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On the ground state wave function of matrix theory

Ying-Hsuan Lin and Xi Yin

Jefferson Physical Laboratory, Harvard University, Cambridge, MA 02138 U.S.A.

E-mail: yhlin@physics.harvard.edu, xiyin@fas.harvard.edu

Abstract: We propose an explicit construction of the leading terms in the asymptotic expansion of the ground state wave function of BFSS SU(N) matrix quantum mechanics. Our proposal is consistent with the expected factorization property in various limits of the Coulomb branch, and involves a different scaling behavior from previous suggestions. We comment on some possible physical implications.

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Contents

1 Introduction

The matrix theory of Banks-Fischler-Susskind-Shenker [1, 2] was formulated by [4–6] along the lines of the AdS/CFT correspondence [3] as a duality between the 16-supercharge SU(N) gauged matrix quantum mechanics and the decoupling limit of the 0-brane geometry in type IIA string theory, which admits an M-theory lift to an asymptotically null-compactified spacetime. Though the matrix quantum mechanics may appear to be a (deceivingly) simple theory, it has been difficult to extract bulk physics from it. Perturbative computations in matrix theory beyond one-loop suffers from infrared divergences that are regularized through non-perturbative effects [8]. It is expected that semi-classical gravity in the bulk can only be recovered through strong coupling dynamics at large N. Relatively little is known regarding the strong coupling/low energy dynamics of matrix quantum mechanics beyond Monte Carlo simulations. Attempts of analytically understanding the strong coupling dynamics of matrix theory include the use of truncated Schwinger-Dyson equations, with limited success.

Various indirect arguments, as well as a careful computation of the supersymmetric index, indicate that the theory has a unique, normalizable, SO(9) rotationally invariant supersymmetric ground state [11–16]. There is a continuum of scattering states above the ground state. It is commonly believed (though not often stated explicitly) that there are no normalizable energy eigenstates of nonzero energy; in other words, all excited energy eigenstates are scattering states. This is consistent with the bulk picture that black holes can decay by radiating D0-branes [29], which are the only particles in the bulk that can escape to infinity. The bulk picture on the other hand also suggests the existence of an exponentially large number of metastable states with exponentially long life time.[1] These metastable states are the dual description of the microstates of the black hole at finite temperature.

An outstanding question is to describe these metastable states directly in the framework of matrix quantum mechanics. The first step is to understand the structure of the ground state wave function. An asymptotic expansion for the ground state wave function in the SU(2) case has been studied in [17, 18], and subsequent proposals for $N \geq 3$ were made in [19, 20]. In this paper we extend the study of the asymptotic expansion to the general SU(N) matrix theory. We will demonstrate that, first of all, the leading term in the asymptotic ground state wave function is governed by a set of 16 supercharges that describe N or $N-1$ free non-relativistic superparticles on $\mathbb{R}^{9|16}$. This is intuitive from the perspective of effective field theory on the Coulomb branch, though in the EFT approach it was unclear how to carry out a systematic expansion in $1/r$, particularly due to trouble with infrared divergences.

We then propose an explicit form of the leading asymptotic ground state wave function, based on a structure that involves a summation over trees that successively group the N particles. Our proposed form solves the supercharge constraint exactly, and obeys the expected factorization property in various limits on the Coulomb branch of the theory. There is a small ambiguity in our wave function, encoded in a simple set of constant "two-body coefficients", which are not determined by any simple argument we know of. Our proposal differs from previous suggestions in the SU(3) case [19]; in particular, the overall scaling power with r is different (the proposal of [19] tails off faster at large distances by a factor of r^{-14}). We also compute the next-to-leading order correction to the asymptotic wave function, and show how we can go to higher orders.

Let us begin by recalling the Hamiltonian of matrix theory,

$$H = \frac{1}{2}\mathrm{Tr}\left(P_i^2 - \frac{1}{2}[X^i, X^j]^2 - \widehat{\Theta}^T\Gamma^i[X^i, \widehat{\Theta}]\right), \tag{1.1}$$

where the bosonic and fermionic matrices can be written as $X^i = X^i_A T_A$, $\widehat{\Theta}_\alpha = \widehat{\Theta}_{\alpha A} T_A$, with T_A the SU(N) generators, normalized by $\mathrm{Tr}(T_A T_B) = \delta_{AB}$. Here $i = 1, 2, \cdots, 9$ and $\alpha = 1, \cdots, 16$ are vector and spinor indices of SO(9). P_i are the canonical momenta conjugate to X^i, while $\widehat{\Theta}_{\alpha A}$ obey canonical anti-commutation relations

$$\{\widehat{\Theta}_{\alpha A}, \widehat{\Theta}_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB}. \tag{1.2}$$

[1] This is a peculiar feature of the bulk geometry, in that only the D0-branes can approach asymptotic infinity at a finite cost of energy. It is in contrast to Schwarzschild black holes in flat spacetime whose lifetime scales like a power of its mass.

Gauging the SU(N) means that we restrict the Hilbert space to consist of SU(N) invariant states. The 16 supercharges are written as

$$Q_\alpha = \mathrm{Tr}\left(P_i(\Gamma^i\widehat{\Theta})_\alpha - \frac{i}{2}[X^i, X^j](\Gamma^{ij}\widehat{\Theta})_\alpha\right), \tag{1.3}$$

which obey the supersymmetry algebra up to a gauge rotation

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}H + 2\Gamma^i_{\alpha\beta}X^i_A C_A. \tag{1.4}$$

Here C_A are the operator realization of SU(N) generators,

$$C = C_A T_A = -i[X^i, P_i] - \frac{1}{2}\{\widehat{\Theta}_\alpha, \widehat{\Theta}_\alpha\}. \tag{1.5}$$

Our objective is to find the SO(9) invariant ground state wave function annihilated by all Q_α. The idea is to begin with a Born-Oppenheimer-type approximation, by starting at a generic point on the Coulomb branch where the X^i's are close to being commuting with one another, and treat the off-diagonal components as internal degrees of freedom. In the next section we will formulate an expansion of the wave function in powers of $r^{-\frac{3}{2}}$ where r is essentially the distance between eigenvalues on the Coulomb branch. A (so far) consistent proposal for the leading term in the asymptotic expansion of the ground state is given in section 3. The next-to-leading order correction is computed in section 4, and a systematic way of going to higher orders is presented. We conclude with discussions on the physical implications of our result and some speculations.

2 The asymptotic expansion

In this section we explain the method for solving the supersymmetry constraint equations on the wave function based on an asymptotic expansion, closely following the approach of [18] (see also [20]).

2.1 Removing the gauge redundancy

We are after the SU(N)-invariant ground state wave function which is annihilated by the supercharges Q_α, namely

$$\mathrm{Tr}\left\{\frac{\partial}{\partial X^i}\Gamma^i_{\alpha\beta}\widehat{\Theta}_\beta + \frac{1}{2}[X^i, X^j]\Gamma^{ij}_{\alpha\beta}\widehat{\Theta}_\beta\right\}\Psi = 0. \tag{2.1}$$

In analyzing the asymptotic form of the wave function, we will expand near a generic point at large distances on the Coulomb branch, and put the bosonic matrices X^i in the form

$$UX^iU^{-1} = \begin{pmatrix} r^i_1 & 0 & & \\ 0 & r^i_2 & & \\ & & \ddots & \\ & & & r^i_N \end{pmatrix} + \begin{pmatrix} 0 & q^i_{12} & & \\ (q^i_{12})^* & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \tag{2.2}$$

for some SU(N) matrix U. We write $\vec{r}_a = (r_a^1, \cdots, r_a^9)$, $\vec{q}_{ab} = (q_{ab}^1, \cdots, q_{ab}^9)$, and work in the regime of large $|\vec{r}_a - \vec{r}_b|$ such that q_{ab}^i are very massive. To ensure that this is the case, namely that the q_{ab}^i's are transverse to the valley of the scalar potential, we must choose U in such a way that $\vec{q}_{ab} \cdot (\vec{r}_a - \vec{r}_b) = 0$ for all a, b. This condition fixes U up to the diagonal $\mathrm{U}(1)^{N-1}$ that rotates the phases of $\vec{q}_{ab}$. We will leave these degrees of freedom in U unfixed. This is acceptable because it still allows us to work in the regime of small q_{ab}^i in the large r_a^i limit. Since in this limit q_{ab}^i are described as harmonic oscillators in a potential $|\vec{r}_a - \vec{r}_b|^2 (q_{ab}^i)^2$, it is convenient to define

$$y_{ab}^i = |\vec{r}_a - \vec{r}_b|^{\frac{1}{2}} q_{ab}^i \tag{2.3}$$

so that $y_{ab}^i \sim \mathcal{O}(1)$.

Similarly, we separate $\widehat{\Theta}_\alpha$, after the appropriate SU(N) rotation, into diagonal and off-diagonal modes, according to

$$U\widehat{\Theta}_\alpha U^{-1} = \begin{pmatrix} \theta_{\alpha 1} & 0 & & \\ 0 & \theta_{\alpha 2} & & \\ & & \ddots & \\ & & & \theta_{\alpha N} \end{pmatrix} + \begin{pmatrix} 0 & (\Theta_\alpha)_{12} & & \\ (\Theta_\alpha)_{12}^* & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}. \tag{2.4}$$

From now the unhatted notation $(\Theta_\alpha)_{ab}$ will always refer to these off-diagonal components of $U\widehat{\Theta}_\alpha U^{-1}$. Note that the overall SU(N) gauge rotation, which acts on both X^i and $\widehat{\Theta}_\alpha$, only acts by rotating U and does not act on $(r^i, q^i, \theta_\alpha, \Theta_\alpha)$.

The next step is to write $\partial/\partial X^i$ in terms of derivatives on r_a^i and y_{ab}^i. The details are given in appendix A, with the result

$$\begin{aligned}\left[U\frac{\partial}{\partial X^i}U^{-1}\right]_{ba} &= \delta_{ab}\frac{\partial}{\partial r_a^i} + \Pi_{ab}^{ij}\frac{\partial}{\partial q_{ab}^j} - \frac{\widehat{r}_{ab}^i}{|r_{ab}|}\sum_{c\neq a,b}\left(\frac{y_{ca}^k}{|r_{ca}|^{\frac{1}{2}}}\Pi_{cb}^{kj}\frac{\partial}{\partial q_{cb}^j} - \frac{y_{bc}^k}{|r_{bc}|^{\frac{1}{2}}}\Pi_{ac}^{kj}\frac{\partial}{\partial q_{ac}^j}\right)\\ &\quad + \frac{\widehat{r}_{ab}^i}{|\vec{r}_{ab}|}\left[U\frac{\partial}{\partial U}\right]_{ba} + \mathcal{O}(r^{-\frac{5}{2}}),\end{aligned} \tag{2.5}$$

where $r_{ab}^i \equiv r_a^i - r_b^i$, and $\Pi_{ab}^{ij} \equiv \delta^{ij} - \widehat{r}_{ab}^i \widehat{r}_{ab}^j$. Next, we need to change coordinate on the fermions $\widehat{\Theta}_\alpha$ into $(\theta_\alpha, \Theta_\alpha)$ as well. In doing so, we must make the replacement

$$\left[U\frac{\partial}{\partial U}\right]_{ab} \to R_{ab} + M_{ab}, \tag{2.6}$$

where R_{ab} is the overall SU(N) gauge rotation generator that only acts on U but not on $(r^i, q^i, \theta_\alpha, \Theta_\alpha)$, and M_{ab} is the SU(N) generator acting on the fermions.[2]

Now we can write

$$\begin{aligned}\left[U\frac{\partial}{\partial X^i}U^{-1}\right]_{ba} &= \delta_{ab}\frac{\partial}{\partial r_a^i} + \Pi_{ab}^{ij}\frac{\partial}{\partial q_{ab}^j} - \frac{\widehat{r}_{ab}^i}{|r_{ab}|}\sum_{c\neq a,b}\left(\frac{y_{ca}^k}{|r_{ca}|^{\frac{1}{2}}}\Pi_{cb}^{kj}\frac{\partial}{\partial q_{cb}^j} - \frac{y_{bc}^k}{|r_{bc}|^{\frac{1}{2}}}\Pi_{ac}^{kj}\frac{\partial}{\partial q_{ac}^j}\right)\\ &\quad + \frac{\widehat{r}_{ab}^i}{|r_{ab}|}(R_{ba} + M_{ba}) + \mathcal{O}(r^{-\frac{5}{2}}).\end{aligned} \tag{2.8}$$

[2] Explicitly,

$$M_{ab} = \frac{1}{2}[(\Theta_\alpha)_{ae}, (\Theta_\alpha)_{ec}] + (\theta_{\alpha a} - \theta_{\alpha b})(\Theta_\beta)_{ab}. \tag{2.7}$$

In the application below, we will take this expression for $\partial/\partial X^i$ to act on an SU(N) invariant wave function, that is, a wave function that is invariant under the SU(N) action simultaneously on the original bosons and fermions X^i and Θ_α. In the new coordinate system $(U, r^i, q^i, \theta_\alpha, \Theta_\alpha)$, it only acts on U. The upshot is that R_{ab} annihilates the SU(N) invariant wave function and can be dropped from now, and U will no longer appear explicitly in our computations below.

2.2 The asymptotic expansion of the supercharge

After dropping the R_{ab} term and changing variables from q^i_{ab} to y^i_{ab}, we can now write the supercharge as an expansion in $r^{-\frac{3}{2}}$, in the form

$$\begin{aligned} iQ_\alpha =& \sum_{a\neq b} |\vec{r}_{ab}|^{\frac{1}{2}} \left[\Pi^{ij}_{ab} \frac{\partial}{\partial y^j_{ba}} \Gamma^i_{\alpha\beta}(\Theta_\beta)_{ba} + \frac{1}{2}\widehat{r}^i_{ab} y^j_{ab} \Gamma^{ij}_{\alpha\beta}(\Theta_\beta)_{ba} \right] \\ &+ \sum_a \frac{\partial}{\partial r^i_a}\Gamma^i_{\alpha\beta}\theta_{\beta a} + \sum_{a\neq b}\left[\frac{\widehat{r}^i_{ab}}{2|r_{ab}|} y^j_{ab} \frac{\partial}{\partial y^j_{ab}} \Gamma^i_{\alpha\beta}(\theta_{\beta a} - \theta_{\beta b}) \right. \\ &+ \sum_{c\neq a,b} \frac{y^i_{ac} y^j_{cb}}{|r_{ac}|^{\frac{1}{2}}|r_{bc}|^{\frac{1}{2}}} \Gamma^{ij}_{\alpha\beta}(\Theta_\beta)_{ba} + \frac{y^i_{ab}y^j_{ba}}{2|r_{ab}|}\Gamma^{ij}_{\alpha\beta}(\theta_{\beta a} - \theta_{\beta b}) - \frac{\widehat{r}^i_{ab}}{|r_{ab}|}\Gamma^i_{\alpha\beta}(\Theta_\beta)_{ba} M_{ab} \\ &\left. - \sum_{c\neq a,b} \left(\frac{|r_{bc}|^{\frac{1}{2}}}{|r_{ac}|^{\frac{1}{2}}} y^k_{ca} \Pi^{kj}_{cb} \frac{\partial}{\partial y^j_{cb}} - \frac{|r_{ac}|^{\frac{1}{2}}}{|r_{bc}|^{\frac{1}{2}}} y^k_{bc}\Pi^{kj}_{ac}\frac{\partial}{\partial y^j_{ac}} \right) \frac{\widehat{r}^i_{ab}}{|r_{ab}|}\Gamma^i_{\alpha\beta}(\Theta_\beta)_{ab} \right] + \mathcal{O}(r^{-\frac{5}{2}}). \end{aligned} \tag{2.9}$$

We will write the first line after the equal sign as Q^0_α and[3] the next two lines as Q^1_α. Q^0_α scales like $r^{\frac{1}{2}}$ while Q^1_α scales like r^{-1}. The wave function will take the following form

$$\Psi = \Psi_0 + \Psi_1 + \Psi_2 + \cdots , \tag{2.10}$$

where Ψ_n scales like $r^{-\kappa-\frac{3}{2}n}$. Our goal is to determine κ. Separating the equations according to the scaling degree in r, we have a series of equations

$$\begin{aligned} &Q^0_\alpha \Psi_0 = 0, \\ &Q^0_\alpha \Psi_1 + Q^1_\alpha \Psi_0 = 0, \quad \text{etc.} \end{aligned} \tag{2.11}$$

The first equation is a differential equation in y^i_{ab} only. The solution Ψ_0 takes the form

$$\Psi_0 = f(\vec{r}_a)|\psi_0(\widehat{r})\rangle_{y,\Theta}, \tag{2.12}$$

where $|\psi_0(\widehat{r})\rangle_{y,\Theta}$ is the ground state wave function of an $\widehat{r}_{ab}$-dependent (denoted here collectively by $\widehat{r}$) supersymmetric harmonic oscillator in the off-diagonal (y,Θ) sector, obeying

$$Q^0_\alpha|\psi_0(\hat{r})\rangle_{y,\Theta} \equiv \sum_{a\neq b}|r_{ab}|^{\frac{1}{2}} \left[\Pi^{ij}_{ab}\frac{\partial}{\partial y^j_{ba}}\Gamma^i_{\alpha\beta}(\Theta_\beta)_{ba} + \frac{1}{2}\widehat{r}^i_{ab} y^j_{ab}\Gamma^{ij}_{\alpha\beta}(\Theta_\beta)_{ba}\right] |\psi_0(\hat{r})\rangle_{y,\Theta} = 0. \tag{2.13}$$

$f(\vec{r}_a)$ is a so far undetermined wave function that has some overall scaling $r^{-\kappa}$, and includes the fermionic wave function in the diagonal θ sector.

[3]Our convention for Q^n_α's differs from that of the usual supercharge by a factor of i.

The next key step is to consider a projection P_0 onto the zero-eigenspace of Q^0_α. Since iQ^0_α is Hermitian, any state of the form $Q^0_\alpha\Psi_1$ must be orthogonal to the zero-eigenspace of Q^0_α, and is thus annihilated by P_0. Consequently, the next-to-leading order equation in the asymptotic expansion implies

$$P_0 Q^1_\alpha \Psi_0 = 0. \tag{2.14}$$

Since Q^1_α involves an r-derivative, this equation will provide nontrivial constraints on $f(\vec{r}_a)$.

2.3 Reducing to the Cartan wave function

Q^0_α has an anti-commutator of the form

$$\{Q^0_\alpha, Q^0_\beta\} = \delta_{\alpha\beta}\sum_{a\neq b}|r_{ab}|\left[\Pi^{ij}_{ab}\frac{\partial}{\partial y^i_{ab}}\frac{\partial}{\partial y^j_{ba}} - \frac{1}{4}y_{ab}\cdot y_{ba} - \frac{1}{2}\widehat{r}^k_{ab}\Gamma^k_{\gamma\delta}(\Theta_\gamma)_{ab}(\Theta_\delta)_{ba}\right] + \Gamma^k_{\alpha\beta}\mathcal{M}_k. \tag{2.15}$$

For each pair a, b, consider the matrix $\widehat{r}^k_{ab}\Gamma^k_{\alpha\beta}$ that acts on SO(9) spinors. This matrix has eight $+1$ eigenvalues and eight -1 eigenvalues. Let $\Pi^\pm_{ab}$ be the projection operators onto the positive and negative spinor eigenspaces of $\widehat{r}^i_{ab}\Gamma^i$. By definition, $\Pi^\pm_{ab} = \Pi^\mp_{ba}$.

Given a fixed pair a, b, let $|F_{ab}(\widehat{r}_{ab})\rangle$ be a unit norm state in the Θ_{ab} sector, that is annihilated by $(\Theta^-_{ab})_\alpha \equiv (\Pi^-_{ab})_{\alpha\beta}(\Theta_\beta)_{ab}$ and $(\Theta^-_{ba})_\alpha \equiv (\Pi^-_{ba})_{\alpha\beta}(\Theta_\beta)_{ba} = (\Pi^+_{ab})_{\alpha\beta}(\Theta_\beta)^*_{ab}$ for all α, and is invariant under simultaneous SO(9) rotations on $\widehat{r}_{ab}$, Θ_{ab} *and* Θ_{ba}. Such a state is unique up an overall ($\widehat{r}$-independent) phase. We will write $|F(\widehat{r})\rangle = \bigotimes_{a<b}|F_{ab}(\widehat{r}_{ab})\rangle$ for such a zeroth-order fermion ground state in the entire off-diagonal Θ sector (again, the notation here is such that $\widehat{r}$ stands collectively for the set of all $\widehat{r}_{ab}$'s). We can then construct $|\psi_0(\widehat{r})\rangle$ by combining $|F(\widehat{r})\rangle$ with the harmonic oscillator ground state wave function for the y^i_{ab}'s,

$$|\psi_0(\widehat{r})\rangle = e^{-\frac{1}{4}\sum_{a\neq b}|y_{ab}|^2}|F(\widehat{r})\rangle. \tag{2.16}$$

There are $8N(N-1)$ independent y^i_{ab}'s, and the ground state energy of the harmonic oscillator precisely cancels with the fermionic contribution in the coefficient of $\delta_{\alpha\beta}$. One can verify that $|\psi_0(\widehat{r})\rangle$ is annihilated by $\mathcal{M}_k$ as well.

Now we can write

$$\Psi_0 = e^{-\frac{1}{4}\sum_{a\neq b}|y_{ab}|^2}\sum_s f_s(\vec{r}_a)|s\rangle\otimes|F(\widehat{r})\rangle, \tag{2.17}$$

for a set of functions $f_s(\vec{r}_a)$, where s labels states in the Clifford module of the $16(N-1)$ diagonal θ_{aa}'s ($s = 1, \cdots, 2^{8(N-1)}$). Let us inspect the action of

$$\begin{aligned}Q^1_\alpha = \sum_a \frac{\partial}{\partial r^i_a}\Gamma^i_{\alpha\beta}\theta_{\beta a} + \sum_{a\neq b}\Bigg[&\frac{\widehat{r}^i_{ab}}{2|r_{ab}|}y^j_{ab}\frac{\partial}{\partial y^j_{ab}}\Gamma^i_{\alpha\beta}(\theta_{\beta a}-\theta_{\beta b}) + \sum_{c\neq a,b}\frac{y^i_{ac}y^j_{cb}}{|r_{ac}|^{\frac{1}{2}}|r_{bc}|^{\frac{1}{2}}}\Gamma^{ij}_{\alpha\beta}(\Theta_\beta)_{ba}\\ &+\frac{y^i_{ab}y^j_{ba}}{2|r_{ab}|}\Gamma^{ij}_{\alpha\beta}(\theta_{\beta a}-\theta_{\beta b}) - \frac{\widehat{r}^i_{ab}}{|r_{ab}|}\Gamma^i_{\alpha\beta}(\Theta_\beta)_{ba}M_{ab}\\ &-\sum_{c\neq a,b}\left(\frac{|r_{bc}|^{\frac{1}{2}}}{|r_{ac}|^{\frac{1}{2}}}y^k_{ca}\Pi^{kj}_{cb}\frac{\partial}{\partial y^j_{cb}} - \frac{|r_{ac}|^{\frac{1}{2}}}{|r_{bc}|^{\frac{1}{2}}}y^k_{bc}\Pi^{kj}_{ac}\frac{\partial}{\partial y^j_{ac}}\right)\frac{\widehat{r}^i_{ab}}{|r_{ab}|}\Gamma^i_{\alpha\beta}(\Theta_\beta)_{ab}\Bigg]\end{aligned} \tag{2.18}$$

on Ψ_0. Keep in mind that $\partial/\partial r_a^i$ which appears in Q_α^1 acts not only on the functions $f_s(\vec{r}_a)$ but on $|F(\widehat{r})\rangle$ as well.

Under the projection P_0, we can replace $y_{ab}\cdot\partial_{y_{ab}}$ and $y^i_{ac}y^j_{cb}$ in Q^1_α by their expectation values in the harmonic oscillator ground state wave function $e^{-\frac{1}{4}\sum_{a,b}|y_{ab}|^2}=e^{-\frac{1}{2}\sum_{a<b}|y_{ab}|^2}$. Furthermore, any term that involves the product of an odd number of Θ's when acting on Ψ_0 cannot preserve the fermion ground state in the Θ sector, and the result will be annihilated by P_0. Note that the projector P_0 does not touch the $\theta_{\alpha a}$ degrees of freedom. Let us define $(\Theta^\pm_\alpha)_{ab}\equiv\Pi^+_{ab}(\Theta_\alpha)_{ab}$. All states that survive the P_0 projection are annihilated by Θ^-_α, while any state obtained by acting with Θ^+_α is killed by P_0. Using the relation

$$P_0(\Theta_\beta)_{ba}M_{ab}\Psi_0=P_0[(\Theta_\beta)^-_{ba},M_{ab}]\Psi_0=-(\Pi^+_{ab}(\theta_a-\theta_b))_\beta\Psi_0, \tag{2.19}$$

we can replace Q^1_α by a simplified operator

$$\begin{aligned}\widetilde{Q}^1_\alpha&=\sum_a\frac{\partial}{\partial r^i_a}\Gamma^i_{\alpha\beta}\theta_{\beta a}-\sum_{a\neq b}\frac{2}{|r_{ab}|}\widehat{r}^i_{ab}\Gamma^i_{\alpha\beta}(\theta_{\beta a}-\theta_{\beta b})+\sum_{a\neq b}\frac{1}{|r_{ab}|}(\Pi^+_{ab})_{\alpha\beta}(\theta_{\beta a}-\theta_{\beta b})\\&=\sum_a\frac{\partial}{\partial r^i_a}\Gamma^i_{\alpha\beta}\theta_{\beta a}-\sum_{a\neq b}\frac{3}{|r_{ab}|}\widehat{r}^i_{ab}\Gamma^i_{\alpha\beta}\theta_{\beta a},\end{aligned} \tag{2.20}$$

in the sense that

$$P_0Q^1_\alpha\Psi_0=P_0\widetilde{Q}^1_\alpha\Psi_0. \tag{2.21}$$

Furthemore, the r^i_a dependence of Ψ_0 may be expressed as dependence on $|r_{ab}|$ and $\widehat{r}_{ab}$. Under a variation δr^i_a, we have

$$\begin{aligned}\delta|r_{ab}|&=\widehat{r}_{ab}\cdot(\delta\vec{r}_a-\delta\vec{r}_b),\\\delta\widehat{r}_{ab}&=\frac{\delta\vec{r}_{ab}-\widehat{r}_{ab}(\widehat{r}_{ab}\cdot\delta\vec{r}_{ab})}{|r_{ab}|}.\end{aligned} \tag{2.22}$$

Thus we can write

$$\frac{\partial}{\partial r^i_a}=\sum_{b\neq a}\left(\widehat{r}^i_{ab}\frac{\partial}{\partial|r_{ab}|}+\frac{\widehat{r}^j_{ab}}{|r_{ab}|}R^{ji}_{ab}\right), \tag{2.23}$$

where R^{ij}_{ab} is the generator of SO(9) rotation on $\widehat{r}_{ab}$ for each pair a,b. Note that it does not act on the fermions, by definition. In the Θ sector, the zeroth order ground state wave function $|F(\widehat{r})\rangle$ by construction is invariant under the SO(9) rotation on $\widehat{r}_{ab}$, Θ_{ab}, *and* Θ_{ba}. Let us denote by F^{ij}_{ab} the SO(9) rotation generator on Θ_{ab} and Θ_{ba}, namely

$$F^{ij}_{ab}=\frac{1}{4}(\Theta_{ab}\Gamma^{ij}\Theta_{ba}). \tag{2.24}$$

Thus when acting on $|F(\widehat{r})\rangle$ with R^{ij}_{ab}, we can replace R^{ij}_{ab} by $-F^{ij}_{ab}$. Note that $F^{ij}_{ab}|F(\widehat{r})\rangle=\frac{1}{4}(\Theta^+_{ab}\Gamma^{ij}\Theta^+_{ba})|F(\widehat{r})\rangle$, and is thus annihilated by the projector P_0. In other words, we can ignore the $\widehat{r}_{ab}$-dependence of $|F(\widehat{r})\rangle$ in computing $P_0\widetilde{Q}^1_\alpha\Psi_0$. For this purpose, we might as well replace $\widetilde{Q}^1_\alpha$ by an operator[4] of the same form as (2.20), but now acting entirely on the "Cartan wave function"

$$\Psi_0^C = \sum_s f_s(\vec{r}_a)|s\rangle \tag{2.25}$$

that is just in the (r, θ) sector. Now the projector P_0 is no longer needed; the equation $P_0 Q^1_\alpha \Psi_0 = 0$ simply reduces to

$$\widetilde{Q}^1_\alpha \Psi_0^C = 0. \tag{2.26}$$

2.4 Treating the Cartan fermions

In the simplest SU(2) case, the indices a, b take values 1 and 2 (and $\vec{r}_2 = -\vec{r}_1$). There are 16 θ_α's, giving rise to $2^8 = 256$ states in the θ sector. With respect to the SO(9) rotation on the θ_α's, these 256 states branch into

$$\mathbf{44} \oplus \mathbf{84} \oplus \mathbf{128}. \tag{2.27}$$

Here the **44** is the traceless symmetric tensor representation of SO(9). The other two irreducible representations of SO(9) cannot form a singlet by tensoring with a power of the vector representation (coming from $\widehat{r}$). The fermion part of the SO(9) invariant ground state wave function, $|s\rangle$, must thus be constructed from the **44**. Such a state is unique up to the overall factor, namely, it is $|\widehat{r}\widehat{r}\rangle \equiv \widehat{r}^i \widehat{r}^j |s_{ij}\rangle$, where $|s_{ij}\rangle$ is a basis for the **44**. The SO(9) invariance of the wave function allows us to replace R^{ij} by $-\frac{1}{4}(\theta\Gamma^{ij}\theta)$ that rotates θ instead of $\vec{r}$. One can show that

$$\widehat{r}^j(\Gamma^i\theta)_\alpha(\theta\Gamma^{ij}\theta)|\widehat{r}\widehat{r}\rangle = 36\widehat{r}^i(\Gamma^i\theta)_\alpha|\widehat{r}\widehat{r}\rangle. \tag{2.28}$$

One then finds that $\widetilde{Q}^1_\alpha \Psi_0^C = 0$ is solved by $\Psi_0^C = r^{-6}|\widehat{r}\widehat{r}\rangle$.

The case of general SU(N) gauge group will be treated in the next section. Note that the integration measure for our wave function Ψ at large r takes the form[5]

$$\int \prod_{a=1}^{N-1} d^9\vec{r}_a \left(\prod_{a<b} r_{ab}^2\right) r^{-4N(N-1)} \int \prod_{a\neq b} d^9\vec{y}_{ab}\ \delta(\vec{y}_{ab}\cdot\widehat{r}_{ab}). \tag{2.29}$$

If the leading asymptotic wave function Ψ_0 has an overall scaling $r^{-\kappa}$, normalizability then demands $\kappa > -\frac{3}{2}(N-3)(N-1)$.

3 The leading ground state wave function

3.1 Reducing to free superparticles

We are seeking an $S_N \times \mathrm{SO}(9)$ invariant Cartan wave function Ψ_0^C that is annihilated by

$$\widetilde{Q}^1_\alpha = \sum_a \left(\frac{\partial}{\partial r_a^i} - \sum_{b\neq a} \frac{3}{r_{ab}^2} r_{ab}^i\right) \Gamma^i_{\alpha\beta}\theta_{\beta a}. \tag{3.1}$$

[4]By a slight abuse of notation we will still denote this operator by $\widetilde{Q}^1_\alpha$.

[5]Here r_{ab}^2 come from the gauge-fixing, and $r^{-4N(N-1)}$ comes from the change of variables from q to y.

It is convenient to define

$$\Psi^{\rm new} \equiv \prod_{a<b} |r_{ab}|^{-3} \Psi_0^C. \tag{3.2}$$

Then the equation for $\Psi^{\rm new}$ becomes simply $Q_\alpha^{\rm new}\Psi^{\rm new} = 0$, where $Q_\alpha^{\rm new}$ take the form of the supercharges for a set of free superparticles,

$$Q_\alpha^{\rm new} = \sum_a \frac{\partial}{\partial r_a^i} \Gamma^i_{\alpha\beta}\theta_{\beta a}. \tag{3.3}$$

We immediately learn that $\Psi^{\rm new}$ takes the form

$$\Psi^{\rm new} = \sum_s F_s(r_a^i)|s\rangle, \tag{3.4}$$

where $F_s(r_a^i)$ for each internal fermion state $|s\rangle$ is a harmonic function on $\mathbb{R}^{9(N-1)}$. Indeed, in the SU(2) case, $\Psi^{\rm new} = r^{-9}\widehat{r}^i\widehat{r}^j|s_{ij}\rangle \propto \partial_i\partial_j r^{-7}|s_{ij}\rangle$ is of such form.

3.2 The SU(N) proposal

So far we have been writing the supercharges and the Hamiltonian as if we were dealing with the U(N) theory. In dealing with the SU(N) matrix theory, we need to factor out the center of mass degrees of freedom. This is straightforward in the bosonic sector: the wave function when viewed as a function of $\vec{x}_1, \cdots, \vec{x}_N$ is taken to be invariant under the overall translation $\vec{P} = \sum_{a=1}^N \vec{p}_a$. Care must be taken in the fermion sector, however, since we have quantized the $\theta_{\alpha a}$ independently, with

$$\{\theta_{\alpha a}, \theta_{\beta b}\} = \delta_{ab}\delta_{\alpha\beta}. \tag{3.5}$$

We should factor out $\overline{\theta} = (\theta_1 + \theta_2 + \cdots + \theta_N)/N$, and only work with the combinations of θ's (for instance, $\theta_a - \overline{\theta}$) that anti-commute with $\overline{\theta}$. In the expression for the supercharge Q_α in terms of $r_a^i, q_{ab}^i, \theta_{\alpha a}, (\Theta_\alpha)_{ab}$, the only term that involves the center of mass position and fermionic coordinate $\overline{\theta}$ is $\sum_{a=1}^N p_a^i\Gamma^i\theta_a$, where $p_a^i = -i\partial/\partial r_a^i$. In passing to the SU($N$) system, we can separate

$$\sum_{a=1}^N p_a^i \Gamma^i\theta_a = P^i\Gamma^i\overline{\theta} + \sum_{a=1}^N \left(p_a^i - \frac{1}{N}P^i\right)\Gamma^i(\theta_a - \overline{\theta}), \tag{3.6}$$

and simply drop the first term $P^i\Gamma^i\overline{\theta}$, since P^i and $\overline{\theta}$ commute with the remaining terms of the supercharge. The ground state wave function will depend on the relative bosonic coordinates $\vec{x}_a - \vec{x}_b$, and its fermionic component may be constructed as an element of the Clifford module coming from $\theta_a - \overline{\theta}$. Be aware that $\theta_a - \overline{\theta}$ do not anti-commute with $\theta_b - \overline{\theta}$ for $a \neq b$. Rather, we have

$$\left\{\theta_a - \overline{\theta}, \theta_b - \overline{\theta}\right\} = \delta_{ab} - \frac{1}{N}. \tag{3.7}$$

One can in principle go to a basis in which the anti-commutators become diagonal, and quantize the theory using that basis. However, such a basis is rather inconvenient to work with. Below we will employ a different approach.

Though the problem of finding Ψ_0 is reduced to the free problem of finding Ψ^C or Ψ^{new}, this problem doesn't have a unique solution in the general $\mathrm{SU}(N)$ case, even after imposing $S_N \times \mathrm{SO}(9)$ invariance. It is possible that there are more constraints coming from the smoothness of the wave function at small r^i_{ab} when all order corrections are included. For now, we will constrain Ψ_0 further by some physical intuition. Namely, we expect that in a limit on the Coulomb branch where (r^i_a, θ_α) are separated into two clusters centered at (x^i, θ_α) and (y^i, η_α), and the $\mathrm{SU}(N)$ broken into $\mathrm{SU}(M) \times \mathrm{SU}(N)$, Ψ^{new} should be approximately proportional to the $\mathrm{SU}(2)$ wave function in the relative bosonic and fermionic coordinates $(x^i - y^i, \theta_\alpha - \eta_\alpha)$. Motivated by this, we now make a proposal for Ψ_0 (or equivalently for Ψ^{new}) which will be an exact solution of $P_0 Q^1_\alpha \Psi_0 = 0$, and satisfies this factorization criterion.

We will in fact define recursively a weighted n-body asymptotic wave function,

$$\Psi^{(n)}_{k_1,k_2,\cdots,k_n}(\vec{r}_1, \widehat{\theta}_{1\alpha}; \vec{r}_2, \widehat{\theta}_{2\alpha}; \cdots; \vec{r}_n, \widehat{\theta}_{n\alpha}). \tag{3.8}$$

Here k_a are a set of positive integers. By writing $\widehat{\theta}_{\alpha a}$ in the argument, we simply mean that the fermionic component of the wave function is built by quantization of $\widehat{\theta}_{\alpha a}$ according to their appropriate anti-commutators. We will see in the construction below that $\widehat{\theta}_{\alpha a}$ obey the anti-commutation relations

$$\{\widehat{\theta}_{\alpha a}, \widehat{\theta}_{b\beta}\} = \frac{1}{k_a}\delta_{ab}\delta_{\alpha\beta}. \tag{3.9}$$

In fact, by construction $\Psi_{k_1,\cdots,k_n}$ will be a function of the relative positions $\vec{r}_a - \vec{r}_b$ only, and its fermion component will be built out of $\widehat{\theta}_{\alpha a} - \widehat{\theta}_{\alpha b}$ only.

First of all, we define a two-body wave function,

$$\Psi^{(2)}_{k_1,k_2}(\vec{r}_1, \widehat{\theta}_{\alpha 1}; \vec{r}_2, \widehat{\theta}_{\alpha 2}) = C_{k_1,k_2}\Psi^{\text{new}}_{\mathrm{SU}(2)}\left(\frac{\vec{r}_1 - \vec{r}_2}{\sqrt{k_1^{-1} + k_2^{-1}}}, \frac{\widehat{\theta}_{\alpha 1} - \widehat{\theta}_{\alpha 2}}{\sqrt{k_1^{-1} + k_2^{-1}}}\right). \tag{3.10}$$

Here $\Psi^{\text{new}}_{\mathrm{SU}(2)}(\vec{r}, \theta)$ is as in the $\mathrm{SU}(2)$ case,

$$\Psi^{\text{new}}_{\mathrm{SU}(2)}(\vec{r}, \theta) = \sum_{i,j=1}^{9} \partial_i \partial_j |\vec{r}|^{-7} |s_{ij}\rangle_\theta. \tag{3.11}$$

$C_{k_1,k_2} = C_{k_2,k_1}$ is a normalization constant that may depend on k_1, k_2, which is so far undetermined. Note that the two-body wave function factor is invariant under exchanging the two bodies $(\vec{r} \to -\vec{r}, \theta \to -\theta)$.

Now we define the recursive relation between the n-body wave function and the $(n-1)$-body wave function

$$\begin{aligned}\Psi^{(n)}_{k_1,k_2,\cdots,k_n}(\vec{r}_1, \widehat{\theta}_{1\alpha}; \vec{r}_2, \widehat{\theta}_{2\alpha}; \cdots; \vec{r}_n, \widehat{\theta}_{n\alpha}) &= \sum_{1 \le i < j \le n} C_{k_i,k_j}\Psi^{\text{new}}_{\mathrm{SU}(2)}\left(\frac{\vec{r}_i - \vec{r}_j}{\sqrt{k_i^{-1} + k_j^{-1}}}, \frac{\widehat{\theta}_i - \widehat{\theta}_j}{\sqrt{k_i^{-1} + k_j^{-1}}}\right) \\ &\times \Psi^{(n-1)}_{k_i+k_j,k_1,\cdots,\cancel{k_i},\cdots,\cancel{k_j},\cdots,k_n}\left(\frac{k_i r_i + k_j r_j}{k_i + k_j}, \frac{k_i\widehat{\theta}_i + k_j\widehat{\theta}_j}{k_i + k_j}; r_1, \widehat{\theta}_1; \cdots; \cancel{r_i}, \cancel{\widehat{\theta}_i}; \cdots; \cancel{r_j}, \cancel{\widehat{\theta}_j}; \cdots; r_n, \widehat{\theta}_n\right).\end{aligned} \tag{3.12}$$

Note that by our construction, $\frac{\widehat{\theta}_i - \widehat{\theta}_j}{\sqrt{k_i^{-1}+k_j^{-1}}}$ anti-commutes with $\frac{k_i\widehat{\theta}_i + k_j\widehat{\theta}_j}{k_i+k_j}$ and with all other $\widehat{\theta}_k$, $k \neq i, j$.

It is then straightforward to verify that

$$\Psi^{\rm new} = \Psi^{(N)}_{1,1,\cdots,1}(\vec{r}_1, \theta_{1\alpha}; \vec{r}_2, \theta_{2\alpha}; \cdots; \vec{r}_N, \theta_{N\alpha}) \tag{3.13}$$

is an exact solution for the asymptotic ground state Cartan wave function, namely the corresponding Ψ_0^C is annihilated by $\widetilde{Q}^1_\alpha$.[6]

The proposed $\Psi^{\rm new}$ is also manifestly invariant under the permutation (Weyl group action) by S_N, and is SO(9) rotationally invariant. And it satisfies the factorization property in various limits of the Coulomb branch with the symmetry breaking pattern $\mathrm{SU}(N) \to \mathrm{SU}(k) \times \mathrm{SU}(N-k)$. To see the latter, consider the limit where say a cluster $\vec{r}_1, \cdots, \vec{r}_k \sim \vec{R}_1$ are far separated from $\vec{r}_{k+1}, \cdots, \vec{r}_N \sim \vec{R}_2$. In this limit $\Psi^{\rm new}$ is dominated by

$$\begin{aligned}\Psi^{\rm new} \longrightarrow\ & C_{k,N-k}\Psi^{\rm new}_{\mathrm{SU}(2)}\Bigg(\sqrt{\frac{k(N-k)}{N}}\,(\vec{R}_1 - \vec{R}_2), \sqrt{\frac{N-k}{kN}}\,(\widehat{\theta}_1 + \cdots + \widehat{\theta}_k) \\ & \qquad - \sqrt{\frac{k}{(N-k)N}}\,(\widehat{\theta}_{k+1} + \cdots + \widehat{\theta}_N)\Bigg) \\ & \times \Psi^{(k)}_{1,\cdots,1}(r_1, \widehat{\theta}_1; \cdots; r_k, \widehat{\theta}_k)\, \Psi^{(N-k)}_{1,\cdots,1}(r_{k+1}, \widehat{\theta}_{k+1}; \cdots; r_N, \widehat{\theta}_N),\end{aligned} \tag{3.16}$$

which scales like $|\vec{R}_1 - \vec{R}_2|^{-9}$ at large separations between the two clusters. The contributions from other terms in the recursive sum die off like $|\vec{R}_1 - \vec{R}_2|^{-18}$ or faster in this limit.

$\Psi^{\rm new}$ may also be expressed as a summation over all trees that join the N particles, the product of two-body wave functions associated with each bifurcation of the tree, weighed by the coefficient $\prod_{bifurcation} C_{k_i,k_j}$.

Note that the asymptotic wave function Ψ_0 is not normalizable, obviously, since it is homogeneous under the simultaneous rescaling of all $\vec{r}_a$. We don't have an a priori argument to fix the coefficients C_{k_1,k_2}. It is perhaps tempting to suggest that $C_{k_1,k_2} = 1$ for all k_1, k_2, but this need not be the case. Even though the full two-body wave function has a natural normalization, $\Psi^{(2)}_{k_1,k_2}$ only captures its tail at large distances.

[6]This is easily seen from the simple identity under the change of variables

$$\begin{aligned} r^- &= \frac{r_1 - r_2}{\sqrt{k_1^{-1} + k_2^{-1}}}, \qquad r^+ = \frac{k_1 r_1 + k_2 r_2}{k_1 + k_2}, \\ \theta^- &= \frac{\widehat{\theta}_1 - \widehat{\theta}_2}{\sqrt{k_1^{-1} + k_2^{-1}}}, \qquad \theta^+ = \frac{k_1\widehat{\theta}_1 + k_2\widehat{\theta}_2}{k_1 + k_2}, \end{aligned} \tag{3.14}$$

that

$$\widehat{\theta}_1 \frac{\partial}{\partial r_1} + \widehat{\theta}_2 \frac{\partial}{\partial r_2} = \theta^- \frac{\partial}{\partial r^-} + \theta^+ \frac{\partial}{\partial r^+}. \tag{3.15}$$

The normalization factors are needed in order to preserve the desired normalization of the anti-commutators of $\widehat{\theta}$'s.

This proposal would easily answer the question of the overall scaling exponent in r. Ψ^{new} scales like $r^{-9(N-1)}$, and therefore

$$\kappa = -\frac{3}{2}N(N-1) + 9(N-1). \tag{3.17}$$

The power of convergence in the integration of the squared wave function at large r is then $r^{-9(N-1)}$. This is different from the previous proposal of [19] in the SU(3) case, for instance. The ansatz of [19] is constructed by taking an $\vec{r}_a$-independent SO(9) singlet fermion wave function, multiplied by the scalar harmonic function $r^{-9(N-1)+2}$, and then acted on by all 16 free supercharges Q_α^{new}. The resulting wave function falls off faster than our proposal by a factor of r^{-14} at large distances.

4 Going to higher orders

4.1 The general structure

Now that we have found a solution for Ψ_0 that obeys

$$P_0 Q_\alpha^1 \Psi_0 = 0, \tag{4.1}$$

we can then determine Ψ_1 as

$$\Psi_1 = \frac{1}{16H^0} Q_\alpha^0 Q_\alpha^1 \Psi_0 + \mathcal{K}_1, \tag{4.2}$$

where $-16H^0 = Q_\alpha^0 Q_\alpha^0$ (this comes from $\{Q_\alpha^0, Q_\beta^0\} = -2H^0\delta_{\alpha\beta} + \Gamma^k_{\alpha\beta}\mathcal{M}_k^0$), and $\mathcal{K}_1$ is a yet to be determined wave function in the kernel of H^0 (or of the Q_α^0's). It follows from the Jacobi identity on the Q_α's expanded to first order that (4.2) indeed solves the equation $Q_\beta^0\Psi_1 + Q_\beta^1\Psi_0 = 0$.

The next equation in the $r^{-\frac{3}{2}}$ expansion is

$$Q_\alpha^0\Psi_2 + Q_\alpha^1\Psi_1 + Q_\alpha^2\Psi_0 = 0. \tag{4.3}$$

Not knowing Ψ_2, we can again project by P_0, and consider

$$P_0 Q_\alpha^1 \Psi_1 + P_0 Q_\alpha^2 \Psi_0 = 0. \tag{4.4}$$

This may be expressed as an equation for $\mathcal{K}_1$,

$$P_0 Q_\alpha^1 \mathcal{K}_1 = -P_0\left(Q_\alpha^1 \frac{1}{16H^0} Q_\beta^0 Q_\beta^1 + Q_\alpha^2\right)\Psi_0. \tag{4.5}$$

The situation here is similar to the equations for Ψ_0. We could demand $\mathcal{K}_1$ to be a Cartan wave function tensored with $|\psi_0(\hat{r})\rangle$ (the unique ground state of H^0 in the (y,Θ) sector), and then try to solve a Dirac-like equation for free superparticles, but now with a source term.

In fact, the r.h.s. of (4.5) vanishes. This can be seen by inspecting the general structure of the r.h.s. of (4.5). $Q_\beta^0 Q_\beta^1 \Psi_0$ is a linear combination of states in the (y,Θ) sector that

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has H^0 eigenvalues $\frac{1}{2}|r_{ab}|$, $|r_{ab}|$, or $\frac{3}{2}|r_{ab}|$. It is straightforward to compute $(H^0)^{-1}Q^0_\beta Q^1_\beta\Psi_0$ explicitly, which we defer to the next subsection. When we act on it further with $P_0Q^1_\alpha$, only the (y,Θ)-sector lowering operators in Q^1_α contribute. In the end, we can write $P_0Q^1_\alpha(H^0)^{-1}Q^0_\beta Q^1_\beta\Psi_0$ in a way such that no $\vec{r}_a$-derivatives are taken on Ψ_0. Now Q^0_α changes the total level in the y-sector by an odd amount, while Q^1_α contains only terms that change the total y-level by an even amount. Thus $Q^1_\alpha(H^0)^{-1}Q^0_\beta Q^1_\beta\Psi_0$ must be excited in the y-sector and is annihilated by P_0.

As for the term $P_0Q^2_\alpha\Psi_0$ on the r.h.s. of (4.5), once again we need only consider the terms in Q^2_α that leave the (y,Θ) sector in its ground state. It is not hard to see that Q^2_α has the schematic form $\theta y\partial_r + \theta y^2\partial_y + \Theta y\partial_y + y\Theta^3 + y\theta\Theta^2 + y^3\partial_y\Theta$. The last term comes from expanding $\partial q^j_{ab}/\partial X^i$ to one order higher than what is computed explicitly in appendix A. We don't need its explicit form nonetheless. None of these terms could keep both y and Θ sectors in their ground states. We conclude that $P_0Q^2_\alpha\Psi_0 = 0$.

So in the end $\mathcal{K}_1$ obeys exactly the same equations as that of Ψ_0, and can be set to zero.[7]

4.2 Solving for Ψ_1

The next-to-leading order asymptotic wave function Ψ_1 is thus given by $\frac{1}{16}(H^0)^{-1}Q^0_\alpha Q^1_\alpha\Psi_0$. We can put $Q^1_\alpha\Psi_0 = (1-P_0)Q^1_\alpha\Psi_0$ into the form

$$\begin{aligned}
Q^1_\alpha\Psi_0 = \sum_{a\neq b}\Bigg[& \frac{\widehat{r}^j_{ab}}{4|r_{ab}|}(\Theta^+_{ab}\Gamma^{ij}\Theta^+_{ba})\Gamma^i_{\alpha\beta}\theta_{\beta a} \\
& + \frac{\widehat{r}^i_{ab}}{2|r_{ab}|}\left(y^j_{ab}\frac{\partial}{\partial y^j_{ab}} + 4\right)\Gamma^i_{\alpha\beta}(\theta_{\beta a}-\theta_{\beta b}) + \sum_{c\neq a,b}\frac{y^i_{ac}y^j_{cb}}{|r_{ac}|^{\frac{1}{2}}|r_{bc}|^{\frac{1}{2}}}\Gamma^{ij}_{\alpha\beta}(\Theta^+_\beta)_{ba} \\
& + \frac{y^i_{ab}y^j_{ba}}{2|r_{ab}|}\Gamma^{ij}_{\alpha\beta}(\theta_{\beta a}-\theta_{\beta b}) - \frac{\widehat{r}^i_{ab}}{|r_{ab}|}(1-P_0)\Gamma^i_{\alpha\beta}(\Theta_\beta)_{ba}M_{ab} \\
& - \sum_{c\neq a,b}\left(\frac{|r_{bc}|^{\frac{1}{2}}}{|r_{ac}|^{\frac{1}{2}}}y^k_{ca}\Pi^{kj}_{cb}\frac{\partial}{\partial y^j_{cb}} - \frac{|r_{ac}|^{\frac{1}{2}}}{|r_{bc}|^{\frac{1}{2}}}y^k_{bc}\Pi^{kj}_{ac}\frac{\partial}{\partial y^j_{ac}}\right)\frac{\widehat{r}^i_{ab}}{|r_{ab}|}\Gamma^i_{\alpha\beta}(\Theta^+_\beta)_{ab}\Bigg]\Psi_0.
\end{aligned} \tag{4.6}$$

It is straightforward though tedious to compute $Q^0_\alpha Q^1_\alpha\Psi_0$. By inspecting the excitation levels in the (y,Θ)-sector, we can easily act $(H^0)^{-1}$ on it and obtain, after some simplification,

$$\begin{aligned}
\Psi_1 = -\frac{5}{8}\sum_{a\neq b}\frac{1}{|r_{ab}|^{\frac{3}{2}}}(\Theta^+_{ba}\not{y}_{ab}(\theta_a-\theta_b))\Psi_0 + \sum_{a\neq b}\sum_{c\neq a,b}\frac{1}{|r_{ab}|+|r_{ac}|+|r_{bc}|}\Bigg[\frac{15}{8}\frac{(\Theta^+_{bc}\not{y}_{ca}\Theta^+_{ab})}{|r_{ac}|^{\frac{1}{2}}} \\
+ \frac{1}{16}\left(\frac{1}{|r_{bc}|}-\frac{1}{|r_{ab}|}\right)\frac{(\vec{r}_{ab}\cdot\vec{y}_{ca})(\Theta^+_{ab}\Theta^+_{bc})}{|r_{ac}|^{\frac{1}{2}}} - 2\frac{(\vec{r}_{cb}\cdot\vec{y}_{ac})(\vec{y}_{cb}\cdot\vec{y}_{ba})}{|r_{ab}|^{\frac{1}{2}}|r_{ac}|^{\frac{1}{2}}|r_{bc}|^{\frac{1}{2}}}\Bigg]\Psi_0.
\end{aligned} \tag{4.7}$$

[7]More precisely, it can be absorbed into Ψ_0, which isn't a priori homogeneous. Though our proposal for Ψ_0 is homogeneous with respect to the simultaneous rescaling of all $\vec{r}_a$, in principle there could be corrections of subleading power in r, for instance the type of solution considered in [19].

4.3 Higher orders in the $r^{-\frac{3}{2}}$ expansion

While the first order correction Ψ_1 is determined algebraically from Ψ_0, this is a priori not the case at higher orders. For instance, in order to solve for Ψ_2, we need to consider the following two equations. The first one is

$$\begin{aligned} &Q^0_\alpha \Psi_2 + Q^1_\alpha \Psi_1 + Q^2_\alpha \Psi_0 = 0 \\ &\qquad \Rightarrow \quad \Psi_2 = \frac{1}{16H^0}\left(Q^0_\alpha Q^1_\alpha \Psi_1 + Q^0_\alpha Q^2_\alpha \Psi_0\right) + \mathcal{K}_2, \end{aligned} \tag{4.8}$$

where $\mathcal{K}_2$ obeys $Q^0_\alpha \mathcal{K}_2 = 0$. Here we are separating Ψ_2 into a piece that involves excited states in the off-diagonal (y, Θ) sector, and a piece $\mathcal{K}_2$ that involves only the ground state in the off-diagonal sector. The second equation we need to consider is

$$\begin{aligned} &Q^0_\alpha \Psi_3 + Q^1_\alpha \Psi_2 + Q^2_\alpha \Psi_1 + Q^3_\alpha \Psi_0 = 0 \\ &\qquad \Rightarrow \quad P_0 Q^1_\alpha \Psi_2 + P_0 Q^2_\alpha \Psi_1 + P_0 Q^3_\alpha \Psi_0 = 0. \end{aligned} \tag{4.9}$$

$\mathcal{K}_2$ can now be determined from

$$P_0 Q^1_\alpha \mathcal{K}_2 = -P_0 Q^1_\alpha \frac{1}{16H^0}\left(Q^0_\beta Q^1_\beta \Psi_1 + Q^0_\beta Q^2_\beta \Psi_0\right) - P_0\left(Q^2_\alpha \Psi_1 + Q^3_\alpha \Psi_0\right). \tag{4.10}$$

The r.h.s. of (4.10) appears to be nontrivial, and now we need to solve a Dirac-like equation for the wave function of $N-1$ superparticles with a source. Note that while we demand $\mathcal{K}_2$ to fall off like r^{-3} faster than Ψ_0 at large distances, $\mathcal{K}_2$ is of course not normalizable and such a solution generally exists.

5 Discussion

The observation that the leading asymptotic ground state wave function Ψ_0 is governed by supercharges for free superparticles has been pointed out previously in [6, 20]. This is perhaps obvious already from the perspective of effective field theory, though in the effective field theory approach it may not have been clear how to construct a systematic asymptotic expansion. In the well known perturbative computation of scattering at large impact parameters [7–10], beyond one-loop order one encounters infrared divergences, which have been mostly ignored.[8]

The condition $P_0 Q^1_\alpha \Psi_0 = 0$ does not uniquely determine Ψ_0, however. If we had started with the wrong ansatz for Ψ_0, in principle there could be obstructions in solving the recursive equations for the asymptotic expansion at higher orders, or it could also be that the inconsistency is not visible at the level of the asymptotic expansion, but rather may be seen only after summing up the entire series in some way. It would also be tricky to guess a solution that is consistent with all symmetries of the problem. Our proposal is the simplest one that is consistent with all symmetries of the problem and the expected factorization property when the eigenvalues/D0-branes are divided into clusters on the

[8]The point is that an IR divergence due to propagators at near zero frequency would have been cut off non-perturbatively, essentially due to the normalizability of the ground state wave function itself.

Coulomb branch. There could be corrections to this proposal already at leading order, namely in Ψ_0 itself, but it does not seem easy to construct another solution with the desired symmetry properties. [19] suggested a different form of Ψ_0, which in principle could enter as a correction to our proposal, but it has a different scaling in r and dies off faster at large distances. Even if such corrections are present in Ψ_0, it would not be possible to determine it based on the asymptotic expansion alone, as it would render Ψ_0 inhomogeneous under the overall scaling of r.[9]

The structure of the proposed Ψ_0 may provide some hints on the semi-classical nature of the bulk spacetime, at distances $r \ll N^{\frac{1}{3}}$ (in M-theory Planck units) from the origin. While $N^{\frac{1}{7}} \ll r \ll N^{\frac{1}{3}}$ is the weakly curved type IIA string theory regime, and $1 \ll r \ll N^{\frac{1}{7}}$ is the weakly curved 11-dimensional M-theory regime, both lie in the strong 't Hooft coupling domain of the matrix quantum mechanics, and within the expected spatial spread of the ground state wave function. It has been mysterious why a probe eigenvalue that comes in from the asymptotic region (corresponding to a highly stringy regime in the bulk IIA picture) and interact with the ground state wave function of the remaining, say, $\mathrm{SU}(N-1)$ part of the matrix quantum mechanics, would behave like a semi-classical particle governed by the Born-Infeld action in the bulk geometry. In our proposal for Ψ_0, which takes the form of a sum over products of two-body wave functions, one could hope the answer to be already approximately valid for r_{ab}'s that are parameterically large compared to 1 (or the scale set by g_{YM} in the QM), as opposed to $N^{\frac{1}{3}}$ (or $N^{\frac{1}{7}}$ for that matter), though this is not at all obvious. Based on this form of Ψ_0 and its subleading corrections, perhaps a more reliable computation can be performed for the scattering of eigenvalues/D0-branes off the ground state wave function at impact parameters less than $N^{\frac{1}{3}}$, extending the results of [8, 10] to the seemingly non-perturbative regime.[10]

Eventually, we would like to count and understand the structure of long-lived metastable states of the matrix quantum mechanics at large N, which are supposed to be dual to microstates of the black hole in the bulk, either in the weakly coupled IIA regime or in the M-theory regime. Despite some numerical success based on Monte Carlo study of the thermal free energy [25–30], there is little analytic understanding of the structure of such nonzero energy states. Some encouraging results are obtained using truncated Schwinger-Dyson equations and extrapolating to the low temperature regime [21–24]. We hope a more precise understanding of the ground state wave function will provide insight on how to construct the general metastable excited states and ultimately a way to study Lorentzian observables relevant to the physics of black holes.

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[9] Note in particular that r^{-14} is not an integer power of $r^{-\frac{3}{2}}$.

[10] Even in the eikonal regime, taking into account the infrared modification of the propagators due to the ground state wave function already pollutes the structure of an analytic series expansion of the effective potential in v^2/r^4 and in $1/r^3$. This starts at v^8 order where the r dependence is no longer fixed by supersymmetry.

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A Change of variables in the asymptotic expansion

Due to the constraint

$$\vec{q}_{ab} \cdot \vec{r}_{ab} = 0, \tag{A.1}$$

we are only allowed to use

$$(\delta^{ij} - \widehat{r}^i_{ab}\widehat{r}^j_{ab})\frac{\partial}{\partial q^j_{ab}}, \tag{A.2}$$

where $\widehat{r}_{ab} \equiv \vec{r}_{ab}/|\vec{r}_{ab}|$.

Writing $X^i = U^{-1}(r^i_a E_a + q^i_{ab} T_{ab})U$, we have

$$UdX^iU^{-1} = [r^i_a E_a + q^i_{ab}T_{ab}, dUU^{-1}] + dr^i_a E_a + dq^i_{ab}T_{ab}. \tag{A.3}$$

Taking the trace of both sides multiplied by $(r^i_c - r^i_d)T_{dc} = r^i_{cd}T_{dc}$ (not summing over c, d), we have

$$\begin{aligned} r^i_{cd}\left(UdX^iU^{-1}\right)_{cd} &= r^i_{cd}\mathrm{Tr}\left(T_{dc}[r^i_a E_a + q^i_{ab}T_{ab}, dUU^{-1}]\right) \\ &= |r_{cd}|^2(dUU^{-1})_{cd} + r^i_{cd}\left[q^i_{cb}(dUU^{-1})_{bd} - (dUU^{-1})_{cb}q^i_{bd}\right] \end{aligned} \tag{A.4}$$

The second term on the r.h.s. is down by a factor of $r^{-\frac{3}{2}}$ compared to the first term on the r.h.s. , once we make the change of variables $q^i_{ab} = |r_{ab}|^{-\frac{1}{2}}y^i_{ab}$ and maintain $y \sim \mathcal{O}(1)$. We can then express

$$\begin{aligned} dUU^{-1} = &\sum_{c\neq d}\frac{r^i_{cd}}{r^2_{cd}}(UdX^iU^{-1})_{cd}T_{cd} \\ &- \sum_{c\neq d}\frac{r^i_{cd}}{r^2_{cd}}\sum_{b\neq c,d}\left[\frac{q^i_{cb}r^j_{bd}}{r^2_{bd}}(UdX^jU^{-1})_{bd} - \frac{q^i_{bd}r^j_{cb}}{r^2_{cb}}(UdX^jU^{-1})_{cb}\right]T_{cd} + \mathcal{O}(r^{-4}) \end{aligned} \tag{A.5}$$

Note that the diagonal components of dUU^{-1} are unconstrained and are simply set to zero. Plugging this back into (A.3), we have

$$\begin{aligned} UdX^iU^{-1} = &\frac{r^i_{ab}r^j_{ab}}{r^2_{ab}}(UdX^iU^{-1})_{ab}T_{ab} + dr^i_a E_a + dq^i_{ab}T_{ab} + \frac{q^i_{ab}r^j_{ba}}{|r_{ab}|^2}(UdX^jU^{-1})_{ba}(E_a - E_b) \\ &+ \sum_{c\neq a,b}\Pi^{ij}_{ab}\left[\frac{q^j_{ac}r^k_{cb}}{r^2_{cb}}(UdX^kU^{-1})_{cb} - \frac{q^j_{cb}r^k_{ac}}{r^2_{ac}}(UdX^kU^{-1})_{ac}\right]T_{ab} + \mathcal{O}(r^{-3}). \end{aligned} \tag{A.6}$$

From this, we then solve for dr_a^i and dq_{ab}^i in terms of dX^i up to $\mathcal{O}(r^{-3})$ terms.

$$\begin{aligned}
dr_a^i &= \mathrm{Tr}(E_a U dX^i U^{-1}) + \sum_{b\neq a} \frac{r_{ab}^j}{|r_{ab}|^{\frac{5}{2}}} \left[y_{ab}^i (U dX^j U^{-1})_{ba} + y_{ba}^i (U dX^j U^{-1})_{ab}\right] + \mathcal{O}(r^{-3}), \\
dq_{ab}^i &= \Pi_{ab}^{ij} (U dX^i U^{-1})_{ab} \\
&\quad - \sum_{c\neq a,b} \Pi_{ab}^{ij} \left[\frac{y_{ac}^j}{|r_{ac}|^{\frac{1}{2}}} \frac{\hat{r}_{cb}^k}{|r_{cb}|} (U dX^k U^{-1})_{cb} - \frac{y_{cb}^j}{|r_{cb}|^{\frac{1}{2}}} \frac{\hat{r}_{ac}^k}{|r_{ac}|} (U dX^k U^{-1})_{ac}\right] + \mathcal{O}(r^{-3}).
\end{aligned} \tag{A.7}$$

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