac{\partial}{\partial z^{++}} \Rightarrow \partial_w^{++} - (z^{++})^2\frac{\partial}{\partial z^{++}}\,. \tag{2.41}$$

In construction of the superfield particle actions we will need the expressions for the invariant integration measures over the full harmonic and the harmonic analytic superspaces [3]:

$$d\zeta_H = dw\, dt_A\, d\bar{\theta}^-d\theta^-d\bar{\theta}^+d\theta^+\left(1 + m\,\theta^+\bar{\theta}^- - m\,\theta^-\bar{\theta}^+\right) \tag{2.42}$$

and

$$d\zeta_A^{--} = dw\, dt_A\, d\bar{\theta}^+d\theta^+\,, \qquad \delta d\zeta_A^{--} = 0\,. \tag{2.43}$$

3 Coupling of dynamical multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ with semi-dynamical multiplet $(\mathbf{4}, \mathbf{4}, \mathbf{0})$

3.1 The multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$

The multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ is described by the Grassmann-even real superfield $\mathcal{X}$ subjected to the conditions (2.25)–(2.29),

$$\mathcal{D}^0\,\mathcal{X} = 0\,, \qquad \left(\mathcal{D}_z^{--} - \mathcal{D}^{--}\right)\mathcal{X} = 0\,, \qquad \tilde{I}^0\,\mathcal{X} = \tilde{F}\,\mathcal{X} = \tilde{I}^{++}\,\mathcal{X} = 0\,, \tag{3.1}$$

and additional constraints

$$\mathcal{D}^{++}\,\mathcal{X} = 0\,, \tag{3.2}$$

$$\mathcal{D}^-\mathcal{D}^+\,\mathcal{X} = 0\,, \qquad \bar{\mathcal{D}}^-\bar{\mathcal{D}}^+\,\mathcal{X} = 0\,, \qquad \left(\mathcal{D}^-\bar{\mathcal{D}}^+ + \bar{\mathcal{D}}^-\mathcal{D}^+\right)\mathcal{X} = 2m\mathcal{X}\,. \tag{3.3}$$

The set of the constraints (3.1)–(3.3) is invariant with respect to SU(2|1) transformations. Indeed, $\delta\left(\mathcal{D}^-\mathcal{D}^+\,\mathcal{X}\right) = -2m\left(\epsilon^-\bar{\theta}^+ + \bar{\epsilon}^-\theta^+\right)\mathcal{D}^-\mathcal{D}^+\,\mathcal{X}$, etc. The constraints (3.1)–(3.3) are solved by[1]

$$\begin{aligned}\mathcal{X} &= x + \theta^-\psi^+ + \bar{\theta}^-\bar{\psi}^+ - \theta^+\psi^- - \bar{\theta}^+\bar{\psi}^- \\ &\quad + \theta^-\bar{\theta}^-N^{++} + \theta^+\bar{\theta}^+N^{--} + \theta^-\bar{\theta}^+N - \theta^+\bar{\theta}^-\bar{N} \\ &\quad + \theta^-\theta^+\bar{\theta}^-\Omega^+ + \bar{\theta}^-\bar{\theta}^+\theta^-\bar{\Omega}^+ + \theta^-\theta^+\bar{\theta}^+\Omega^- + \bar{\theta}^-\bar{\theta}^+\theta^+\bar{\Omega}^- + \theta^-\bar{\theta}^-\theta^+\bar{\theta}^+D\,.\end{aligned} \tag{3.4}$$

Here,

$$N^{\pm\pm} = N^{ik}w_i^{\pm}w_k^{\pm}\,, \quad N = -i\partial_{t_A}x - N^{ik}w_i^+w_k^- + mx\,, \quad \bar{N} = i\partial_{t_A}x + N^{ik}w_i^+w_k^- + mx\,, \tag{3.5}$$

$$D = 2\left(\partial_{t_A}\partial_{t_A}x + m^2x - i\partial_{t_A}N^{ik}w_i^+w_k^-\right), \tag{3.6}$$

$$\psi^{\pm} = \psi^i w_i^{\pm}\,, \qquad \bar{\psi}^{\pm} = \bar{\psi}^i w_i^{\pm}\,, \qquad \Omega^- = m\psi^-\,, \qquad \bar{\Omega}^- = m\bar{\psi}^-\,, \tag{3.7}$$

$$\Omega^+ = -2i\partial_{t_A}\psi^+ - 2m\psi^+\,, \qquad \bar{\Omega}^+ = 2i\partial_{t_A}\bar{\psi}^+ - 2m\bar{\psi}^+ \tag{3.8}$$

and $x(t_A)$, $N^{ik} = N^{(ik)}(t_A)$, $\psi^i(t_A)$, $\bar{\psi}_i(t) = \overline{(\psi^i)}$ are d=1 fields.

[1] Note that $\mathcal{D}^-\mathcal{D}^+\,\mathcal{X} = \left(-\frac{\partial}{\partial\theta^+} - 2i\,\bar{\theta}^-\partial_{t_A} - m\,\bar{\theta}^-\theta^-\frac{\partial}{\partial\theta^-} + m\,\bar{\theta}^+\frac{\partial}{\partial z^{++}}\right)\mathcal{D}^+\,\mathcal{X} - 2m\,\bar{\theta}^-\,\mathcal{D}^+\,\mathcal{X}$, etc., because of (2.14) and (2.21).

After passing to the central basis coordinates by (2.4), we observe that the θ expansion of the superfield (3.4) in the central basis takes the form [1]

$$\begin{aligned}\mathcal{X}(t,\theta_i,\bar\theta^i) = x &+ \theta_k\psi^k - \bar\theta^k\bar\psi_k + m\,\theta_k\bar\theta^k\,x + \theta^k\bar\theta^j N_{kj} \\ &+ \frac{1}{2}\,(\theta)^2\bar\theta^k\left(i\dot\psi_k + 2m\psi_k\right) - \frac{1}{2}\,(\bar\theta)^2\theta_k\left(i\dot{\bar\psi}^k - 2m\bar\psi^k\right) \\ &+ (\theta)^2(\bar\theta)^2\left(\frac{1}{4}\,\ddot x + m^2 x\right),\end{aligned} \tag{3.9}$$

where the component fields $x(t)$, $N^{ik} = N^{(ik)}(t)$, $\psi^i(t)$, $\bar\psi_i(t) = (\overline{\psi^i})$ are the functions of real time t and $(\theta)^2 \equiv \theta_i\theta^i$, $(\bar\theta)^2 \equiv \bar\theta^i\bar\theta_i$, $\dot x = \partial_{t_A}x$, etc.

The fermionic SU(2|1) transformations of component fields are the following

$$\begin{aligned}\delta x &= -\,\epsilon_k\psi^k + \bar\epsilon^k\bar\psi_k\,,\\ \delta\psi^k &= i\,\bar\epsilon^k\dot x - \bar\epsilon_j N^{kj} - m\,\bar\epsilon^k x\,, \qquad \delta\bar\psi_k = -i\,\epsilon_k\dot x - \epsilon^j N_{kj} - m\,\epsilon_k x\,,\\ \delta N^{kj} &= -2i\Big(\epsilon^{(k}\dot\psi^{j)} + \bar\epsilon^{(k}\dot{\bar\psi}^{j)}\Big) - 2m\Big(\epsilon^{(k}\psi^{j)} - \bar\epsilon^{(k}\bar\psi^{j)}\Big)\,.\end{aligned} \tag{3.10}$$

The free $\mathcal{X}$-action reads

$$S_{\mathcal{X}} = -\frac{1}{4}\int d\zeta_H\,\mathcal{X}^2\,. \tag{3.11}$$

Integrating in it over the θ-variables and harmonics,[2] we obtain the component action [1]

$$S_{\mathcal{X}} = \frac{1}{2}\int dt\left[\dot x\dot x + i\left(\bar\psi_k\dot\psi^k - \dot{\bar\psi}_k\psi^k\right) - m^2x^2 + 2m\,\bar\psi_k\psi^k - \frac{1}{2}\,N^{ik}N_{ik}\right]. \tag{3.12}$$

Another description of the multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ is through an analytic real prepotential $\mathcal{V}(\zeta_A)$ ($\mathcal{D}^+\,\mathcal{V} = \bar{\mathcal{D}}^+\,\mathcal{V} = 0$). Its pregauge freedom

$$\delta\mathcal{V} = \mathcal{D}^{++}\lambda^{--}\,, \qquad \lambda^{--} = \lambda^{--}(\zeta_A)\,, \tag{3.13}$$

can be exploited to show that $\mathcal{V}(\zeta_A)$ describes just the multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ (by choosing the appropriate WZ gauge). The superfield $\mathcal{V}(\zeta_A)$ is related to the superfield $\mathcal{X}$ in the central basis by the harmonic integral transform

$$\mathcal{X}(t,\theta_i,\bar\theta^i) = \int dw\left(1 + m\,\theta^+\bar\theta^- - m\,\theta^-\bar\theta^+\right)^{-1}\mathcal{V}\left(t_A,\theta^+,\bar\theta^+,w^\pm\right)\Big|\,, \tag{3.14}$$

where the vertical bar $\Big|$ means that the expressions $t_A = t + i\left(\theta^+\bar\theta^- + \theta^-\bar\theta^+\right)$, $\theta^- = \theta^i w_i^-$, $\bar\theta^- = \bar\theta^k w_k^-$, $\theta^+ = \theta^i w_i^+\left(1 + m\,\theta^k w_k^-\bar\theta^l w_l^+\right)$, $\bar\theta^+ = \bar\theta^k w_k^+\left(1 - m\,\theta^k w_k^+\bar\theta^l w_l^-\right)$ defined in (2.4) should be substituted into the integrand. Then, from (3.14) we can identify the fields appearing in the WZ gauge for $\mathcal{V}$ with the fields in (3.4)

$$\mathcal{V}(\zeta_A) = x(t_A) - 2\,\theta^+\psi^i(t_A)w_i^- - 2\,\bar\theta^+\bar\psi^i(t_A)w_i^- + 3\,\theta^+\bar\theta^+N^{ik}(t_A)w_i^-w_k^-\,. \tag{3.15}$$

The representation (3.14) generalizes the analogous transform in the "flat" non-deformed $\mathcal{N}$=4 supersymmetric mechanics [14, 15, 17, 18].

[2] We use $\int dw\,w^{+i}w_k^- = \frac{1}{2}\,\delta_k^i$, $\int dw\,w^{+(i_1}w^{+i_2)}w^-_{(k_1}w^-_{k_2)} = -2\int dw\,w^{+(i_1}w^{-i_2)}w^+_{(k_1}w^-_{k_2)} = \frac{1}{3}\,\delta^{(i_1}_{(k_1}\delta^{i_2)}_{k_2)}$.

The passive SU(2|1) transformation of the prepotential field $\mathcal{V}$ has the form

$$\delta\mathcal{V} = -2m\left(\epsilon^-\bar\theta^+ + \bar\epsilon^-\theta^+\right)\mathcal{V}\,, \tag{3.16}$$

and the compensating gauge transformations for preserving the WZ gauge (3.15) are

$$\delta_{\rm comp}\mathcal{V} = \mathcal{D}^{++}\Lambda^{--}_{\rm comp}\,,\quad \Lambda^{--}_{\rm comp} = -\left(\epsilon^i\psi^j + \bar\epsilon^i\bar\psi^j\right)w^-_i w^-_j + \left(\theta^+\bar\epsilon^i - \bar\theta^+\epsilon^i\right)N^{jk}w^-_i w^-_j w^-_k\,. \tag{3.17}$$

Applying (3.16) and (3.17) to the WZ gauge expression (3.15), we reproduce the component field transformations (3.10).

Note that (3.16) agrees with the general transformation law (2.24) with $\tilde I^{++}\mathcal{V}=\tilde F\mathcal{V}=0$, $\tilde I^0\mathcal{V}=2$.[3] Using the transformation of the harmonic measure $\delta\, dw = \partial^{--}_w(\eta^+\bar\theta^+ + \bar\eta^+\theta^+)\,dw$ in the central basis, it is straightforward to be convinced that (3.16) just reproduces the transformation $\delta\mathcal{X}=0$ for $\mathcal{X}$ defined in (3.14).

3.2 The multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ and SU(2|1) invariant WZ term

The multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ is described by the superfield $\mathcal{Z}^+(t_A,\theta^\pm,\bar\theta^\pm,z^{++},u^\pm)$ possessing the unity U(1) charge,

$$\mathcal{D}^0\,\mathcal{Z}^+ = \mathcal{Z}^+\,, \tag{3.18}$$

and satisfying the SU(2|1) covariant constraints

$$\left(\mathcal{D}^{--}_z - \mathcal{D}^{--}\right)\mathcal{Z}^+ = 0\,,\qquad \tilde I^0\,\mathcal{Z}^+ = -\mathcal{Z}^+\,,\qquad \tilde F\,\mathcal{Z}^+ = \tilde I^{++}\,\mathcal{Z}^+ = 0\,, \tag{3.19}$$

as well as

$$\mathcal{D}^{++}\,\mathcal{Z}^+ = 0\,, \tag{3.20}$$

$$\mathcal{D}^+\mathcal{Z}^+ = \bar{\mathcal{D}}^+\mathcal{Z}^+ = 0\,. \tag{3.21}$$

The constraints (3.21) together with $\tilde I^{++}\,\mathcal{Z}^+ = 0$ imply the superfield $\mathcal{Z}^+$ to be analytic, that is

$$\mathcal{Z}^+(t_A,\theta^+,\bar\theta^+,u^\pm,z^{++}) = \mathscr{Z}^+(t_A,\theta^+,\bar\theta^+,w^\pm) = \mathscr{Z}^+(\zeta_A)\,. \tag{3.22}$$

The general solution of the full set of the constraints (3.18)–(3.21) is represented by the component expansion of the harmonic superfield (3.22) in the following form [3]

$$\mathscr{Z}^+(t_A,\theta^+,\bar\theta^+,w^\pm) = z^i w^+_i + \theta^+\varphi + \bar\theta^+\phi - 2i\theta^+\bar\theta^+\dot Z^i w^-_i\,. \tag{3.23}$$

The fermionic SU(2|1) transformation of $\mathcal{Z}^+$ is a particular case of the general transformation law (2.38),

$$\delta\mathcal{Z}^+ = m\left(\epsilon^-\bar\theta^+ + \bar\epsilon^-\theta^+\right)\mathcal{Z}^+\,. \tag{3.24}$$

[3] The superfield $\mathcal{V}$ supplies an example of analytic SU(2|1) superfield not satisfying the constraint (2.26). This property is harmless because $\mathcal{V}$ is not subject to any extra harmonic constraints. One can formally define $\mathcal{D}^{++}\mathcal{V}$, and it is a covariant SU(2|1) analytic superfield living on the superspace $\hat\zeta_A$ (2.9) and having a linear dependence on z^{++} (in the $(w^\pm_i, z^{++})$ basis).

It implies the following transformations for the component fields

$$\begin{aligned}
&\delta z^i = -\,\epsilon^i\varphi - \bar\epsilon^i\phi\,, \quad \delta\varphi = 2i\bar\epsilon^k\dot z_k + m\,\bar\epsilon^k z_k\,, \quad \delta\phi = 2i\epsilon_k\dot z^k - m\,\epsilon_k z^k\,,\\
&\delta\bar z_i = \epsilon_i\bar\phi - \bar\epsilon_i\bar\varphi\,, \qquad \delta\bar\varphi = 2i\epsilon_k\dot{\bar z}^k - m\,\epsilon_k\bar z^k\,, \quad \delta\bar\phi = -\,2i\bar\epsilon^k\dot{\bar z}_k - m\,\bar\epsilon^k\bar z_k\,.
\end{aligned} \tag{3.25}$$

It has been shown in [3] that the Wess-Zumino type actions enjoying SU(2|1) supersymmetry cannot be constructed for the single multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$. However, if we couple the multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$ (3.22) to the multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ (3.4), (3.15) the SU(2|1)-invariant WZ action can be set up.

Such WZ action is given by the following integral over the analytic subspace

$$S_{\rm WZ}(\mathcal{V},\mathcal{Z}^+) = \frac{1}{2}\int d\zeta_A^{--}\,\mathcal{V}\,\mathcal{Z}^+\tilde{\mathcal{Z}}^+\,, \tag{3.26}$$

where $\tilde{\mathcal{Z}}^+$ is generalized harmonic conjugate of $\mathcal{Z}^+$ (see [3, 24] for definition of such conjugation). As a consequence of (2.43), (3.16) and (3.24), the action (3.26) is SU(2|1) invariant. The corresponding component action $S_{\rm WZ} = \displaystyle\int dt\, L_{\rm WZ}$ with the component Lagrangian

$$\begin{aligned}
L_{\rm WZ} = &-\frac{i}{2}\,x\left(\bar z_k\dot z^k - \dot{\bar z}_k z^k\right) - \frac{1}{2}\,N^{kj}z_k\bar z_j\\
&+\frac{1}{2}\,\psi^k\left(z_k\bar\varphi + \bar z_k\phi\right) + \frac{1}{2}\,\bar\psi^k\left(z_k\bar\phi - \bar z_k\varphi\right) + \frac{1}{2}\,x\left(\varphi\bar\varphi + \phi\bar\phi\right)
\end{aligned} \tag{3.27}$$

is invariant under the SU(2|1) transformations (3.10), (3.25).

3.3 Total action

Now we consider a system with the action given by the sum $S_{\mathcal{X}} + S_{\rm WZ}$. Making use of the component form of these actions defined in (3.12) and (3.27), eliminating the auxiliary fields ϕ, $\bar\phi$, φ, $\bar\varphi$, N^{ik} from this sum by their algebraic equations of motion

$$N^{ik} = -z^{(i}\bar z^{k)}\,, \qquad \varphi = -\psi^k z_k/x\,, \quad \bar\varphi = -\bar\psi^k\bar z_k/x\,, \qquad \phi = -\bar\psi^k z_k/x\,, \quad \bar\phi = \psi^k\bar z_k/x \tag{3.28}$$

and, finally, redefining $z^k \to z^k/\sqrt{x}$, we obtain

$$\begin{aligned}
S_{\mathcal{X}} + S_{\rm WZ} = \int dt\,\Big\{&\frac{1}{2}\,\dot x\dot x + \frac{i}{2}\,\left(\bar\psi_k\dot\psi^k - \dot{\bar\psi}_k\psi^k\right) - \frac{i}{2}\left(\bar z_k\dot z^k - \dot{\bar z}_k z^k\right)\\
&-\frac{1}{2}\,m^2x^2 + m\,\bar\psi_k\psi^k - \frac{1}{x^2}\,\left[\frac{1}{8}\,(z^k\bar z_k)^2 + \psi^i\bar\psi^k z_{(i}\bar z_{k)}\right]\Big\}.
\end{aligned} \tag{3.29}$$

In contrast to the analogical model of the $\mathcal{N}=4$ supersymmetric mechanics [14, 15], the action (3.29) contains mass term (oscillator term) for the component field x. But the spinning variables z^i prove to be not restricted by any constraint besides the second class constraints produced by the first order kinetic term for these variables. As a result, the quantum spectrum of this composite model involves an infinite number of the states, like in its "flat" prototype.

For getting the finite number of physical states it is necessary to impose an additional constraint which amounts to the gauging procedure described in the next section.

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4 Gauging of coupled dynamical multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ and semi-dynamical multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$

The WZ action (3.26) and the total action $S_{\mathcal{X}}+S_{\rm WZ}$ are invariant with respect to the global U(1) transformations

$$\mathcal{Z}^{+\prime}=e^{i\lambda}\mathcal{Z}^{+},\qquad \tilde{\mathcal{Z}}^{+\prime}=e^{-i\lambda}\tilde{\mathcal{Z}}^{+}\,. \tag{4.1}$$

Now we require local invariance of this action, with the parameter in (4.1) being promoted to an analytic superfield $\lambda=\lambda(\zeta_A)$ satisfying the conditions

$$\mathcal{D}^{+}\lambda=\bar{\mathcal{D}}^{+}\lambda=0\,,\qquad \left(\mathcal{D}_z^{--}-\mathcal{D}^{--}\right)\lambda=0\,,\qquad \mathcal{D}^{0}\lambda=\tilde{I}^{0}\lambda=\tilde{F}\lambda=\tilde{I}^{++}\lambda=0\,. \tag{4.2}$$

To secure this local symmetry in the considered system we introduce the Grassmann-even analytic gauge superfield V^{++}, which satisfies the conditions

$$\mathcal{D}^{+}V^{++}=\bar{\mathcal{D}}^{+}V^{++}=0\,,\qquad \tilde{I}^{++}V^{++}=0\,, \tag{4.3}$$
$$\left(\mathcal{D}_z^{--}-\mathcal{D}^{--}\right)V^{++}=0\,,\qquad \mathcal{D}^{0}V^{++}=-\tilde{I}^{0}V^{++}=2V^{++}\,,\qquad \tilde{F}V^{++}=0 \tag{4.4}$$

and is defined up to the gauge transformations

$$V^{++\prime}=V^{++}-D^{++}\lambda\,. \tag{4.5}$$

The gauge superfield V^{++} covariantizes the derivative $\mathcal{D}^{++}$. As a result, the complex analytic superfield $\mathcal{Z}^{+},\tilde{\mathcal{Z}}^{+}$, instead of the constraints (3.21), gets subjected to the covariantized harmonic constraints

$$\nabla^{++}\mathcal{Z}^{+}\equiv\left(\mathcal{D}^{++}+i\,V^{++}\right)\mathcal{Z}^{+}=0\,,\qquad \nabla^{++}\tilde{\mathcal{Z}}^{+}\equiv\left(\mathcal{D}^{++}-i\,V^{++}\right)\tilde{\mathcal{Z}}^{+}=0\,. \tag{4.6}$$

We can also add to the total action the gauge-invariant Fayet-Iliopoulos (FI) term

$$S_{\rm FI}=\frac{i}{2}\,c\int\mu_A^{(-2)}\,V^{++}\,. \tag{4.7}$$

So, we will consider the action

$$S=S_{\mathcal{X}}+S_{\rm WZ}+S_{\rm FI}\,. \tag{4.8}$$

Using the U(1) gauge freedom (4.5), (4.1) we can choose the WZ gauge

$$V^{++}=2i\,\theta^{+}\bar{\theta}^{+}A(t_A)\,. \tag{4.9}$$

Then

$$S_{\rm FI}=-c\int dt\,A\,. \tag{4.10}$$

The solution of the constraint (4.6) in the WZ gauge (4.9) is

$$\begin{aligned}\mathcal{Z}^{+}(t_A,\theta^{+},\bar{\theta}^{+},u^{\pm},z^{++})&=z^{i}w_i^{+}+\theta^{+}\varphi+\bar{\theta}^{+}\phi-2i\,\theta^{+}\bar{\theta}^{+}\nabla_{t_A}z^{i}w_i^{-}\,,\\ \tilde{\mathcal{Z}}^{+}(t_A,\theta^{+},\bar{\theta}^{+},u^{\pm},z^{++})&=\bar{z}_i w^{+i}+\theta^{+}\bar{\phi}-\bar{\theta}^{+}\bar{\varphi}-2i\,\theta^{+}\bar{\theta}^{+}\nabla_{t_A}\bar{z}_i w^{-i}\,,\end{aligned} \tag{4.11}$$

where

$$\nabla z^k = \dot{z}^k + iA\, z^k\,, \qquad \nabla \bar{z}_k = \dot{\bar{z}}_k - iA\, \bar{z}_k\,. \tag{4.12}$$

Plugging the expressions (4.11) and (3.15) into the action (3.26) and integrating there over θs and harmonics, we obtain the component form of the WZ action

$$\begin{aligned} S_{\rm WZ} = &-\frac{i}{2}\int dt\,\Big(\bar{z}_k\nabla z^k - \nabla\bar{z}_k\, z^k\Big)x - \frac{1}{2}\int dt\, N^{ik}\bar{z}_i z_k \\ &+\frac{1}{2}\int dt\,\Big[\psi^k\Big(\bar{\varphi}\, z_k + \bar{z}_k\phi\Big) + \bar{\psi}^k\Big(\bar{\phi}\, z_k - \bar{z}_k\varphi\Big) - x\Big(\bar{\phi}\,\phi + \bar{\varphi}\,\varphi\Big)\Big]\,. \end{aligned} \tag{4.13}$$

The fermionic fields ϕ, φ are auxiliary. The action is invariant under the residual local U(1) transformations

$$A' = A - \dot{\lambda}_0\,, \qquad z^{i\prime} = e^{i\lambda_0} z^i\,, \quad \bar{z}_i{}' = e^{-i\lambda_0}\bar{z}_i \tag{4.14}$$

(and similar phase transformations of the fermionic fields).

The total action (4.8) in the WZ gauge takes the following on-shell form (like in (3.29), we make the redefinition $z^k \to z^k/\sqrt{x}$)

$$S = S_b + S_f\,, \tag{4.15}$$

$$S_b = \frac{1}{2}\int dt\,\left[\dot{x}\dot{x} - m^2x^2 + i\left(\dot{\bar{z}}_k z^k - \bar{z}_k\dot{z}^k\right) - \frac{(\bar{z}_k z^k)^2}{4x^2} + 2A\left(\bar{z}_k z^k - c\right)\right], \tag{4.16}$$

$$S_f = \int dt\Big[\frac{i}{2}\Big(\bar{\psi}_k\dot{\psi}^k - \dot{\bar{\psi}}_k\psi^k\Big) + m\bar{\psi}_k\psi^k\Big] - \int dt\,\frac{\psi^i\bar{\psi}^k z_{(i}\bar{z}_{k)}}{x^2}\,. \tag{4.17}$$

The last term in the bosonic action (4.16) produces first class constraint $\bar{z}_k z^k - c \approx 0$ restricting the quantum spectrum to a single supermultiplet.

5 Matrix model

Now we are going to generalize the model of the previous section to the U(n), d=1 gauge theory following the papers [13, 16].

The matrix model to be constructed involves the following U(n) entities:

- n^2 commuting superfields $\mathcal{X}^a_b = \widetilde{(\mathcal{X}^b_a)}$, $a, b = 1, \ldots, n$, forming the hermitian $n{\times}n$-matrix superfield $\mathcal{X} = (\mathcal{X}^b_a)$ in adjoint representation of U(n);
- n commuting complex superfields $\mathcal{Z}^+_a$ forming the U(n) spinor $\mathcal{Z}^+ = (\mathcal{Z}^+_a)$, $\tilde{\mathcal{Z}}^+ = (\tilde{\mathcal{Z}}^{+a})$;
- n^2 non-propagating "gauge superfields" $V^{++} = (V^{++b}{}_a)$, $\widetilde{(V^{++b}{}_a)} = V^{++a}{}_b$.

The local U(n) transformations are given by

$$\mathcal{X}' = e^{i\lambda}\mathcal{X}e^{-i\lambda}, \qquad \mathcal{Z}^{+\prime} = e^{i\lambda}\mathcal{Z}^+, \qquad V^{++\prime} = e^{i\lambda}\, V^{++}\, e^{-i\lambda} - i\, e^{i\lambda}(D^{++}e^{-i\lambda}), \tag{5.1}$$

where $\lambda^b_a(\zeta_A) \in u(n)$ is the "hermitian" analytic matrix parameter, $\widetilde{\lambda} = \lambda$.

The SU(2|1) supersymmetric matrix model with U(n) gauge symmetry is described by the action

$$\mathcal{S}_{\rm matrix} = \mathcal{S}_{\mathcal{X}} + \mathcal{S}_{\rm WZ} + \mathcal{S}_{\rm FI}\,. \tag{5.2}$$

The first term in (5.2),

$$\mathcal{S}_{\mathcal{X}} = -\frac{1}{4}\int \mu_H \mathrm{Tr}\,(\mathcal{X}^{\,2}\,)\,, \tag{5.3}$$

is the gauged action of the $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplets. Now the superfields $\mathcal{X} = (\mathcal{X}_a^b)$ are subjected to the constraints (3.1) and

$$\nabla^{++}\,\mathcal{X} = \mathcal{D}^{++}\mathcal{X} + i\,[V^{++},\mathcal{X}] = 0\,, \tag{5.4}$$

$$\nabla^{-}\nabla^{+}\,\mathcal{X} = 0\,, \qquad \bar\nabla^{-}\bar\nabla^{+}\,\mathcal{X} = 0\,, \qquad (\nabla^{-}\bar\nabla^{+} + \bar\nabla^{-}\nabla^{+})\,\mathcal{X} = 2m\mathcal{X}\,, \tag{5.5}$$

which are gauge-covariantization of the constraints (3.2), (3.3). The constraint (5.4) involves the covariant harmonic derivative $\nabla^{++} = \mathcal{D}^{++} + i\,V^{++}$, where the gauge matrix connection $V^{++}(\zeta,w)$ is an analytic superfield.[4] The gauge connections entering the spinor covariant derivatives in (5.5) are properly expressed through $V^{++}(\zeta,u)$. The parameters of the $\mathrm{U}(n)$ gauge group are analytic, so $\nabla^{+} = \mathcal{D}^{+}\,,\ \bar\nabla^{+} = \bar{\mathcal{D}}^{+}$.

The last term in (5.2) is the FI term

$$\mathcal{S}_{\rm FI} = \frac{i}{2}\,c\int \mu_A^{(-2)}\,\mathrm{Tr}\,V^{++}\,, \tag{5.6}$$

whereas the second term,

$$\mathcal{S}_{\rm WZ} = \frac{1}{2}\int \mu_A^{(-2)}\mathcal{V}_0\,\widetilde{\mathcal{Z}}^{+a}\mathcal{Z}_a^{+}\,, \tag{5.7}$$

is a WZ action describing coupling of n commuting analytic superfields $\mathcal{Z}_a^{+}$ and the singlet $\mathrm{U}(1)$ part $\mathcal{X}_0 \equiv \mathrm{Tr}\,(\mathcal{X})$. The real analytic superfield $\mathcal{V}_0(\zeta,w)$ is defined by the integral transform (3.14) for the trace part:

$$\mathcal{X}_0(t,\theta_i,\bar\theta^i) = \int dw\Big(1 + m\theta^-\bar\theta^+ - m\theta^+\bar\theta^- - 2m^2\theta^+\theta^-\bar\theta^+\bar\theta^-\Big)\mathcal{V}_0\left(t_A,\theta^+,\bar\theta^+,w^\pm\right)\Big|\,. \tag{5.8}$$

The n multiplets $(\mathbf{4},\mathbf{4},\mathbf{0})$ are described by the superfields $\mathcal{Z}_a^{+}$ defined by the constraints (3.19)–(3.21) in which the constraint $\mathcal{D}^{++}\mathcal{Z}^{+} = 0$ is gauge-covariantized:

$$\nabla^{++}\mathcal{Z}^{+} = \left(\mathcal{D}^{++} + iV^{++}\right)\mathcal{Z}^{+} = 0\,. \tag{5.9}$$

Using the gauge freedom (5.1) we can choose the WZ gauge

$$V^{++} = 2i\,\theta^+\bar\theta^+A(t_A)\,, \tag{5.10}$$

where now $A(t_A)$ is an $n{\times}n$ matrix field. In this gauge we have

$$\nabla^{\pm\pm} = \mathcal{D}^{\pm\pm} - 2\,\theta^\pm\bar\theta^\pm A, \qquad \nabla^{-} = \mathcal{D}^{-} + 2\,\bar\theta^{-}A, \qquad \bar\nabla^{-} = \bar{\mathcal{D}}^{-} + 2\,\theta^{-}A\,. \tag{5.11}$$

The solution to the constraints (3.1) and the constraints (5.4), (5.5) for matrix field $\mathcal{X}$ is similar to (5.5) and it is as follows:

$$\begin{aligned}\mathcal{X} = {}& X + \theta^-\Psi^+ + \bar\theta^-\bar\Psi^+ - \theta^+\Psi^- - \bar\theta^+\bar\Psi^- \\ & + \theta^-\bar\theta^-N^{++} + \theta^+\bar\theta^+N^{--} + \theta^-\bar\theta^+N - \theta^+\bar\theta^-\bar N \\ & +\,\theta^-\theta^+\bar\theta^-\Omega^+ + \bar\theta^-\bar\theta^+\theta^-\bar\Omega^+ + \theta^-\theta^+\bar\theta^+\Omega^- + \bar\theta^-\bar\theta^+\theta^+\bar\Omega^- + \theta^-\bar\theta^-\theta^+\bar\theta^+D\,.\end{aligned} \tag{5.12}$$

[4] Besides the covariant derivative ∇^{++} which commutes with $\mathcal{D}^{+},\bar{\mathcal{D}}^{+}$ and so preserves the analyticity, one can define the derivative $\nabla^{--} = \mathcal{D}^{--} + i\,V^{--}$, so that $[\nabla^{++},\nabla^{--}] = \mathcal{D}^{0}$. The non-analytic connection V^{--} is expressed through V^{++} from this commutation relation [24].

Here,

$$N^{\pm\pm} = N^{ik} w_i^{\pm} w_k^{\pm} \,, \; N = -i\nabla_{t_A} X - N^{ik} w_i^{+} w_k^{-} + mX \,, \; \bar{N} = i\nabla_{t_A} X + N^{ik} w_i^{+} w_k^{-} + mX \,, \tag{5.13}$$

$$D = 2\left(\nabla_{t_A}\nabla_{t_A} X + m^2 x - i\nabla_{t_A} N^{ik} w_i^{+} w_k^{-}\right), \tag{5.14}$$

$$\Psi^{\pm} = \Psi^{i} w_i^{\pm} \,, \qquad \bar{\Psi}^{\pm} = \bar{\Psi}^{i} w_i^{\pm} \,, \qquad \Omega^{-} = m\Psi^{-} \,, \qquad \bar{\Omega}^{-} = m\bar{\Psi}^{-} \,, \tag{5.15}$$

$$\Omega^{+} = -2i\nabla_{t_A}\Psi^{+} - 2m\psi^{+}, \qquad \bar{\Omega}^{+} = 2i\nabla_{t_A}\bar{\Psi}^{+} - 2m\bar{\Psi}^{+} \,. \tag{5.16}$$

The quantities $X(t_A)$, $N^{ik} = N^{(ik)}(t_A)$, $\Psi^i(t_A)$, $\bar{\Psi}_i(t_A) = (\Psi^i)^\dagger$ in (5.13)–(5.16) are matrix d=1 fields and the covariant derivatives are defined by

$$\begin{array}{ll} \nabla_{t_A} X = \partial_{t_A} X + i[A, X] \,, & \nabla_{t_A} N^{ik} = \partial_{t_A} N^{ik} + i[A, N^{ik}] \,, \\ \nabla_{t_A} \Psi^{i} = \partial_{t_A} \Psi^{i} + i[A, \Psi^{i}] \,, & \nabla_{t_A} \bar{\Psi}_{i} = \partial_{t_A} \bar{\Psi}_{i} + i[A, \bar{\Psi}_{i}] \,. \end{array} \tag{5.17}$$

The solution of the constraints (3.19)–(3.21) with the covariantization (5.9) for U(n) spinor superfield $\mathcal{Z}^+$ is similar to (4.11):

$$\begin{aligned} \mathcal{Z}^{+}(t_A, \theta^{+}, \bar{\theta}^{+}, u^{\pm}, z^{++}) &= Z^{i} w_i^{+} + \theta^{+}\varphi + \bar{\theta}^{+}\phi - 2i\,\theta^{+}\bar{\theta}^{+}\nabla_{t_A} Z^{i} w_i^{-} \,, \\ \tilde{\mathcal{Z}}^{+}(t_A, \theta^{+}, \bar{\theta}^{+}, u^{\pm}, z^{++}) &= \bar{Z}_{i} w^{+i} + \theta^{+}\bar{\phi} - \bar{\theta}^{+}\bar{\varphi} - 2i\,\theta^{+}\bar{\theta}^{+}\nabla_{t_A} \bar{Z}_{i} w^{-i} \,, \end{aligned} \tag{5.18}$$

where

$$\nabla Z^{k} = \dot{Z}^{k} + iA\, Z^{k} \,, \qquad \nabla \bar{Z}_{k} = \dot{\bar{Z}}_{k} - iA\, \bar{Z}_{k} \tag{5.19}$$

are covariant derivatives of U(n) spinor d=1 fields Z^i_a, $\bar{Z}^a_i = (\overline{Z^i_a})$.

Inserting the expressions (5.12), (5.18) in the action (5.2) and eliminating the fields N^{ik}, ϕ, $\bar{\phi}$, φ, $\bar{\varphi}$ by their equations of motion we obtain, in the WZ gauge,

$$\mathcal{S}_{\rm matrix} = \mathcal{S}_b + \mathcal{S}_f, \tag{5.20}$$

$$\begin{aligned} \mathcal{S}_b = {} & \frac{1}{2}\,{\rm Tr}\int dt\left(\nabla X\nabla X - m^2 X^2\right) - c\int dt\,{\rm Tr} A \\ & + \frac{1}{2}\,{\rm Tr}\int dt\left[iX_0\left(\nabla\bar{Z}_k\, Z^k - \bar{Z}_k\nabla Z^k\right) - \frac{n}{4}\,(\bar{Z}^{(i} Z^{k)})(\bar{Z}_i Z_k)\right], \end{aligned} \tag{5.21}$$

$$\mathcal{S}_f = \frac{1}{2}\,{\rm Tr}\int dt\Big[i\left(\bar{\Psi}_k\nabla\Psi^k - \nabla\bar{\Psi}_k\Psi^k\right) + 2m\bar{\Psi}_k\Psi^k\Big] - \int dt\,\frac{\Psi_0^{(i}\bar{\Psi}_0^{k)}(\bar{Z}_i Z_k)}{X_0}\,, \tag{5.22}$$

where

$$X_0 \equiv {\rm Tr}(X), \qquad \Psi^i_0 \equiv {\rm Tr}(\Psi^i), \qquad \bar{\Psi}^i_0 \equiv {\rm Tr}(\bar{\Psi}^i)$$

and $(\bar{Z}_i Z_k) \equiv \bar{Z}^a_i Z_{ka}$, $(\nabla\bar{Z}_k\, Z^k) \equiv \nabla\bar{Z}^a_k\, Z^k_a$.

Let us consider the bosonic limit of $S_{\rm matrix}$, i.e. the action (5.21). Using the residual gauge invariance of the action (5.21), $X' = e^{i\lambda}\, X\, e^{-i\lambda}$, $Z'^k = e^{i\lambda} Z^k$, $A' = e^{i\lambda}\, A\, e^{-i\lambda} - i\, e^{i\lambda}(\partial_t e^{-i\lambda})$, where $\lambda^b_a(t) \in u(n)$ are ordinary d=1 gauge parameters, we can impose the gauge

$$X^b_a = 0\,, \qquad a \neq b\,,$$

i.e. $X_a^b = X_a\delta_a^b$ and $X_0 = \sum_{a=1}^{n} X_a$. As a result of this, and after eliminating A_a^b, $a \neq b$, by the equations of motion, the action (5.21) takes the following form (instead of Z_a^i we introduce the new fields $Z'^i_a = (X_0)^{1/2}\, Z_a^i$ and omit the primes on these fields),

$$\begin{aligned} \mathcal{S}_b = \frac{1}{2}\int dt \Bigg\{ & \sum_a \Big(\dot X_a \dot X_a - m^2 X_a X_a\Big) - \frac{i}{2}\sum_a \Big(\bar Z_k^a \dot Z_a^k - \dot{\bar Z}{}_k^a Z_a^k\Big) + 2\sum_a A_a^a \Big(Z_k^a Z_a^k - c\Big) + \\ & + \sum_{a\neq b} \frac{\mathrm{Tr}(S_a S_b)}{4(X_a - X_b)^2} - \frac{n\,\mathrm{Tr}(\hat S \hat S)}{2(X_0)^2} \Bigg\}, \end{aligned} \tag{5.23}$$

where we used the following notation:

$$(S_a)_k{}^j \equiv \bar Z_k^a Z_a^j, \tag{5.24}$$

$$(\hat S)_k{}^j \equiv \sum_a \left[(S_a)_k{}^j - \frac{1}{2}\delta_k^j (S_a)_l{}^l \right] \tag{5.25}$$

and no sum over the repeated index a in (5.24) is assumed.

The terms $\sum_a A_a^a \left(Z_k^a Z_a^k - c\right)$ in (5.23) produce n constraints (for each index a)

$$\bar Z_k^a Z_a^k - c \approx 0 \tag{5.26}$$

for the fields Z_a^k. The constraints (5.26) generate abelian gauge $[\mathrm{U}(1)]^n$ symmetry, $Z_a^k \to e^{i\varphi_a} Z_a^k$, with local parameters $\varphi_a(t)$.

Due to the constraints (5.26), the fields Z_a^k describe n sets of the target harmonics. After quantization, these variables become purely internal (U(2)-spin) degrees of freedom. So, in the Hamiltonian approach, the kinetic WZ term for Z in (5.23) gives rise to the following Dirac brackets:

$$[Z_a^k, \bar Z_j^b]_D = -i\delta_a^b \delta_j^k. \tag{5.27}$$

With respect to these brackets the quantities (5.24) for each index a form $u(2)$ algebras

$$[(S_a)_i{}^j, (S_b)_k{}^l]_D = i\delta_{ab}\left\{ \delta_i^l (S_a)_k{}^j - \delta_k^j (S_a)_i{}^l \right\}. \tag{5.28}$$

As a result, after quantization the variables Z_a^k describe n sets of fuzzy spheres.

The action (5.23) contains a potential in the center-of-mass sector with the coordinate X_0 (last term in (5.23)). Modulo this extra potential, the bosonic limit of the system constructed is none other than the U(2)-spin Calogero-Moser model which is a massive generalization of the U(2)-spin Calogero model [26, 27] in the formulation of [28–30].

6 Concluding remarks and outlook

In this paper, we proposed new models of SU(2|1) supersymmetric quantum mechanics as a deformation of the corresponding "flat" $\mathcal{N} = 4, d = 1$ supersymmetric models. The characteristic features of these models is the use of different types of supermultiplets: dynamical, semi-dynamical and pure gauge ones. In considered models, dynamical multiplets

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are the $(\mathbf{1},\mathbf{4},\mathbf{3})$ ones. The prepotential superfield description of them has provided an opportunity to build the WZ action for the $(\mathbf{4},\mathbf{4},\mathbf{0})$ multiplets and thereby to use the latter for describing semi-dynamical degrees of freedom. The SU$(2|1)$ version of the superfield gauging procedure of refs. [17, 18] involving the appropriate gauge multiplets allowed us to gauge away some of the dynamical and semi-dynamical fields on shell.

We have studied these new SU$(2|1)$ supersymmetric mechanics models both in the one-particle case and in the multi-particle one. In the latter case the system is described off shell by the matrix theory with U(n) gauging. After elimination of auxiliary and pure gauge fields this matrix theory yields new $\mathcal{N}=4$ superextensions of the A_{n-1} Calogero-Moser model. The mass (frequency) of the physical states is defined by the deformation parameter of the SU$(2|1)$ supersymmetry.

The $\mathcal{N}=4$ superextensions of the Calogero-Moser model play a crucial role in applying the multiparticle integrable Calogero-type systems to the black hole physics. As was argued in [31], $\mathcal{N}=4$ supersymmetric extension of the conformal Calogero model can provide a microscopic description of the extreme Reissner-Nordström black hole in the near-horizon limit. At the same time, the corresponding physical states are identified with the eigenstates of the Calogero-Moser Hamiltonian. The deformed $\mathcal{N}=4$ supersymmetric generalization of the Calogero-Moser system found here can shed more light on these issues. One can expect, e.g., that this new multiparticle SU$(2|1)$ model exhibits a trigonometric realization of the $d=1$ superconformal group $D(2,1;\alpha)$ along the lines of refs. [32–34].

Finally, it is worth pointing out that we have obtained $\mathcal{N}=4$ supersymmetric extension of the A_{n-1} Calogero-Moser system by dealing with the matrix model with the U(n) gauging. Superextensions of the Calogero-Moser models corresponding to other root systems could presumably be obtained by choosing other gauge groups and/or representations for the matrix and WZ superfields.

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A Master SU(2|1) harmonic formalism

A.1 Extended harmonic setting

The formalism below is very similar to the bi-harmonic approach developed in [25] for the harmonic space description of quaternion-Kähler manifolds. The difference is that in [25] all three extra co-ordinates $z^0, z^{\pm\pm}$ were introduced, while in our case it will be enough to deal with two such coordinates $z^{\pm\pm}$.

Let us consider an extended SU$(2|1)$ harmonic superspace in the w-parametrization of harmonic variables

$$\left(t_A, \theta^{\pm}, \bar{\theta}^{\pm}, w_i^{\pm}, z^{++}, z^{--}\right) = \left(\hat{\zeta}_H, z^{--}\right), \tag{A.1}$$

where z^{--} is an additional coordinate with the following SU$(2|1)$ transformation properties

$$\delta z^{--} = \lambda^{--} - 2\lambda^{+-} z^{--}, \quad \lambda^{--} = m(\eta^- \bar{\theta}^- + \bar{\eta}^- \theta^-)\,, \;\lambda^{+-} = m(\eta^- \bar{\theta}^+ + \bar{\eta}^- \theta^+)\,. \tag{A.2}$$

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All other coordinates are transformed as in section 2. We assume that only generators I^0 and F form the stability subgroup and hence correspond to the homogeneous transformations of coordinates. Respectively, the general superfield given on (A.1), $\Phi(t,\theta,w,z,)$, is assumed to transform as (we consider passive transformations)

$$\delta\Phi = -\lambda^{+-}\tilde{I}^0\Phi + 2\omega^{+-}\tilde{F}\Phi\,, \quad \omega^{+-} = m(\eta^-\bar{\theta}^+ - \bar{\eta}^-\theta^+)\,, \tag{A.3}$$

where $\tilde{I}^0$ and $\tilde{F}$ are just the "matrix parts" of the U(1) generators I^0 and F counting two independent external U(1) charges of Φ. For sake of brevity we do not indicate these two charges explicitly. In general, Φ possesses also the standard harmonic U(1) charge q,

$$\begin{aligned}
&\mathcal{D}^0\Phi = q\,\Phi\,,\\
&\mathcal{D}^0 = D^0_w + 2z^{++}\frac{\partial}{\partial z^{++}} - 2z^{--}\frac{\partial}{\partial z^{--}}\,,\\
&D^0_w = \partial^0_w + \theta^+\frac{\partial}{\partial\theta^+} + \bar{\theta}^+\frac{\partial}{\partial\bar{\theta}^+} - \theta^-\frac{\partial}{\partial\theta^-} - \bar{\theta}^-\frac{\partial}{\partial\bar{\theta}^-}\,.
\end{aligned} \tag{A.4}$$

The covariant derivatives are defined by the following formulas

$$\begin{aligned}
\mathcal{D}^{++}_z &= D^{++}_w - (z^{++})^2\frac{\partial}{\partial z^{++}} + z^{++}(\mathcal{D}^0 + \tilde{I}^0) + [1 + m(\theta^+\bar{\theta}^- - \theta^-\bar{\theta}^+)]\frac{\partial}{\partial z^{--}}\,,\\
D^{++}_w &= \partial^{++}_w + 2i\theta^+\bar{\theta}^+\partial_t + \theta^+\frac{\partial}{\partial\theta^-} + \bar{\theta}^+\frac{\partial}{\partial\bar{\theta}^-}\\
&\quad + m\theta^+\bar{\theta}^+\left(\theta^-\frac{\partial}{\partial\theta^-} - \bar{\theta}^-\frac{\partial}{\partial\bar{\theta}^-}\right) - 2m\theta^+\bar{\theta}^+\tilde{F},
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
\mathcal{D}^{--}_z &= D^{--}_w + [1 + m(\theta^+\bar{\theta}^- - \theta^-\bar{\theta}^+)]\frac{\partial}{\partial z^{++}} - (z^{--})^2\frac{\partial}{\partial z^{--}} + z^{--}\tilde{I}^0\,,\\
D^{--}_w &= [1{+}m(\theta^+\bar{\theta}^- {-}\theta^-\bar{\theta}^+)]\partial^{--}_w + 2i\theta^-\bar{\theta}^-\partial_t + \theta^-\frac{\partial}{\partial\theta^+} + \bar{\theta}^-\frac{\partial}{\partial\bar{\theta}^+} - 2m\theta^-\bar{\theta}^-\tilde{F}\,,
\end{aligned} \tag{A.6}$$

$$\mathcal{D}^+ = \frac{\partial}{\partial\theta^-} - m\bar{\theta}^-\frac{\partial}{\partial z^{--}}\,, \quad \bar{\mathcal{D}}^+ = -\frac{\partial}{\partial\bar{\theta}^-} + m\theta^-\frac{\partial}{\partial z^{--}}\,. \tag{A.7}$$

One should add to this set two more independent covariant derivatives

$$\frac{\partial}{\partial z^{--}}\,, \quad \frac{\partial}{\partial z^{++}}\,, \quad \delta\frac{\partial}{\partial z^{--}} = 2\lambda^{+-}\frac{\partial}{\partial z^{--}}\,,\ \delta\frac{\partial}{\partial z^{++}} = 0\,. \tag{A.8}$$

It is also easy to define the covariant spinor derivatives $\mathcal{D}^-$ and $\bar{\mathcal{D}}^-$,

$$\mathcal{D}^-_z := [\mathcal{D}^{--}, \mathcal{D}^+]\,, \quad \bar{\mathcal{D}}^-_z := [\mathcal{D}^{--}, \bar{\mathcal{D}}^+]\,. \tag{A.9}$$

For brevity, we will not present here their explicit form.

Now it is direct to be convinced that the quantities

$$\mathcal{D}^{\pm\pm}_z\Phi\,, \quad \frac{\partial}{\partial z^{\pm\pm}}\Phi\,, \quad \mathcal{D}^+\Phi\,, \quad \bar{\mathcal{D}}^+\Phi \tag{A.10}$$

(as well as $\mathcal{D}^-_z\Phi\,, \bar{\mathcal{D}}^-_z\Phi$) transform according to the generic superfield transformation law (A.3), with taking into account that the covariant derivatives (A.5)–(A.8) themselves possess non-trivial $\tilde{I}^0$ and $\tilde{F}$ charges[5]

$$
\begin{aligned}
&\tilde{I}^0\left(\mathcal{D}_z^{++},\, \mathcal{D}_z^{--},\, \mathcal{D}^+,\, \bar{\mathcal{D}}^+,\, \frac{\partial}{\partial z^{++}},\, \frac{\partial}{\partial z^{--}}\right) = \left(0,\, 2\,\mathcal{D}_z^{--},\, -\mathcal{D}^+,\, -\bar{\mathcal{D}}^+,\, 0,\, -2\,\frac{\partial}{\partial z^{--}}\right),\\
&\tilde{F}\left(\mathcal{D}_z^{++},\, \mathcal{D}_z^{--},\, \mathcal{D}^+,\, \bar{\mathcal{D}}^+,\, \frac{\partial}{\partial z^{++}},\, \frac{\partial}{\partial z^{--}}\right) = \left(0,\, 0,\, -\frac{1}{2}\,\mathcal{D}^+,\, \frac{1}{2}\,\bar{\mathcal{D}}^+,\, 0,\, 0\right). \qquad \text{(A.11)}
\end{aligned}
$$

Note the useful (anti)commutation relations

$$
\{\mathcal{D}^+, \bar{\mathcal{D}}^+\} = 2m\,\frac{\partial}{\partial z^{--}}\,, \quad [\mathcal{D}_z^{++}, \mathcal{D}^+] = [\mathcal{D}_z^{++}, \bar{\mathcal{D}}^+] = 0\,, \quad [\mathcal{D}_z^{++}, \mathcal{D}_z^{--}] = 0\,, \qquad \text{(A.12)}
$$

$$
\begin{aligned}
&\left[\frac{\partial}{\partial z^{++}},\, \mathcal{D}_z^{++}\right] = \mathcal{D}^0 + \tilde{I}^0\,, \qquad && \left[\frac{\partial}{\partial z^{++}},\, \mathcal{D}_z^{--}\right] = 0\,,\\
&\left[\frac{\partial}{\partial z^{--}},\, \mathcal{D}_z^{++}\right] = 0\,, && \left[\frac{\partial}{\partial z^{--}},\, \mathcal{D}_z^{--}\right] = \tilde{I}^0\,. \qquad \text{(A.13)}
\end{aligned}
$$

Defining

$$
\mathcal{D}^{\pm\pm} = \mathcal{D}_z^{\pm\pm} - \frac{\partial}{\partial z^{\mp\mp}}\,, \qquad \text{(A.14)}
$$

we also find

$$
[\mathcal{D}^{++}, \mathcal{D}^{--}] = \mathcal{D}^0\,. \qquad \text{(A.15)}
$$

While checking (A.12), (A.13), one should take into account the matrix U(1) charges assignment (A.11). Also note that the SU(2|1) transformations of objects $\mathcal{D}^{\pm\pm}\Phi$, as distinct from $\mathcal{D}_z^{\pm\pm}\Phi$, reveal some deviations from the generic superfield law (A.3). For instance, $\mathcal{D}^{++}\Phi$, with $\tilde{I}^0\Phi = p\Phi$, $\tilde{F}\Phi = l\Phi$, transforms as

$$
\delta\mathcal{D}^{++}\Phi = -\lambda^{+-}p\,\mathcal{D}^{++}\Phi + 2\,\omega^{+-}l\,\mathcal{D}^{++}\Phi - 2\lambda^{+-}\frac{\partial}{\partial z^{--}}\Phi\,. \qquad \text{(A.16)}
$$

A.2 Eliminating z dependence

We wish to deal with the superfields containing no dependence on the extra coordinates $z^{\pm\pm}$. As the first step, we impose the manifestly covariant conditions

$$
\text{a)}\ \ (\mathcal{D}^0 + \tilde{I}^0)\Phi = 0\,, \qquad \text{b)}\ \ \frac{\partial}{\partial z^{++}}\Phi = 0\,, \qquad \text{(A.17)}
$$

which eliminate the dependence on z^{++} from both the superfield Φ and covariant derivatives.[6] Now

$$
\begin{aligned}
&\Phi \to \Phi(t, \theta, w, z^{--}) =: \Phi_{(z)}\,, \qquad && \mathcal{D}_z^{++} \to D_w^{++} + [1 + m(\theta^+\bar{\theta}^- - \theta^-\bar{\theta}^+)]\frac{\partial}{\partial z^{--}}\,,\\
&\mathcal{D}_z^{--} \to D_w^{--} - (z^{--})^2\frac{\partial}{\partial z^{--}} + z^{--}\tilde{I}^0\,, && \mathcal{D}^0 \to D_w^0 - 2z^{--}\frac{\partial}{\partial z^{--}}\,. \qquad \text{(A.18)}
\end{aligned}
$$

[5] And of course the standard harmonic U(1) charges in accord with the numbers of + and − indices.

[6] In some cases there is no need to impose (A.17a), still dealing with the z^{++}-independent superfields (see footnote 3).

Eliminating z^{--} dependence is more subtle and admits three different possibilities. Before explaining this, let us pass to another form of the transformation law (A.3) for $\Phi_{(z)}$, such that it is chosen to be active with respect to $\delta z^{--} = \lambda^{--} - 2\lambda^{+-} z^{--}$

$$\hat{\delta}\Phi_{(z)} = \lambda^{+-} D^0_w \Phi_{(z)} + 2\omega^{+-} \tilde{F}\Phi_{(z)} - \lambda^{--} \frac{\partial}{\partial z^{--}} \Phi_{(z)}\,, \tag{A.19}$$

where we made use of (A.18) and the constraint (A.17a).

Now we are prepared to discuss three options for eliminating z^{--} dependence.

I. The simplest possibility is to put

$$\frac{\partial}{\partial z^{--}}\Phi_{(z)} = 0\,, \qquad \Phi_{(z)} \Rightarrow \phi(t,\theta,w)\,, \qquad \hat{\delta}\phi = \lambda^{+-} D^0_w \phi + 2\,\omega^{+-}\tilde{F}\phi\,,$$
$$\mathcal{D}^{++}_z \Rightarrow D^{++}_w\,, \qquad \mathcal{D}^{--}_z \Rightarrow D^{--}_w - z^{--} D^0_w\,, \qquad \mathcal{D}^0 \to D^0_w\,. \tag{A.20}$$

In this case $\mathcal{D}^+ = \frac{\partial}{\partial\theta^-}\,, \quad \bar{\mathcal{D}}^+ = -\frac{\partial}{\partial\bar{\theta}^-}$ and one can impose the SU(2|1) covariant Grassmann analyticity conditions $\frac{\partial}{\partial\theta^-}\phi = \frac{\partial}{\partial\bar{\theta}^-}\phi = 0$ without any need for the constraint $D^{++}_w\phi = 0$, as opposed to the harmonic formalism of [3], in which Grassmann analyticity conditions imply the vanishing of the ++ harmonic derivative of the analytic superfield. We also note that the action of the second covariant harmonic derivative $\mathcal{D}^{--}_z$ on ϕ produces a superfield with a linear dependence on z^{--}, $\mathcal{D}^{--}_z\phi = D^{--}_w\phi - z^{--}D^0_w\phi$, unless $D^0_w\phi = 0$. Correspondingly, $D^{--}_w\phi$ transforms through the superfield ϕ itself. One can show that the same subtleties take place for the spinor derivatives $\mathcal{D}^-\phi$ and $\bar{\mathcal{D}}^-\phi$.

II. The harmonic formalism of [3] is recovered, when the z^{--} dependence of $\Phi_{(z)}$ is fixed in a more sophisticated way, by imposing the constraint

$$\mathcal{D}^{++}_z\Phi_{(z)} = 0 \to \frac{\partial}{\partial z^{--}}\Phi_{(z)} = -[1 + m(\theta^+\bar{\theta}^- - \theta^-\bar{\theta}^+)]^{-1} D^{++}_w \Phi_{(z)}\,. \tag{A.21}$$

This condition expresses all the coefficients in the z^{--} power series expansion of $\Phi_{(z)} = \phi(t,\theta,w) + z^{--}\phi^{++}(t,\theta,w) + \ldots$ in terms of powers of $\tilde{D}^{++}_w := [1 + m(\theta^+\bar{\theta}^- - \theta^-\bar{\theta}^+)]^{-1} D^{++}_w$ acting on the lowest coefficient, i.e. on ϕ. The transformation law (A.19) is reduced to

$$\hat{\delta}\phi = \lambda^{+-} D^0_w\phi + 2\,\omega^{+-}\tilde{F}\phi + \lambda^{--}\tilde{D}^{++}_w\phi\,, \tag{A.22}$$

that is precisely the generic superfield SU(2|1) transformation law postulated in [3]. The harmonic derivatives $\tilde{D}^{++}_w$ and D^{--}_w coincide with those defined in [3], $[\tilde{D}^{++}_w, D^{--}_w] = D^0_w$. The objects $\tilde{D}^{++}_w\phi$, $D^{--}_w\phi$ and $\mathcal{D}^+\phi = (\frac{\partial}{\partial\theta^-} + m\bar{\theta}^-\tilde{D}^{++}_w)\phi$, $\bar{\mathcal{D}}^+\phi = (-\frac{\partial}{\partial\bar{\theta}^-} - m\theta^-\tilde{D}^{++}_w)\phi$ are transformed according to (A.22).[7] The harmonic Grassmann analyticity for ϕ implies the constraint $\tilde{D}^{++}_w\phi = 0$.

[7] The same is true for the z-independent parts of the covariant spinor derivatives $\mathcal{D}^-, \bar{\mathcal{D}}^-$ in which the substitution (A.21) has been made.

III. Yet one more way to fix the z^{--} dependence of $\Phi_{(z)}$ is to impose the condition like the well-known Scherk-Schwarz reduction condition

$$\Phi_{(z)} = e^{z^{--}\tilde{I}^{++}}\,\phi'(t,\theta,w)\,,\quad \frac{\partial}{\partial z^{--}}\Phi_{(z)} = e^{z^{--}\tilde{I}^{++}}\,(\tilde{I}^{++}\phi')\,,\quad [\tilde{I}^0,\tilde{I}^{++}] = 2\tilde{I}^{++}\,, \tag{A.23}$$

$$\hat{\delta}\phi' = \lambda^{+-}D^0_w\phi' + 2\,\omega^{+-}\tilde{F}\phi' - \lambda^{--}\tilde{I}^{++}\phi'\,. \tag{A.24}$$

The corresponding version of the SU(2|1) harmonic formalism is just the one constructed and discussed in section 2. In particular, $\mathcal{D}^{++}_z = \mathcal{D}^{++}_w + \tilde{I}^{++}$, where $\mathcal{D}^{++}_w$ is now just (2.18) written in the $(w^{\pm}_i, z^{++})$ basis and restricted to the superfields satisfying the conditions (A.17).[8] The covariant derivative $\mathcal{D}^{--}_z$ defined in (2.16) coincides, on the same subclass of SU(2|1) superfields, with D^{--}_w. Actually, the option **III** is very similar to the option **I**. Like in the latter case, the Grassmann analyticity requires $\tilde{I}^{++}\phi = 0$, but not $\tilde{D}^{++}_w\phi = 0$ as in [3].

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