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The landscape of G-structures in eight-manifold compactifications of M-theory

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Abstract: We consider spaces of "virtual" constrained generalized Killing spinors, i.e. spaces of Majorana spinors which correspond to "off-shell" s-extended supersymmetry in compactifications of eleven-dimensional supergravity based on eight-manifolds M. Such spaces naturally induce two stratifications of M, called the chirality and stabilizer stratification. For the case $s = 2$, we describe the former using the canonical Whitney stratification of a three-dimensional semi-algebraic set $\mathcal{R}$. We also show that the stabilizer stratification coincides with the rank stratification of a cosmooth generalized distribution $\mathcal{D}_0$ and describe it explicitly using the Whitney stratification of a four-dimensional semi-algebraic set $\mathfrak{P}$. The stabilizer groups along the strata are isomorphic with SU(2), SU(3), G_2 or SU(4), where SU(2) corresponds to the open stratum, which is generically non-empty. We also determine the rank stratification of a larger generalized distribution $\mathcal{D}$ which turns out to be integrable in the case of compactifications down to AdS_3.

Keywords: Differential and Algebraic Geometry, Flux compactifications, M-Theory

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Introduction

General compactifications of M-theory on eight-manifolds provide a rich class of geometries which are of physical interest due to their relation to F-theory [1–3]. They can serve to test ideas such as exceptional generalized geometry [4–10], since eight is the first dimension for which the problem of "dual gravitons" [11–16] appears. Given these aspects, it is rather surprising that current understanding of such backgrounds is quite limited. The notable exception is the class of compactifications down to 3-dimensional Minkowski space, which were studied intensively following the seminal work of [17] (for the $\mathcal{N} = 1$ case) and [18] (for the $\mathcal{N} = 2$ case). Such backgrounds are obtained by constraining the internal part of the supersymmetry generators to be Majorana-Weyl rather than merely Majorana. As expected from no-go theorems (first used within this setting in [19]), such Minkowski compactifications cannot support a flux at the classical level. However, they can support *small* fluxes at the quantum level, which are suppressed by inverse powers of the size of the compactification manifold. Since such fluxes are difficult to control beyond leading order [20, 21], a natural idea is to consider instead compactifications down to AdS_3 spaces.

As pointed out in [19], compactifications of M-theory down to AdS_3 *do* support classical fluxes, which are therefore not suppressed. This happens because the internal parts of the supersymmetry generators are no longer required to be Majorana-Weyl. This seemingly innocuous extension leads to a surprisingly intricate geometry, as already apparent in the case of $\mathcal{N} = 1$ unbroken supersymmetry [19, 22], which can be described using the theory of singular foliations [23, 24]. By comparison, little is known[1] about $\mathcal{N} = 2$ compactifications down to AdS_3. In this paper, we consider certain aspects of the geometry of $\mathcal{N} = 2$ eight-dimensional backgrounds by working directly in eight dimensions. Namely, we solve the question of classifying the stratified reductions of structure group which arise on the internal eight-manifold M, showing that the full picture is considerably richer than has been previously presumed. Pointwise positions of internal supersymmetry generators as well as their stabilizer groups are described by stratifications of the internal space M: the first by the *chirality stratification* and the second by the *stabilizer stratification*. Unlike the case $\mathcal{N} = 1$, the two stratifications need not agree. We find that these stratifications can be described explicitly using the preimages through certain maps $b : M \to \mathbb{R}^3$ and $B : M \to \mathbb{R}^4$ of the connected refinements of the canonical Whitney stratifications [28, 29] of semi-algebraic [30–32] subsets $\mathcal{R} \subset \mathbb{R}^3$ and $\mathfrak{P} \subset \mathbb{R}^4$, where $\mathcal{R}$ is obtained from $\mathfrak{P}$ by projection on the three-dimensional space corresponding to the first three coordinates of $\mathbb{R}^4$. The maps b and B are constructed from bilinears in the internal supersymmetry generators, while the semi-algebraic set $\mathfrak{P}$ can be described explicitly using algebraic constraints implied by the Fierz identities. This gives a geometric picture of such backgrounds which shows how they can be approached using the theory of stratified manifolds. We classify the stabilizer groups for each stratum, thus giving a complete description of the "stratified G-structure" which arises in such backgrounds. In particular, we find that a generic eight-manifold M of this type contains an open stratum on which the structure group reduces to $\mathrm{SU}(2)$. In a certain sense, this stratum is the "largest", but it was not considered previously. We also classify the amount of supersymmetry preserved by an M2-brane transverse to M along each stratum.

Since the classification results mentioned above are independent of the precise form of the supersymmetry equations, they hold more generally than the case of compactifications down to AdS_3. To highlight this, we develop the formalism required to describe the "topological part" of the conditions for supersymmetry, characterizing those finite-dimensional spaces of globally-defined Majorana spinors which can be spanned by solutions of constrained generalized Killing equations (so-called "virtual CGK spaces"). We show that such spaces must obey a local non-degeneracy condition which puts them in bijection with trivial sub-bundles of the bundle of Majorana spinors, endowed with a trivial flat connection. This formulation clarifies some aspects of the mathematical description of so-called "off-shell supersymmetric" backgrounds.

The paper is organized as follows. Section 1 gives the general description of virtual CGK spaces $\mathcal{K}$ and of the chirality and stabilizer stratifications which they induce on M and shows how this framework arises in the case of compactifications down to AdS_3. We also treat the case $\mathcal{N} = 1$ as a warm-up, pointing out its differences with the case $\mathcal{N} = 2$. The rest of the paper is devoted to the detailed study of the latter case. Section 2 discusses the scalar and one-form bilinears which can be constructed using a basis of $\mathcal{K}$ when $\dim \mathcal{K} = 2$ and introduces two cosmooth generalized distributions $\mathcal{D}$ and $\mathcal{D}_0$ (where $\mathcal{D}_0 \subset \mathcal{D}$) which

[1] Such $\mathcal{N} = 2$ backgrounds were considered in [25] using a nine-dimensional formalism and were also discussed in [26] with similar methods, but without carefully studying the corresponding geometry of the eight-manifold. Certain $\mathcal{N} = 1$ compactifications down to three-dimensional Minkowski space but with torsion-full $\mathrm{SU}(4)$ structure were studied in [27, section 3].

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are naturally associated to the one-form bilinears. The rank stratification of $\mathcal{D}_0$ turns out to coincide with the stabilizer stratification, thus providing a way to identify the latter. In the case of compactifications to AdS_3, the distribution $\mathcal{D}_0$ need not be integrable, but one can show that the larger distribution $\mathcal{D}$ integrates to a singular foliation in the sense of Haefliger (topologically, this is a Haefliger structure [33] which may be non-regular). Section 3 discusses the chirality stratification, giving its explicit description in terms of a convex three-dimensional semi-algebraic body $\mathcal{R}$ and a smooth map $b : M \to \mathbb{R}^3$ whose image is contained in $\mathcal{R}$. Section 4 discusses the algebraic constraints on zero- and one-form spinor bilinears which are induced by Fierz identities, showing how they can be described using a four-dimensional semi-algebraic set $\mathfrak{P}$. In the same section, we discuss the geometry of $\mathfrak{P}$ and of its boundary, its canonical Whitney stratification and the preimage of $\partial\mathcal{R}$ inside $\partial\mathfrak{P}$ through the map which projects on the first three coordinates. Section 5 shows that the rank stratifications of $\mathcal{D}$ and $\mathcal{D}_0$ (where the latter coincides with the stabilizer stratification) are different coarsenings of the B-preimage of the connected refinement of the canonical Whitney stratification of $\mathfrak{P}$, where B is a map from M to $\mathbb{R}^4$ with image contained in $\mathfrak{P}$. The two coarsenings are given explicitly, leading to the classification of stabilizer groups. In the same section, we show how the chirality stratification fits into this picture, while in section 6 we conclude. The appendices contain various proofs as well as other technical details. The main results of this paper are Theorems 1, 2, 3 and 4, which can be found in subsections 3.6, 5.1 and 5.2. For ease of reference, various results are summarized in tables and figures. The notations and conventions used in the paper are explained in appendix A.

1 Virtual CGK spaces

The eight-manifold M can be used in various ways to construct a supersymmetric background $\mathbf{M}$ of eleven-dimensional supergravity [34], for example by taking $\mathbf{M}$ to be foliated in eight-manifolds with typical leaf M or by taking it to be a (warped) product between M and some non-compact 3-manifold N endowed with a metric of Minkowski signature. In such backgrounds, supersymmetry generators can be constructed starting from globally-defined solutions $\xi \in \Gamma(M, S)$ of equations of the type:

$$\mathbb{D}\xi = 0\,, \quad Q\xi = 0\,, \tag{1.1}$$

which we shall call *constrained generalized Killing (CGK) spinor equations*. Here $\mathbb{D} : \Gamma(M, S) \to \Omega^1(M, S)$ is a connection on the bundle S of Majorana spinors over M and $Q \in \Gamma(M, \mathrm{End}(S))$ is a globally-defined endomorphism of S. Such equations encode the condition that a supersymmetry transformation whose generator has ξ as its "internal part" preserves the background. The explicit forms of $\mathbb{D}$ and Q depend on the precise background under consideration and will generally involve the metric of M as well as various differential forms defined on M. We let $\mathcal{K}(\mathbb{D}, Q)$ denote the (finite-dimensional) space of solutions to (1.1).

Definition. A finite-dimensional subspace $\mathcal{K}$ of $\Gamma(M, S)$ is called a *virtual CGK space* if there exists a connection $\mathbb{D}$ on S and a globally-defined endomorphism $Q \in \Gamma(M, \mathrm{End}(S))$ such that $\mathcal{K} = \mathcal{K}(\mathbb{D}, Q)$.

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Definition. A virtual CGK space $\mathcal{K}$ is called *$\mathscr{B}$-compatible* if there exists a $\mathscr{B}$-compatible connection $\mathbb{D}$ on S and a global endomorphism $Q \in \Gamma(M, \mathrm{End}(S))$ such that $\mathcal{K} = \mathcal{K}(\mathbb{D}, Q)$.

First remarks. The physics literature of flux compactifications sometimes makes a distinction between:[2]

(a) The *topological condition* for supersymmetry, namely that the given background admits a number s of independent and globally-defined spinors $\xi_1, \ldots, \xi_s$ of the desired type;

(b) The *algebro-differential conditions* for supersymmetry, namely that the spinors at (a) satisfy an equation of the form (1.1).

To clarify this, let $\xi_1, \ldots, \xi_s \in \Gamma(M, S)$ be s globally defined Majorana spinors on M. Recall that $\Gamma(M, S)$ has a canonical structure of module over $\mathcal{C}^\infty(M, \mathbb{R})$. Since the latter is an $\mathbb{R}$-algebra, this also endows $\Gamma(M, S)$ with a structure of (infinite-dimensional) vector space over $\mathbb{R}$.

Definition. The globally-defined spinors $\xi_1, \ldots, \xi_s$ are called *weakly linearly independent* if they are linearly independent over the field $\mathbb{R}$ of real numbers, i.e. linearly independent as elements of the infinite-dimensional $\mathbb{R}$-vector space $\Gamma(M, S)$ of smooth globally-defined sections of S. They are called *strongly linearly independent* if they are linearly independent over $\mathcal{C}^\infty(M, \mathbb{R})$, i.e. linearly independent as elements of the $\mathcal{C}^\infty(M, \mathbb{R})$-module $\Gamma(M, S)$.

Weak linear independence of $\xi_1, \ldots, \xi_s$ means that the relation:

$$c_1 \xi_1(p) + \ldots + c_s \xi_s(p) = 0 \quad \forall p \in M\,,$$

where $c_1, \ldots, c_s$ are real *constants*, implies $c_1 = \ldots = c_s = 0$. Strong linear independence means that the relation:

$$c_1(p) \xi_1(p) + \ldots + c_s(p) \xi_s(p) = 0 \quad \forall p \in M\,,$$

where $c_1, \ldots, c_s \in \mathcal{C}^\infty(M, \mathbb{R})$ are smooth real-valued *functions* defined on M, implies $c_1(p) = \ldots = c_s(p) = 0$ for all $p \in M$. Since constant real-valued functions are smooth, it is clear that strong linear independence implies weak linear independence. It is also clear that strong linear independence amounts to the condition that $\xi_1(p), \ldots, \xi_s(p)$ are linearly independent inside the vector space S_p for all $p \in M$. As we show below, condition (b) implies that the independence condition at (a) should be understood as *strong* linear independence.

The supersymmetry equations (1.1) do not specify precise choices of globally-defined spinors but only a subspace $\mathcal{K}$ of $\Gamma(M, S)$, namely the space $\mathcal{K}(\mathbb{D}, Q)$ of all globally-defined solutions of (1.1). Hence we need a formulation of the strong linear independence condition which does not rely on choosing a basis for $\mathcal{K}$. Since this is a pointwise condition, it

[2] The topological conditions are sometimes called "algebraic conditions" [35, 36] while the supersymmetry conditions are called "differential conditions", but this terminology is inaccurate for our purpose. In this paper, we are interested in supersymmetry conditions for the "internal part" of spinors, hence the equations on the internal manifold M will generally have both a differential and an algebraic part as in (1.1). On the other hand, existence of a certain number of globally-defined independent spinors is clearly a topological, rather than algebraic, condition.

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can be formulated in a frame-free manner using the evaluation map. This leads to the notion of locally non-degenerate subspaces of $\Gamma(M, S)$. As we show below, a subspace $\mathcal{K}$ of $\Gamma(M, S)$ is a virtual CGK space iff it obeys this non-degeneracy condition. When $\mathcal{K}$ is $\mathscr{B}$-compatible, the freedom to change an orthonormal basis of $\mathcal{K}$ is related to the R-symmetry of supersymmetric effective actions built using such backgrounds.

Remark. The fact that some subspace $\mathcal{K} \subset \Gamma(M, S)$ is a virtual CGK space does *not* mean that $\mathcal{K}$ consists of internal parts of supersymmetry generators for any specific background of eleven-dimensional supergravity built on M. To know whether this is the case, one has to study which pairs $(\mathbb{D}, Q)$ can arise in a given class of backgrounds. The notion of virtual CGK space encodes the "topological part" of the supersymmetry conditions, which is much weaker than the full supersymmetry conditions in a given background or class of backgrounds.

1.1 Locally non-degenerate subspaces of $\boldsymbol{\Gamma(M, S)}$

For any $p \in M$, let $\mathrm{ev}_p : \Gamma(M, S) \to S_p$ be the evaluation map at p:

$$\mathrm{ev}_p(\xi) \stackrel{\text{def.}}{=} \xi(p)\,, \quad \forall \xi \in \Gamma(M, S)\,.$$

Notice that ev_p is $\mathbb{R}$-linear and surjective. Any subspace $\mathcal{K} \subset \Gamma(M, S)$ induces a generalized linear sub-bundle $\mathrm{ev}_*(\mathcal{K}) \stackrel{\text{def.}}{=} \sqcup_{p\in M}\mathrm{ev}_p(\mathcal{K})$ of S, which is smooth in the sense of [37].

Definition. A subspace $\mathcal{K} \subset \Gamma(M, S)$ is *locally non-degenerate* if the restriction $\mathrm{ev}_p|_{\mathcal{K}} : \mathcal{K} \to S_p$ is injective for all $p \in M$.

The local non-degeneracy condition means that any element $\xi \in \mathcal{K}$ is either the zero section of S or a section of S which does not vanish anywhere on M. A locally non-degenerate subspace $\mathcal{K}$ of $\Gamma(M, S)$ has finite dimension $s \stackrel{\text{def.}}{=} \dim \mathcal{K} \leq \mathrm{rk} S = 16$. In this case, it is easy to see that $\mathrm{ev}_*(\mathcal{K})$ is an ordinary sub-bundle of S which is topologically trivial, because any basis $\xi_1, \ldots, \xi_s$ of $\mathcal{K}$ obviously forms a frame of K. Let $\mathrm{Grn}_s(M, S)$ denote the set of locally non-degenerate s-dimensional subspaces of $\Gamma(M, S)$; notice that $\mathrm{Grn}_s(M, S)$ can be viewed as an infinite-dimensional manifold. Let $\mathrm{Trivf}_s(M, S)$ denote the set of pairs $(K, \mathbf{D})$, where K is a trivial (in the sense of globally trivializable) smooth rank s sub-bundle of S and $\mathbf{D}$ is a trivial flat connection on K.

Remark. Given a trivial rank s sub-bundle K of S and a point $p \in M$, trivial flat connections on K can be identified (using parallel transport) with bundle isomorphisms $\varphi_p : K \stackrel{\sim}{\to} M \times S_p$, so $\mathrm{Trivf}_s(M, S)$ can be identified with the set of all pairs (K, φ_p). Notice that this identification depends on the choice of $p \in M$ and hence it is natural only if we work with pointed manifolds (M, p). A natural description which does not require the choice of a base point is given below.

Proposition. There exists a natural bijection $\Phi_s : \mathrm{Grn}_s(M, S) \stackrel{\sim}{\longrightarrow} \mathrm{Trivf}_s(M, S)$, whose inverse is given by $\Phi_s^{-1}(K, \mathbf{D}) = \Gamma_{\text{flat}}(K, \mathbf{D})$, where:

$$\Gamma_{\text{flat}}(K, \mathbf{D}) \stackrel{\text{def.}}{=} \{\xi \in \Gamma(M, K) | \mathbf{D}\xi = 0\}$$

is the space of all $\mathbf{D}$-flat sections of K.

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Proof. Let $\Pi_1(M)$ be the first homotopy groupoid of M and $A(K)$ be the isomorphism groupoid of K (the groupoid whose objects are the points of M and whose Hom-set from p to q is the set of linear isomorphisms from K_p to K_q). The map which assigns the pair (p,q) to curves starting at p and ending at q induces a functor $E : \Pi_1(M) \to M \times M$, where $M \times M$ is the trivial groupoid whose objects are the points of M. Given $\mathcal{K} \in \mathrm{Grn}_s(M,S)$, the rank s bundle $K \stackrel{\text{def.}}{=} \mathrm{ev}_*(\mathcal{K})$ is trivial, as pointed out above. The corestriction:

$$e_p \stackrel{\text{def.}}{=} \mathrm{ev}_p|_{\mathcal{K}}^{K_p} : \mathcal{K} \to K_p \tag{1.2}$$

of $\mathrm{ev}_p|_{\mathcal{K}}$ to its image is bijective for all $p \in M$. Given $p,q \in M$, consider the bijection:

$$\mathbf{U}_{pq} \stackrel{\text{def.}}{=} e_q \circ e_p^{-1} : K_p \xrightarrow{\sim} K_q\,. \tag{1.3}$$

This satisfies:

$$\mathbf{U}_{qr} \circ \mathbf{U}_{pq} = \mathbf{U}_{pr} \quad \text{and} \quad \mathbf{U}_{pp} = \mathrm{id}_{K_p}\,, \quad \forall p,q,r \in M$$

and hence defines a functor $\mathbf{U} : M \times M \to A(K)$ whose image is a trivial sub-groupoid of $A(K)$ (the Hom-sets of the image being singleton sets). There exists a unique flat connection $\mathbf{D}$ on K whose holonomy functor $\mathrm{Hol}_{\mathbf{D}}$ (the functor which associates to every morphism of the groupoid $\Pi_1(M)$ the parallel transport of $\mathbf{D}$ along curves belonging to that homotopy class) coincides with the composition $\mathbf{U} \circ E : \Pi_1(M) \to A(K)$. This flat connection is trivial since the image of $\mathrm{Hol}_{\mathbf{D}} = \mathbf{U} \circ E$ (which coincides with the image of $\mathbf{U}$) is a trivial groupoid. This construction gives a natural map $\Phi_s : \mathrm{Grn}_s(M,S) \to \mathrm{Trivf}_s(M,S)$ given by $\Phi_s(\mathcal{K}) = (K, \mathbf{D})$. Relation (1.3) implies that any $\xi \in \mathcal{K}$ satisfies:

$$\xi(q) = \mathbf{U}_{pq}(\xi(p))\,, \quad \forall p,q \in M\,, \tag{1.4}$$

which implies $\mathbf{D}\xi = 0$. Hence $\mathcal{K}$ is contained in the space $\Gamma_{\text{flat}}(K,\mathbf{D})$. Since $\dim \Gamma_{\text{flat}}(K,\mathbf{D}) = \mathrm{rk}K = s = \dim \mathcal{K}$, we must have $\mathcal{K} = \Gamma_{\text{flat}}(K,\mathbf{D})$. This shows that $\mathcal{K}$ is uniquely determined by $(K,\mathbf{D})$ and hence that Φ_s is injective. Consider now a pair $(K,\mathbf{D}) \in \mathrm{Trivf}_s(M,S)$ and set $\mathcal{K} \stackrel{\text{def.}}{=} \Gamma_{\text{flat}}(K,\mathbf{D})$. We have $\dim \mathcal{K} = \mathrm{rk}K = s$. The map $\mathrm{ev}_p|_{\mathcal{K}}$ is injective with image equal to K_p. Thus $\mathcal{K}$ is locally non-degenerate and $K = \mathrm{ev}_*(\mathcal{K})$. Since $\mathbf{D}$ is a trivial flat connection, its parallel transport along curves from p to q depends only on p and q, being given by (1.3). Thus $(K,\mathbf{D}) = \Phi_s(\mathcal{K})$, which shows that Φ_s is surjective. □

1.2 $\mathscr{B}$-compatible locally non-degenerate subspaces of $\Gamma(M,S)$

Definition. A locally nondegenerate subspace $\mathcal{K} \subset \Gamma(M,S)$ is *$\mathscr{B}$-compatible* if the following condition is satisfied:

$$\mathscr{B}(\xi,\xi') = \text{constant on } M\,, \quad \forall \xi,\xi' \in \mathcal{K}\,. \tag{1.5}$$

Any $\mathscr{B}$-compatible locally nondegenerate subspace $\mathcal{K}$ is endowed with a Euclidean metric $\mathscr{B}_0 : \mathcal{K}\times\mathcal{K} \to \mathbb{R}$ which is defined through $\mathscr{B}(\xi,\xi') = \mathscr{B}_0(\xi,\xi')1_M$, where $1_M \in \mathcal{C}^\infty(M,\mathbb{R})$ is the constant function equal to one on M. For simplicity, we will not distinguish notationally between $\mathscr{B}_0$ and the $\mathcal{C}^\infty(M,\mathbb{R})$-valued bilinear form $\mathscr{B}|_{\mathcal{K}\otimes\mathcal{K}} = \mathscr{B}_0 1_M$. Condition (1.5) is an invariant way of saying that $\mathcal{K}$ admits a basis $\xi_1, \ldots, \xi_s$ having the property that the scalar products $\mathscr{B}_p(\xi_i(p), \xi_j(p))$ are independent of the point $p \in M$ for all $i, j = 1 \ldots s$. Using the Gram-Schmidt algorithm for $\mathscr{B}_0$, it is easy to see that this amounts to the condition that $\mathcal{K}$ admits a basis which is everywhere orthonormal in the sense $\mathscr{B}_p(\xi_i(p), \xi_j(p)) = \delta_{ij}$ for all $i, j = 1 \ldots s$ and all $p \in M$.

Let $\mathrm{Grn}_s(M, S, \mathscr{B})$ be the subset of $\mathrm{Grn}_s(M, S)$ consisting of $\mathscr{B}$-compatible locally nondegenerate subspaces of dimension s and $\mathrm{Trivf}_s(M, S, \mathscr{B})$ be the subset of $\mathrm{Trivf}_s(M, S)$ consisting of those pairs $(K, \mathbf{D}) \in \mathrm{Trivf}_s(M, S)$ for which $\mathbf{D}$ is a $\mathscr{B}$-compatible connection.

Corollary. Φ_s restricts to a bijection between $\mathrm{Grn}_s(M, S, \mathscr{B})$ and $\mathrm{Trivf}_s(M, S, \mathscr{B})$.

Proof. Let $\mathcal{K} \in \mathrm{Grn}_s(M, S)$ and $(K, \mathbf{D}) \stackrel{\text{def.}}{=} \Phi_s(\mathcal{K})$. Condition (1.5) is equivalent with:

$$\mathscr{B}_q \circ (e_q \otimes e_q) = \mathscr{B}_p \circ (e_p \otimes e_p) \, , \ \forall \ p, q \in M \, , \tag{1.6}$$

where the map e_p was defined in (1.2). Since $e_p : \mathcal{K} \to K_p$ is bijective for all p, the relation $e_q = \mathbf{U}_{pq} \circ e_p$ (which follows from (1.3)) shows that (1.6) is equivalent with the condition:

$$\mathscr{B}_q|_{K_q} \circ (\mathbf{U}_{pq} \otimes \mathbf{U}_{pq}) = \mathscr{B}_p|_{K_p} \, , \tag{1.7}$$

which amounts to the requirement that $\mathbf{U}_{pq}$ be an isometry from $(K_p, \mathscr{B}_p|_{K_p})$ to $(K_q, \mathscr{B}_q|_{K_q})$ for all $p, q \in M$. In turn, this is equivalent with the requirement that the trivial flat connection $\mathbf{D}$ be $\mathscr{B}$-compatible. □

Let $\mathcal{K} \in \mathrm{Grn}_s(M, S)$ and $(K, \mathbf{D}) \stackrel{\text{def.}}{=} \Phi_s(\mathcal{K})$. The following statement is obvious in view of the above:

Proposition. Let $\xi_1, \ldots, \xi_s \in \mathcal{K}$ and $\Xi \stackrel{\text{def.}}{=} (\xi_1, \ldots, \xi_s)$. Then:

1. Ξ is a basis of $\mathcal{K}$ iff it is a $\mathbf{D}$-flat global frame of K.
2. When $\mathcal{K}$ is $\mathscr{B}$-compatible, Ξ is an orthonormal basis of $\mathcal{K}$ iff it is an everywhere-orthonormal $\mathbf{D}$-flat global frame of K.

1.3 Relation to virtual CGK spaces

Let $\mathcal{K}(\mathbb{D}, Q)$ denote the space of solutions to (1.1) and $s \stackrel{\text{def.}}{=} \dim \mathcal{K}(\mathbb{D}, Q)$.

Proposition. $\mathcal{K}(\mathbb{D}, Q)$ is a locally non-degenerate subspace of $\Gamma(M, S)$.

Proof. For ease of notation, let $\mathcal{K} \stackrel{\text{def.}}{=} \mathcal{K}(\mathbb{D}, Q)$. Let $\mathcal{P}_{pq}(M)$ denote the set of curves in M starting at p and ending at q. For any $\gamma \in \mathcal{P}_{pq}(M)$, let:

$$U_{pq}(\gamma) : S_p \xrightarrow{\sim} S_q$$

denote the parallel transport of $\mathbb{D}$ along γ. Since the connection $\mathbb{D}$ need not be flat, the isomorphisms $U_{pq}(\gamma)$ may depend on γ and not only on its homotopy class. For any $\xi \in \mathcal{K}$, the first equation in (1.1) implies:

$$\xi(q) = U_{pq}(\gamma)\xi(p) \, , \quad \forall p, q \in M \, , \quad \forall \gamma \in \mathcal{P}_{pq}(M) \, . \tag{1.8}$$

When $\xi \in \ker(\mathrm{ev}_p)$ (i.e. $\xi(p) = 0$), relation (1.8) gives $\xi(q) = 0$ for all $q \in M$ and hence $\xi = 0$. This shows that the restriction $\mathrm{ev}_p|_{\mathcal{K}} : \mathcal{K} \to S_p$ is injective for all $p \in M$ and thus that $\mathcal{K}$ is a locally non-degenerate subspace of $\Gamma(M, S)$.

□

Let $(K, \mathbf{D}) \stackrel{\text{def.}}{=} \Phi_s(\mathcal{K}(\mathbb{D}, Q))$.

Proposition. The bundle K is $\mathbf{D}$-invariant, thus:

$$\mathbb{D}(\Gamma(M, K)) \subset \Omega^1(M, K)\,. \tag{1.9}$$

Furthermore, the restriction of $\mathbb{D}$ to K is a trivial flat connection on K which coincides with $\mathbf{D}$:

$$\mathbb{D}\xi = \mathbf{D}\xi\,, \quad \forall \xi \in \Gamma(M, K)\,. \tag{1.10}$$

Proof. Defining e_p as in (1.2), relation (1.8) implies:

$$U_{pq}(\gamma)|_{K_p} = \mathrm{ev}_q \circ e_p^{-1}\,, \tag{1.11}$$

showing that $U_{pq}(\gamma)(K_p) = K_q$ for all $p, q \in M$ and $\gamma \in \mathcal{P}_{pq}(M)$. This means that $\mathbb{D}$ preserves the bundle K, i.e. relation (1.9) holds. Corestricting U_{pq} to its codomain, (1.11) gives the parallel transport of the connection $\mathbb{D}_0$ induced by $\mathbb{D}$ on the sub-bundle K:

$$U_{pq}(\gamma)|_{K_p}^{K_q} = e_q \circ e_p^{-1} = \mathbf{U}_{pq}\,,$$

where in the last line we used formula (1.3) for the parallel transport $\mathbf{U}_{pq}$ of the trivial flat connection $\mathbf{D}$ of K. This shows that $\mathbf{D}$ coincides with the restriction of $\mathbb{D}$ to K. □

Remark. Let us fix $p \in M$. Using relations (1.8), it is easy to see that K_p can be written as:

$$K_p = \cap_{\gamma \in \mathcal{P}_{pp}(M)} \ker(U_{pp}(\gamma) - \mathrm{id}_{K_p}) \cap \cap_{q \in M, \gamma \in \mathcal{P}_{pq}(M)} \ker(U_{pq}(\gamma)^{-1} \circ Q_q \circ U_{pq}(\gamma))\,. \tag{1.12}$$

Given $\xi(p) \in K_p$, the element $U_{pq}(\gamma)\xi(p) \in S_q$ is independent of the choice of $\gamma \in \mathcal{P}_{pq}(M)$ and ξ can be recovered using (1.8). Thus (1.1) is equivalent with the condition $\xi(p) \in K_p$, where K_p is given by (1.12).

Proposition. Assume that $\mathbb{D}$ is $\mathscr{B}$-compatible. Then $\mathcal{K}(\mathbb{D}, Q)$ is a $\mathscr{B}$-compatible locally non-degenerate subspace of $\Gamma(M, S)$.

Proof. When $\mathbb{D}$ is $\mathscr{B}$-compatible, its parallel transport satisfies:

$$\mathscr{B}_q(U_{pq}(\gamma) \otimes U_{pq}(\gamma)) = \mathscr{B}_p\,, \quad \forall p, q \in M\,, \quad \forall \gamma \in \mathcal{P}_{pq}(M)\,.$$

Restricting this to K_p shows that $\mathbf{U}_{pq} \stackrel{\text{def.}}{=} U_{pq}(\gamma)|_{K_p}$ is an isometry from $(K_p, \mathscr{B}_p)$ to $(K_q, \mathscr{B}_q)$ for all $p, q \in M$, i.e. relation (1.7) is satisfied. This implies the conclusion since (1.7) is equivalent with (1.5). □

Proposition. Let $\mathcal{K}$ be an s-dimensional subspace of $\Gamma(M,S)$. Then the following statements are equivalent:

(a) $\mathcal{K}$ is a virtual CGK space.

(b) $\mathcal{K}$ is locally non-degenerate.

Proof. The implication $(a) \Rightarrow (b)$ was proved before. To prove the inverse implication, let $\mathcal{K} \in \mathrm{Grn}_s(M,S)$ and $(K,\mathbf{D}) \stackrel{\text{def.}}{=} \Phi_s(\mathcal{K})$. Choosing a complement K' of K inside S gives a direct sum decomposition:

$$S = K \oplus K' .$$

We have $\mathcal{K} = \Gamma_{\text{flat}}(K,\mathbf{D}) \subset \Gamma(M,K)$ and hence $\mathbf{D}\xi = 0$ for all $\xi \in \mathcal{K}$. Let $Q \in \Gamma(M,\mathrm{End}(S))$ denote the projector of S onto K' parallel to K. Then $K = \ker Q$ and hence $Q\xi = 0$ for any $\xi \in \mathcal{K}$. Let D' be any connection on K'. Then the direct sum $\mathbb{D} \stackrel{\text{def.}}{=} \mathbf{D} \oplus D'$ is a connection on S which satisfies $\mathbb{D}\xi = 0$ for all $\xi \in \mathcal{K}$. It follows that we have $\mathcal{K} \subset \mathcal{K}(\mathbb{D},Q)$. To show the inverse inclusion, let $\xi \in \mathcal{K}(\mathbb{D},Q)$. Then $Q\xi = 0$ and hence $\xi \in \Gamma(M,K)$. The equation $\mathbb{D}\xi = 0$ is thus equivalent with $\mathbf{D}\xi = 0$. It follows that we have $\xi \in \Gamma_{\text{flat}}(K,\mathbf{D}) = \mathcal{K}$ and hence $\mathcal{K}(\mathbb{D},Q) \subset \mathcal{K}$. □

Proposition. Let $\mathcal{K}$ be an s-dimensional subspace of $\Gamma(M,S)$. Then the following statements are equivalent:

(a) $\mathcal{K}$ is a $\mathscr{B}$-compatible virtual CGK space.

(b) $\mathcal{K}$ is a $\mathscr{B}$-compatible locally non-degenerate subspace of $\Gamma(M,S)$.

Proof. The implication $(a) \Rightarrow (b)$ was proved before. For the inverse implication, let $\mathcal{K} \in \mathrm{Grn}_s(M,S,\mathscr{B})$ and $(K,\mathbf{D}) \stackrel{\text{def.}}{=} \Phi_s(\mathcal{K})$. Let $K^\perp$ be the $\mathscr{B}$-orthocomplement of K inside S and Q the $\mathscr{B}$-orthoprojector on $K^\perp$. Let D' be any $\mathscr{B}$-compatible connection on $K^\perp$ and let $\mathbb{D} = \mathbf{D} \oplus D'$. The same argument as in the proof of the previous proposition shows that we have $\mathcal{K} = \mathcal{K}(\mathbb{D},Q)$. Since $\mathcal{K}$ is $\mathscr{B}$-compatible, the connection $\mathbf{D}$ is $\mathscr{B}$-compatible and hence $\mathbb{D}$ is $\mathscr{B}$-compatible as well. □

1.4 The chirality stratification

Let:

$$P_\pm \stackrel{\text{def.}}{=} \frac{1}{2}(1 \pm \gamma(\nu)) \in \Gamma(M,\mathrm{Hom}(S,S^\pm))$$

be the $\mathscr{B}$-orthogonal projectors of S onto $S^\pm$ and let $(K,\mathbf{D}) = \Phi_s(\mathcal{K})$ for some locally non-degenerate subspace $\mathcal{K} \subset \Gamma(M,S)$.

Definition. The *chiral projections* of K are the smooth generalized sub-bundles of $S^\pm$ defined through:

$$K_\pm \stackrel{\text{def.}}{=} P_\pm K \subset S^\pm .$$

The *chiral rank functions* $r_\pm$ of K are the rank functions of $K_\pm$:

$$r_\pm \stackrel{\text{def.}}{=} \mathrm{rk} K_\pm : M \to \mathbb{N} .$$

Notice that $r_\pm$ are lower semicontinuous and that they satisfy:

$$r_\pm \leq s\,, \quad r_+ + r_- \geq s\,, \tag{1.13}$$

where the last inequality follows from the fact that K is a sub-bundle of the generalized bundle $K_+ \oplus K_-$.

Definition. The *chiral slices* of K are the following cosmooth generalized sub-bundles of K:

$$K^\pm \stackrel{\text{def.}}{=} S^\pm \cap K\,.$$

The identity $S^\pm = \ker P_\mp$ implies $K^\pm = \ker(P_\mp|_K)$, hence we have exact sequences of generalized sub-bundles of S:

$$0 \to K^\mp \hookrightarrow K \stackrel{P_\pm|_K}{\longrightarrow} K_\pm \to 0\,,$$

which give the relations:

$$\sigma_\pm \stackrel{\text{def.}}{=} \mathrm{rk} K^\pm = s - r_\mp\,. \tag{1.14}$$

Definition. We say that $p \in M$ is a K*-special point* if $(r_-(p), r_+(p)) \neq (s, s)$. The K*-special locus* is the following subset of M:

$$\mathcal{S} \stackrel{\text{def.}}{=} \{p \in M | p \text{ is a } K\text{−special point}\}\ . \tag{1.15}$$

The open complement:

$$\mathcal{G} \stackrel{\text{def.}}{=} M \setminus \mathcal{S} = \{p \in M | r_-(p) = r_+(p) = s\}$$

will be called the *non-special locus* of K; its elements are the *non-special points*. The special locus admits a stratification induced by the chiral rank functions:

$$\mathcal{S} = \sqcup_{\substack{0 \leq k, l \leq s \\ k+l \geq s \\ (k,l) \neq (s,s)}} \mathcal{S}_{kl}\,,$$

where:

$$\mathcal{S}_{kl} \stackrel{\text{def.}}{=} \{p \in \mathcal{S} | r_-(p) = k \;\&\; r_+(p) = l\}\,.$$

Definition. The *chirality stratification* of M induced by $\mathcal{K}$ is the decomposition:

$$M = \mathcal{G} \sqcup \sqcup_{\substack{0 \leq k, l \leq s \\ k+l \geq s \\ (k,l) \neq (s,s)}} \mathcal{S}_{kl}\,.$$

1.5 The stabilizer stratification

For any $p \in M$, consider the natural representation of the group $\mathrm{Spin}(T_pM, g_p) \simeq \mathrm{Spin}(8)$ on S_p.

Definition. The *stabilizer group of* K *at* p is the closed subgroup of $\mathrm{Spin}(T_pM, g_p)$ consisting of those elements which act trivially on the subspace $K_p \subset S_p$:

$$H_p \stackrel{\text{def.}}{=} \{h \in \mathrm{Spin}(T_pM, g_p)|\ hu = u \ \ \forall u \in K_p\}\,. \tag{1.16}$$

Definition. Let $\mathcal{K}$ be an s-dimensional locally-nondegenerate subspace of $\Gamma(M,S)$. The *stabilizer stratification of M induced by $\mathcal{K}$* is the stratification of M given by the isomorphism type of H_p. Two points $p,q\in M$ belong to the same stratum of this stratification iff H_p and H_q are isomorphic.

Remark. Given a frame $(\xi_1,\ldots,\xi_s)$ of $\mathcal{K}$, the group H_p coincides with the common stabilizer of $\xi_i(p)$:

$$H_p=\mathrm{Stab}_{\mathrm{Spin}(T_pM,g_p)}(\xi_1(p),\ldots,\xi_s(p))=\{h\in\mathrm{Spin}(T_pM,g_p)|\ h\xi_i(p)=\xi_i(p)\ \ \forall i=1\ldots s\}\,.$$

When $\mathcal{K}$ is $\mathscr{B}$-compatible, we can formulate this as follows. Let $V_p^{(s)}(S_p,\mathscr{B}_p)$ be the Stiefel manifold of orthonormal s-frames of the Euclidean space $(S_p,\mathscr{B}_p)$ and $V^{(s)}(S,\mathscr{B})$ be the fiber bundle over M having $V_p^{(s)}(S_p,\mathscr{B}_p)$ as its fiber at p. Since the action of $\mathrm{Spin}(T_pM,S_p)$ on S_p preserves $\mathscr{B}_p$, it induces an action on $V^{(s)}(S_p,\mathscr{B}_p)$:

$$(u_1,\ldots u_s)\to(hu_1,\ldots,hu_s)\,,\quad\forall h\in\mathrm{Spin}(T_pM,g_p)\,,\quad\forall(u_1,\ldots,u_s)\in V^{(s)}(S_p,\mathscr{B}_p)\,.\tag{1.17}$$

An orthonormal basis $\Xi\stackrel{\text{def.}}{=}(\xi_1,\ldots,\xi_s)$ of $\mathcal{K}$ can be viewed as a smooth section of the fiber bundle $V^{(s)}(S,\mathscr{B})$. Then H_p coincides with the stabilizer of the value $\Xi(p)$ of this section under the action (1.17). The Stiefel manifold $V^{(s)}(S_p,\mathscr{B}_p)$ has a stratification by the isomorphism type of stabilizers inside $\mathrm{Spin}(T_pM,g_p)$. Similarly, there is a stratification $\Sigma^{(s)}$ of the total space of $V^{(s)}(S,\mathscr{B})$ by the isomorphism type of stabilizers. Since H_p is independent of the choice of Ξ, the Ξ-preimage of the stratification $\Sigma^{(s)}$ is independent of Ξ and coincides with the stabilizer stratification of M induced by $\mathcal{K}$. A similar formulation exists when $\mathcal{K}$ is not $\mathscr{B}$-compatible, if one replaces $V^{(s)}(S,\mathscr{B})$ by the bundle $V^{(s)}(S)$ whose fiber at $p\in M$ is the Stiefel manifold $V^{(s)}(S_p)$ of all s-frames of the fiber S_p.

Assuming $\mathrm{rk}K\geq 1$, let $\mathfrak{q}_p:\mathrm{Spin}(T_pM,g_p)\to\mathrm{SO}(T_pM,g_p)$ denote the double covering morphism. The image $G_p\stackrel{\text{def.}}{=}\mathfrak{q}_\mathfrak{p}(H_p)$ is a subgroup of $\mathrm{SO}(T_pM,g_p)$. The $\mathfrak{q}_p$-preimage of the unit element id_{T_pM} of $\mathrm{SO}(T_pM,g_p)$ is a two-point set which consists of the unit element of $\mathrm{Spin}(T_pM,g_p)$ and another element which we denote by ϵ_p. The latter acts on S_p as minus the identity and hence it cannot be contained in H_p. It follows that the restriction of $\mathfrak{q}_p$ to H_p is injective and hence it gives an isomorphism from H_p to G_p. Thus the stabilizer stratification coincides with the stratification of M by the isomorphism type of $G_p\simeq H_p$. Let T be a stratum of the connected refinement of this stratification and let G_T denote the isomorphism type of $G_p\simeq H_p$ for $p\in T$. Endow T with the topology induced from M. The restriction $\mathrm{Fr}_+(M)|_T$ of the oriented frame bundle $\mathrm{Fr}_+(M)$ of M is a principal $\mathrm{SO}(8)$ bundle (in the sense of general topology) defined over the connected topological space T. Picking specific G_p-orbits inside the fibers $\mathrm{Fr}_p(M)$ for $p\in T$ specifies a G_T-reduction of structure group of $\mathrm{Fr}(M)|_T$ and such reductions for all connected strata T fit together into a "stratified G-structure" defined on M.

Remark. In the Physics literature, what we call a stratified G-structure is sometimes called a "local G-structure". In Mathematics, the word "local" refers to a structure or property which is defined/which holds for all points of some *open* subset of a topological space. Since most strata of the stabilizer stratification are not open subsets of M, it is clear that a stratified G-structure cannot be a local G-structure in the sense used in Mathematics.

1.6 The case of compactifications to $\mathbf{AdS_3}$

As an example, consider compactifications down to an AdS_3 space of cosmological constant $\Lambda = -8\kappa^2$, where κ is a positive parameter. In this case, the eleven-dimensional background $\mathbf{M}$ is diffeomorphic with $N \times M$, where N is an oriented 3-manifold diffeomorphic with $\mathbb{R}^3$ and carrying the AdS_3 metric g_3. The metric on $\mathbf{M}$ is taken to be a warped product:

$$\mathrm{d}\mathbf{s}^2 = e^{2\Delta}\mathrm{d}s^2 \quad \text{where} \quad \mathrm{d}s^2 = \mathrm{d}s_3^2 + g_{mn}\mathrm{d}x^m\mathrm{d}x^n\,. \tag{1.18}$$

The warp factor Δ is a smooth real-valued function defined on M while $\mathrm{d}s_3^2$ is the squared length element of the AdS_3 metric g_3. The Ansatz for the field strength $\mathbf{G}$ of eleven-dimensional supergravity is:

$$\mathbf{G} = \nu_3 \wedge \mathbf{f} + \mathbf{F}\,, \qquad \text{with } \mathbf{F} \stackrel{\text{def.}}{=} e^{3\Delta}F\,, \quad \mathbf{f} \stackrel{\text{def.}}{=} e^{3\Delta}f \tag{1.19}$$

where $f \in \Omega^1(M)$, $F \in \Omega^4(M)$ and ν_3 is the volume form of (N, g_3). The Ansatz for the supersymmetry generator is:

$$\boldsymbol{\eta} = e^{\frac{\Delta}{2}}\sum_{i=1}^{s}\zeta_i \otimes \xi_i\,, \tag{1.20}$$

where $\xi_i \in \Gamma(M,S)$ are Majorana spinors of spin 1/2 on the internal space (M,g) and ζ_i are Majorana spinors on (N,g_3) which satisfy the Killing equation with positive Killing constant.[3] Assuming that ζ_i are Killing spinor on the AdS_3 space (N,g_3), the supersymmetry condition is satisfied if ξ_i satisfies (1.1), where:

$$\mathbb{D}_X = \nabla^S_X + \frac{1}{4}\gamma(X\lrcorner F) + \frac{1}{4}\gamma((X_\sharp \wedge f)\nu) + \kappa\gamma(X\lrcorner\nu)\,, \quad X \in \Gamma(M, TM)$$

and:

$$Q = \frac{1}{2}\gamma(\mathrm{d}\Delta) - \frac{1}{6}\gamma(\iota_f\nu) - \frac{1}{12}\gamma(F) - \kappa\gamma(\nu)\,.$$

Here ∇^S is the connection induced on S by the Levi-Civita connection of (M,g), while ν is the volume form of (M,g). Neither Q nor the connection $\mathbb{D}$ preserve the chirality decomposition $S = S^+ \oplus S^-$ of S when $\kappa \neq 0$:

$$\mathbb{D}(S^\pm) \not\subseteq T^*M \otimes S^\pm\,, \quad Q(S^\pm) \not\subseteq S^\pm\,.$$

It is not hard to check [38] that $\mathbb{D}$ is $\mathscr{B}$-compatible:

$$\mathrm{d}\mathscr{B}(\xi',\xi'') = \mathscr{B}(\mathbb{D}\xi',\xi'') + \mathscr{B}(\xi',\mathbb{D}\xi'')\,, \quad \forall \xi',\xi'' \in \Gamma(M,S)\,. \tag{1.21}$$

This implies that any $\xi,\xi' \in \mathcal{K}(\mathbb{D},Q)$ satisfy $\mathscr{B}(\xi,\xi') = \text{constant}$, i.e. $\mathcal{K}$ is a $\mathscr{B}$-compatible flat subspace of $\Gamma(M,S)$. The restriction $\mathbf{D} = \mathbb{D}|_K$ is a $\mathscr{B}$-compatible trivial flat connection on $\mathcal{K}(\mathbb{D},Q)$.

[3]With our conventions (see appendix A), gamma matrices in signature $(-1,2)$ can be taken to be real, for example $\gamma_0 = i\sigma_2, \gamma_1 = \sigma_1, \gamma_2 = \sigma_3$ where σ_k are the Pauli matrices. In the Mathematics convention for Clifford algebras, γ_k are replaced by $\hat{\gamma}_k = i\gamma_k$. A Killing Majorana spinor on AdS_3 satisfies $\nabla_k\zeta = \lambda\gamma_k\zeta$, with a real Killing constant $\lambda = \pm\kappa$. In the Mathematics convention, this corresponds to $\nabla_k\zeta = \hat{\lambda}\hat{\gamma}_k\zeta$, with imaginary $\hat{\lambda} = -i\lambda = \mp i\kappa$; these are known as "imaginary Killing spinors". In the Ansatz, we choose $\lambda = +\kappa$.

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Remarks.

1. An equivalent formulation of the Ansatz (1.20) is that the supersymmetry generators of the background span the space $\mathcal{K}_3 \otimes \mathcal{K}$, where $\mathcal{K}_3$ is the two-dimensional space of real Killing spinors on AdS_3 with positive Killing constant. Then ξ_i in the Ansatz can be taken to form an orthonormal basis of $\mathcal{K}$, while ζ_i are *arbitrary* elements of $\mathcal{K}_3$, so that the Ansatz describes the general element of $\mathcal{K}_3 \otimes \mathcal{K}$. Notice that one does not gain anything by decomposing ξ_i into their positive and negative chirality parts in the Ansatz since $\mathbb{D}$ and Q do not preserve the sub-bundles $S^\pm$ and hence $\mathcal{K}$ need not equal the direct sum of the intersections $\mathcal{K} \cap \Gamma(M, S^+)$ and $\mathcal{K} \cap \Gamma(M, S^-)$.
2. The amount $\mathcal{N}$ of supersymmetry preserved by the background may be larger than s in the limit $\Lambda = 0$, when AdS_3 reduces to the three-dimensional Minkowski space. In that limit, the results of [19, 24] imply that all fluxes must vanish, thus $F = f = \kappa = 0$ and that $\mathrm{d}\Delta = 0$, which imply $\mathbb{D} = \nabla^S$ and $Q = 0$, hence both $\mathbb{D}$ and Q preserve the sub-bundles S^+ and S^- of S. A discussion of this phenomenon for the case $s = 1$ (which gives $\mathcal{N} = 1$ for $\Lambda < 0$ and $\mathcal{N} = 2$ for $\Lambda = 0$) can be found in [24, appendix B.1].

1.7 A toy example: the case $s = 1$

Let us illustrate the discussion above with the case $s = 1$. Then $\mathcal{K}$ is a one-dimensional locally non-degenerate subspace of $\Gamma(M, S)$ while $(K, \mathbf{D})$ is a trivial flat line sub-bundle of S. Assume that $\mathcal{K}$ is $\mathscr{B}$-compatible. Then a $\mathscr{B}$-compatible frame of $\mathcal{K}$ is given by a single Majorana spinor $\xi \in \Gamma(M, S)$ which is everywhere of norm one; the same spinor gives a global normalized frame of K. The chiral projections $K_\pm$ are the generalized sub-bundles of S generated by the positive and negative chirality parts $\xi^\pm \stackrel{\text{def.}}{=} P_\pm \xi$ of ξ. The chiral rank functions are given by:

$$r_\pm(p) = \dim\langle \xi^\pm(p) \rangle = \begin{cases} 0 & \text{if } \xi(p) \in S_p^\mp \\ 1 & \text{if } \xi(p) \not\in S_p^\mp \end{cases} .$$

The chiral slices are:

$$K^\pm(p) = \begin{cases} 0 & \text{if } \xi(p) \not\in S_p^\pm \\ \langle \xi(p) \rangle \simeq \mathbb{R} & \text{if } \xi(p) \in S_p^\pm \end{cases} .$$

Since $\xi(p)$ is everywhere non-vanishing, we have $r_+ + r_- \geq 1$, thus the allowed values are $(r_-(p), r_+(p)) \in \{(0,1), (1,0), (1,1)\}$. Hence the chirality stratification takes the form:

$$M = \mathcal{U} \sqcup \mathcal{W}_- \sqcup \mathcal{W}_+$$

where:

$$\begin{aligned}
\mathcal{U} &\equiv \mathcal{G} \stackrel{\text{def.}}{=} \{p \in M | r_-(p) = r_+(p) = 1\} = \{p \in M | \xi(p) \not\in S_p^+ \sqcup S_p^-\} \\
\mathcal{W}_- &\equiv \mathcal{S}_{10} \stackrel{\text{def.}}{=} \{p \in M | r_-(p) = 1\,,\ r_+(p) = 0\} = \{p \in M | \xi(p) \in S_p^-\} \\
\mathcal{W}_+ &\equiv \mathcal{S}_{01} \stackrel{\text{def.}}{=} \{p \in M | r_-(p) = 0\,,\ r_+(p) = 1\} = \{p \in M | \xi(p) \in S_p^+\}\,.
\end{aligned}$$

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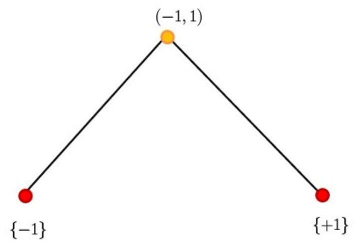


Figure 1. Hasse diagram of the incidence poset (see appendix C) of the connected refinement of the Whitney stratification of the interval $[-1, 1]$. The b-preimages of the strata represented by red and yellow dots correspond to the Spin(7) and G_2 loci of M.

Thus $\mathcal{U}$ is the non-chiral locus while $\mathcal{S}_{10}$ and $\mathcal{S}_{01}$ are the negative and positive chirality loci of [24]. The union of the latter is the chiral locus $\mathcal{W} = \mathcal{W}_- \sqcup \mathcal{W}_+ = \mathcal{S}_{10} \sqcup \mathcal{S}_{01}$ of loc. cit. In this case, the stabilizer stratification is a coarsening of the chirality stratification, namely we have $H_p \simeq \mathrm{Spin}(7)$ for $p \in \mathcal{W}$ and $H_p \simeq \mathrm{G}_2$ for $p \in \mathcal{U}$. The stabilizer stratification coincides with the rank stratification of the cosmooth generalized distribution $\mathcal{D} \stackrel{\text{def.}}{=} \ker V$, where $V \stackrel{\text{def.}}{=}_U \mathscr{B}(\xi, \gamma_a \xi) e^a$ is the one-form bilinear constructed from ξ, where the expressions are given on an open subset $U \subset M$ which supports a local coframe (e^a). Namely, we have $\dim \mathcal{D}(p) = 7$ for $p \in \mathcal{U}$ and $\dim \mathcal{D}(p) = 8$ for $p \in \mathcal{W}$. The group $G_p = \mathfrak{q}_p(H_p)$ is a subgroup of $\mathrm{SO}(\mathcal{D}(p), g_p)$ for any $p \in M$.

Let $b \stackrel{\text{def.}}{=}_U \mathscr{B}(\xi, \gamma(\nu)\xi) \in \mathcal{C}^\infty(\mathbb{R})$ denote the scalar bilinear constructed from ξ. It was shown in [19, 38] that the Fierz identities for ξ imply $1 - b^2 = ||V||^2$ and hence $b^2 \leq 1$. Thus the image of the map b is contained within the interval $[-1, 1]$. This interval is a semi-algebraic set given by the single polynomial inequality $b^2 \leq 1$ for a variable $b \in \mathbb{R}$. Its canonical Whitney stratification has a 0-dimensional stratum given by the two-point set $\{-1, 1\}$ and a one-dimensional stratum given by the open interval $(-1, 1)$. The connected refinement of the Whitney stratification has two connected 0-dimensional strata given by the one-point sets $\{+1\}$ and $\{-1\}$ and a connected 1-dimensional stratum given by the open interval $(-1, 1)$. The Hasse diagram of the incidence poset of this stratification is depicted in figure 1. It was shown in [24] that the rank/stabilizer stratification coincides with the b-preimage of the canonical Whitney stratification of $[-1, 1]$:

$$\mathcal{W} = b^{-1}(\{-1, 1\}), \quad \mathcal{U} = b^{-1}((-1, 1)).$$

On the other hand, the chirality stratification coincides with the b-preimage of the connected refinement of the Whitney stratification:

$$\mathcal{W}_\pm = b^{-1}(\{\pm 1\}), \quad \mathcal{U} = b^{-1}((-1, 1)).$$

It was also shown in [24] that, for compactifications down to AdS_3, the supersymmetry conditions (1.1) imply that the singular distribution $\mathcal{D}$ integrates to a singular foliation in the sense of Haefliger [33].

Remark. The compactifications studied in [17] correspond to the case $M = \mathcal{W}_+$.

As we shall see in the next sections, the situation is much more complicated when $s = 2$. In that case (assuming that $\mathcal{K}$ is $\mathscr{B}$-compatible):

1. The chirality and stabilizer stratifications do not agree, in the sense that neither of them is a refinement of the other.
2. There exists a cosmooth singular distribution $\mathcal{D}$ (determined by the intersection of the kernels of three one-form valued spinor bilinears V_1, V_2 and V_3) which integrates to a Haefliger foliation in the AdS_3 case. The rank stratification of $\mathcal{D}$ does not agree with the chirality stratification or with the stabilizer stratification.
3. The stabilizer stratification coincides with the rank stratification of a cosmooth singular sub-distribution $\mathcal{D}_0 \subset \mathcal{D}$ (given by the intersection of $\mathcal{D}$ with the kernel of a fourth one-form spinor bilinear W), but $\mathcal{D}_0$ need not be integrable in the case of compactifications down to AdS_3. The group $G_p = \mathfrak{q}_p(H_p)$ is a subgroup of $\mathrm{SO}(\mathcal{D}_0(p), g_p)$ (and hence also a subgroup of $\mathrm{SO}(\mathcal{D}(p), g_p)$) for any $p \in M$.
4. The chirality stratification coincides with the b-preimage of the connected refinement of the Whitney stratification of a three-dimensional semi-algebraic set $\mathcal{R}$, where $b \in \mathcal{C}^\infty(M, \mathcal{R})$ is a map constructed using scalar spinor bilinears defined by an orthonormal basis of $\mathcal{K}$. We have $\mathrm{im} b \subset \mathcal{R}$.
5. The stabilizer stratification and the rank stratification of $\mathcal{D}$ are different coarsenings of the B-preimage of the connected refinement of the canonical Whitney stratification of a four-dimensional semi-algebraic set $\mathfrak{P}$, where $B : M \to \mathbb{R}^4$ is another map constructed using an orthonormal basis of $\mathcal{K}$. We have $\mathrm{im} B \subset \mathfrak{P}$.

2 The generalized distributions $\mathcal{D}$ and $\mathcal{D}_0$ in the case $s = 2$

Throughout this section, $\mathcal{K}$ denotes a $\mathscr{B}$-compatible locally non-degenerate subspace of $\Gamma(M, S)$.

2.1 Functions and one-forms defined by a basis of $\mathcal{K}$

An orthonormal basis (ξ_1, ξ_2) of $\mathcal{K}$ induces three smooth functions $b_i \in \mathcal{C}^\infty(M, \mathbb{R})$ ($i = 1, 2, 3$), namely:

$$b_1 =_U \mathscr{B}(\xi_1, \gamma(\nu)\xi_1)\,, \ b_2 =_U \mathscr{B}(\xi_2, \gamma(\nu)\xi_2)\,, \ b_3 =_U \mathscr{B}(\xi_1, \gamma(\nu)\xi_2)\,. \tag{2.1}$$

It will be convenient to work with the combinations:

$$b_\pm \stackrel{\text{def.}}{=} \frac{1}{2}(b_1 \pm b_2)\ . \tag{2.2}$$

Also consider the one-forms $V_i, V_3, W \in \Omega^1(M)$ (with $i = 1, 2$) given by:

$$V_i =_U \mathscr{B}(\xi_i, \gamma_a \xi_i) e^a \quad , \quad V_3 \stackrel{\text{def.}}{=}_U \mathscr{B}(\xi_1, \gamma_a \xi_2) e^a\,, \quad W \stackrel{\text{def.}}{=}_U \mathscr{B}(\xi_1, \gamma_a \gamma(\nu) \xi_2) e^a\,, \tag{2.3}$$

where the relations hold in any local coframe (e^a) defined above an open subset $U \subset M$. It will be convenient to work with the linear combinations:

$$V_\pm \stackrel{\text{def.}}{=} \frac{1}{2}(V_1 \pm V_2)\,, \quad V_3^\pm = \frac{1}{2}(V_3 \pm W)\,. \tag{2.4}$$

We have:

$$V_1 = V_+ + V_- \,, \quad V_2 = V_+ - V_- \,, \quad V_3 = V_3^+ + V_3^- \,, \quad W = V_3^+ - V_3^- \,.$$

Decomposing ξ_i into their positive and negative chirality parts gives:

$$V_1 =_U 2\mathscr{B}(\xi_1^-, \gamma_a \xi_1^+)e^a \,, \quad V_2 =_U 2\mathscr{B}(\xi_2^-, \gamma_a \xi_2^+)e^a \,, \quad V_3^\pm =_U \mathscr{B}(\xi_1^\mp, \gamma_a \xi_2^\pm)e^a \,. \tag{2.5}$$

2.2 The distributions $\mathcal{D}$ and $\mathcal{D}_0$

The 1-forms V_1, V_2, V_3 generate a smooth generalized sub-bundle $\mathcal{V}$ (in the sense of [37]) of the cotangent bundle of M, which is also generated by V_+, V_-, V_3. Let:

$$\mathcal{D} \stackrel{\text{def.}}{=} \ker V_1 \cap \ker V_2 \cap \ker V_3 = \ker V_+ \cap \ker V_- \cap \ker V_3$$

denote the polar of $\mathcal{V}$, which is a cosmooth generalized distribution on M, i.e. a cosmooth generalized sub-bundle of TM in the sense of [37]. Its orthogonal complement $\mathcal{D}^\perp$ inside TM is a smooth generalized sub-bundle of TM which is spanned by the three vector fields obtained from V_+, V_-, V_3 by applying the musical isomorphism. Notice that $\mathcal{D}$ contains the cosmooth generalized distribution:

$$\mathcal{D}_0 \stackrel{\text{def.}}{=} \ker V_+ \cap \ker V_- \cap \ker V_3^+ \cap \ker V_3^- = \mathcal{D} \cap \ker W \subset \mathcal{D} \,.$$

Remark. When considering compactifications to AdS_3, one can show that the supersymmetry conditions imply that $\mathcal{D}$ is an integrable distribution (namely, it integrates to a singular foliation in the sense of Haefliger) while $\mathcal{D}_0$ may fail to be integrable. This is one reason for considering the generalized distribution $\mathcal{D}$.

2.3 Behavior under changes of orthonormal basis of $\mathcal{K}$

An orthonormal basis (ξ'_1, ξ'_2) of $\mathcal{K}$ having the same orientation as (ξ_1, ξ_2) has the form:

$$\begin{aligned} \xi'_1 &= \cos\left(\frac{u}{2}\right)\xi_1 + \sin\left(\frac{u}{2}\right)\xi_2 \,, \\ \xi'_2 &= -\sin\left(\frac{u}{2}\right)\xi_1 + \cos\left(\frac{u}{2}\right)\xi_2 \end{aligned} \tag{2.6}$$

(where $u \in \mathbb{R}$) and defines the following 0-forms and 1-forms, where $i = 1, 2$:

$$\begin{aligned} &b'_i =_U \mathscr{B}(\xi'_i, \gamma(\nu)\xi'_i) \,, \quad b'_3 =_U \mathscr{B}(\xi'_1, \gamma(\nu)\xi'_2) \\ &V'_i =_U \mathscr{B}(\xi'_i, \gamma_a \xi'_i)e^a \,, \quad V'_3 \stackrel{\text{def.}}{=}_U \mathscr{B}(\xi'_1, \gamma_a \xi'_2)e^a \,, \quad W' \stackrel{\text{def.}}{=}_U \mathscr{B}(\xi'_1, \gamma_a \gamma(\nu)\xi'_2)e^a \,. \end{aligned} \tag{2.7}$$

Substituting (2.6) into these expressions, we find that b_+, V_+ and W are invariant while each of the pairs b_-, b_3 and V_-, V_3 transforms in the fundamental representation of $\mathrm{SO}(2)$:

$$\begin{aligned} b'_+ &= b_+ \,, & V'_+ &= V_+ \,, & W' &= W \\ b'_- &= \cos(u)b_- + \sin(u)b_3 \,, & V'_- &= \cos(u)V_- + \sin(u)V_3 & & \\ b'_3 &= -\sin(u)b_- + \cos(u)b_3 \,, & V'_3 &= -\sin(u)V_- + \cos(u)V_3 \,. & & \end{aligned} \tag{2.8}$$

The improper rotation:

$$\begin{bmatrix} \xi_1' \\ \xi_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{2.9}$$

which permutes ξ_1 and ξ_2 induces permutations $b_1 \leftrightarrow b_2$ and $V_1 \leftrightarrow V_2$ while V_3, b_3 remain unchanged and W changes sign (to arrive at these conclusions, one uses the relations $\gamma(\nu)^t = \gamma(\nu)$, $\gamma_a^t = \gamma_a$ and the fact that $\gamma(\nu)$ anticommutes with γ_a). Hence (2.9) induces the transformations:

$$\begin{aligned} b_+ &\to b_+\,, \quad V_+ \to V_+\,, \quad W \to -W \\ b_- &\to -b_-\,, \quad V_- \to -V_- \\ b_3 &\to b_3\,, \quad V_3 \to V_3\,. \end{aligned} \tag{2.10}$$

It follows that b_+ and V_+ depend only on $\mathcal{K}$ while W depends on $\mathcal{K}$ and on a choice of orientation of $\mathcal{K}$. On the other hand, b_- and V_- change sign while b_3 and V_3 are invariant under a change of orientation of $\mathcal{K}$. It also follows from the above that $\mathcal{D}$ and $\mathcal{D}_0$ depend only on the space $\mathcal{K}$ and do not depend on the choice of basis (ξ_1, ξ_2) for $\mathcal{K}$.

2.4 The rank stratification of $\mathcal{D}$

The compact manifold M decomposes into a disjoint union according to the rank of $\mathcal{D}$:

$$M = \mathcal{U} \sqcup \mathcal{W}\,, \tag{2.11}$$

where the open set:

$$\mathcal{U} \stackrel{\text{def.}}{=} \{p \in M | \mathrm{rk}\mathcal{D}(p) = 5\} = \{p \in M | V_+(p), V_-(p), V_3(p) \text{ are linearly independent}\}$$

will be called the *generic locus* while its closed complement:

$$\mathcal{W} \stackrel{\text{def.}}{=} \{p \in M | \mathrm{rk}\mathcal{D}(p) > 5\} = \{p \in M | V_+(p), V_-(p), V_3(p) \text{ are linearly dependent}\}$$

will be called the *degeneration locus*. The latter admits a stratification according to the corank of $\mathcal{D}(p)$:

$$\mathcal{W} = \sqcup_{k=0}^2 \mathcal{W}_k\,,$$

whose locally closed strata are given by:

$$\mathcal{W}_k \stackrel{\text{def.}}{=} \{p \in \mathcal{W} | \dim \mathcal{V}_p = k\} = \{p \in \mathcal{W} | \mathrm{rk}\mathcal{D}(p) = 8 - k\}\,. \tag{2.12}$$

Combining everything gives the *rank stratification of* $\mathcal{D}$:

$$M = \mathcal{U} \sqcup \mathcal{W}_2 \sqcup \mathcal{W}_1 \sqcup \mathcal{W}_0\,. \tag{2.13}$$

Definition. $\mathcal{K}$ is called *generic* if $\mathcal{U} \neq \emptyset$ and *non-generic* otherwise.

Notice that $\mathcal{K}$ is non-generic iff $\mathrm{rk}\mathcal{D}(p) \geq 6$ for all $p \in M$, i.e. iff $V_1(p), V_2(p)$ and $V_3(p)$ are linearly dependent for all $p \in M$.

Remark. For any $p \in M$, let $A_p \in \mathrm{Hom}(\mathbb{R}^3, T_p^*M)$ denote the linear map which takes the canonical basis ϵ_i of $\mathbb{R}^3$ into $V_i(p)$:

$$A_p(\epsilon_i) = V_i(p)\,, \quad \forall i = 1 \dots 3\,.$$

This defines a smooth section $A \in \Gamma(M, \mathrm{Hom}(\underline{\mathbb{R}}^3, T^*M))$, where $\underline{\mathbb{R}}^3 = M \times \mathbb{R}^3$ is the trivial rank 3 vector bundle over M. Each space $\mathrm{Hom}(\mathbb{R}^3, T_p^*M) \simeq \mathrm{Mat}(3, 8, \mathbb{R})$ admits a Whitney stratification (the so-called canonical stratification [39, 40]) whose strata are the Stiefel manifolds $V^{(k)}(T_p^*M) = \{A \in \mathrm{Hom}(\mathbb{R}^3, T_p^*M) | \mathrm{rk} A = k\} \simeq \{\hat{A} \in \mathrm{Mat}(3, 8, \mathbb{R}) | \mathrm{rk}\hat{A} = k\}$, where $k = 0, 1, 2, 3$. This induces a stratification of the total space of the bundle $\mathrm{Hom}(\underline{\mathbb{R}}^3, T^*M)$, whose preimage through the section A is the stratification (2.13). The preimage of the stratum defined by $\mathrm{rk}A = 3$ is the set $\mathcal{U}$ while the preimages of the strata defined by $\mathrm{rk}A = k$ with $k = 0, 1, 2$ are the sets $\mathcal{W}_k$.

2.5 The rank stratification of $\mathcal{D}_0$

The generalized distribution $\mathcal{D}_0$ induces a decomposition:

$$M = \mathcal{U}_0 \sqcup \mathcal{Z}\,, \tag{2.14}$$

where:

$$\mathcal{U}_0 \stackrel{\text{def.}}{=} \{p \in M | \mathrm{rk}\mathcal{D}_0(p) = 4\} = \{p \in M | V_+(p), V_-(p), V_3(p), W(p) \text{ are linearly independent}\}$$

is an open subset of M while:

$$\mathcal{Z} \stackrel{\text{def.}}{=} \{p \in M | \mathrm{rk}\mathcal{D}_0(p) > 4\} = \{p \in M | V_+(p), V_-(p), V_3(p), W(p) \text{ are linearly dependent}\}$$

is closed. The latter stratifies according to the corank of $\mathcal{D}_0$:

$$\mathcal{Z} = \sqcup_{k=0}^{3} \mathcal{Z}_k\,,$$

with locally closed strata given by:

$$\mathcal{Z}_k \stackrel{\text{def.}}{=} \{p \in \mathcal{Z} | \mathrm{rk}\mathcal{D}_0(p) = 8 - k\}\,. \tag{2.15}$$

We shall see later[4] that we always have:

$$\mathcal{U}_0 = \mathcal{U} \quad \text{and} \quad \mathcal{Z}_3 = \emptyset\,,$$

so in particular $\mathrm{rk}\mathcal{D}_0(p)$ can never equal five. We thus obtain the *rank stratification of* $\mathcal{D}_0$:

$$M = \mathcal{U} \sqcup \mathcal{Z}_2 \sqcup \mathcal{Z}_1 \sqcup \mathcal{Z}_0\,. \tag{2.16}$$

[4]This follows from the algebraic constraints satisfied by V_i and W — see Theorem 4 of subsection 5.2.

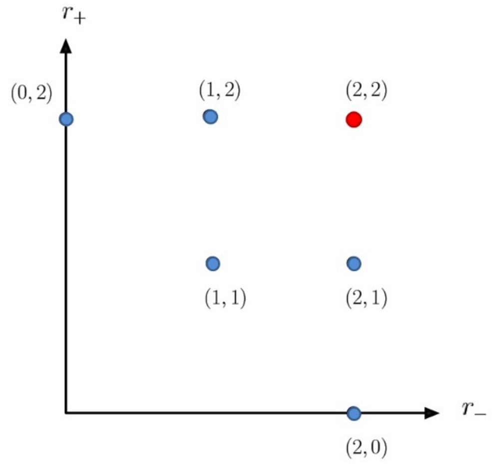


Figure 2. Allowed values for the pair $(r_-(p), r_+(p))$. The values corresponding to K-special points are shown in blue, while the remaining value is shown as a red dot.

2.6 Constraints on the stabilizer stratification

Since the action of $\mathrm{Spin}(T_pM, g_p)$ on S_p commutes with $\gamma_p(\nu_p)$, relations (2.3) imply:

$$H_p \subset \mathrm{Stab}_{\mathrm{Spin}(T_pM,g_p)}(V_+(p), V_-(p), V_3(p), W(p)) , \tag{2.17}$$

where $\mathrm{Spin}(T_pM, g_p)$ acts on T_p^*M by the dual of the vector representation. The action of $\mathrm{Spin}(T_pM, g_p)$ on T_p^*M is obtained from that of $\mathrm{SO}(T_pM, g_p)$ by pre-composing with the covering morphism $\mathfrak{q}_p : \mathrm{Spin}(T_pM, g_p) \to \mathrm{SO}(T_pM, g_p)$. Hence (2.17) implies:

$$G_p \subset \mathrm{Stab}_{\mathrm{SO}(T_pM,g_p)}(V_+(p), V_-(p), V_3(p), W(p)) \simeq \mathrm{SO}(\mathcal{D}_0(p), g_p) . \tag{2.18}$$

In particular, we have:

$$G_p \subset \mathrm{Stab}_{\mathrm{SO}(T_pM,g_p)}(V_+(p), V_-(p), V_3(p)) \simeq \mathrm{SO}(\mathcal{D}(p), g_p) . \tag{2.19}$$

3 The chirality stratification for $s = 2$

Let $\mathcal{K}$ be a two-dimensional $\mathscr{B}$-compatible locally-nondegenerate subspace of $\Gamma(M, S)$ and $(K, \mathbf{D})$ be the associated trivial flat sub-bundle of S. Relations (1.13) imply (see figure 2):

$$(r_-(p), r_+(p)) \in \{(0, 2), (2, 0), (1, 1), (1, 2), (2, 1), (2, 2)\} , \quad \forall p \in M . \tag{3.1}$$

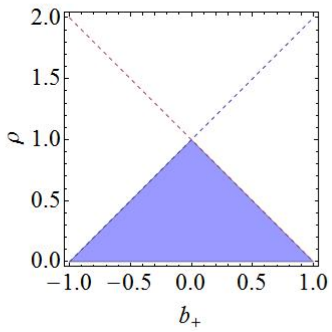
A point $p \in M$ is K-special if $(r_-(p), r_+(p)) \neq (2, 2)$ (the blue dots in figure 2). The special locus decomposes as:

$$\mathcal{S} = \mathcal{S}_{12} \sqcup \mathcal{S}_{21} \sqcup \mathcal{S}_{11} \sqcup \mathcal{S}_{02} \sqcup \mathcal{S}_{20} ,$$

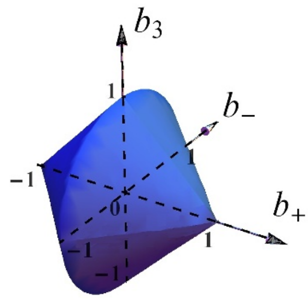
where $\mathcal{S}_{kl} = \{p \in M | r_-(p) = k,\ r_+(p) = l\}$, while the chirality stratification is given by:

$$M = \mathcal{G} \sqcup \mathcal{S}_{12} \sqcup \mathcal{S}_{21} \sqcup \mathcal{S}_{11} \sqcup \mathcal{S}_{02} \sqcup \mathcal{S}_{20} ,$$

where $\mathcal{G}$ is the non-special locus.



(a) The region Δ in the (b_+, ρ) plane.



(b) The body $\mathcal{R}$ is the solid of revolution obtained by rotating Δ around its hypotenuse, which lies on the b_+ axis; it is the union of two compact right-angled cones whose bases coincide.

Figure 3. The region Δ (blue) and the body $\mathcal{R}$.

3.1 The semi-algebraic body $\mathcal{R}$

Consider the compact convex body (see figure 3):

$$\mathcal{R} = \left\{(b_+, b_-, b_3) \in [-1,1]^3 \mid \sqrt{b_-^2 + b_3^2} \leq 1 - |b_+|\right\}, \tag{3.2}$$

which is contained in the three-dimensional compact unit ball. Setting:

$$\rho \stackrel{\text{def.}}{=} \sqrt{b_-^2 + b_3^2} \in [0,1],$$

one finds that $\mathcal{R}$ is the solid of revolution obtained by rotating the following isosceles right triangle around its hypothenuse:

$$\Delta \stackrel{\text{def.}}{=} \{(b_+, \rho) \in [-1,1] \times [0,1] \mid \rho \leq 1 - |b_+|\}. \tag{3.3}$$

The compact interval:

$$I \stackrel{\text{def.}}{=} \{(b_+, 0, 0) | b_+ \in [-1,1]\} = \{b \in \mathcal{R} | b_- = b_3 = 0\} \tag{3.4}$$

will be called the *axis* of $\mathcal{R}$ while the compact disk:

$$D \stackrel{\text{def.}}{=} \{(0, b_-, b_3) | b_-^2 + b_3^2 \leq 1\} = \{b \in \mathcal{R} | b_+ = 0\} \tag{3.5}$$

will be called the *median disk* of $\mathcal{R}$. The boundary ∂D of the median disk will be called the *median circle* (see figure 4).

Notice that $\mathcal{R}$ is a semi-algebraic set, since it can be described by polynomial inequalities:

$$\mathcal{R} = \left\{(b_+, b_-, b_3) \in \mathbb{R}^3 | b_+^2 + b_-^2 + b_3^2 \leq 1 \ \& \ b_-^2 + b_3^2 \leq \frac{1}{4}(1 + b_-^2 + b_3^2 - b_+^2)^2\right\}.$$

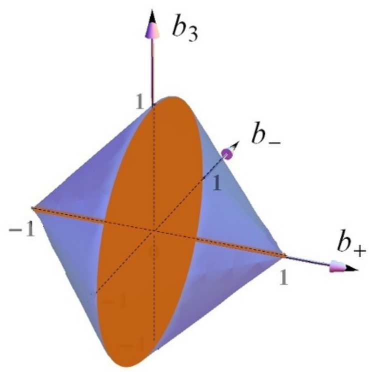


Figure 4. The axis I and the median disk D, depicted in orange.

Hence both $\mathcal{R}$ and its frontier $\partial\mathcal{R}$ (which is again a semi-algebraic set) admit [28] canonical[5] stratifications by semi-algebraic sets. Namely, the frontier:

$$\partial\mathcal{R}=\{b\in\mathcal{R}|\rho=1-|b_+|\}=\left\{(b_+,b_-,b_3)\in\mathbb{R}^3|b_+^2+b_-^2+b_3^2\leq 1\ \&\ b_-^2+b_3^2=\frac{1}{4}(1+b_-^2+b_3^2-b_+^2)^2\right\}$$

decomposes into borderless manifolds $\partial_k\mathcal{R}$ of dimensions $k=0,1,2$:

$$\partial\mathcal{R}=\partial_0\mathcal{R}\sqcup\partial_1\mathcal{R}\sqcup\partial_2\mathcal{R}\,, \tag{3.6}$$

where:

$$\partial_0\mathcal{R}\stackrel{\text{def.}}{=}\partial I\,,\ \partial_1\mathcal{R}\stackrel{\text{def.}}{=}\partial D\,,\ \partial_2\mathcal{R}\stackrel{\text{def.}}{=}\partial\mathcal{R}\setminus(\partial D\cup\partial I)\,. \tag{3.7}$$

The set $\partial_1\mathcal{R}$ coincides with the median circle and hence it is connected. The set $\partial_0\mathcal{R}$ is disconnected, being a disjoint union of two singleton sets:

$$\partial_0\mathcal{R}=\partial_0^-\mathcal{R}\sqcup\partial_0^+\mathcal{R}\,,$$

where:

$$\partial_0^-\mathcal{R}\stackrel{\text{def.}}{=}\{(-1,0,0)\}\,,\quad \partial_0^+\mathcal{R}\stackrel{\text{def.}}{=}\{(1,0,0)\}$$

will be called the *left and right tips* of $\mathcal{R}$. We have:

$$\partial_0\mathcal{R}\sqcup\partial_1\mathcal{R}=\mathcal{R}\cap S^2=\left\{(b_+,b_-,b_3)\in\mathbb{R}^3|b_+^2+b_-^2+b_3^2=1\ \&\ b_-^2+b_3^2=\frac{1}{4}(1+b_-^2+b_3^2-b_+^2)^2\right\},$$

where S^2 denotes the unit sphere in the space $\mathbb{R}^3$.

The set $\partial_2\mathcal{R}$ is relatively open in $\partial\mathcal{R}$, being a disjoint union of two connected components:

$$\partial_2\mathcal{R}=\partial_2^-\mathcal{R}\sqcup\partial_2^+\mathcal{R}\,,$$

[5]Recall that the canonical Whitney stratification of a semi-algebraic set is the coarsest stratification which satisfies the frontier conditions as well as Whitney's regularity condition (b). The strata of this stratification need not be connected. The general algorithm through which such stratifications can be obtained is due to [28] and is discussed in detail in [41].

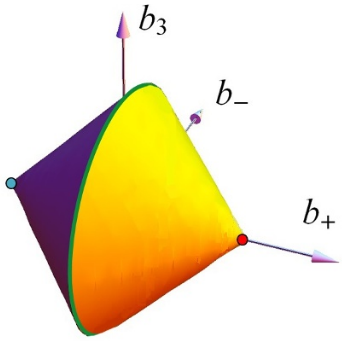


Figure 5. The connected refinement of the canonical Whitney stratification of $\partial\mathcal{R}$. We use green for the median circle $\partial_1\mathcal{R} = \partial D$, purple for $\partial_2^-\mathcal{R}$, yellow for $\partial_2^+\mathcal{R}$, blue for $\partial_0^-\mathcal{R}$ and red for $\partial_0^+\mathcal{R}$. Theorem 1 of subsection 3.6 shows that the b-preimage of $\partial_1\mathcal{R}$ equals $\mathcal{S}_{11}$, while the b-preimages of $\partial_2^+\mathcal{R}$ and $\partial_2^-\mathcal{R}$ equal $\mathcal{S}_{12}$ and $\mathcal{S}_{21}$ respectively. The b-preimages of $\partial_0^+\mathcal{R}$ and $\partial_0^-\mathcal{R}$ are the sets $\mathcal{S}_{02}$ and $\mathcal{S}_{20}$.

connected stratum	dimension	component of	topology	b_+	ρ
$\partial_0^\pm\mathcal{R}$	0	$\partial_0\mathcal{R}$	point	± 1	0
$\partial_1\mathcal{R}$	1	$\partial_1\mathcal{R}$	circle	0	1
$\partial_2^\pm\mathcal{R}$	2	$\partial_2\mathcal{R}$	open annulus	$\pm(1-\rho)$	$(0,1)$

Table 1. Connected strata of $\partial\mathcal{R}$.

where:

$$\partial_2^-\mathcal{R} \stackrel{\text{def.}}{=} \{b \in \partial_2\mathcal{R} |\ b_+ \in (-1,0)\}\,, \quad \partial_2^+\mathcal{R} \stackrel{\text{def.}}{=} \{b \in \partial_2\mathcal{R} |\ b_+ \in (0,1)\}$$

will be called the *left and right components* of $\partial_2\mathcal{R}$. The canonical Whitney stratification of $\partial\mathcal{R}$ has strata given by $\partial_0\mathcal{R}$, $\partial_1\mathcal{R}$ and $\partial_2\mathcal{R}$ and corresponds to the decomposition (3.6), while its connected refinement (see appendix C) has strata given by $\partial_0^\pm\mathcal{R}$, $\partial_1\mathcal{R}$ and $\partial_2^\pm\mathcal{R}$ and corresponds to the decomposition:

$$\partial\mathcal{R} = \partial_0^-\mathcal{R} \sqcup \partial_0^+\mathcal{R} \sqcup \partial_1\mathcal{R} \sqcup \partial_2^-\mathcal{R} \sqcup \partial_2^+\mathcal{R}\,. \tag{3.8}$$

The connected strata appearing in (3.8) are depicted in figure 5, while the values of b_+ and ρ on those strata are summarized in table 1. Together with $\mathrm{Int}\mathcal{R}$, the strata $\partial_k\mathcal{R}$ give the canonical Whitney stratification of $\mathcal{R}$, whose connected refinement has strata $\mathrm{Int}\mathcal{R}, \partial_0^\pm\mathcal{R}, \partial_1\mathcal{R}$ and $\partial_2^\pm\mathcal{R}$.

For later reference, let:

$$\mathcal{R}^- \stackrel{\text{def.}}{=} \{b \in \mathcal{R} | b_+ \leq 0\}\,, \quad \mathcal{R}^+ \stackrel{\text{def.}}{=} \{b \in \mathcal{R} | b_+ \geq 0\} \tag{3.9}$$

denote the two closed halves of $\mathcal{R}$ lying to the left and right of the median disk. Notice that $\mathcal{R}^\pm$ are three-dimensional compact full cones. We have a disjoint union decomposition:

$$\mathcal{R} = \partial\mathcal{R} \sqcup D \sqcup \mathrm{Int}(\mathcal{R}^+) \sqcup \mathrm{Int}(\mathcal{R}^-)\,.$$

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We also define:

$$I^- \stackrel{\text{def.}}{=} I \cap \mathcal{R}^- = [-1,0] \times \{(0,0)\}\,, \quad I^+ \stackrel{\text{def.}}{=} I \cap \mathcal{R}^+ = [0,1] \times \{(0,0)\}\,,$$

which give the decomposition:

$$I = \mathrm{Int}(I^+) \sqcup \mathrm{Int}(I^-) \sqcup \{(0,0,0)\}\,.$$

3.2 The map b

Define the function $b \in \mathcal{C}^\infty(M, \mathbb{R}^3)$ through:

$$b(p) \stackrel{\text{def.}}{=} (b_+(p), b_-(p), b_3(p))\,. \tag{3.10}$$

Proposition. The image of b is a subset of $\mathcal{R}$.

Proof. Let us separate ξ_i into positive and negative chirality parts:

$$\xi_i = \xi_i^+ + \xi_i^-\,, \text{ with } \xi_i^\pm \stackrel{\text{def.}}{=} P_\pm \xi_i \text{ and } i = 1,2\,.$$

The condition $\mathscr{B}(\xi_i, \xi_j) = \delta_{ij}$ and the definitions of b_1, b_2 and b_3 give the equations:

$$||\xi_i^+||^2 + ||\xi_i^-||^2 = 1\,, \qquad ||\xi_i^+||^2 - ||\xi_i^-||^2 = b_i\,,$$
$$\mathscr{B}(\xi_1^+, \xi_2^+) + \mathscr{B}(\xi_1^-, \xi_2^-) = 0\,, \qquad \mathscr{B}(\xi_1^+, \xi_2^+) - \mathscr{B}(\xi_1^-, \xi_2^-) = b_3\,,$$

which can be solved to give:

$$||\xi_i^\pm||^2 = \frac{1}{2}(1 \pm b_i)\,, \quad \mathscr{B}(\xi_1^\pm, \xi_2^\pm) = \pm\frac{1}{2} b_3\,. \tag{3.11}$$

The Gram matrix Γ of the ordered system $(\xi_1^+, \xi_2^+, \xi_1^-, \xi_2^-)$ takes the block diagonal form:[6]

$$\Gamma = \begin{bmatrix} \Gamma_+ & 0 \\ 0 & \Gamma_- \end{bmatrix} \tag{3.12}$$

where:

$$\Gamma_\pm \stackrel{\text{def.}}{=} \begin{bmatrix} ||\xi_1^\pm||^2 & \mathscr{B}(\xi_1^\pm, \xi_2^\pm) \\ \mathscr{B}(\xi_2^\pm, \xi_1^\pm) & ||\xi_2^\pm||^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(1 \pm b_1) & \pm\frac{1}{2} b_3 \\ \pm\frac{1}{2} b_3 & \frac{1}{2}(1 \pm b_2) \end{bmatrix}$$

are the Gram matrices of the pairs $(\xi_1^\pm, \xi_2^\pm)$. A simple computation gives:

$$\det \Gamma_\pm = \frac{1}{4}(1 + b_1 b_2 \pm b_1 \pm b_2 - b_3^2) = \frac{1}{4}\left[(1 \pm b_+)^2 - \rho^2\right]\,. \tag{3.13}$$

The conclusion now follows from (3.13) upon using the fact that $\Gamma_\pm$ are semipositive, which by Sylvester's theorem amounts to the conditions $(\Gamma_\pm)_{11} \geq 0$, $(\Gamma_\pm)_{22} \geq 0$ and $\det \Gamma_\pm \geq 0$. $\square$

[6] The Gram matrices considered here are defined at every point $p \in M$, hence they are matrix-valued functions defined on M.

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Remarks.

1. We have $r_\pm = \mathrm{rk}\Gamma_\pm$ and $\mathrm{rk}(K_+ \oplus K_-) = \mathrm{rk}\Gamma$.
2. The determinant $\det\Gamma = \det\Gamma_+ \det\Gamma_-$ vanishes iff one of $\det\Gamma_\pm$ vanishes. The equality $\det\Gamma_\pm = 0$ is attained on the locus where $r_\pm \leq 1$.
3. $\Gamma_\pm$ is symmetric under the exchange $\xi_1 \leftrightarrow \xi_2$.

3.3 The map b'

Consider the determinant line bundle $\det K = \wedge^2 K$. The scalar product $\mathscr{B}|_K$ induces a norm on $\det K$ which we denote by $|| \ ||$. Since (ξ_1, ξ_2) is an orthonormal frame of $(K, \mathscr{B})$, we have $||\xi_1 \wedge \xi_2|| = 1$ and hence $\xi_1 \wedge \xi_2$ is an orthonormal frame of $\det K$. The generalized bundles $K_\pm \subset S_\pm$ inherit the Euclidean scalar product $\mathscr{B}$ from S and hence $\wedge^2 K_\pm$ are normed generalized vector bundles of rank at most one. The generalized bundle morphisms $P_\pm^K \stackrel{\text{def.}}{=} P_\pm|_K^{K_\pm} : K \to K_\pm$ induce generalized bundle morphisms $\wedge^2 P_\pm^K : \det K \to \wedge^2 K_\pm$.

Proposition. We have:

$$\det\Gamma_\pm = || \wedge^2 P_\pm^K ||_{\mathrm{op}} ,$$

where $|| \ ||_{\mathrm{op}}$ denotes the fiberwise operator norm on the generalized bundle $\mathrm{Hom}(\wedge^2 K, \wedge^2 K_\pm)$. In particular, $\det\Gamma_\pm$ depend only on the subspace $\mathcal{K} \subset \Gamma(M, S)$ and are independent of the choice of orthonormal basis for $\mathcal{K}$.

Proof. By definition of $\wedge^2 P_\pm^K$, we have $(\wedge^2 P_\pm^K)(\xi_1 \wedge \xi_2) = P_\pm(\xi_1) \wedge P_\pm(\xi_2) = \xi_1^\pm \wedge \xi_2^\pm$. Using the Gram identity, this gives $||(\wedge^2 P_\pm^K)(\xi_1 \wedge \xi_2)||^2 = ||\xi_1^\pm \wedge \xi_2^\pm||^2 = \det\Gamma_\pm$, which implies the conclusion. ☐

Remark. The proposition allows one to give a different proof of the fact that the functions $b_+, \rho^2 \in \mathcal{C}^\infty(M, \mathbb{R})$ depend only on $\mathcal{K}$. This follows by taking the sum and difference of equations (3.13), which allows one to express ρ and b_+ in terms of $\det\Gamma_+$ and $\det\Gamma_-$.

The map $b' \stackrel{\text{def.}}{=} (b, \rho) : M \to \mathbb{R}^2$ depends only on $\mathcal{K}$. Since the image of b is contained inside $\mathcal{R}$, we find:

Proposition. The image of b' is a subset of Δ.

3.4 Relation to the rank stratifications of $\mathcal{D}$ and $\mathcal{D}_0$

Lemma. Let $p \in \mathcal{S}$ be a K-special point. Then:

1. When $p \in \mathcal{S}_{11} \sqcup \mathcal{S}_{12}$, we can rotate the orthonormal basis of $\mathcal{K}$ such that either of the following holds, at our choice:

   (a) $\xi_1(p) \in S_p^+$, in which case $V_1(p) = V_3^+(p) = 0$, $V_3(p) = V_3^-(p)$ and $W(p) = -V_3(p)$
   (b) $\xi_2(p) \in S_p^+$, in which case $V_2(p) = V_3^-(p) = 0$, $V_3(p) = V_3^+(p)$ and $W(p) = V_3(p)$

2. When $p \in \mathcal{S}_{11} \sqcup \mathcal{S}_{21}$, we can rotate the orthonormal basis of $\mathcal{K}$ such that either of the following holds, at our choice:

(a) $\xi_1(p) \in S_p^-$, in which case $V_1(p) = V_3^-(p) = 0$, $V_3(p) = V_3^+(p)$ and $W(p) = V_3(p)$

(b) $\xi_2(p) \in S_p^-$, in which case $V_2(p) = V_3^+(p) = 0$, $V_3(p) = V_3^-(p)$ and $W(p) = -V_3(p)$

3. When $p \in \mathcal{S}_{11}$, we can rotate the orthonormal basis of $\mathcal{K}$ such that either of the following holds, at our choice:

(a) $\xi_1(p) \in S_p^+$ and $\xi_2(p) \in S_p^-$, in which case $V_1(p) = V_2(p) = V_3^+(p) = 0$, $V_3(p) = V_3^-(p)$ and $W(p) = -V_3(p)$

(b) $\xi_1 \in S_p^-$ and $\xi_2(p) \in S_p^+$, in which case $V_1(p) = V_2(p) = V_3^-(p) = 0$, $V_3(p) = V_3^+(p)$ and $W(p) = V_3(p)$.

Proof. 1. The condition $p \in \mathcal{S}_{11} \sqcup \mathcal{S}_{12}$ implies $r_-(p) = 1$ and hence $\det \Gamma_-(p) = 0$. Then $\xi_1^-(p) = \lambda_1 w$ and $\xi_2^-(p) = \lambda_2 w$ for some $w \in S_p^- \setminus \{0\}$, where λ_1 and λ_2 are real numbers, one of which may be zero. Under a rotation (2.6) of the basis of $\mathcal{K}$, we have:

$$(\xi_1')^-(p) = \lambda_1' w\,, \quad (\xi_2')^-(p) = \lambda_2' w$$

with:

$$\lambda_1' = \lambda_1 \cos\left(\frac{u}{2}\right) + \lambda_2 \sin\left(\frac{u}{2}\right)\,, \quad \lambda_2' = -\lambda_1 \sin\left(\frac{u}{2}\right) + \lambda_2 \cos\left(\frac{u}{2}\right)\,.$$

It is easy to see that we can choose u such that either of the combinations $\lambda_1 \cos\left(\frac{u}{2}\right) + \lambda_2 \sin\left(\frac{u}{2}\right)$ or $-\lambda_1 \sin\left(\frac{u}{2}\right) + \lambda_2 \cos\left(\frac{u}{2}\right)$ vanishes, at our choice. The statements about the 1-form spinor bilinears follow immediately from the forms of ξ_i after such a rotation.

2. The case $p \in \mathcal{S}_{11} \sqcup \mathcal{S}_{21}$ proceeds similarly.

3. The condition $p \in \mathcal{S}_{11}$ implies $\det \Gamma_+(p) = \det \Gamma_-(p) = 0$. Using the result at point 2., perform a rotation of the orthonormal basis of $\mathcal{K}$ such that $\xi_2^+(p) = 0$ for the new basis. Then $\xi_2^-(p) = \xi_2(p)$ and hence $||\xi_2^-(p)|| = ||\xi_2(p)|| = 1$ and $\mathscr{B}_p(\xi_1^-(p), \xi_2^-(p)) = -\mathscr{B}_p(\xi_1^+(p), \xi_2^+(p)) = 0$, where the last relation follows from $\mathscr{B}(\xi_1, \xi_2) = 0$. Thus:

$$\det \Gamma_-(p) = ||\xi_1^-(p)||^2\,. \tag{3.14}$$

Since $\det \Gamma_-(p)$ is invariant under (2.6), we also have $\det \Gamma_-(p) = 0$ after the rotation, which implies $\xi_1^-(p) = 0$ by (3.14). Thus $\xi_1(p) \in S_p^+$ and $\xi_2(p) \in S_p^-$ after the rotation. Had we rotated such that $\xi_1^+(p) = 0$, we would have similarly concluded that $\xi_1(p) \in S_p^-$ and $\xi_2(p) \in S_p^+$. The statements about the 1-form spinor bilinears follow immediately.

□

Remark. For $p \in \mathcal{S}_{02} \sqcup \mathcal{S}_{20}$, we obviously have $V_1(p) = V_2(p) = V_3(p) = W(p) = 0$. The compactifications studied in [18] correspond to the case $M = \mathcal{S}_{02}$.

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Proposition. Let $p \in \mathcal{S}$ be a K-special point. Then $\mathcal{D}_0(p) = \mathcal{D}(p)$ and we can rotate the basis of $\mathcal{K}$ such that either $V_3(p) = W(p)$ or $V_3(p) = -W(p)$, at our choice. Moreover:

- For $p \in \mathcal{S}_{20} \sqcup \mathcal{S}_{02}$, we have $\mathcal{D}(p) = T_pM$, hence $\mathrm{rk}\mathcal{D}(p) = 8$
- For $p \in \mathcal{S}_{11}$, we have $\mathrm{rk}\mathcal{D}(p) = 7$
- For $p \in \mathcal{S}_{12} \sqcup \mathcal{S}_{21}$ we have $\mathrm{rk}\mathcal{D}(p) = 6$.

Proof. Follows from the Lemma and from the remark above upon using the fact that $\mathcal{D}$ and $\mathcal{D}_0$ are invariant under rotations of the basis of $\mathcal{K}$. The proposition also follows from Theorem 1 below and from the results of subsection 4.3 and of appendix E. □

Remark. It is shown in appendix E that, for $p \in \mathcal{G}$, we have $\mathrm{rk}\mathcal{D}(p) \in \{5, 6, 7\}$ and $\mathrm{rk}\mathcal{D}_0(p) \in \{4, 6\}$, hence $\mathcal{D}_0(p)$ and $\mathcal{D}(p)$ may differ; in fact, their ranks cannot be determined only from the value of $b(p)$. Together with the Proposition, this gives:

$$\mathcal{S}_{02} \sqcup \mathcal{S}_{20} = \mathcal{W}_0 = \mathcal{Z}_0\,, \quad \mathcal{S}_{11} = \mathcal{Z}_1\,, \quad \mathcal{S}_{12} \sqcup \mathcal{S}_{21} \subset \mathcal{Z}_2\,.$$

A precise description of the relation between the chirality stratification and the rank stratifications of $\mathcal{D}$ and $\mathcal{D}_0$ can be found in section 5.

3.5 Relation to the stabilizer group

Proposition. Let p be any point of M. Then the following statements hold:

1. When $p \in \mathcal{S}_{02} \sqcup \mathcal{S}_{20}$ we have $H_p \simeq \mathrm{SU}(4)$
2. When $p \in \mathcal{S}_{11}$, we have $H_p \simeq \mathrm{G}_2$
3. When $p \in \mathcal{S}_{12} \sqcup \mathcal{S}_{21}$, we have $H_p \simeq \mathrm{SU}(3)$
4. When $p \in \mathcal{G}$, we have either $H_p \simeq \mathrm{SU}(2)$ or $H_p \simeq \mathrm{SU}(3)$, according to whether $\dim \mathcal{D}_0(p) = 4$ or $\dim \mathcal{D}_0(p) = 6$.

Proof.
1. In this case, ξ_1 and ξ_2 are chiral and of the same chirality at p, so their stabilizer inside $\mathrm{Spin}(8)$ equals $\mathrm{SU}(4)$.
2. After a rotation as in the Lemma given in the previous subsection, we have two non-vanishing spinors ξ_1 and ξ_2 of opposite chirality at p, whose stabilizer inside $\mathrm{Spin}(8)$ is isomorphic with G_2.
3. Consider the case $p \in \mathcal{S}_{12}$. The Lemma shows that (up to a rotation) we can assume $\xi_1(p) = \xi_1^+(p)$, $\xi_2^-(p) \neq 0$ and that $\xi_1^+(p), \xi_2^+(p)$ are linearly independent. Since S_+ and S_- are $\mathscr{B}$-orthogonal sub-bundles of S, orthogonality of ξ_1 and ξ_2 implies that $\xi_1^+(p)$ and $\xi_2^+(p)$ are $\mathscr{B}_p$-orthogonal. The stabilizer H'_p of the pair $(\xi_2^+(p), \xi_2^-(p))$ inside $\mathrm{Spin}(8)$ is isomorphic with G_2 and $S_p^\pm$ have the $\mathscr{B}$-orthogonal decompositions:

$$S_p^\pm = \Sigma_1^\pm(p) \oplus \Sigma_7^\pm(p)\,,$$

   where $\Sigma_1^\pm(p)$ are one-dimensional subspaces carrying trivial irreps while $\Sigma_7^\pm(p)$ are subspaces carrying the seven-dimensional irreps of $H'_p \simeq \mathrm{G}_2$. We have $\xi_2^\pm(p) \in \Sigma_1^\pm(p)$. Since $\xi_1^+(p)$ is $\mathscr{B}_p$-orthogonal to $\xi_2^+(p)$, we have $\xi_1^+(p) \in \Sigma_7^+(p)$. H_p is isomorphic with the stabilizer of the non-zero element $\xi_1^+(p) \in \Sigma_7^+(p)$ inside H'_p, which is known[7] to

[7] The action of G_2 on S^6 induced by this irrep. is transitive with stabilizer isomorphic with $\mathrm{SU}(3)$.

be isomorphic with SU(3). This shows that $H_p \simeq \mathrm{SU}(3)$. The case $p \in \mathcal{S}_{21}$ proceeds similarly.

4. When $p \in \mathcal{G}$, we have $H_p \simeq \mathrm{Stab}_{H''_p}(\xi_1^-(p))$, where $H''_p \stackrel{\text{def.}}{=} \mathrm{Stab}_{\mathrm{Spin}(8)}(\xi_1^+(p), \xi_2^+(p), \xi_2^-(p))$. By point 3. above, we have $H''_p \simeq \mathrm{SU}(3)$. The spaces $S_p^\pm$ decompose as:

$$S_p^\pm = \Sigma_1^\pm(p) \oplus \Sigma_1'^\pm(p) \oplus \Xi^\pm(p)\,,$$

where $\Sigma_1^\pm(p)$ and $\Sigma_1'^\pm(p)$ are trivial irreps while $\Xi^\pm(p) \simeq \mathbb{C}^3$ are fundamental irreps of SU(3) such that $\Sigma_7^\pm(p) = \Sigma_1'^\pm(p) \oplus \Xi^\pm(p)$ while $\xi_1^+(p) \in \Sigma_1'^+(p)$ and $\xi_2^\pm(p) \in \Sigma_1^\pm(p)$. Notice that $\xi_1^-(p)$ and $\xi_2^-(p)$ need not be $\mathscr{B}_p$-orthogonal. We have $H_p \simeq \mathrm{Stab}_{H''_p}(\zeta(p))$, where $\zeta(p)$ denotes the $\mathscr{B}_p$-orthogonal projection of $\xi_1^-(p)$ onto the subspace $\Xi^-(p)$. We distinguish the cases:

- $\zeta(p) = 0$. Then $H_p = H''_p \simeq \mathrm{SU}(3)$.
- $\zeta(p) \neq 0$. Then $H_p \simeq \mathrm{SU}(2)$, since it is known[8] that SU(3) acts transitively on the sphere S^5, with stabilizer SU(2).

The results of appendix F show that the first case arises iff $\mathrm{rk}\mathcal{D}_0(p) = 6$ while the second case arises iff $\mathrm{rk}\mathcal{D}_0(p) = 4$.

□

Remark. Appendix F gives an explicit construction of a one-parameter deformation of the pair (ξ_1, ξ_2) which breaks the stabilizer group from SU(3) to SU(2).

Corollary. The stabilizer stratification coincides with the rank stratification of $\mathcal{D}_0$.

3.6 Characterizing the chirality stratification

Theorem 1. The K-special locus is given by:

$$\mathcal{S} = b^{-1}(\partial\mathcal{R}) = \{p \in M | b(p) \in \partial\mathcal{R}\}\,. \tag{3.15}$$

Furthermore, we have:

- $\mathcal{S}_{11} = b^{-1}(\partial_1\mathcal{R}) = b^{-1}(\partial D)$
- $\mathcal{S}_{12} = b^{-1}(\partial_2^+\mathcal{R})$ and $\mathcal{S}_{21} = b^{-1}(\partial_2^-\mathcal{R})$
- $\mathcal{S}_{02} = b^{-1}(\partial_0^+\mathcal{R})$ and $\mathcal{S}_{20} = b^{-1}(\partial_0^-\mathcal{R})$.

Moreover, we have $\mathcal{G} = b^{-1}(\mathrm{Int}\mathcal{R})$ and hence the chirality stratification of M coincides with the b-preimage of the connected refinement of the canonical Whitney stratification of $\mathcal{R}$.

[8] For any $n \geq 2$, the action of SU(n) on S^{2n-1} induced from the fundamental representation of SU(n) on $\mathbb{C}^n = \mathbb{R}^{2n}$ is transitive and has stabilizer isomorphic with SU($n-1$).

stratum	$\mathcal{R}$-description	$r_-(p)$	$r_+(p)$	$\mathrm{rk}\mathcal{D}$	$\mathrm{rk}\mathcal{D}_0$	b_+	ρ	H_p	$\sigma_+(p)$	$\sigma_-(p)$
$\mathcal{S}_{02}$	$b^{-1}(\partial_0^+\mathcal{R})$	0	2	8	8	$+1$	0	SU(4)	2	0
$\mathcal{S}_{20}$	$b^{-1}(\partial_0^-\mathcal{R})$	2	0	8	8	-1	0	SU(4)	0	2
$\mathcal{S}_{11}$	$b^{-1}(\partial_1\mathcal{R})$	1	1	7	7	0	1	G_2	1	1
$\mathcal{S}_{12}$	$b^{-1}(\partial_2^+\mathcal{R})$	1	2	6	6	$1-\rho$	$(0,1)$	SU(3)	1	0
$\mathcal{S}_{21}$	$b^{-1}(\partial_2^-\mathcal{R})$	2	1	6	6	$-(1-\rho)$	$(0,1)$	SU(3)	0	1
$\mathcal{G}$	$b^{-1}(\mathrm{Int}\mathcal{R})$	2	2	5, 6, 7	4, 6	$(-1,1)$	$< 1-\lvert b_+\rvert$	SU(2) or SU(3)	0	0

Table 2. Chirality stratification for $s = 2$. The quantities $\sigma_\pm$ are defined through $\sigma_\pm(p) = \dim K^\pm(p) = 2 - r_\mp(p)$ (see (1.14)).

Proof. Relation (3.13) implies:

$$\det \Gamma_\pm(p) = 0 \Longleftrightarrow \rho(p) = 1 \pm b_+(p)\ . \tag{3.16}$$

Since $\det \Gamma_\pm(p) \geq 0$, we have $\rho(p) \leq 1 \pm b_+(p)$ and hence $\rho(p) \leq 1 - |b_+(p)|$. Thus $\det \Gamma_+(p) = 0$ can be realized only for $b_+(p) \leq 0$ and $\det \Gamma_-(p) = 0$ can be realized only for $b_+(p) \geq 0$ and in both cases we have $\rho(p) = 1 - |b_+(p)|$ i.e. $b(p) \in \partial\mathcal{R}$. The case $\det \Gamma_+(p) = \det \Gamma_-(p) = 0$ occurs for $b_+(p) = 0$ and $\rho(p) = 1$, i.e. on the median circle ∂D. We also have:

$$\Gamma_\pm(p) = 0 \iff b_3(p) = 0\ \&\ b_1(p) = b_2(p) = \mp 1 \iff \rho(p) = 0\ \&\ b_+(p) = \mp 1\,.$$

Hence $\Gamma_+(p) = 0$ or $\Gamma_-(p) = 0$ corresponds to $b(p) \in \partial I$, namely $\Gamma_+(p) = 0$ corresponds to the left tip $(b_+(p), \rho(p)) = (-1, 0)$ of $\mathcal{R}$ while $\Gamma_-(p) = 0$ corresponds to the right tip $(b_+(p), \rho(p)) = (+1, 0)$ of $\mathcal{R}$. The remaining statements follow since $b(M) \subset \mathcal{R}$. $\square$

The situation is summarized in table 2 and figure 5.

Remarks.

1. Theorem 1 implies a similar characterization of the stratification $\mathcal{S}$ as the b'-preimage of the obvious stratification with connected strata of the set $\Delta \setminus (-1, 1) \times \{0\}$; we leave the details of this to the reader.

2. For every $p \in M$, the dimensions $\sigma_\pm(p) \stackrel{\text{def.}}{=} \dim K^\pm(p) = 2 - r_\mp(p)$ of the chiral slices of K_p count the number of linearly independent spinors inside the space K_p which have chirality ± 1. In the case of compactifications down to AdS_3, $\sigma_+(p)$ can be interpreted [26] as the number of supersymmetries of the background which are preserved by a space-time filling M2-brane placed at p, while $\sigma_-(p)$ counts the number of supersymmetries preserved by a space-time filling M2-antibrane placed at p; these numbers are indicated in the last column of the table.

4 Algebraic constraints

The Fierz identities for $\xi_1^\pm, \xi_2^\pm$ imply that the following relations hold (see appendix B):

$$\begin{aligned} &||V_-||^2 + b_-^2 = ||V_3||^2 + b_3^2\,, \quad ||V_+||^2 + b_+^2 = 1 - (||V_3||^2 + b_3^2) \\ &\langle V_+, V_-\rangle + b_+ b_- = \langle V_+, V_3\rangle + b_+ b_3 = \langle V_-, V_3\rangle + b_- b_3 = 0 \\ &||W||^2 + ||V_3||^2 = 1 + b_-^2 - b_+^2 \\ &\langle W, V_+\rangle = 0\,, \quad \langle W, V_-\rangle = b_3\,, \quad \langle W, V_3\rangle = -b_-\,. \end{aligned} \tag{4.1}$$

In particular, the first two rows of (4.1) form the following system for V_r, b_r:

$$\begin{aligned} ||V_-||^2 + b_-^2 &= ||V_3||^2 + b_3^2 \\ ||V_+||^2 + b_+^2 &= 1 - (||V_3||^2 + b_3^2) \\ \langle V_+, V_-\rangle + b_+ b_- &= \langle V_+, V_3\rangle + b_+ b_3 = \langle V_-, V_3\rangle + b_- b_3 = 0\,. \end{aligned} \tag{4.2}$$

Relations (4.2) constrain the norms $||V_r||^2$ and the angles $\theta_{rs} = \theta_{sr}$ between V_r and V_s (a total of six quantities) in terms of the three quantities b_r. Fixing the latter generally fails to completely determine the former.

Remark. For a general choice of V_r, one cannot find b_r such that (4.2) is satisfied. The conditions on V_r under which it is possible to solve for b_r are given in appendix D.

4.1 Reduction to a semipositivity problem

Let us define:

$$\beta \stackrel{\text{def.}}{=} \sqrt{b_3^2 + ||V_3||^2} = \sqrt{b_-^2 + ||V_-||^2} \tag{4.3}$$

as well as:

$$\rho \stackrel{\text{def.}}{=} \sqrt{b_-^2 + b_3^2} \tag{4.4}$$

and consider the map $B : M \to \mathbb{R}^4$ defined through:

$$B(p) \stackrel{\text{def.}}{=} (b(p), \beta(p))\,, \quad p \in M\,. \tag{4.5}$$

The second line in (4.2) gives:

$$||V_+||^2 = 1 - b_+^2 - \beta^2\,, \tag{4.6}$$

which shows that β contains the same information as the norm of V_+, provided that b_+ is known.

When β is fixed, the constraints (4.2) amount to the condition that the Gram matrix of V_+, V_-, V_3 be given by:

$$G(b, \beta) = \begin{bmatrix} 1 - \beta^2 - b_+^2 & -b_+ b_- & -b_+ b_3 \\ -b_- b_+ & \beta^2 - b_-^2 & -b_- b_3 \\ -b_3 b_+ & -b_3 b_- & \beta^2 - b_3^2 \end{bmatrix}. \tag{4.7}$$

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The system given by (4.3) and the last two rows of (4.2) has solutions V_r iff the symmetric matrix $G(b, \beta)$ is positive semidefinite; in this case, V_r are determined by β and b_r up to a common action of the group $\Gamma(M, \mathrm{O}(TM, g))$. Furthermore, V_+, V_- and V_3 are linearly independent at $p \in M$ iff $G(p) \stackrel{\text{def.}}{=} G(b(p), \beta(p))$ is positive definite. Similarly, the system (4.1) amounts to the condition that the Gram matrix of V_+, V_-, V_3, W be given by:

$$\hat{G}(b, \beta) = \begin{bmatrix} 1-\beta^2-b_+^2 & -b_+b_- & -b_+b_3 & 0 \\ -b_-b_+ & \beta^2-b_-^2 & -b_-b_3 & b_3 \\ -b_3b_+ & -b_3b_- & \beta^2-b_3^2 & -b_- \\ 0 & b_3 & -b_- & 1-\beta^2-b_+^2+\rho^2 \end{bmatrix} . \tag{4.8}$$

Notice that $V_+ \perp W$ and $||W||^2 = ||V_+||^2 + \rho^2$.

Remark. Relation (4.6) and the observations of subsection 2.3 imply that β is invariant under any proper or improper rotation of the orthonormal basis of $\mathcal{K}$. Hence b_+, ρ and β depend only on $\mathcal{K}$. Relations (4.7) and (4.8) show that all scalar invariants under the transformations (2.6) which can be constructed from V_+, V_-, V_3 and W can be expressed as functions of b_+, ρ and β.

The semipositivity conditions for $G(b, \beta)$ can be analyzed using Sylvester's criterion, leading to a nonlinear programming problem whose solution is given in appendix D. To state the results concisely, we introduce a compact four-dimensional semi-algebraic body $\mathfrak{P}$ which can be viewed as a singular segment fibration over $\mathcal{R}$.

4.2 The four-dimensional body $\mathfrak{P}$

Recall that the image of b is contained in $\mathcal{R}$. The determinant of the Gram matrix (4.7) takes the form:

$$\det G = -\beta^2 P(b, \beta) , \tag{4.9}$$

where:

$$P(b, \beta) \stackrel{\text{def.}}{=} \beta^4 - \beta^2(1 + b_3^2 + b_-^2 - b_+^2) + b_3^2 + b_-^2 = \beta^4 - \beta^2(1 + \rho^2 - b_+^2) + \rho^2 . \tag{4.10}$$

Thus $\det G(b, \beta)$ vanishes for $\beta = 0$ or $\beta = \sqrt{f_\pm(b)}$, where the functions $f_\pm : \mathcal{R} \to \mathbb{R}$ (which give the roots of the second order polynomial $x^2 - (1 + \rho^2 - b_+^2)x + \rho^2$) are defined through:

$$f_\pm(b_+, b_-, b_3) = f_\pm(b_+, \rho) \stackrel{\text{def.}}{=} \frac{1}{2}\left(1 - b_+^2 + \rho^2 \pm \sqrt{h(b_+, \rho)}\right) . \tag{4.11}$$

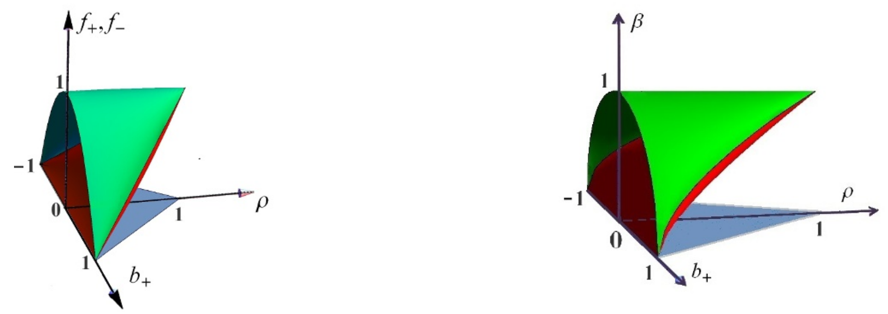
The discriminant:

$$h(b) = h(b_+, \rho) \stackrel{\text{def.}}{=} (1 + b_+ + \rho)(1 - b_+ + \rho)(1 + b_+ - \rho)(1 - b_+ - \rho) \tag{4.12}$$

is non-negative on Δ and vanishes only for $\rho = 1 - |b_+|$, i.e. on the left and right sides of Δ. The functions $f_\pm$ satisfy:

$$0 \le f_-(b) \le f_+(b) \le 1 , \quad \forall b \in \mathcal{R} ,$$

where:



(a) Graphs of the functions $f_+(b_+, \rho)$ (green) and $f_-(b_+, \rho)$ (red) for $(b_+, \rho) \in \Delta$ (blue).

(b) Graphs of the functions $\sqrt{f_+(b_+, \rho)}$ (green) and $\sqrt{f_-(b_+, \rho)}$ (red) for $(b_+, \rho) \in \Delta$ (blue).

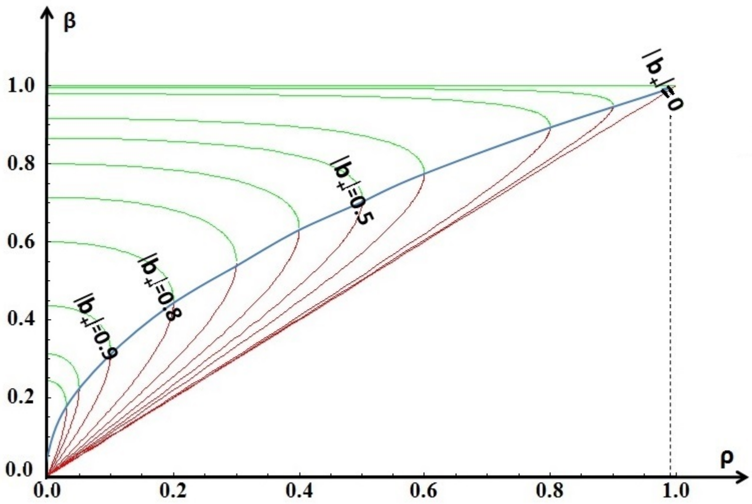


Figure 7. Graphs of $\beta = \sqrt{f_+(b_+, \rho)}$ (green) and $\beta = \sqrt{f_-(b_+, \rho)}$ (red) for various fixed values of $|b_+| \in [0, 1]$. Notice that the two graphs match each other smoothly at $|b_+| = 1 - \rho < 1$ (corresponding to $\partial\mathcal{R}$), where both f_+ and f_- equal $\sqrt{\rho}$. The matching is non-smooth only when $\rho = \beta = 1,\ b_+ = 0$, which corresponds to the circle $\partial\mathfrak{D}$ defined below.

- the equality $f_-(b) = f_+(b)$ is attained iff $b \in \partial\mathcal{R}$, where we have $f_+|_{\partial\mathcal{R}} = f_-|_{\partial\mathcal{R}} = \rho$;
- the equality $f_-(b) = 0$ is attained iff $b \in I$;
- the equality $f_+(b) = 1$ is attained iff $b \in D$.

Notice that $f_\pm$ depend only on b_+ and ρ and hence they can be viewed as functions defined on Δ (see figures 6 and 7). In fact, they are symmetric under $b_+ \to -b_+$, so they depend only on $|b_+|$ and ρ. Various special values of $f_\pm$ are summarized in table 3.

For every $b \in \mathcal{R}$, consider the closed interval:

$$J(b) = J(b_+, \rho) \stackrel{\text{def.}}{=} [\sqrt{f_-(b)}, \sqrt{f_+(b)}] \subset \left[\sqrt{b_-^2 + b_3^2}, \sqrt{1 - b_+^2}\right]. \qquad (4.13)$$

	$b\in \mathrm{Int} I$ $(b_+\in(-1,1),\ \rho=0)$	$b\in\partial I$ $(b_+=\pm 1,\ \rho=0)$	$b\in \mathrm{Int} D\setminus\{(0,0,0)\}$ $(b_+=0,\ \rho\in(0,1))$	$b\in\partial D$ $(b_+=0\,,\ \rho=1)$	$b\in\partial\mathcal{R}$ $(\rho=1-\lvert b_+\rvert)$
$f_+(b_+,\rho)$	$1-b_+^2$	0	1	1	ρ
$f_-(b_+,\rho)$	0	0	ρ^2	1	ρ
β	$[0,\sqrt{1-b_+^2}]$	0	$[\rho,1]$	1	$\sqrt{\rho}$
$\mathrm{rk}G$	$\{1,2,3\}$	0	$\{2,3\}$	1	$\{0,1,2\}$

Table 3. Special values for f_+ and f_-. The values allowed for $\mathrm{rk}G$ on each region follow from Theorem 2 of subsection 5.1.

This interval degenerates to a single point for $b\in\partial\mathcal{R}$, namely $J|_{\partial\mathcal{R}}=\{\sqrt{\rho}\}$. Finally, consider the following four-dimensional compact body:

$$\mathfrak{P}\stackrel{\text{def.}}{=}\{(b,\beta)\in\mathbb{R}^4|b\in\mathcal{R}\ \&\ \beta\in J(b)\}\,, \tag{4.14}$$

which is fibered over $\mathcal{R}$ via the projection $(b,\beta)\xrightarrow{\pi}b$. The fiber over $b\in\mathcal{R}$ is the segment $J(b)$, which, as mentioned above, degenerates to a point over $\partial\mathcal{R}$.

The frontier of $\mathfrak{P}$. Let:

$$C\stackrel{\text{def.}}{=}\left\{(b_-,b_3,\beta)\in\mathbb{R}^3|\ 0\leq\sqrt{b_-^2+b_3^2}\leq\beta\leq 1\right\} \tag{4.15}$$

be the full compact cone in $\mathbb{R}^3$ with apex at the origin and base given by the disk $D^2\times\{1\}$ and let:

$$F\stackrel{\text{def.}}{=}\partial C=\left\{(b_-,b_3,\sqrt{b_-^2+b_3^2})|(b_-,b_3)\in\mathrm{Int}(D^2)\right\}\sqcup\{(b_-,b_3,1)|(b_-,b_3)\in D^2\}$$

denote its frontier. Let $\dot{C}\stackrel{\text{def.}}{=}C\setminus\{(0,0,0)\}$ and $\dot{F}\stackrel{\text{def.}}{=}F\setminus\{(0,0,0)\}$. Notice that C is homeomorphic with the compact 3-dimensional ball, F is homeomorphic with S^2 and $\dot{F}$ is homeomorphic with $\mathbb{R}^2$ (and hence with the interior of the unit disk D^2). Consider the function $g:\dot{C}\to\mathbb{R}$ given by (see figure 8):

$$g(b_-,b_3,\beta)=g(\rho,\beta)\stackrel{\text{def.}}{=}\frac{1}{\beta}\sqrt{(1-\beta^2)(\beta^2-\rho^2)}\,. \tag{4.16}$$

The quantity under the square root is non-negative for $(b_-,b_3,\beta)\in C$ and we have $0\leq g(\rho,\beta)\leq\frac{\sqrt{\beta^2-\rho^2}}{\beta}\leq 1$ for $(b_-,b_3,\beta)\in\dot{C}$. Notice that g vanishes on $\dot{F}$ and is strictly positive in the interior of C.

Consider the following three-dimensional subsets of $\mathbb{R}^4$, each of which is homeomorphic with $\dot{C}$:

$$\mathfrak{C}^\pm\stackrel{\text{def.}}{=}\{(\pm g(b_-,b_3,\beta),b_-,b_3,\beta)|\ (b_-,b_3,\beta)\in\dot{C}\} \tag{4.17}$$

and the following compact interval sitting inside $\mathbb{R}^4$:

$$\mathfrak{I}\stackrel{\text{def.}}{=}[-1,1]\times\{(0,0,0)\}=I\times\{0\}\,. \tag{4.18}$$

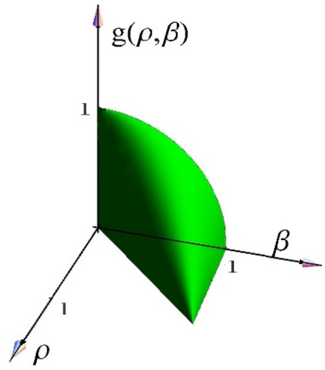


Figure 8. Graph of the function $g(\rho, \beta)$ for (ρ, β) belonging to the triangular region defined by the inequalities $0 < \rho \leq \beta \leq 1$. Notice that the directional limits of $g(\rho, \beta)$ at the point $\rho = \beta = 0$ (taken from within this triangular region) take any value within the interval $[0, 1]$.

The intersection of the sets $\mathfrak{C}^{\pm}$ is given by:

$$\mathfrak{F} \stackrel{\text{def.}}{=} \mathfrak{C}^{+} \cap \mathfrak{C}^{-} = \{0\} \times \dot{F} \tag{4.19}$$

and $\mathfrak{C}^{\pm}$ are disjoint from $\mathfrak{I}$ (since $\beta \neq 0$ on $\mathfrak{C}^{\pm}$ while $\beta = 0$ on $\mathfrak{I}$). Notice that $\mathrm{Int}\mathfrak{C}^{+}$ and $\mathrm{Int}\mathfrak{C}^{-}$ are homeomorphic with $\mathrm{Int}\dot{C} = \mathrm{Int}C$ and hence with the interior of the unit 3-ball while $\mathfrak{F}$ is homeomorphic with the interior of the two-dimensional disk. Let:

$$\mathfrak{I}^{+} \stackrel{\text{def.}}{=} [0, 1] \times \{0_{\mathbb{R}^3}\} = I^{+} \times \{0\}\,, \quad \mathfrak{I}^{-} \stackrel{\text{def.}}{=} [-1, 0] \times \{0_{\mathbb{R}^3}\} = I^{-} \times \{0\}$$

be the compact right and left halves of $\mathfrak{I}$, which satisfy $\mathfrak{I}^{+} \cap \mathfrak{I}^{-} = \{0_{\mathbb{R}^4}\}$. Figure 9 shows the sections of $\partial\mathfrak{P}$ with the hyperplane $b_3 = 0$.

Proposition. The frontier of $\mathfrak{P}$ is given by:

$$\partial\mathfrak{P} = \mathfrak{C}^{+} \cup \mathfrak{C}^{-} \cup \mathfrak{I} = \mathrm{Int}\mathfrak{C}^{+} \sqcup \mathrm{Int}\mathfrak{C}^{-} \sqcup \mathfrak{F} \sqcup \mathfrak{I}\,, \tag{4.20}$$

where the components can be identified as:

$$\begin{aligned} \mathrm{Int}\mathfrak{C}^{\pm} &= \{(b, \beta) \in \partial\mathfrak{P} |\ \beta > 0\ \&\ \pm b_{+} > 0\} \\ \mathfrak{F} &= \{(b, \beta) \in \partial\mathfrak{P} |\ \beta > 0\ \&\ b_{+} = 0\} \\ \mathfrak{I} &= \{(b, \beta) \in \partial\mathfrak{P} |\ \beta = 0\}\,. \end{aligned} \tag{4.21}$$

Moreover, $\mathfrak{I}$ is closed (thus $\mathrm{fr}\mathfrak{I} = \emptyset$), while:[9]

$$\mathrm{fr}(\mathrm{Int}\mathfrak{C}^{\pm}) = \mathfrak{F} \sqcup \mathfrak{I}^{\pm}\,, \quad \mathrm{fr}\mathfrak{F} = \{0_{\mathbb{R}^4}\}\,. \tag{4.22}$$

[9]Since $\partial\mathfrak{P}$ is a closed subset of $\mathbb{R}^4$, the small frontier of a subset $A \subset \mathfrak{P}$ taken with respect to the topology induced on $\partial\mathfrak{P}$ from $\mathbb{R}^4$ coincides with the small frontier $\mathrm{fr}A$ of A in $\mathbb{R}^4$.

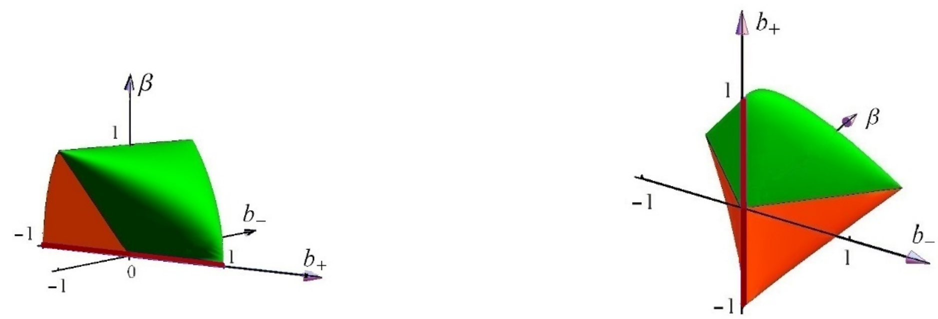


Figure 9. Section of $\partial\mathfrak{P}$ with the hyperplane $b_3 = 0$, where the corresponding sections of $\mathfrak{C}^-$, $\mathfrak{C}^+$ and $\mathfrak{I}$ are represented in orange, green and brown.

Remark. Topologically, $\mathrm{fr}(\mathrm{Int}\mathfrak{C}^\pm) = \mathrm{fr}(\mathfrak{C}^\pm)$ is obtained from the compact disk upon picking two opposite points on the boundary circle and identifying the resulting halves of the boundary to a segment corresponding to $\mathfrak{I}^\pm$; the result is of course homeomorphic to a sphere.

Proof. We have:

$$P(b, \beta) = (1 - \beta^2)(\rho^2 - \beta^2) + b_+^2 \beta^2 .$$

The frontier of $\mathfrak{P}$ is the semi-algebraic set obtained by intersecting $\mathcal{R}$ with the hypersurface $P(b, \beta) = 0$. This equation can be written as:

$$b_+^2 \beta^2 = (1 - \beta^2)(\beta^2 - \rho^2) \tag{4.23}$$

and requires that the right hand side be non-negative, which for $b \in \mathcal{R}$ is equivalent with the condition $\beta \in [\rho, 1]$ i.e. $(b_-, b_3, \beta) \in C$. To study the solutions of (4.23), assume that this condition is satisfied and consider the cases:

- $\beta = 0$. Then (4.23) requires $\rho = 0$ while b_+ is undetermined within the interval $[-1, 1]$, which means that (b, β) belongs to the interval $\mathfrak{I}$.
- $\beta > 0$. Then (4.23) requires $(b_-, b_3, \beta) \in \dot{C}$ as well as $b_+ = \pm g(b_-, b_3, \beta)$, where the function $g : \dot{C} \to \mathbb{R}$ was defined in (4.16).

The above shows that $\partial\mathfrak{P}$ has the decomposition (4.20) and that (4.21) holds. The remaining statements follow from (4.21).

□

The body $\mathfrak{P}$ is a semi-algebraic set, hence it admits a canonical Whitney stratification by smooth semi-algebraic subsets. To describe this stratification, notice that the set defined in (4.19) decomposes as:

$$\mathfrak{F} = \mathrm{Int}\mathfrak{D} \sqcup \partial\mathfrak{D} \sqcup \mathfrak{A} , \tag{4.24}$$

where:

$$\begin{aligned} \mathfrak{D} &\stackrel{\text{def.}}{=} \{(0, b_-, b_3, 1) | (b_-, b_3) \in D^2\} = D \times \{1\} \\ \mathfrak{A} &\stackrel{\text{def.}}{=} \{(0, b_-, b_3, \sqrt{b_-^2 + b_3^2}) | (b_-, b_3) \in \mathrm{Int}D^2 \setminus \{0_{\mathbb{R}^2}\}\} \end{aligned} \tag{4.25}$$

are homeomorphic with the compact disk and with an open annulus, respectively. We have:

$$\partial\mathfrak{D} = \{(0, b_-, b_3, 1)|(b_-, b_3) \in \partial D^2\} = \partial D \times \{1\}\,.$$

The frontier $\partial\mathfrak{P}$ has the following decomposition into borderless manifolds of dimensions $k = 0, 1, 2, 3$:

$$\partial\mathfrak{P} = \partial_3\mathfrak{P} \sqcup \partial_2\mathfrak{P} \sqcup \partial_1\mathfrak{P} \sqcup \partial_0\mathfrak{P}\,, \tag{4.26}$$

where the k-dimensional pieces are the following unions of connected components:

$$\begin{aligned}
\partial_3\mathfrak{P} &= \mathrm{Int}\mathfrak{C}^+ \sqcup \mathrm{Int}\mathfrak{C}^- \\
\partial_2\mathfrak{P} &= \mathrm{Int}\mathfrak{D} \sqcup \mathfrak{A} \\
\partial_1\mathfrak{P} &= \mathrm{Int}\mathfrak{I}^+ \sqcup \mathrm{Int}\mathfrak{I}^- \sqcup \partial\mathfrak{D} \\
\partial_0\mathfrak{P} &= \partial_0^+\mathfrak{P} \sqcup \partial_0^0\mathfrak{P} \sqcup \partial_0^-\mathfrak{P}
\end{aligned}\,, \tag{4.27}$$

with:

$$\partial_0^\pm\mathfrak{P} \stackrel{\text{def.}}{=} \partial_0^\pm\mathcal{R} \times \{0\} = \{(\pm 1, 0, 0, 0)\}\,, \quad \partial_0^0\mathfrak{P} \stackrel{\text{def.}}{=} \{0_{\mathbb{R}^4}\}\,. \tag{4.28}$$

The ten connected components listed above give the connected refinement of the canonical Whitney stratification of $\partial\mathfrak{P}$, whose incidence poset is depicted in figure 10. Using relations (4.22) and (4.24), we find:

$$\begin{aligned}
\mathrm{fr}(\mathrm{Int}\mathfrak{C}^\pm) &= \mathrm{Int}\mathfrak{D} \sqcup \partial\mathfrak{D} \sqcup \mathfrak{A} \sqcup \mathrm{Int}\mathfrak{I}^\pm \sqcup \partial_0^0\mathfrak{P} \sqcup \partial_0^\pm\mathfrak{P} \\
\mathrm{fr}(\mathrm{Int}\mathfrak{D}) &= \partial\mathfrak{D}\,, \quad \mathrm{fr}\mathfrak{A} = \partial\mathfrak{D} \sqcup \partial_0^0\mathfrak{P} \\
\mathrm{fr}(\mathrm{Int}\mathfrak{I}^\pm) &= \partial_0^0\mathfrak{P} \sqcup \partial_0^\pm\mathfrak{P}\,, \quad \mathrm{fr}(\partial\mathfrak{D}) = \emptyset\,,
\end{aligned} \tag{4.29}$$

which imply:

$$\begin{aligned}
\mathrm{fr}(\partial_3\mathfrak{P}) &= \partial_2\mathfrak{P} \sqcup \partial_1\mathfrak{P} \sqcup \partial_0\mathfrak{P} = \mathfrak{F} \sqcup \mathfrak{I} \\
\mathrm{fr}(\partial_2\mathfrak{P}) &= \partial\mathfrak{D} \sqcup \partial_0^0\mathfrak{P} \\
\mathrm{fr}(\partial_1\mathfrak{P}) &= \partial_0\mathfrak{P} = \partial\mathfrak{I} \sqcup \partial_0^0\mathfrak{P}\,.
\end{aligned} \tag{4.30}$$

Notice that $\mathrm{fr}\mathfrak{A} = \partial\mathfrak{D} \sqcup \partial_0^0\mathfrak{P}$.

Remark. The canonical Whitney stratification of $\partial\mathfrak{P}$ has six strata given by $\partial_3\mathfrak{P}$, $\partial_2\mathfrak{P}$, $\partial\mathfrak{D}$, $\mathrm{Int}\mathfrak{I}^+ \sqcup \mathrm{Int}\mathfrak{I}^-$, $\partial\mathfrak{I} = \partial_0^+\mathfrak{P} \sqcup \partial_0^-\mathfrak{P}$ and $\partial_0^0\mathfrak{P}$. The canonical Whitney stratification of $\mathfrak{P}$ is obtained from this by adding the stratum $\mathrm{Int}\mathfrak{P}$ and similarly for its connected refinement.

The values of b_+, ρ and β on the connected strata of $\partial\mathfrak{P}$ are summarized in table 4.

The following statement follows from the results of appendix D:

Proposition. The locus $\beta = 0$ on $\mathfrak{P}$ coincides with the compact segment $\mathfrak{I}$, while the locus $\beta = 1$ on $\mathfrak{P}$ coincides with the compact disk $\mathfrak{D} = D \times \{1\}$. The locus $\beta = \rho$ on $\mathfrak{P}$ coincides with $\bar{\mathfrak{A}} = \mathfrak{A} \sqcup \partial\mathfrak{D} \sqcup \partial_0^0\mathfrak{P}$. In particular, the only locus on $\mathcal{R}$ where the value $\beta = 0$ can be attained is the interval I while the only locus on $\mathcal{R}$ where $\beta = 1$ can be attained is the median disk D.

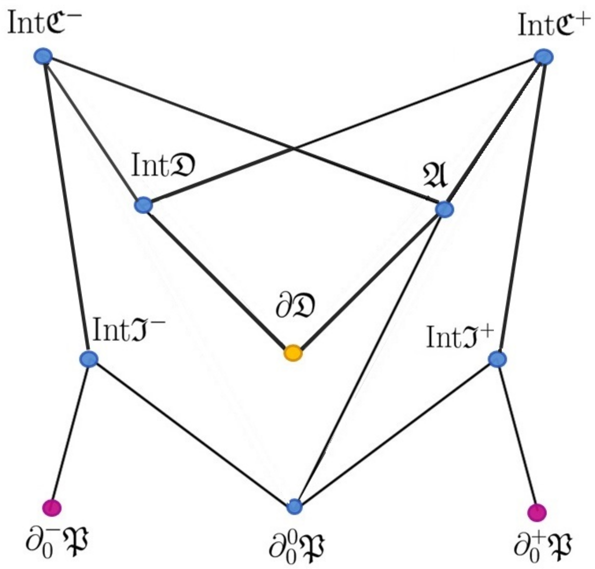


Figure 10. The Hasse diagram of the incidence poset (see appendix C) of the connected refinement of the Whitney stratification of $\partial\mathfrak{P}$. The B-preimages of the connected components depicted as points colored in magenta, yellow and cyan are strata of SU(4), G_2 and SU(3) structure in M (see table 5 in subsection 5.2). The diagram depicts the covering relation of the incidence poset, namely an element of that poset covers another iff it sits above it in the diagram and there is an edge connecting the two elements. The small frontier of each connected Whitney stratum is the disjoint union of the strata covered by it in the diagram.

connected stratum	dimension	component of	topology	b_+	ρ	β
$\partial_0^-\mathfrak{P}$	0	$\partial_0\mathfrak{P}$	point	-1	0	0
$\partial_0^+\mathfrak{P}$	0	$\partial_0\mathfrak{P}$	point	$+1$	0	0
$\partial_0^0\mathfrak{P}$	0	$\partial_0\mathfrak{P}$	point	0	0	0
Int$\mathfrak{I}^-$	1	$\partial_1\mathfrak{P}$	open interval	$(-1,0)$	0	0
Int$\mathfrak{I}^+$	1	$\partial_1\mathfrak{P}$	open interval	$(0,1)$	0	0
$\partial\mathfrak{D}$	1	$\partial_1\mathfrak{P}$	circle	0	1	1
Int$\mathfrak{D}$	2	$\partial_2\mathfrak{P}$	open disk	0	$[0,1)$	1
$\mathfrak{A}$	2	$\partial_2\mathfrak{P}$	open annulus	0	$(0,1)$	ρ
Int$\mathfrak{C}^-$	3	$\partial_3\mathfrak{P}$	open full cone	$-g(\rho,\beta)$	$(0,1)$	$(\rho,1)$
Int$\mathfrak{C}^+$	3	$\partial_3\mathfrak{P}$	open full cone	$+g(\rho,\beta)$	$(0,1)$	$(\rho,1)$

Table 4. Connected refinement of the Whitney stratification of $\partial\mathfrak{P}$. The colors used in this table (magenta, yellow and cyan) correspond to loci of SU(4), G_2 and SU(3) structures on M.

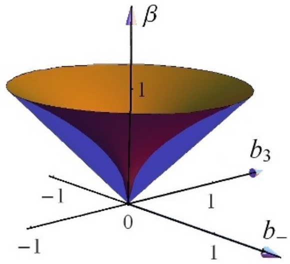


Figure 11. The loci $\mathfrak{S}^\pm$ correspond to the hypersurface $\beta = \sqrt{\rho}$ (brown) inside Int$C$ (blue).

4.3 The preimage of $\partial\mathcal{R}$ inside $\partial\mathfrak{P}$

Consider the surjection $\pi : \mathfrak{P} \to \mathcal{R}$ given by $\pi(b, \beta) = b$ (the projection on the first three coordinates). Since $J|_{\partial\mathcal{R}} = \{\sqrt{\rho}\}$, we have:

$$\pi^{-1}(b) = \{(b, \sqrt{\rho})\} \quad \text{for} \quad b \in \partial\mathcal{R}\,.$$

Hence the restriction of π to the subset $\pi^{-1}(\partial\mathcal{R}) \subset \partial\mathfrak{P}$ is a bijection onto $\partial\mathcal{R}$. It is clear that $\partial\mathfrak{I} \cup \partial\mathfrak{D}$ is contained in $\pi^{-1}(\partial\mathcal{R})$ while Int$\mathfrak{I}$, $\mathfrak{A}$ and Int$\mathfrak{D}$ are disjoint from $\pi^{-1}(\partial\mathcal{R})$. Using (4.27), this gives:

$$\pi^{-1}(\partial\mathcal{R}) = \partial\mathfrak{I} \sqcup \partial\mathfrak{D} \sqcup \mathfrak{S}^+ \sqcup \mathfrak{S}^- \tag{4.31}$$

where:

$$\mathfrak{S}^\pm \stackrel{\text{def.}}{=} \mathrm{Int}\mathfrak{C}^\pm \cap \pi^{-1}(\partial\mathcal{R})$$

We have:

$$\begin{aligned}
\pi(\partial\mathfrak{D}) &= \partial D \\
\pi(\partial\mathfrak{I}) &= \partial I \quad \text{namely} \quad \pi(\partial_0^\pm\mathfrak{P}) = \partial_0^\pm\mathcal{R} \\
\pi(\mathfrak{S}^\pm) &= \partial_2^\pm\mathcal{R} \quad \text{hence} \quad \pi(\partial_3\mathfrak{P}) = \partial_2\mathcal{R}. \\
\pi(\partial_2\mathfrak{P}) &\subset \mathrm{Int}D\,, \qquad \pi(\mathrm{Int}\mathfrak{I}) \subset \mathrm{Int}I\,.
\end{aligned} \tag{4.32}$$

In particular, $\pi(\partial_0^0\mathfrak{P})$ and $\pi(\mathrm{Int}\mathfrak{I}^\pm)$ are contained in Int$\mathcal{R}$.

Proposition. $\mathfrak{S}^\pm$ are the following hypersurfaces contained in Int$\mathfrak{C}^\pm$ (see figure 9):

$$\mathfrak{S}^\pm = \left\{(\pm(1-\rho), b_-, b_3, \sqrt{\rho}) | \rho \stackrel{\text{def.}}{=} \sqrt{b_-^2 + b_3^2} \in (0,1)\right\} = \{(b, (b_-^2 + b_3^2)^{1/4}) | b \in \partial_2^\pm\mathcal{R}\}\,. \tag{4.33}$$

Proof. For $(b, \beta) \in \mathrm{Int}\mathfrak{C}^\pm$, we have $b_+ = \pm g(\rho, \beta)$, where $g(\rho, \beta)$ was defined in (4.16). The condition $(b, \beta) \in \mathfrak{S}^\pm$ further requires $b \in \partial\mathcal{R}$, i.e. $b_+ = \pm(1-\rho)$. This gives $1 - \rho = g(\rho, \beta)$, which (upon squaring both sides) is easily seen to be equivalent with $\beta = \sqrt{\rho}$. The condition $(b, \beta) \in \mathrm{Int}\mathfrak{C}^\pm$ excludes the values $\beta = \rho = 0$ and $\beta = \rho = 1$, hence we must have $\rho \in (0, 1)$. $\square$

Remark. Since $\rho \leq 1$, we have $\beta|_{\partial\mathcal{R}} = \sqrt{\rho} \geq \rho$, with equality iff $b \in \partial D$, which corresponds to $(b, \beta) \in \partial\mathfrak{D}$.

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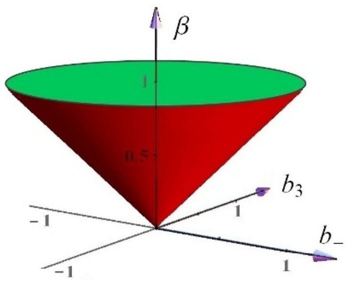
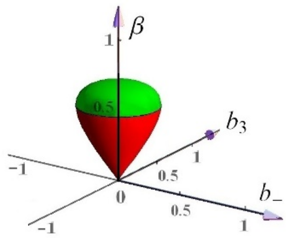


(a) Plot of $\sqrt{f_+(0,b_-,b_3)}$ (green) and $\sqrt{f_-(0,b_-,b_3)}$ (red) for (b_-,b_3) belonging to the unit disk. The section of $\mathfrak{P}$ with the hyperplane $b_+=0$ is the compact full cone $\mathfrak{K}=\{0\}\times C$ contained between these two graphs, whose basis is the disk $\mathfrak{D}$ (green). This disk coincides with the locus on $\mathfrak{P}$ where $\beta=1$. The apex of the cone is the midpoint of the interval $\mathfrak{I}$.



(b) Plot of $\sqrt{f_+(b)}$ and $\sqrt{f_-(b)}$ for $b\in\mathcal{R}$ with $b_+=0.5$ (thus $\rho\leq 1-0.5$). The section of $\mathfrak{P}$ with the hyperplane $b_+=0.5$ is the body of revolution contained between these two graphs. The boundary of this body is the union of a cone with a "cap" (a curved disk).

Figure 12. Presentation of $\mathfrak{P}$ as a singular fibration over the interval $[-1,1]$. The sections of $\mathfrak{P}$ with planes $b_+=$ const. $\neq\pm1$ are 3-dimensional bodies of revolution around the β-axis, obtained by rotating the graphs of figure 7. The points of Int$\mathfrak{I}$ are conical singularities for these bodies. The bodies degenerate to points for $b_+=\pm1$.

Sections of $\mathfrak{P}$ with the hyperplanes $b_+=$ const. The sections of $\mathfrak{P}$ with such hyperplanes are depicted in figure 12; they allow one to present $\mathfrak{P}$ as a fibration over the interval $[-1,1]$. In particular, the section with the hyperplane $b_+=0$ is the compact full 3-dimensional cone $\mathfrak{K}=\{0\}\times C$, whose frontier equals $\mathfrak{F}$.

5 Description of the rank stratifications of $\mathcal{D}$ and $\mathcal{D}_0$

5.1 Description of the rank stratification of $\mathcal{D}$

The following result shows that the map B has image contained in $\mathfrak{P}$ and that the rank stratification of $\mathcal{D}$ is a certain coarsening of the B-preimage of the connected refinement of the Whitney stratification of $\mathfrak{P}$.

Theorem 2. The image of the map B defined in (4.5) is contained in $\mathfrak{P}$:

$$\mathrm{im}B\subset\mathfrak{P}$$

Furthermore, the following hold for $p\in M$:

- $\mathrm{rk}\mathcal{D}(p)=5$ iff $B(p)\in\mathrm{Int}\mathfrak{P}$
- $\mathrm{rk}\mathcal{D}(p)=6$ iff $B(p)\in\partial_2\mathfrak{P}\cup\partial_3\mathfrak{P}=\mathrm{Int}\mathfrak{D}\sqcup\mathfrak{A}\sqcup\mathrm{Int}\mathfrak{C}^+\sqcup\mathrm{Int}\mathfrak{C}^-$
- $\mathrm{rk}\mathcal{D}(p)=7$ iff $B(p)\in\partial_0^0\mathfrak{P}\sqcup\partial_1\mathfrak{P}=\partial\mathfrak{D}\sqcup\mathrm{Int}\mathfrak{I}$
- $\mathrm{rk}\mathcal{D}(p)=8$ iff $B(p)\in\partial_0^+\mathfrak{P}\sqcup\partial_0^-\mathfrak{P}=\partial\mathfrak{I}$.

In particular, the rank stratification of $\mathcal{D}$ is given by:

$$\mathcal{U}=B^{-1}(\mathrm{Int}\mathfrak{P})\,,\quad \mathcal{W}_2=B^{-1}(\partial_2\mathfrak{P}\cup\partial_3\mathfrak{P})\,,\quad \mathcal{W}_1=B^{-1}(\partial\mathfrak{D}\sqcup\mathrm{Int}\mathfrak{I})\,,\quad \mathcal{W}_0=B^{-1}(\partial\mathfrak{I})$$

and we have $\mathcal{W}=B^{-1}(\partial\mathfrak{P})$.

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Proof. See appendix D. $\square$

Remark. The map b of (3.10) is related to the map B of (4.5) by:

$$b = \pi \circ B\,.$$

Using relations (4.31) and (4.32), this implies:

$$b^{-1}(\partial\mathcal{R}) = B^{-1}(\pi^{-1}(\partial\mathcal{R}))\,,$$

namely:

$$b^{-1}(\partial_0^\pm\mathcal{R}) = B^{-1}(\partial_0^\pm\mathfrak{P})\,,\; b^{-1}(\partial D) = B^{-1}(\partial\mathfrak{D})\,,\; b^{-1}(\partial_2^\pm\mathcal{R}) = B^{-1}(\mathfrak{S}^\pm)\,. \tag{5.1}$$

The behavior of the one-forms V_r on the locus $\mathcal{W}$ is given by the following result, whose proof can be found in appendix D:

Theorem 3. Let $p \in \mathcal{W}$ and write:

$$b_-(p) = \rho(p)\cos\psi\,,\; b_3(p) = \rho(p)\sin\psi$$

with $\psi \in [0, 2\pi)$. Then V_r and b_r behave as follows:

1. When $p \in \mathcal{W}_2$, we have:

   (a) For $p \in b^{-1}(\mathrm{Int}\mathfrak{D})$ we have:

$$\begin{aligned} &\beta(p) = 1\,, && b_+(p) = 0\,, && \rho(p) \in [0,1) \\ &V_+(p) = 0\,, && \|V_-(p)\| = \sqrt{1-\rho(p)^2\cos^2\psi} && \\ &\|V_3(p)\| = \sqrt{1-\rho(p)^2\sin^2\psi}\,, && \cos\theta_{-3} = -\frac{\rho^2(p)\sin\psi\cos\psi}{\|V_-(p)\|\;\|V_3(p)\|}\,. && \end{aligned}$$

   (b) When $p \in B^{-1}(\mathfrak{A})$, we have:

$$\begin{aligned} &\beta(p) = \rho(p)\,, && b_+(p) = 0\,, && \rho(p) \in (0,1) \\ &\|V_+(p)\| = \sqrt{1-\rho(p)^2}\,, && V_-(p) = (\rho(p)\sin\psi)v\,, && V_3(p) = -(\rho(p)\cos\psi)v \end{aligned}$$

   with $v \in T_p^*M$ an arbitrary 1-form of unit norm such that $V_+(p) \perp v$.

   (c) When $p \in B^{-1}(\mathrm{Int}\mathfrak{C}^\pm)$, we have:

$$\begin{aligned} &b_+(p) = \pm g(\rho(p), \beta(p))\,, && 0 < \rho(p) < \beta(p) < 1 \\ &V_+(p) = -\frac{\|V_+\|^2}{b_+\rho(p)}(\cos\psi V_-(p) + \sin\psi V_3(p)) && \\ &\|V_-(p)\| = \sqrt{\beta^2 - \rho(p)^2\cos^2\psi}\,, && \|V_3(p)\| = \sqrt{\beta^2 - \rho(p)^2\sin^2\psi} \\ &\cos\theta_{-3}(p) = -\frac{\rho(p)^2\sin 2\psi}{2\|V_-(p)\|\;\|V_3(p)\|}\,. && \end{aligned}$$

2. When $p \in \mathcal{W}_1$, we have:

   (a) For $p \in B^{-1}(\partial\mathfrak{D})$ we have:

$$\beta(p) = 1\,, \qquad b_+(p) = 0\,, \qquad \rho(p) = 1$$
$$V_+(p) = 0\,, \qquad V_-(p) = (\sin\psi)v\,, \qquad V_3(p) = -(\cos\psi)v\,,$$

   where $v \in T_p^*M$ is an arbitrary 1-form of unit norm.

   (b) For $p \in B^{-1}(\mathrm{Int}\mathfrak{I})$ we have:

$$\beta(p) = 0\,, \qquad b_+(p) \in (-1,1)\,, \qquad \rho(p) = 0$$
$$||V_+(p)|| = \sqrt{1 - b_+(p)^2}\,, \qquad V_-(p) = V_3(p) = 0\,.$$

3. When $p \in \mathcal{W}_0$ we have:

$$\beta(p) = 0\,, \qquad b_+(p) = \pm 1\,, \qquad \rho(p) = 0$$
$$V_+(p) = V_-(p) = V_3(p) = 0\,.$$

5.2 Description of the rank stratification of $\mathcal{D}_0$ and of the stabilizer stratification

The following result shows that the rank stratification of $\mathcal{D}_0$ (which coincides with the stabilizer stratification) is given by another coarsening of the B-preimage of the connected refinement of the canonical Whitney stratification of $\mathfrak{P}$.

Theorem 4. For $p \in M$, we have:

- $\mathrm{rk}\mathcal{D}_0(p) = 4$ iff $B(p) \in \mathrm{Int}\mathfrak{P}$ i.e. iff $p \in \mathcal{U}$
- $\mathrm{rk}\mathcal{D}_0(p) = 6$ iff $B(p) \in \mathrm{Int}\mathfrak{I} \sqcup \mathrm{Int}\mathfrak{D} \sqcup \mathfrak{A} \sqcup \mathrm{Int}\mathfrak{C}^+ \sqcup \mathrm{Int}\mathfrak{C}^- = \mathrm{Int}\mathfrak{I} \sqcup \partial_2\mathfrak{P} \sqcup \partial_3\mathfrak{P}$
- $\mathrm{rk}\mathcal{D}_0(p) = 7$ iff $B(p) \in \partial\mathfrak{D}$
- $\mathrm{rk}\mathcal{D}_0(p) = 8$ (i.e. $\mathcal{D}(p) = T_pM$) iff $B(p) \in \partial\mathfrak{I}$.

Hence the rank stratification of $\mathcal{D}_0$ is given by:

$$\mathcal{U}_0 = \mathcal{U}\,,\ \mathcal{Z}_3 = \emptyset\,,\ \mathcal{Z}_2 = B^{-1}(\mathrm{Int}\mathfrak{I} \sqcup \partial_2\mathfrak{P} \sqcup \partial_3\mathfrak{P})\,,\ \mathcal{Z}_1 = B^{-1}(\partial\mathfrak{D})\,,\ \mathcal{Z}_0 = B^{-1}(\partial\mathfrak{I}) = \mathcal{W}_0$$

and the stabilizer group H_p is given by:

- $H_p \simeq \mathrm{SU}(2)$ if $p \in \mathcal{U}_0 = \mathcal{U}$
- $H_p \simeq \mathrm{SU}(3)$ if $p \in \mathcal{Z}_2$
- $H_p \simeq \mathrm{G}_2$ if $p \in \mathcal{Z}_1$
- $H_p \simeq \mathrm{SU}(4)$ if $p \in \mathcal{Z}_0$.

Proof. Follows immediately from Theorem 1 of section 3 together with the Lemma of appendix E. □

The situation is summarized in table 5.

The b-image of the G-structure stratification is depicted in figure 13.

$\mathfrak{P}$-description	$\mathcal{D}$-stratum	$\mathcal{D}_0$-stratum	rk$\mathcal{D}$	rk$\mathcal{D}_0$	H_p
$B^{-1}(\partial\mathfrak{I})$	$\mathcal{W}_0$	$\mathcal{Z}_0$	8	8	SU(4)
$B^{-1}(\partial\mathfrak{D})$	$\mathcal{W}_1^1$	$\mathcal{Z}_1$	7	7	G_2
$B^{-1}(\mathrm{Int}\mathfrak{I})$	$\mathcal{W}_1^0$	$\subset \mathcal{Z}_2$	7	6	SU(3)
$B^{-1}(\partial_2\mathfrak{P} \sqcup \partial_3\mathfrak{P})$	$\mathcal{W}_2$	$\subset \mathcal{Z}_2$	6	6	SU(3)
$\mathrm{Int}\mathfrak{P}$	$\mathcal{U}$	$\mathcal{U}_0$	5	4	SU(2)

Table 5. The ranks of $\mathcal{D}$ and $\mathcal{D}_0$ on various loci and the isomorphism type of H_p.

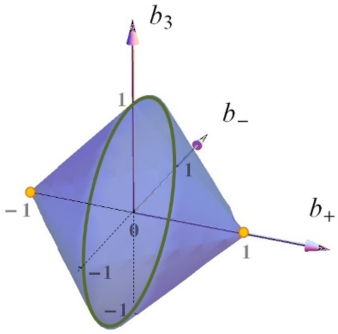

Figure 13. The b-image of the SU(4) locus is contained in ∂I (orange). The b-image of the G_2 locus is contained in ∂D (green). The b-image of the SU(3) locus is contained in $\mathcal{R} \setminus (\partial I \cup \partial D)$ (blue), while the b-image of the SU(2) locus is contained in Int$\mathcal{R}$ (blue).

5.3 Comparing the rank stratifications of $\mathcal{D}$ and $\mathcal{D}_0$

Using relations (4.27), Theorem 2 shows that $\mathcal{W}_k$ decompose as follows:

$$\begin{aligned}
\mathcal{W}_0 &= \mathcal{W}_0^+ \sqcup \mathcal{W}_0^- \quad \text{where} \quad \mathcal{W}_0^\pm = B^{-1}(\partial_0^\pm \mathfrak{P}) \\
\mathcal{W}_1 &= \mathcal{W}_1^0 \sqcup \mathcal{W}_1^1 \quad \text{where} \quad \mathcal{W}_1^0 \stackrel{\text{def.}}{=} B^{-1}(\mathrm{Int}\mathfrak{I})\,, \; \mathcal{W}_1^1 \stackrel{\text{def.}}{=} B^{-1}(\partial\mathfrak{D}) \\
\mathcal{W}_2 &= \mathcal{W}_2^2 \sqcup \mathcal{W}_2^3 \quad \text{where} \quad \mathcal{W}_2^2 \stackrel{\text{def.}}{=} B^{-1}(\partial_2\mathfrak{P})\,, \; \mathcal{W}_2^3 \stackrel{\text{def.}}{=} B^{-1}(\partial_3\mathfrak{P})\,,
\end{aligned}$$

where $\mathcal{W}_2^2$ and $\mathcal{W}_2^3$ decompose further as:

$$\begin{aligned}
\mathcal{W}_2^2 &= \mathcal{W}_2^{2+} \sqcup \mathcal{W}_2^{2-} \quad \text{with} \quad \mathcal{W}_2^{2+} \stackrel{\text{def.}}{=} B^{-1}(\mathrm{Int}\mathfrak{D})\,, \quad \mathcal{W}_2^{2-} \stackrel{\text{def.}}{=} B^{-1}(\mathfrak{A}) \\
\mathcal{W}_2^3 &= \mathcal{W}_2^{3+} \sqcup \mathcal{W}_2^{3-} \quad \text{with} \quad \mathcal{W}_2^{3\pm} \stackrel{\text{def.}}{=} B^{-1}(\mathrm{Int}\mathfrak{C}^\pm)\,,
\end{aligned}$$

so that:

$$\mathcal{W}_2 = \mathcal{W}_2^{2+} \sqcup \mathcal{W}_2^{2-} \sqcup \mathcal{W}_2^{3+} \sqcup \mathcal{W}_2^{3-}\,.$$

Finally, $\mathcal{W}_1^0$ decomposes as:

$$\mathcal{W}_1^0 = \mathcal{W}_1^{0+} \sqcup \mathcal{W}_1^{0-} \sqcup \mathcal{W}_1^{00} \quad \text{with} \quad \mathcal{W}_1^{0\pm} = B^{-1}(\mathrm{Int}\mathfrak{I}^\pm)\,, \quad \mathcal{W}_1^{00} = B^{-1}(\partial_0^0\mathfrak{P})\,.$$

The components listed above give the B-preimage of the connected refinement of the canonical Whitney stratification of $\partial\mathfrak{P}$, to which we can add $B^{-1}(\mathrm{Int}\mathfrak{P})$ to obtain the V-preimage

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	$\mathfrak{P}$-description	b-image	$\mathcal{D}$-stratum	$\mathcal{D}_0$-stratum	b_+	ρ	β	H_p
$\mathcal{W}_0^+$	$B^{-1}(\partial_0^+\mathfrak{P})$	$\partial_0^+\mathcal{R}$	$\mathcal{W}_0$	$\mathcal{Z}_0$	$+1$	0	0	SU(4)
$\mathcal{W}_0^-$	$B^{-1}(\partial_0^-\mathfrak{P})$	$\partial_0^-\mathcal{R}$	$\mathcal{W}_0$	$\mathcal{Z}_0$	-1	0	0	SU(4)
$\mathcal{W}_1^1$	$B^{-1}(\partial\mathfrak{D})$	$\partial_1\mathcal{R}=\partial D$	$\mathcal{W}_1$	$\mathcal{Z}_1$	0	1	1	G_2
$\mathcal{W}_1^{0+}$	$B^{-1}(\mathrm{Int}\mathfrak{I}^+)$	$\mathrm{Int}(I^+)$	$\mathcal{W}_1$	$\mathcal{Z}_2$	$(0,+1)$	0	0	SU(3)
$\mathcal{W}_1^{0-}$	$B^{-1}(\mathrm{Int}\mathfrak{I}^-)$	$\mathrm{Int}(I^-)$	$\mathcal{W}_1$	$\mathcal{Z}_2$	$(-1,0)$	0	0	SU(3)
$\mathcal{W}_1^{00}$	$B^{-1}(\partial_0^0\mathfrak{P})$	$\{0_{\mathbb{R}^3}\}$	$\mathcal{W}_1$	$\mathcal{Z}_2$	0	0	0	SU(3)
$\mathcal{W}_2^{2+}$	$B^{-1}(\mathrm{Int}\mathfrak{D})$	$\mathrm{Int}D$	$\mathcal{W}_2$	$\mathcal{Z}_2$	0	$[0,1)$	1	SU(3)
$\mathcal{W}_2^{2-}$	$B^{-1}(\mathfrak{A})$	$\mathrm{Int}D\setminus\{0\}$	$\mathcal{W}_2$	$\mathcal{Z}_2$	0	$(0,1)$	ρ	SU(3)
$\mathcal{W}_2^{3+}$	$B^{-1}(\mathrm{Int}\mathfrak{C}^+)$	$\mathrm{Int}(\mathcal{R}^+)$	$\mathcal{W}_2$	$\mathcal{Z}_2$	$+g(\rho,\beta)$	$[0,1)$	$(\rho,1)$	SU(3)
$\mathcal{W}_2^{3-}$	$B^{-1}(\mathrm{Int}\mathfrak{C}^-)$	$\mathrm{Int}(\mathcal{R}^-)$	$\mathcal{W}_2$	$\mathcal{Z}_2$	$-g(\rho,\beta)$	$[0,1)$	$(\rho,1)$	SU(3)
$\mathcal{U}$	$B^{-1}(\mathrm{Int}\mathfrak{P})$	$\mathrm{Int}\mathcal{R}$	$\mathcal{U}$	$\mathcal{U}_0$	$(-1,1)$	$[0,1)$	$J(b_+,\rho)$	SU(2)

Table 6. Preimage of the connected refinement of the canonical Whitney stratification of $\mathfrak{P}$.

of the connected refinement of the Whitney stratification of $\mathfrak{P}$ (see table 6). Theorems 2 and 4 give:

$$\mathcal{U}_0=\mathcal{U}\,,\quad \mathcal{Z}_3=\emptyset\,,\quad \mathcal{Z}_2=\mathcal{W}_1^0\sqcup\mathcal{W}_2\,,\quad \mathcal{Z}_1=\mathcal{W}_1^1\,,\quad \mathcal{Z}_0=\mathcal{W}_0\,.$$

In view of the last equality, we define $\mathcal{Z}_0^\pm\stackrel{\text{def.}}{=}\mathcal{W}_0^\pm$.

5.4 Description of the chirality stratification

We saw in section 3 that $\mathcal{S}=b^{-1}(\partial\mathcal{R})$. Since $b=B\circ\pi$, this gives $\mathcal{S}=B^{-1}(\pi^{-1}(\partial\mathcal{R}))$. The set $\pi^{-1}(\partial\mathcal{R})\subset\partial\mathfrak{P}$ which was discussed in section 4.3. Together with Theorem 1, decomposition (4.31) and relations (4.32) imply:

$$\begin{aligned}
&\mathcal{S}_{02}=B^{-1}(\partial_0^+\mathfrak{P})=\mathcal{W}_0^+\,, && \mathcal{S}_{20}=B^{-1}(\partial_0^-\mathfrak{P})=\mathcal{W}_0^-\\
&\mathcal{S}_{12}=B^{-1}(\mathfrak{S}^+)\subset\mathcal{W}_2^{3+}\,, && \mathcal{S}_{21}=B^{-1}(\mathfrak{S}^-)\subset\mathcal{W}_2^{3-}\\
&\mathcal{S}_{11}=B^{-1}(\partial\mathfrak{D})=\mathcal{W}_1^1=\mathcal{Z}_1\ .
\end{aligned}$$

In particular, we have $\mathcal{S}\subset\mathcal{W}_0\sqcup\mathcal{W}_1^1\sqcup\mathcal{W}_2^3\subset\mathcal{W}$ and

$$\mathcal{G}=\mathcal{U}\sqcup B^{-1}(\mathrm{Int}\mathfrak{I})\sqcup B^{-1}(\partial_2\mathfrak{P})\sqcup B^{-1}(\partial_3\mathfrak{P}\setminus\mathfrak{S})\,,$$

where $\mathfrak{S}\stackrel{\text{def.}}{=}\mathfrak{S}^+\sqcup\mathfrak{S}^-$. The situation is summarized in table 7, where we remind the reader that the restrictions of $\mathcal{D}$ and $\mathcal{D}_0$ to the special locus $\mathcal{S}$ coincide (see section 3).

5.5 Relation to previous work

Some aspects of $\mathcal{N}=2$ compactifications of eleven-dimensional supergravity down to AdS_3 were approached in [26] using a nine-dimensional formalism based on the auxiliary 9-manifold $\hat{M}\stackrel{\text{def.}}{=}M\times S^1$, but without carefully exploring the consequences of that formalism for the geometry of M. Sections 3-5 of [26] also discuss some consequences of the

$\mathfrak{P}$-description	$\mathcal{S}$-stratum	$\mathcal{D}$-stratum	$\mathcal{D}_0$-stratum	rk$\mathcal{D}$	rk$\mathcal{D}_0$	H_p
$B^{-1}(\partial_0^+\mathfrak{P})$	$\mathcal{S}_{02}$	$\mathcal{W}_0^+$	$\mathcal{Z}_0^+$	8	8	SU(4)
$B^{-1}(\partial_0^-\mathfrak{P})$	$\mathcal{S}_{20}$	$\mathcal{W}_0^-$	$\mathcal{Z}_0^-$	8	8	SU(4)
$B^{-1}(\partial\mathfrak{D})$	$\mathcal{S}_{11}$	$\mathcal{W}_1^1$	$\mathcal{Z}_1$	7	7	G_2
$B^{-1}(\mathfrak{S}^+)$	$\mathcal{S}_{12}$	$\subset \mathcal{W}_2^{3+}$	$\subset \mathcal{Z}_2$	6	6	SU(3)
$B^{-1}(\mathfrak{S}^-)$	$\mathcal{S}_{21}$	$\subset \mathcal{W}_2^{3-}$	$\subset \mathcal{Z}_2$	6	6	SU(3)

Table 7. Description of the special strata of the chirality stratification. The table does not show the non-special locus $\mathcal{G}$.

supersymmetry equations (which were also derived in [25]) using the nine-dimensional formalism. Reference [26] makes intensive use of an assumption (equation (3.9) of loc. cit.) which, as we show in appendix G, can only hold when the SU(2) locus $\mathcal{U}$ of M is empty. Since most results of [26] (including the count of the number of supersymmetries preserved by membranes transverse to M as well as the discussion of sections 3-6 of that reference) rely on that assumption, those results can apply only to the highly non-generic case when $\mathcal{U} = \emptyset$. As we explain in detail in forthcoming work, failure of [26, eq. (3.9)] is related to the transversal vs. non-transversal character of the intersection of a certain distribution $\hat{\mathcal{D}}$ defined on $\hat{M}$ with the pullback to $\hat{M}$ of the tangent bundle of M.

6 Conclusions

We studied the conditions for "off-shell" extended supersymmetry in compactifications of eleven-dimensional supergravity on eight-manifolds M. We gave an explicit description of the stabilizer stratification induced by two globally-defined Majorana spinors as a certain coarsening of the preimage of the connected refinement of the Whitney stratification of a four-dimensional compact semi-algebraic set $\mathfrak{P}$ through a map $B : M \to \mathbb{R}^4$ whose image is contained in $\mathfrak{P}$. We also described the chirality stratification as a coarsening of the preimage of the connected refinement of the Whitney stratification of a 3-dimensional compact semi-algebraic set $\mathcal{R}$ through a smooth map $b : M \to \mathbb{R}^3$ whose image is contained in $\mathcal{R}$. Unlike the case of $\mathcal{N} = 1$ compactifications, the stabilizer and chirality stratifications do not agree. We found a rich landscape of reductions of structure group along the various strata, which we classified explicitly. The open strata of the chirality and stabilizer stratifications coincide and correspond to an open subset $\mathcal{U} \subset M$ which carries an SU(2) structure. This locus is present in generic $\mathcal{N} = 2$ flux compactifications of eleven-dimensional supergravity on eight manifolds, for example in generic $\mathcal{N} = 2$ compactifications down to AdS_3 spaces.

We also discussed two natural cosmooth generalized distributions $\mathcal{D}$ and $\mathcal{D}_0$ which exist on M when considering such backgrounds. These are defined by the four one-form spinor bilinears V_1, V_2, V_3 and W which are induced by two independent globally-defined Majorana spinors given on M, namely $\mathcal{D}$ is the intersection of the kernel distributions of V_1, V_2 and V_3 while $\mathcal{D}_0$ is the intersection of $\mathcal{D}$ with the kernel distribution of W. We showed that the rank stratification of $\mathcal{D}_0$ coincides with the stabilizer stratification, while the rank stratification of $\mathcal{D}$ is another coarsening of the B-preimage of the connected refinement of the Whitney stratification of $\mathfrak{P}$. The restriction of $\mathcal{D}$ to the open stratum $\mathcal{U}$ is a rank five regular Frobenius distribution which carries an SU(2) structure in the sense of [42], while the restriction of $\mathcal{D}_0$ to $\mathcal{U}$ is a rank four Frobenius distribution (the almost contact distribution of [43]). Since the image $G_p = \mathfrak{q}_p(H_p) \subset \mathrm{SO}(T_pM, g_p)$ of the pointwise stabilizer group H_p of two independent Majorana spinors fixes the forms $V_1(p), V_2(p), V_3(p)$ and $W(p)$, the distribution $\mathcal{D}_0|_\mathcal{U}$ carries the SU(2) structure of $\mathcal{D}|_\mathcal{U}$ in the sense that G_p is contained in the group $\mathrm{SO}(\mathcal{D}_0(p), g_p) \simeq \mathrm{SO}(4)$ for any point $p \in \mathcal{U}$. In this paper, we focused on the classification of spinor positions and stabilizer groups, which we treated in detail given its complexity. We mention that considerably more can be said about the chirality and stabilizer stratifications provided that one makes appropriate Thom-Boardman type genericity assumptions which allow one to apply results from the singularity theory of differentiable maps [44–48].

Since the manifolds M considered in this paper are eight-dimensional, it is not entirely clear how a description of such backgrounds may be given within the framework of exceptional generalized geometry [4–10], similar to the one given in [7–9] for 7-dimensional backgrounds of eleven-dimensional supergravity and in [35, 36, 49, 50] for six-dimensional type II backgrounds. This stems from difficulties[10] in building an appropriate generalized connection in eight dimensions, which in turn relates to the presence of Kaluza-Klein monopoles in the U-duality algebra and hence to the problem of including "dual gravitons" at the nonlinear level in $E_{8(8)}$-covariant formulations of eleven-dimensional supergravity [11–14] (which is obstructed by the no-go results of [15, 16]). A solution to this problem was recently proposed in [51] within the framework of exceptional field theory but, as pointed out in [52], that solution may be incomplete. It would be interesting to understand what light may be shed on our results by exceptional generalized geometry.

The results of this paper show that the rich landscape of G-structures arising in $\mathcal{N} = 2$ flux compactifications of eleven-dimensional supergravity on eight-manifolds admits a natural description using stratification theory and standard constructions of real semi-algebraic geometry [30–32], thus giving clues about the mathematical tools required for general treatments of flux backgrounds. We note that the approach via cosmooth generalized distributions, stratified G-structures and semi-algebraic sets appears to be quite general and thus could be applied to flux backgrounds of any supergravity theory. In general, the complexity of the stratifications involved grows rather fast with the number of spinors (as implied by the results of [41]), but such stratifications can be computed algorithmically. We mention that powerful algorithms exist [32] for the study of semi-algebraic sets.

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[10]The precise problem (see [7]) is that one wants to consider generalized connections which are compatible with the generalized metric as well as torsion-free in an appropriate sense, however one does not have a natural definition of the torsion of a generalized connection when $\dim M > 7$.

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A Notations and conventions

Throughout this paper, (M, g) denotes a connected and compact smooth Riemannian eight-manifold, which we assume to be oriented and spin. The unital commutative $\mathbb{R}$-algebra of smooth real-valued functions on M is denoted by $\mathcal{C}^\infty(M, \mathbb{R})$. The fact that M is orientable and spin means that its first two Stiefel-Whitney classes vanish, i.e. $w_1(M) = w_2(M) = 0$. All fiber bundles we consider are smooth.[11] We use freely the results and notations of [25, 38, 54], with the same conventions as there.

Recall that the set of isomorphism classes of spin structures of M is a torsor for the finite group $H^1(M, \mathbb{Z}_2)$. Let $(T^*M, \diamond)$ denote the Kähler-Atiyah bundle of (M, g), which is a bundle of unital associative $\mathbb{R}$-algebras. Consider the set $\mathcal{A}$ consisting of all pairs (S, γ), where S is a vector bundle of rank 16 over M and $\gamma : (T^*M, \diamond) \xrightarrow{\sim} (\mathrm{End}(S), \circ)$ is a unital isomorphism of bundles of $\mathbb{R}$-algebras. Two pairs $(S, \gamma), (S', \gamma')$ are called equivalent (and we write $(S, \gamma) \sim (S', \gamma')$) if there exists an isomorphism of $\mathbb{Z}_2$-graded vector bundles $f : S \xrightarrow{\sim} S'$ such that $\gamma' = \tilde{f} \circ \gamma$, where $\tilde{f} : \mathrm{End}(S) \to \mathrm{End}(S')$ is the unital isomorphism of bundles of algebras corresponding to $\tilde{f}(Q) \stackrel{\text{def.}}{=} f \circ Q \circ f^{-1}$ for all $Q \in \Gamma(M, \mathrm{End}(S))$. Given a spin structure on M, let $S^\pm$ be the corresponding bundles of spinors of positive and negative chirality and $S \stackrel{\text{def.}}{=} S^+ \oplus S^-$ denote the corresponding bundle of real pinors (a.k.a. Majorana spinors). Then S is a bundle of modules over the Kähler-Atiyah bundle $(T^*M, \diamond)$ whose structure morphism is an isomorphism of bundles of algebras $\gamma : (T^*M, \diamond) \xrightarrow{\sim} (\mathrm{End}(S), \circ)$ and hence the pair (S, γ) is an element of $\mathcal{A}$. This gives a map which associates an element of $\mathcal{A}$ to every spin structure of M. It is easy to see that two spin structures are equivalent iff the corresponding pairs (S, γ) and (S', γ') are equivalent in the sense described above, hence we have a bijection between $H^1(M, \mathbb{Z}_2)$ and the set $\mathcal{A}/_\sim$. Throughout the paper, we assume that a spin structure has been chosen for M and we work with the corresponding pair $(S, \gamma) \in \mathcal{A}$.

Up to rescalings by smooth nowhere-vanishing real-valued functions defined on M, the bundle S of Majorana spinors has two admissible pairings $\mathscr{B}_\pm$ (see [54–56]), both of which are symmetric. These pairings are distinguished by their types $\epsilon_{\mathscr{B}_\pm} = \pm 1$. Throughout the paper, we work with $\mathscr{B} \stackrel{\text{def.}}{=} \mathscr{B}_+$, which we can take to be a scalar product on S, denoting the induced norm on S by $|| \ ||$.

Our convention for the Clifford algebra $\mathrm{Cl}(h)$ of a bilinear form h is that common in Physics, i.e. the generators satisfy $e_k e_l + e_l e_k = 2h_{kl}$; the convention common in Mathe-

[11] The generalized bundles [37, 53] considered in this paper are *not* fiber bundles and they will be either smooth or cosmooth.

matics has a minus on the right hand side. One recovers the Mathematics convention by multiplying all e_k with the imaginary unit i; accordingly, the Killing constant of a Killing spinor is multiplied by i. Unlike in some of the literature on flux compactifications, we reserve the name "Killing spinor" for the mathematically consecrated notion, i.e. for a spinor ξ which satisfies $\nabla_k \xi = \lambda e_k \xi$, where λ is the Killing constant and the right hand side involves Clifford multiplication; spinors which satisfy generalizations of this equation in which the right hand side contains a polynomial in e_i are called *generalized* Killing spinors, as usual in the Mathematics literature.

The generalized distributions [37, 53] $\mathcal{D}$ and $\mathcal{D}_0$ considered in this paper are cosmooth in the sense of [37] rather than smooth. As explained in appendix D of [24], their integrability theory (see [57]) is in some sense "orthogonal" to that of smooth generalized distributions [58–61]. When integrable, a cosmooth generalized distribution integrates to a Haefliger structure (a.k.a. a singular foliation in the sense of Haefliger) while a smooth generalized distribution integrates to a singular foliation in the sense of [62, 63].

We use the "mostly plus" convention for pseudo-Riemannian metrics of Minkowski signature. Given a subset A of M, we let $\bar{A}$ denote the closure of A in M (taken with respect to the manifold topology of M). The *frontier* (also called *topological boundary*) of A is defined as $\partial A \stackrel{\text{def.}}{=} \bar{A} \setminus \mathrm{Int}A$, where $\mathrm{Int}A$ denotes the interior of A. The *small topological frontier* is $\mathrm{fr}A \stackrel{\text{def.}}{=} \bar{A} \setminus A$. When considering the canonical Whitney stratification of a semi-algebraic set, we always work with its connected refinement (see appendix B). In some references (such as [41]) it is this connected refinement which is called the canonical Whitney stratification of that semi-algebraic set.

B Algebraic constraints for V_r, W and b

Relations (4.1) can be obtained through direct computation using Fierz identities. Here, we give a proof which relies on reducing (4.2) to a Fierz identity satisfied by a single spinor. Consider the Majorana spinor:

$$\xi(x) \stackrel{\text{def.}}{=} x_{1+}\xi_1^+ + x_{1-}\xi_1^- + x_{2+}\xi_2^+ + x_{2-}\xi_2^- \in \Gamma(M, S)$$

and the corresponding one-form:

$$V(x) \stackrel{\text{def.}}{=}_U \mathscr{B}(\xi(x), \gamma_a \xi(x)) e^a\,,$$

where $U \subset M$ and $x_{i\pm}$ are arbitrary real numbers. This satisfies the relation [38]:

$$||V(x)||^2 = ||\xi(x)||^4 - b(x)^2\,, \tag{B.1}$$

where:

$$b(x) \stackrel{\text{def.}}{=}_U \mathscr{B}(\xi(x), \gamma(\nu)\xi(x))\,.$$

The relations $\gamma_a^t = \gamma_a$ and $\gamma(\nu)^t = \gamma(\nu)$ give:

$$\mathscr{B}(\xi_i^\alpha, \gamma_a \xi_j^\beta) = \mathscr{B}(\xi_j^\beta, \gamma_a \xi_i^\alpha)\,, \quad \mathscr{B}(\xi_i^\alpha, \gamma(\nu)\xi_j^\beta) = \mathscr{B}(\xi_j^\beta, \gamma(\nu)\xi_i^\alpha)$$

for all $i, j = 1, 2$ and all $\alpha, \beta \in \{-, +\}$. Using these as well as $\mathscr{B}(\xi_i^{\pm}, \xi_j^{\mp}) = 0$, we find:

$$\begin{aligned}
V(x) &= x_{1+}x_{1-}V_1 + x_{2+}x_{2-}V_2 + 2x_{1-}x_{2+}V_3^+ + 2x_{1+}x_{2-}V_3^- \\
||\xi(x)||^2 &= x_{1+}^2||\xi_1^+||^2 + x_{2+}^2||\xi_2^+||^2 + x_{1-}^2||\xi_1^-||^2 + x_{2-}^2||\xi_2^-||^2 + 2x_{1+}x_{2+}\mathscr{B}(\xi_1^+, \xi_2^+) + 2x_{1-}x_{2-}\mathscr{B}(\xi_1^-, \xi_2^-) \\
b(x) &= x_{1+}^2||\xi_1^+||^2 + x_{2+}^2||\xi_2^+||^2 - x_{1-}^2||\xi_1^-||^2 - x_{2-}^2||\xi_2^-||^2 + 2x_{1+}x_{2+}\mathscr{B}(\xi_1^+, \xi_2^+) - 2x_{1-}x_{2-}\mathscr{B}(\xi_1^-, \xi_2^-) \,.
\end{aligned}$$

Using (3.11), these relations become:

$$\begin{aligned}
V(x) &= x_{1+}x_{1-}V_1 + x_{2+}x_{2-}V_2 + 2x_{1-}x_{2+}V_3^+ + 2x_{1+}x_{2-}V_3^- \\
||\xi(x)||^2 &= \frac{1}{2}\left[x_{1+}^2(1+b_1) + x_{2+}^2(1+b_2) + x_{1-}^2(1-b_1) + x_{2-}^2(1-b_2)\right] + x_{1+}x_{2+}b_3 - x_{1-}x_{2-}b_3 \\
b(x) &= \frac{1}{2}\left[x_{1+}^2(1+b_1) + x_{2+}^2(1+b_2) - x_{1-}^2(1-b_1) - x_{2-}^2(1-b_2)\right] + x_{1+}x_{2+}b_3 + x_{1-}x_{2-}b_3 \,.
\end{aligned}$$

Substituting these expressions into (B.1) gives an algebraic equation which must hold for all $x_{i\alpha}$, i.e. a certain polynomial in the variables $x_{i\alpha}$ must vanish identically. This means that the coefficients of all monomials in $x_{i\alpha}$ in that polynomial must vanish, giving the relations:

$$\begin{aligned}
&||V_1||^2 = 1 - b_1^2\,, && ||V_2||^2 = 1 - b_2^2\,, \\
&||V_3^+||^2 = \frac{1}{4}(1 - b_1 + b_2 - b_1b_2)\,, && ||V_3^-||^2 = \frac{1}{4}(1 + b_1 - b_2 - b_1b_2)\,, \\
&\langle V_1, V_2\rangle + 4\langle V_3^-, V_3^+\rangle = 2b_3^2\,, && \\
&\langle V_1, V_3^+\rangle = 1/2(1-b_1)b_3\,, && \langle V_1, V_3^-\rangle = -1/2(1+b_1)b_3\,, \\
&\langle V_2, V_3^+\rangle = -1/2(1+b_2)b_3\,, && \langle V_2, V_3^-\rangle = 1/2(1-b_2)b_3\,.
\end{aligned} \tag{B.2}$$

Using $V_3 = V_3^+ + V_3^-$, $W = V_3^+ - V_3^-$ and $V_\pm = \frac{1}{2}(V_1 \pm V_2)$, we can write (B.2) in the form (4.1). The system (4.1) can also be written as:

$$\begin{aligned}
&||V_1||^2 = 1 - b_1^2\,, && ||V_2||^2 = 1 - b_2^2\,, && ||V_3||^2 + ||W||^2 = 1 - b_1b_2\,, \\
&\langle V_1, V_2\rangle + 2||V_3||^2 = 1 - b_1b_2 - 2b_3^2\,, && \langle V_1, V_3\rangle = -b_1b_3\,, && \langle V_2, V_3\rangle = -b_2b_3\,, \\
&\langle V_1, W\rangle = b_3\,, && \langle V_2, W\rangle = -b_3\,, && \langle V_3, W\rangle = \frac{1}{2}(b_2 - b_1)\,.
\end{aligned} \tag{B.3}$$

Using (3.11), we find:

$$\begin{aligned}
1 - b_i^2 &= (1-b_i)(1+b_i) = 4||\xi_i^+||^2||\xi_i^-||^2 \quad (i = 1, 2) \\
1 \mp b_1 \pm b_2 - b_1b_2 &= (1 \mp b_1)(1 \pm b_2) = 4||\xi_1^{\mp}||^2||\xi_2^{\pm}||^2\,.
\end{aligned} \tag{B.4}$$

This allows us to write the norms of $V_i, V_3^{\pm}$ given in (4.1) in the form:

$$||V_i|| = 2||\xi_i^+||\;||\xi_i^-||\,, \quad ||V_3^{\pm}|| = ||\xi_1^{\mp}||\;||\xi_2^{\pm}||\,. \tag{B.5}$$

Proposition. Assume that $\xi_j^{\pm}$ does not vanish anywhere on the open subset $U \subset M$ which supports the local orthonormal coframe $(e^a)_{a=1\ldots8}$ of (M, g). Then $(\gamma^a\xi_j^{\pm})_{a=1\ldots8}$ is an orthogonal frame of $S^{\mp}$ defined above U which satisfies $||\gamma^a\xi_j^{\pm}||^2 = ||\xi_j^{\pm}||^2$ and we have:

$$\xi_i^{\mp} =_U \frac{1}{||\xi_j^{\pm}||^2}\sum_{a=1}^{8}\mathscr{B}(\xi_i^{\mp}, \gamma_a\xi_j^{\pm})\gamma^a\xi_j^{\pm}\,.$$

In particular, if $\xi_1^+, \xi_1^-, \xi_2^+$ and ξ_2^- are all non-vanishing on U then:

$$\xi_1^+ =_U \frac{1}{2||\xi_1^-||^2}\gamma(V_1)\xi_1^- = \frac{1}{||\xi_2^-||^2}\gamma(V_3^-)\xi_2^- \,, \quad \xi_2^+ =_U \frac{1}{2||\xi_2^-||^2}\gamma(V_2)\xi_2^- = \frac{1}{||\xi_1^-||^2}\gamma(V_3^+)\xi_1^-$$
$$\xi_1^- =_U \frac{1}{2||\xi_1^+||^2}\gamma(V_1)\xi_1^+ = \frac{1}{||\xi_2^+||^2}\gamma(V_3^+)\xi_2^+ \,, \quad \xi_2^- =_U \frac{1}{2||\xi_2^+||^2}\gamma(V_2)\xi_2^+ = \frac{1}{||\xi_1^+||^2}\gamma(V_3^-)\xi_1^+ \,. \tag{B.6}$$

Proof. Follows immediately by applying a result proved in [24, section 2.6] (the Corollary on page 14 of loc. cit.). $\square$

Remark. Under the assumption of the second part of the proposition, relations (B.5) show that V_1, V_2 and $V_3^\pm$ are nowhere-vanishing on U and that the following rescaled 1-forms have unit norm everywhere on U, where $i = 1, 2$:

$$\hat{V}_i \stackrel{\text{def.}}{=} \frac{1}{2||\xi_i^+||\,||\xi_i^-||}V_i \,, \quad \hat{V}_3^\pm \stackrel{\text{def.}}{=} \frac{1}{||\xi_1^\mp||\,||\xi_2^\pm||}V_3^\pm \,.$$

Using these normalized 1-forms, relations (B.6) can be written as:

$$\frac{\xi_1^+}{||\xi_1^+||} =_U \gamma(\hat{V}_1)\frac{\xi_1^-}{||\xi_1^-||} = \gamma(\hat{V}_3^-)\frac{\xi_2^-}{||\xi_2^-||} \,, \quad \frac{\xi_2^+}{||\xi_2^+||} =_U \gamma(\hat{V}_2)\frac{\xi_2^-}{||\xi_2^-||} = \gamma(\hat{V}_3^+)\frac{\xi_1^-}{||\xi_1^-||}$$
$$\frac{\xi_1^-}{||\xi_1^-||} =_U \gamma(\hat{V}_1)\frac{\xi_1^+}{||\xi_1^+||} = \gamma(\hat{V}_3^+)\frac{\xi_2^+}{||\xi_2^+||} \,, \quad \frac{\xi_2^-}{||\xi_2^-||} =_U \gamma(\hat{V}_2)\frac{\xi_2^+}{||\xi_2^+||} = \gamma(\hat{V}_3^-)\frac{\xi_1^+}{||\xi_1^+||} \,. \tag{B.7}$$

Notice $\hat{V}_1, \hat{V}_2, \hat{V}_3^\pm$ square to one in the Kähler-Atiyah algebra of (U, g) and hence the endomorphisms $\gamma(\hat{V}_i)$, $\gamma(\hat{V}_3^\pm)$ square to the identity automorphism of the bundle $S|_U$. Also note that the second part of the proposition applies to any open subset of the non-special locus $\mathcal{G} \subset M$ which supports a local orthonormal coframe of (M, g).

Remark. The sub-system (4.2) can be obtained more directly as follows. An arbitrary norm one element ξ of $\mathcal{K}$ has the form:

$$\xi(u) = \cos\left(\frac{u}{2}\right)\xi_1 + \sin\left(\frac{u}{2}\right)\xi_2 \tag{B.8}$$

where $u \in \mathbb{R}$ is constant on M. This induces a function $b(u) \in \mathcal{C}^\infty(M, \mathbb{R})$ and a one-form $V(u) \in \Omega^1(M)$ given by:

$$b(u) =_U \mathscr{B}(\xi(u), \gamma(\nu)\xi(u)) \,, \quad V(u) =_U \mathscr{B}(\xi(u), \gamma_a\xi(u))e^a \,. \tag{B.9}$$

Relation (B.8) gives:

$$b(u) \ = b_+ + b_- \cos u + b_3 \sin u \,, \quad V(u) = V_+ + V_- \cos u + V_3 \sin u \,. \tag{B.10}$$

Since $||\xi(u)|| = 1$, relation (B.1) implies that the following equality must hold for all u (cf. [19, 23, 24]):

$$||V(u)||^2 = 1 - b(u)^2 \,. \tag{B.11}$$

Substituting (B.10) into (B.11), we can separate the Fourier components in u, using the fact that $\{1, \cos(u), \sin(u)|n \in \mathbb{N}^*\}$ form an orthogonal basis of the Hilbert space $\mathrm{L}^2(S^1)$ of (complex-valued) square integrable functions on the circle. This leads to a system of algebraic constraints for b_r and V_r which is equivalent with (B.11). Expanding in Fourier components, one finds after some computation that (B.11) is equivalent with (4.2).

C Stratified spaces

We recall some basic notions from stratification theory in order to fix terminology. In this paper, a finite stratification of a topological space X is understood in the most general sense, i.e. as a finite partition of X into non-empty subsets called *strata*. We say that the stratification is *connected* if all strata are connected. We let $\Sigma \subset \mathcal{P}(X)$ (where $\mathcal{P}(X)$ is the power set of X) denote the set of all strata, thus:

$$X = \sqcup_{S\in\Sigma} S\,.$$

C.1 Incidence poset of a stratification

Consider the partial order relation $\leq$ defined on Σ through:

$$S' \leq S \ \text{ iff } \ S' \subseteq \overline{S}\,.$$

Then $(\Sigma, \leq)$ is a finite poset called the *incidence poset* of the stratification. We let $<$ denote the transitive binary relation defined on Σ through:

$$S' < S \ \text{ iff } \ S' \leq S \text{ and } S' \neq S$$

i.e.:

$$S' < S \ \text{ iff } \ S' \subseteq \mathrm{fr}(S)\,,$$

where $\mathrm{fr}(S)$ denotes the *small* frontier of S (see appendix A). For any $S \in \Sigma$, let $\mathcal{C}(S)$ denote the strict lower set of S:

$$\mathcal{C}(S) \stackrel{\text{def.}}{=} \{S' \in \Sigma | S' < S\} = \{S' \in \Sigma | S' \subseteq \mathrm{fr} S\}\ .$$

For all $S \in \Sigma$, we have the obvious inclusion:

$$\sqcup_{S'\in\mathcal{C}(S)} S' \subseteq \mathrm{fr}(S)\,. \tag{C.1}$$

C.2 The adjointness relation

We say that a stratum S' adjoins a stratum S (and write $S' \trianglelefteq S$) if the intersection $S' \cap \bar{S}$ is non-empty. This defines a reflexive (but generally non-transitive) binary relation on Σ. We say that S' *strictly adjoins* S (and write $S' \triangleleft S$) if $S' \trianglelefteq S$ and $S' \neq S$ i.e. if S' intersects $\mathrm{fr} S$. We have:

$$\mathrm{fr} S \subseteq \sqcup_{S' \triangleleft S} S'\,, \quad \forall S \in \Sigma$$

and:

$$\mathcal{C}(S) \subseteq \{S' \in \Sigma | S' \triangleleft S\}\,. \tag{C.2}$$

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C.3 The frontier condition

We say that the stratification satisfies the *frontier condition* if the *small* frontier of each stratum is a union of strata. This amounts to the requirement that equality is always realized in (C.1):

$$\mathrm{fr}(S) = \sqcup_{S' \in \mathcal{C}(S)} S' \,, \quad \forall S \in \Sigma$$

and with the condition that equality is realized in (C.2). This happens iff the binary relations $<$ and $\triangleleft$ coincide, in which case $\leq$ and $\trianglelefteq$ also coincide i.e. iff $S' \cap \bar{S} \neq \emptyset$ implies $S' \subseteq \bar{S}$. When the frontier condition is satisfied, the small frontier of any stratum can be determined immediately by looking at the Hasse diagram of the incidence poset of the stratification.

C.4 Refinements and coarsenings

We say that a stratification Σ' is a *refinement* of Σ if any stratum of Σ is a union of strata of Σ'. In this case, we also say that Σ is a *coarsening* of Σ'. The *connected refinement* of Σ is the refinement whose strata are the connected components of the strata of Σ; it is the coarsest connected stratification which is a refinement of Σ. We say that two stratifications Σ and Σ' *agree* if one of them is a refinement of the other.

D The semipositivity conditions for G

Consider the Gram matrix (4.7). We use the notation $G_{[ij|ij]}$ for the 2 by 2 submatrix of G obtained by keeping only the i-th and j-th rows and columns of G, where $1 \leq i < j \leq 3$. By Sylvester's criterion:

- G is positive semidefinite, iff each of its principal (unsigned) minors:

  $$\det G\,,\ \ \det G_{[12|12]}\,,\ \ \det G_{[23|23]}\,,\ \ \det G_{[13|13]}\,,\ \ G_{11}\,,\ \ G_{22}\,,\ \ G_{33}$$

  is non-negative.
- G is positive definite iff each of its *leading* principal minors $\det G$, $\det G_{[12|12]}$ and G_{11} is positive; in this case, the non-leading principal minors are automatically positive.

Remark. When G is positive semidefinite, Kosteljanski's inequality [64] gives:

$$\det G[I \cup J] \det G[I \cap J] \leq \det G[I] \det G[J]\,,$$

where $G[I]$ denotes the unsigned principal minor defined by keeping only those rows and columns of G indexed by elements of the subset I of the set $\{1, 2, 3\}$. For $I \cap J = \emptyset$, this reduces to Fisher's inequality:

$$\det G[I \cup J] \leq \det G[I] \det G[J] \quad \text{when} \quad I \cap J = \emptyset\,,$$

which gives:

$$\begin{gathered}\det G \leq \min(G_{11} \det G_{[23|23]}, G_{22} \det G_{[13|13]}, G_{33} \det G_{[12|12]}) \qquad \text{(D.1)}\\ \det G_{[12|12]} \leq G_{11} G_{22}\,,\ \ \det G_{[13|13]} \leq G_{11} G_{33}\,,\ \ \det G_{[23|23]} \leq G_{22} G_{33}\,.\end{gathered}$$

To study Sylvester's conditions, we start by computing the determinants of the various submatrices of G. Consider the polynomial (4.10), which we reproduce here for convenience:

$$P(b,\beta) = \beta^4 - \beta^2(1+\rho^2 - b_+^2) + \rho^2\,. \tag{D.2}$$

Notice that:

$$P(b,\rho) = b_+^2\rho^2\,. \tag{D.3}$$

Direct computation gives:

$$\begin{aligned} G_{11} &= 1-\beta^2-b_+^2\,, & G_{22} &= \beta^2 - b_-^2\,, & G_{33} &= \beta^2 - b_3^2 \\ \det G_{[12|12]} &= -P(b_+,b_-,0,\beta)\,, & \det G_{[13|13]} &= -P(b_+,0,b_3,\beta) & & \\ \det G_{[23|23]} &= \beta^2\left[\beta^2-(b_-^2+b_3^2)\right]\,, & \det G &= -\beta^2 P(b_+,b_-,b_3,\beta)\,. & & \end{aligned} \tag{D.4}$$

When viewing P as a quadratic polynomial in β^2, its discriminant equals the function $h(b_+,\rho)$ defined in (4.12).

Proposition. We have $h(b)\geq 0$ for $b\in\mathcal{R}$, with equality iff $b\in\partial\mathcal{R}$.

Proof. The statement follows by noticing that:

$$h(b_+,\rho) = [(1+b_+)^2-\rho^2][(1-b_+)^2-\rho^2] = [(1+|b_+|)^2-\rho^2][(1-|b_+|)^2-\rho^2]$$

and using the fact that $b\in\mathcal{R}$ implies $(b_+,\rho)\in\Delta$, which in turn means that $|b_+|\leq 1$ and $\rho\leq 1-|b_+|$.

□

It follows that for any $b\in\mathcal{R}$ we can factorize $P(b,\beta)$ as:

$$P(b,\beta) = (\beta^2 - f_+(b))(\beta^2-f_-(b))\,, \tag{D.5}$$

where $f_\pm(b)$ are given in (4.11). This allows us to write:

$$\begin{aligned} -\det G_{[12|12]} &= (\beta^2 - f_+(b_+,b_-,0))(\beta^2-f_-(b_+,b_-,0)) \\ \det G_{[23|23]} &= \beta^2(\beta^2-\rho^2) \\ -\det G_{[13|13]} &= (\beta^2 - f_+(b_+,0,b_3))(\beta^2-f_-(b_+,0,b_3))\,. \end{aligned} \tag{D.6}$$

D.1 Proof of Theorem 2

Theorem 2 is an immediate consequence of Lemmas A, B and C proved below.

Proposition. The following inequality holds for $b\in\mathcal{R}$:

$$\sqrt{h(b_+,\rho)} \leq 1 - b_+^2 - \rho^2\,, \tag{D.7}$$

with equality iff $b_+\rho = 0$.

Proof. For $b\in\mathcal{R}$, we have $(b_+,\rho)\in\Delta$ and hence $\rho\leq 1-|b_+|$, which implies $\rho^2\leq (1-|b_+|)^2 \leq (1-|b_+|)(1+|b_+|) = 1-b_+^2$. Hence the right hand side of (D.7) is non-negative for $b\in\mathcal{R}$. It follows that (D.7) is equivalent with the inequality obtained by squaring both of its sides, which can be seen by direct computation to be equivalent with $4b_+^2\rho^2\geq 0$. □

Proposition. For $b \in \mathcal{R}$, we have:

$$\rho^2 \leq f_-(b_+,\rho) \leq f_+(b_+,\rho) \leq 1 - b_+^2 \,. \tag{D.8}$$

The first and third inequalities in (D.8) are both strict unless $b_+\rho = 0$, in which case both of them become equalities. In particular, we have $J(b) \subset [\rho, \sqrt{1-b_+^2}]$, where the interval $J(b)$ was defined in (4.13).

Proof. The middle inequality is obvious, while the first and third inequalities are both equivalent with (D.7). The other statements follow immediately. □

Remark. For $(b,\beta) \in \mathfrak{P}$, we have:

1. $G_{11} = 0$ (i.e. $||V_+|| = 0$) iff one of the following holds:
   - $\beta = 1$ or
   - $b_+ = b_- = 0$ and $\beta = \sqrt{1-b_+^2}$, which requires $||V_-|| = ||V_3|| = \sqrt{1-b_+^2}$ and $\langle V_-, V_3\rangle = 0$
2. $\det G_{[23|23]} = 0$ (i.e. V_- and V_3 are linearly dependent) iff one of the following holds:
   - $\beta = \rho = 0$ and hence $V_- = V_3 = 0$ and $||V_+|| = \sqrt{1-b_+^2}$ or
   - $b_+ = 0$ and $\beta = \rho = \sqrt{b_-^2 + b_3^2}$, which requires $||V_+||^2 = 1 - b_-^2 - b_3^2$, $||V_-|| = |b_3|$, $||V_3|| = |b_-|$, $V_+ \perp (V_-, V_3)$ and $\langle V_-, V_3\rangle = -b_- b_3$.

Lemma A. Let $b \in \mathcal{R}$. Then the condition $\det G(b,\beta) \geq 0$ is equivalent with the condition that $B = (b,\beta)$ belong to the body $\mathfrak{P}$. Furthermore, this condition implies that G_{11}, G_{22}, G_{33} and $\det G_{[23|23]}$ are non-negative.

Proof. Equation (D.5) shows that condition $\det G \geq 0$ is equivalent with:[12]

$$f_-(b) \leq \beta^2 \leq f_+(b) \quad \text{i.e.} \quad \beta \in J(b)\,, \tag{D.9}$$

which is equivalent with $(b_+,\beta) \in \mathfrak{P}$ (see (4.14)). By (D.8), this implies $\rho^2 \leq \beta^2 \leq 1 - b_+^2$, which upon using (D.6) implies that G_{ii} and $\det G_{[23|23]}$ are non-negative. □

Proposition. For each fixed value of $b_+ \in [-1,1]$, $f_-(b_+,\rho)$ is monotonically increasing while $f_+(b_+,\rho)$ is monotonically decreasing as a function of $\rho \in [0, 1-|b_+|]$. Moreover:

- $f_-(b_+,\rho)$ is strictly increasing as a function of $\rho \in (0, 1-|b_+|)$ for any $b_+ \in [-1,1]$
- $f_+(b_+,\rho)$ is strictly decreasing as a function of $\rho \in (0, 1-|b_+|)$ for any $b_+ \in [-1,0) \cup (0,1]$ while $f_+(0,\rho) = 1$ for any $\rho \in [0,1]$.

[12] The case $\beta = 0$ requires $\rho = 0$, which gives $f_+(b_+,0) = 1 - b_+^2$ and $f_-(b_+,0) = 0$, in which case (D.9) is satisfied.

Proof. We have:[13]

$$\frac{\partial f_-(b_+,\rho)}{\partial \rho} = \rho \frac{1+b_+^2-\rho^2+\sqrt{(1+b_+^2-\rho^2)^2-4b_+^2}}{\sqrt{(1+b_+^2-\rho^2)^2-4b_+^2}} \geq 0$$
$$\frac{\partial f_+(b_+,\rho)}{\partial \rho} = -\rho \frac{1+b_+^2-\rho^2-\sqrt{(1+b_+^2-\rho^2)^2-4b_+^2}}{\sqrt{(1+b_+^2-\rho^2)^2-4b_+^2}} \leq 0\,,$$

where the inequalities follow using $\rho^2 \leq 1$. The first inequality is strict unless $\rho = 0$ or $(b_+, \rho) = (0, 1)$. The second inequality is strict unless $\rho = 0$ or $b_+ = 0$. Notice that $f_+(0, \rho) = 1$ for $\rho \in [0, 1]$. $\square$

Proposition. For any $b \in \mathcal{R}$, we have:

$$\begin{aligned} f_-(b_+,b_-,0) &\leq f_-(b_+,b_-,b_3) \leq f_+(b_+,b_-,b_3) \leq f_+(b_+,b_-,0) \\ f_-(b_+,0,b_3) &\leq f_-(b_+,b_-,b_3) \leq f_+(b_+,b_-,b_3) \leq f_+(b_+,0,b_3)\,. \end{aligned} \tag{D.10}$$

In particular, the condition $\beta^2 \in [f_-(b_+,\rho), f_+(b_+,\rho)]$ implies $\det G_{[12|12]} \geq 0$ and $\det G_{[13|13]} \geq 0$. Furthermore, we have:

- $\det G_{[12|12]} = 0$ iff $\beta = 1$ or ($b_3 = 0$ and $\beta^2 \in \{f_-(b_+,b_-,0), f_+(b_+,b_-,0)\}$)
- $\det G_{[13|13]} = 0$ iff $\beta = 1$ or ($b_- = 0$ and $\beta^2 \in \{f_-(b_+,0,b_3), f_+(b_+,0,b_3)\}$).

Proof. Inequalities (D.10) follow immediately from the Lemma. When $\beta^2 \in [f_-(b_+,\rho), f_+(b_+,\rho)]$, these inequalities imply that β lies between the two roots of $P(b_+,b_-,0;\beta))$ and $P(b_+,0,b_3;\beta)$ (viewed as polynomials in β^2), which shows that $\det G_{[12|12]} \geq 0$ and $\det G_{[13|13]} \geq 0$ (see (D.4)). The other statements follow from the strict monotonicity properties listed in the lemma, recalling that $\beta = 1$ requires $b_+ = 0$. $\square$

Lemma B. The determinants $\det G_{[12|12]}$ and $\det G_{[23|23]}$ are non-negative for any $(b,\beta) \in \mathfrak{P}$.

Proof. Follows immediately from the previous proposition upon recalling that the body $\mathfrak{P}$ is a fibration over $\mathcal{R}$ with fiber given by the interval $J(b)$ defined in (4.13). $\square$

Remark. Lemma B implies that we have $f_-(b_+,\rho) \geq 0$, with equality iff $\rho = 0$ and $|b_+| = 1$.

Proposition. For $(b,\beta) \in \mathfrak{P}$, the equality $\beta = \rho$ can be attained only for $(b,\beta) \in \mathfrak{I} \cup \bar{\mathfrak{A}}$.

Proof. For $(b,\beta) \in \mathfrak{P}$, we have $\beta \in J(b)$ and hence $\rho^2 \leq f_-(b) \leq \beta^2$ by (D.8). Thus $\beta = \rho$ means that equality is realized in the first inequality of (D.8), which requires $b_+\rho = 0$, i.e. $b_+ = 0$ or $\rho = 0$. In the first case we have $(b,\beta) \in \bar{\mathfrak{A}}$ while in the second case we have $(b,\beta) \in \mathfrak{I}$. $\square$

[13] These relations should be interpreted in a limiting sense for $\rho = 1 - |b_+|$.

Lemma C. Let $B = (b, \beta) \in \mathfrak{P}$. Then $\mathrm{rk}G(B) \leq 1$ iff $B \in \mathfrak{I} \sqcup \partial\mathfrak{D}$. Furthermore, we have $\mathrm{rk}G = 0$ iff $B \in \partial\mathfrak{I}$.

Proof. (Necessity) The condition $\mathrm{rk}G(B) \leq 1$ requires that all two by two minors of G vanish. Relations (D.6) show that $\det G_{[23|23]} = 0$ implies $\beta = 0$ or $\beta = \rho$. In the first case, the first row of (D.4) and the conditions $G_{22} \geq 0$ and $G_{33} \geq 0$ imply $\rho = 0$, hence the first case is contained in the second. Thus we must have $\beta = \rho$ and the Proposition gives $(b_+, \rho) \in \mathfrak{I} \cup \bar{\mathfrak{A}}$. Consider the case $(b_+, \rho) \in \bar{\mathfrak{A}}$, i.e. $b_+ = 0$. Substituting $\beta = \rho$ and $b_+ = 0$ in (D.4), we find:

$$\det G_{[12|12]} = b_3^2(1 - \rho^2)\,, \qquad \det G_{[13|13]} = b_-^2(1 - \rho^2)\,. \tag{D.11}$$

Hence these two by two minors of $G(B)$ vanish simultaneously iff $\rho = 1$ or $\rho = 0$, i.e. iff (b_+, β) belongs to $\partial_0^0\mathfrak{P} \sqcup \partial\mathfrak{D}$. Since $\partial_0^0\mathfrak{P}$ is the midpoint of $\mathfrak{I}$, we conclude that $\mathrm{rk}G(B) \leq 1$ requires $(b_+, \rho) \in \mathfrak{I} \sqcup \partial\mathfrak{D}$.

(Sufficiency) For $B \in \mathfrak{I}$ (i.e. for $\rho = \beta = 0$), we have:

$$G(B) = \begin{bmatrix} 1 - b_+^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and hence $\mathrm{rk}G \leq 1$. Notice that $\mathrm{rk}G = 0$ iff $b_+ = \pm 1$ i.e. iff $B \in \partial\mathfrak{I}$. For $B \in \partial D$ (i.e. for $b_+ = 0$ and $\beta = \rho = 1$), we have:

$$G(B) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 - b_-^2 & -b_- b_3 \\ 0 & -b_- b_3 & 1 - b_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & b_3^2 & -b_- b_3 \\ 0 & -b_- b_3 & b_-^2 \end{bmatrix},$$

where in the second row we used the relation $b_-^2 + b_3^2 = \rho^2 = 1$. Thus $\mathrm{rk}G \leq 1$, since the two by two minor in the lower right corner has vanishing determinant. In this case, we cannot have $\mathrm{rk}G = 0$ (i.e. $G = 0$) since $b_-^2 + b_3^2 = 1$ and hence b_- and b_3 cannot vanish simultaneously. $\square$

Proof of Theorem 2. The following result follows by combining Lemmas A, B and C:

Theorem 2'. Let $b \in \mathcal{R}$. Then the matrix $G(b, \beta)$ is semipositive iff $\beta \in J(b)$, i.e. iff $B \stackrel{\text{def.}}{=} (b, \beta) \in \mathfrak{P}$. It is strictly positive iff $B \in \mathrm{Int}\mathfrak{P}$. In particular, we have $\mathrm{rk}G(B) < 3$ at a point $p \in M$ iff $B(p) \in \partial\mathfrak{P}$. When $B \in \partial\mathfrak{P}$, we have:

- $\mathrm{rk}G(B) = 0$ iff $B \in \partial_0^+\mathfrak{P} \sqcup \partial_0^-\mathfrak{P} = \partial\mathfrak{I}$
- $\mathrm{rk}G(B) = 1$ iff $B \in \partial_0^0\mathfrak{P} \sqcup \partial_1\mathfrak{P} = \partial\mathfrak{D} \sqcup \mathrm{Int}\mathfrak{I}$
- $\mathrm{rk}G(B) = 2$ iff $B \in \partial_2\mathfrak{P} \cup \partial_3\mathfrak{P} = \mathrm{Int}\mathfrak{D} \sqcup \mathfrak{A} \sqcup \mathrm{Int}\mathfrak{C}^+ \sqcup \mathrm{Int}\mathfrak{C}^-$.

We know from subsection 3.2 that $\mathrm{im}b \subset \mathcal{R}$. Combining this with Theorem 2', we find that the image of B is contained in $\mathfrak{P}$. Theorem 2 now follows immediately.

D.2 Proof of Theorem 3

Theorem 3 is an immediate consequence of Lemma D proved below.

Lemma D. Let $p \in M$. Then:

1. The value $\beta(p) = 0$ is attained iff $B(p) \in \mathfrak{I}$. At such points, we have $b_-(p) = b_3(p) = 0$, $V_-(p) = V_3(p) = 0$ and $||V_+(p)|| = \sqrt{1 - b_+(p)^2}$, thus $\mathcal{D}(p)$ has dimension seven or eight, according to whether $|b_+| < 1$ or $|b_+| = 1$.
2. The value $\beta(p) = 1$ is attained iff $B(p) \in \mathfrak{D}$. At such points, we have $V_+(p) = 0$, $\det G_{[12|12]}(p) = \det G_{[13|13]}(p) = 0$ and:
$$||V_-(p)|| = \sqrt{1 - b_-(p)^2}\,, \quad ||V_3(p)|| = \sqrt{1 - b_3(p)^2}\,, \quad \langle V_-(p), V_3(p)\rangle = -b_-(p)b_3(p)\,.$$
The space $\mathcal{D}(p)$ has dimension six when $B(p) \in \mathrm{Int}\mathfrak{D}$ and dimension seven when $B(p) \in \partial\mathfrak{D}$.
3. When $B(p) \in \partial\mathfrak{D}$ (i.e. when $\beta(p) = \rho(p) = 1$), we have $b_+(p) = 0\,,\ V_+(p) = 0$,
$$V_-(p) = (\sin\psi)v\,, \;\; V_3(p) = -(\cos\psi)v\,, \;\; b_-(p) = \cos\psi\,, \;\; b_3(p) = \sin\psi$$
and $\langle V_+(p), V_-(p), V_3(p)\rangle = \langle v\rangle$, where $\psi \in [0, 2\pi)$ and $v \in T_p^*M$ is an arbitrary 1-form of norm one.
4. When $B(p) \in \bar{\mathfrak{A}}$ (i.e. when $\beta(p) = \rho(p)$), we have $\det G_{[23|23]}(p) = 0$ and $||V_+(p)|| = \sqrt{1 - \rho(p)^2}$, $V_-(p) = (\rho(p)\sin\psi)v$, $V_3(p) = -(\rho(p)\cos\psi)v$ with $\psi \in [0, 2\pi)$ and $v \in T_p^*M$ an arbitrary 1-form of unit norm such that $V_+(p) \perp v$. The space $\mathcal{D}(p)$ has dimension six when $B(p) \in \mathfrak{A}$ and dimension seven when $B(p) \in \mathrm{fr}\mathfrak{A} = \partial_0^0\mathfrak{P} \sqcup \partial\mathfrak{D}$.

Proof. Inequalities (D.8) imply that $\beta = 0$ can be attained only at $\rho = 0$, i.e. only for $B(p) \in \mathfrak{I}$. They also imply that $\beta = 1$ can be attained only at $b_+ = 0$, i.e. only for $B(p) \in \bar{\mathfrak{A}}$. The remaining statements follow immediately using the system (4.2). $\square$

D.3 Solving for b_r in terms of V_r

Notice that $G_{12}G_{23}G_{13} = -(b_+b_-b_3)^2$, so the condition $b_+b_-b_3 \neq 0$ amounts to the requirement that no two of the vectors V_r are orthogonal. In this case, we have:

$$b_+ = \epsilon\frac{\sqrt{-G_{12}G_{23}G_{13}}}{G_{23}}\,, \quad b_- = \epsilon\frac{\sqrt{-G_{12}G_{23}G_{13}}}{G_{13}}\,, \quad b_3 = \epsilon\frac{\sqrt{-G_{12}G_{23}G_{13}}}{G_{12}} \tag{D.12}$$

where $\epsilon \in \{-1, 1\}$ and hence (4.2) can be solved for b_r iff the following conditions are satisfied:

$$0 \leq 1 - G_{11} + \frac{G_{12}G_{13}}{G_{23}} = G_{22} - \frac{G_{12}G_{23}}{G_{13}} = G_{33} - \frac{G_{13}G_{23}}{G_{12}}(= \beta^2)\,. \tag{D.13}$$

Conditions (D.13) show that the triples of vectors allowed by (4.2) are constrained.

E The rank of $\hat{G}$

Direct computation using (4.8) gives:

$$\det \hat{G}(b,\beta) = P(b,\beta)^2\,, \tag{E.1}$$

The determinants of the 3 by 3 principal minors of $\hat{G}$ are given by:

$$\begin{aligned}
\det \hat{G}_{[123|123]} &= \det G = -\beta^2 P(b,\beta)\,,\\
\det \hat{G}_{[124|124]} &= -(1-b_+^2-\beta^2+b_-^2)P(b,\beta)\,,\\
\det \hat{G}_{[134|134]} &= -(1-b_+^2-\beta^2+b_3^2)P(b,\beta)\,,\\
\det \hat{G}_{[234|234]} &= (\rho^2-\beta^2)P(b,\beta)\,,
\end{aligned} \tag{E.2}$$

where $P(b,\beta)$ was defined in (4.10):

$$P(b,\beta) = (1-\beta^2)(\rho^2-\beta^2)+\beta^2 b_+^2 = -\beta^2(1-b_+^2-\beta^2+\rho^2)+\rho^2\,,$$

while the determinants of the 2 by 2 principal minors are:

$$\begin{aligned}
\det \hat{G}_{[12|12]} &= \det G_{[12|12]} = \beta^2(1-b_+^2-\beta^2+b_-^2)-b_-^2 = -P(b,\beta)+b_3^2(1-\beta^2)\,,\\
\det \hat{G}_{[13|13]} &= \det G_{[13|13]} = \beta^2(1-b_+^2-\beta^2+b_3^2)-b_3^2 = -P(b,\beta)+b_-^2(1-\beta^2)\,,\\
\det \hat{G}_{[14|14]} &= (1-b_+^2-\beta^2)(1-b_+^2-\beta^2+\rho^2)\,,\\
\det \hat{G}_{[23|23]} &= \det G_{[23|23]} = \beta^2(\beta^2-\rho^2)\,,\\
\det \hat{G}_{[24|24]} &= (1-b_+^2-\beta^2+\rho^2)(\beta^2-b_-^2)-b_3^2\,,\\
\det \hat{G}_{[34|34]} &= (1-b_+^2-\beta^2+\rho^2)(\beta^2-b_3^2)-b_-^2\,.
\end{aligned} \tag{E.3}$$

Lemma. The rank of $\hat{G}(B)$ is given as follows:

1. For $B\in \mathrm{Int}\mathfrak{P}$, we have $\mathrm{rk}\hat{G}(B)=4$.
2. For $B\in \mathrm{Int}\mathfrak{I}\sqcup\mathrm{Int}\mathfrak{D}\sqcup\mathfrak{A}\sqcup\mathrm{Int}\mathfrak{C}^+\sqcup\mathrm{Int}\mathfrak{C}^-$, we have $\mathrm{rk}\hat{G}(B)=2$.
3. For $B\in\partial\mathfrak{D}$, we have $\mathrm{rk}\hat{G}(B)=1$.
4. For $B\in\partial\mathfrak{I}$, we have $\mathrm{rk}\hat{G}(B)=0$.

Proof. Since $P(B)$ vanishes iff $B\in\mathfrak{P}$, relation (E.1) implies that $\hat{G}$ is non-degenerate on $\mathrm{Int}\mathfrak{P}$ and degenerate on $\partial\mathfrak{P}$. In particular, we have $\mathrm{rk}\hat{G}(B)=4$ for $B\in\mathrm{Int}\mathfrak{P}$. For $B\in\partial\mathfrak{P}$, we have $P(B)=0$ and hence $\det\hat{G}=0$. Furthermore, all 3 by 3 minors of $\hat{G}$ vanish by relations (E.2). We distinguish the cases:

- $B\in\mathfrak{I}$. Then $\beta=\rho=0$ and $\mathrm{rk}G(B)\leq 1$, thus $\det\hat{G}_{[12|12]}=\det\hat{G}_{[13|13]}=\det\hat{G}_{[23|23]}=0$. Relations (E.3) give:

$$\det\hat{G}_{[14|14]} = (1-b_+^2)^2\,,\quad \det\hat{G}_{[24|24]}=\det\hat{G}_{[34|34]}=0\ \ ,$$

which show that $\mathrm{rk}\hat{G}(B)=2$ for $B\in\mathrm{Int}\mathfrak{I}$. The case $B\in\partial\mathfrak{I}=\partial_0^+\mathfrak{P}\sqcup\partial_0^-\mathfrak{P}$ corresponds to $\rho=\beta=b_-=b_3=0$ with $b_+^2=1$. For these values, (4.8) gives $\hat{G}(B)=0$ and hence $\mathrm{rk}\hat{G}(B)=0$.

- $B \in \partial\mathfrak{D}$, i.e. $b_+ = 0$ and $\beta = \rho = 1$. Then (E.3) shows that all 2 by 2 minors of $\hat{G}$ vanish while (4.8) shows that $\hat{G} \neq 0$, which means that we must have $\mathrm{rk}\hat{G}(B) = 1$.
- $B \in \mathrm{Int}\mathfrak{D}$, i.e. $b_+ = 0$, $\beta = 1$ and $\rho \in [0, 1)$. Then (E.3) gives $\det \hat{G}_{[23|23]} = 1 - \rho^2 > 0$ and hence $\mathrm{rk}\hat{G}(B) = 2$.
- $B \in \mathfrak{A}$, i.e. $b_+ = 0$ and $\beta = \rho \in (0, 1)$. Then (E.3) gives $\det \hat{G}_{[14|14]} = 1 - \rho^2 > 0$ and hence $\mathrm{rk}\hat{G}(B) = 2$.
- $B \in \mathrm{Int}\mathfrak{C}^+ \sqcup \mathrm{Int}\mathfrak{C}^-$, i.e. $P(b, \beta) = 0$ with $b_+ = \pm g(\rho, \beta)$ and $0 \leq \rho < \beta < 1$, where g is the function defined in (4.16). Then $\det \hat{G}_{[23|23]} = \det G_{[23|23]} > 0$ and hence $\mathrm{rk}\hat{G}(B) = 2$.

The Lemma follows by combining these results. $\square$

Proposition. For $p \in \mathcal{G}$, we have $\dim \mathcal{D}_0(p) \in \{4, 6\}$.

Proof. Follows immediately from the Lemma upon noticing that $b(\mathcal{G}) \subset \mathrm{Int}\mathcal{R}$ while $\pi(\partial\mathfrak{D}), \pi(\partial\mathfrak{I}) \subset \partial\mathcal{R}$. $\square$

F On certain deformations of (ξ_1, ξ_2)

F.1 A family of special deformations

Consider a locally non-degenerate and $\mathscr{B}$-compatible two-dimensional subspace $\mathcal{K} \subset \Gamma(M, S)$ and let (ξ_1, ξ_2) be an orthonormal basis of $\mathcal{K}$. Thus $\xi_1(p)$ and $\xi_2(p)$ form an orthonormal system of Majorana spinors for any $p \in M$. Let $\mathcal{G}$ denote the non-special locus of $\mathcal{K}$, i.e. the set consisting of those points $p \in M$ such that the positive chirality components $\xi_1^+(p)$ and $\xi_2^+(p)$ are linearly independent and such that the same holds for the negative chirality components $\xi_1^-(p)$ and $\xi_2^-(p)$.

Consider the special class of deformations of the pair (ξ_1, ξ_2) to another pair of Majorana spinors $(\tilde{\xi}_1, \tilde{\xi}_2)$ such that only ξ_1^- changes:

$$\tilde{\xi}_2 = \xi_2 \quad \text{and} \quad \tilde{\xi}_1^+ = \xi_1^+ \,. \tag{F.1}$$

Recall that $\xi_1^\pm$ and $\xi_2^\pm$ generate the chiral projections $K_\pm$ of the spinor sub-bundle K associated to S. Under a special deformation obeying (F.1), the positive chirality projection is invariant while the negative chirality projection may change:

$$\tilde{K}_+ = K_+ \,, \quad K_- \to \tilde{K}_- \,.$$

As a result, the bundle K changes to $\tilde{K}$ and the space $\mathcal{K}$ changes to the space $\tilde{\mathcal{K}} = \mathbb{R}\tilde{\xi}_1 + \mathbb{R}\tilde{\xi}_2 \subset \Gamma(M, S)$. We require that the system $(\tilde{\xi}_1, \tilde{\xi}_2)$ is everywhere orthonormal, so that $\tilde{\mathcal{K}}$ is again a two-dimensional and $\mathscr{B}$-compatible locally-nondegenerate subspace of $\Gamma(M, S)$.

For the remainder of this appendix, consider two Majorana spinors $\tilde{\xi}_1, \tilde{\xi}_2 \in \Gamma(M, S)$ which satisfy (F.1) and are everywhere orthonormal. Let $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3$ and $\tilde{V}_1, \tilde{V}_2, \tilde{V}_3, \tilde{W}$ denote the zero- and one-forms defined by the spinors $\tilde{\xi}_1, \tilde{\xi}_2$ according to relations (2.1) and (2.3) and $\tilde{b}_\pm, \tilde{V}_3^\pm$ denote the associated quantities defined as in section 2. Let $\tilde{\beta} \in \mathcal{C}^\infty(M, \mathbb{R}^+)$ denote the function defined according to (4.3). Notice that $\tilde{\xi}_1^-$ has the form:

$$\tilde{\xi}_1^- = \alpha_1 \xi_1^- + \alpha_2 \xi_2^- + \zeta \in \Gamma(M, S^-)\,, \tag{F.2}$$

where $\alpha_1, \alpha_2 \in \mathcal{C}^\infty(M, \mathbb{R})$ and $\zeta \in \Gamma(M, S^-)$ is the projection of $\tilde{\xi}_1$ onto the $\mathscr{B}$-orthocomplement of K^- inside S^-. Hence ζ is a section of S^- which is everywhere orthogonal to K_- and whose norm we shall denote by:

$$\lambda \stackrel{\text{def.}}{=} ||\zeta||\,. \tag{F.3}$$

Recall that $b(p) \in \mathrm{Int}\mathcal{R}$ for any $p \in \mathcal{G}$.

Lemma. The following inequalities hold for any point $p \in \mathcal{G}$:

$$|b_1(p)| < 1\,, \quad |b_2(p)| < 1\,, \quad \rho(p) < 1 - |b_+(p)| \tag{F.4}$$

Proof. For any point $p \in \mathcal{G}$, we have $b(p) \in \mathrm{Int}\mathcal{R}$ and hence $\rho(p) < 1 - |b_+(p)| \leq 1 - b_+(p)$, which shows that $\det A(p) > 0$. On the other hand, the planes $b_1 = \pm 1 \leftrightarrow b_+ + b_- = 1$ and $b_2 = \pm 1 \leftrightarrow b_+ - b_- = \pm 1$ in the space $\mathbb{R}^3$ with coordinates b_+, b_-, b_3 intersect the body $\mathcal{R}$ along two segments which lie within $\partial\mathcal{R}$ and hence we have $|b_1(p)| < 1$ and $|b_2(p)| < 1$. $\square$

Proposition. We have $\tilde{b}_i = b_i$ for all $i = 1, 2, 3$ and hence $\tilde{b}_\pm = b_\pm$. On the locus $\mathcal{G}$, we have:

$$|\alpha_1| \leq_{\mathcal{G}} 1 \tag{F.5}$$

and:

$$\alpha_2 =_{\mathcal{G}} \frac{b_3}{1 - b_2}(\alpha_1 - 1) \tag{F.6}$$

Furthermore, the norm of ζ has the following form on the locus $\mathcal{G}$:

$$\lambda \stackrel{\text{def.}}{=} ||\zeta|| =_{\mathcal{G}} \lambda_M \sqrt{1 - \alpha_1^2}\,. \tag{F.7}$$

where:

$$\lambda_M \stackrel{\text{def.}}{=} \sqrt{\frac{(1 - b_+)^2 - \rho^2}{2(1 + b_- - b_+)}} = \sqrt{\frac{(1 - b_1)(1 - b_2) - b_3^2}{2(1 - b_2)}} \in \mathcal{C}^\infty(\mathcal{G}, \mathbb{R})\,. \tag{F.8}$$

Proof. Consider the scalars (2.1) defined by the orthonormal Majorana spinors $\tilde{\xi}_1(p)$ and $\xi_2(p)$, namely $\tilde{b}_1 \stackrel{\text{def.}}{=} \mathscr{B}(\tilde{\xi}_1, \gamma(\nu)\tilde{\xi}_1)\,, \quad \tilde{b}_3 \stackrel{\text{def.}}{=} \mathscr{B}(\tilde{\xi}_1, \gamma(\nu)\xi_2)$ and $\tilde{b}_2 \stackrel{\text{def.}}{=} \mathscr{B}(\xi_2, \gamma(\nu)\xi_2) = b_2$. Since (3.11) hold for $\tilde{\xi}_1, \xi_2$ and $\tilde{b}_r$ and since the positive chirality components of $\tilde{\xi}_1$ and ξ_1 coincide, we find $\tilde{b}_3 = 2\mathscr{B}(\xi_1^+, \xi_2^+) = b_3$ and $\tilde{b}_1 = 2||\xi_1^+||^2 - 1 = b_1$. Thus $\tilde{b}_i = b_i$ for all $i = 1, 2, 3$.

It is clear that $\tilde{\xi}_1^-|_{\mathcal{G}}$ has the form (F.2), where $\zeta = (\mathrm{id}_{S^-} - P_-)\tilde{\xi}_1^-$ is the projection of $\tilde{\xi}_1^-|_{\mathcal{G}}$ onto the orthocomplement of $K^-|_{\mathcal{G}}$ inside $S^-|_{\mathcal{G}}$. Since ζ is $\mathscr{B}_p$-orthonormal on $\xi_1^-(p)$ and $\xi_2^-(p)$, we have:

$$||\tilde{\xi}_1^-||^2 = ||\alpha_1 \xi_1^- + \alpha_2 \xi_2^-||^2 + \lambda^2\,,$$

$$\mathscr{B}(\tilde{\xi}_1^-, \xi_2^-) = \alpha_1 \mathscr{B}(\xi_1^-, \xi_2^-) + \alpha_2 ||\xi_2^-||^2 \,,$$

where we set $\lambda \stackrel{\text{def.}}{=} ||\zeta||$. Since (ξ_1, ξ_2) is $\mathscr{B}_p$-orthonormal and since $\tilde{\xi}_1^+ = \xi_1^+$, the condition that $(\tilde{\xi}_1, \xi_2)$ be orthonormal amounts to the constraints:

$$||\tilde{\xi}_1^-||^2 = ||\xi_1^-||^2 \; (= 1 - ||\xi_1^+||^2) \,,$$
$$\mathscr{B}_p(\tilde{\xi}_1^-, \xi_2^-) = \mathscr{B}_p(\xi_1^-, \xi_2^-) \; (= -\mathscr{B}_p(\xi_1^+, \xi_2^+)) \,,$$

which upon using (3.11) gives the system:

$$\begin{aligned} (1-b_1)\alpha_1^2 + (1-b_2)\alpha_2^2 - 2b_3\alpha_1\alpha_2 &= 1 - b_1 - 2\lambda^2 \\ b_3(1-\alpha_1) + (1-b_2)\alpha_2 &= 0 \,. \end{aligned} \tag{F.9}$$

The left hand side of the first equation defines the quadratic form $\alpha^T A(p)\alpha$, where A is the symmetric matrix-valued function:

$$A \stackrel{\text{def.}}{=} \begin{bmatrix} 1-b_1 & -b_3 \\ -b_3 & 1-b_2 \end{bmatrix} ,$$

whose determinant equals:

$$\det A = (1-b_1)(1-b_2) - b_3^2 = (1-b_+)^2 - \rho^2 \,.$$

The inequalities $|b_1| \leq 1$, $|b_2| \leq 1$ and $\rho \leq 1 - |b_+|$ imply that $A(p)$ is a semi-positive matrix for any $p \in M$ while (F.4) imply that $A(p)$ is strictly positive for $p \in \mathcal{G}$. The eigenvalues a_-, a_+ of A are given by:

$$a_\pm = 1 - b_+ \pm \rho \,.$$

Since A is semipositive on M, we have $\alpha^T A \alpha \geq 0$, which shows that the first equation in (F.9) has solutions iff the right hand side is non-negative, i.e. only for $\lambda \leq \lambda_0$, where $\lambda_0 \stackrel{\text{def.}}{=} \sqrt{\frac{1-b_1}{2}} \in \mathcal{C}^\infty(M, \mathbb{R})$. For any $\lambda(p) \leq \lambda_0(p)$ in this interval, the first equation of (F.9) considered at the point $p \in M$ defines an ellipse $E_\lambda(p)$ in the $\alpha(p)$-plane, whose half-axes have length $\frac{1}{\sqrt{a_\pm(p)}}$. This ellipse degenerates to a single point (namely the origin $\alpha_1(p) = \alpha_2(p) = 0$) for $\lambda(p) = \lambda_0(p)$. For $b_2(p) \neq 1$, the second equation in (F.9) (considered at p) defines a line in the $\alpha(p)$-plane which passes through the points $(1, 0)$ and $(0, -\frac{b_3(p)}{1-b_2(p)})$. This equation implies $\alpha_2(1-b_2) = b_3(\alpha_1 - 1)$, which combines with the first relation of (F.9) to give:

$$2\lambda^2(1-b_2) = (1-\alpha_1^2)[(1-b_+)^2 - \rho^2] \,.$$

Since the left hand side is non-negative and since $\rho^2 < (1-b_+)^2$ on the locus $\mathcal{G}$, this implies (F.5). Provided that (F.5) is satisfied, we can solve (F.9) in terms of α_1. This gives (F.6) and (F.7), with λ_M is as in (F.8). The second equation in (F.9) shows that solutions of (F.9) exist only for $\lambda \leq \lambda_M$. $\square$

Proposition. The 1-forms defined by $\tilde{\xi}_1$ and $\tilde{\xi}_2 = \xi_2$ are given by:

$$\begin{aligned}\tilde{V}_1 &= \alpha_1 V_1 + \alpha_2 V_3 - \alpha_2 W + 2U_1 \\ \tilde{V}_2 &= V_2 \\ \tilde{V}_3 &= \tilde{V}_3^+ + V_3^- = \frac{1}{2}\alpha_2 V_2 + \frac{1}{2}(1+\alpha_1)V_3 - \frac{1}{2}(1-\alpha_1)W + U_2 \\ \tilde{W} &= \tilde{V}_3^+ - V_3^- = \frac{1}{2}\alpha_2 V_2 - \frac{1}{2}(1-\alpha_1)V_3 + \frac{1}{2}(1+\alpha_1)W + U_2 \,,\end{aligned} \tag{F.10}$$

where:

$$U_i \overset{\text{def.}}{=} \mathscr{B}(\zeta, \gamma_a \xi_i^+) e^a \in \Omega^1(M) \quad (i = 1, 2) \tag{F.11}$$

and $\tilde{V}_3^- = V_3^-$.

Proof. Using (2.5), we find that the following relations hold on M:

$$\begin{aligned}\tilde{V}_1 &\overset{\text{def.}}{=} 2\mathscr{B}(\tilde{\xi}_1^-, \gamma_a \xi_1^+) e^a = \alpha_1 V_1 + 2\alpha_2 V_3^- + 2U_1 \\ \tilde{V}_3^+ &\overset{\text{def.}}{=} \mathscr{B}(\tilde{\xi}_1^-, \gamma_a \xi_2^+) e^a = \alpha_1 V_3^+ + \frac{1}{2}\alpha_2 V_2 + U_2 \\ \tilde{V}_2 &= V_2 \,, \quad \tilde{V}_3^- = V_3^- \,.\end{aligned} \tag{F.12}$$

Recall that $\tilde{V}_\pm \overset{\text{def.}}{=} \frac{1}{2}(\tilde{V}_1 \pm \tilde{V}_2)$ and $\tilde{W} \overset{\text{def.}}{=} \tilde{V}_3^+ - \tilde{V}_3^- = \mathscr{B}(\tilde{\xi}_1, \gamma_a \gamma(\nu)\xi_2)$. Equations (F.12) give (F.10), where we used (2.4). □

Consider the following open subset of $\mathcal{G}$:

$$\mathcal{G}_0 \overset{\text{def.}}{=} \{p \in \mathcal{G} | \zeta(p) \neq 0\} = \{p \in \mathcal{G} | \lambda(p) \neq 0\} = \{p \in \mathcal{G} | \alpha_1(p) \neq \pm 1\} \,.$$

Proposition. We have $U_1(p) \neq 0$ and $U_2(p) \neq 0$ at any point $p \in \mathcal{G}_0$.

Proof. Since $p \in \mathcal{G}$, the spinors $\xi_1^+(p)$ and $\xi_2^-(p)$ are linearly independent and in particular non-vanishing. It was shown in [24, section 2.6] that, for any non-vanishing spinor $\eta \in S_p^+ \setminus \{0\}$, the spinors $(\gamma_a \eta)_{a=1\ldots8}$ form a basis of S_p^-. Thus $(\gamma_a \xi_1^+(p))_{a=1\ldots8}$ is a basis of S_p^- and the same is true for $(\gamma_a \xi_2^+(p))_{a=1\ldots8}$. Since $\zeta(p)$ is non-zero, this gives the conclusion. □

Proposition. The one-forms U_1 and U_2 satisfy the following relations on the locus $\mathcal{G}$:

$$\begin{aligned}&\langle U_1, V_1\rangle = \langle U_2, V_2\rangle = 0\,, && \langle U_1, U_2\rangle = \frac{b_3}{2}\lambda^2 \\ &||U_1||^2 = \frac{1+b_1}{2}\lambda^2\,, && ||U_2||^2 = \frac{1+b_2}{2}\lambda^2\,, \\ &\langle U_1, V_3\rangle = \langle U_1, W\rangle = -\frac{1}{2}\langle U_2, V_1\rangle\,, && \\ &\langle U_2, V_3\rangle = -\langle U_2, W\rangle = -\frac{1}{2}\langle U_1, V_2\rangle\,. &&\end{aligned} \tag{F.13}$$

while $\tilde{\beta}$ is given by the following expression on the same locus:

$$\tilde{\beta}^2 = \alpha_1 \beta^2 + \frac{1-\alpha_1}{2}(1 + \rho^2 - b_+^2) - \langle U_1, V_2\rangle\,. \tag{F.14}$$

Proof. Since the system $(\tilde{\xi}_1, \tilde{\xi}_2)$ is everywhere-orthonormal, the 1-forms $\tilde{V}_+, \tilde{V}_-, \tilde{V}_3, \tilde{W}$ satisfy (4.1) and hence their Gram matrix $\hat{\tilde{G}}$ must have the form (see (4.8)):

$$\hat{\tilde{G}} = \hat{G}(b, \tilde{\beta}) = \begin{bmatrix} 1 - \tilde{\beta}^2 - b_+^2 & -b_+b_- & -b_+b_3 & 0 \\ -b_-b_+ & \tilde{\beta}^2 - b_-^2 & -b_-b_3 & b_3 \\ -b_3b_+ & -b_3b_- & \tilde{\beta}^2 - b_3^2 & -b_- \\ 0 & b_3 & -b_- & 1 - \tilde{\beta}^2 - b_+^2 + \rho^2 \end{bmatrix}, \tag{F.15}$$

where we used the fact that $\tilde{b}_i = b_i$ and thus $\tilde{\beta}^2 = b_3 + ||V_3||^2$. The Gram determinant is given by (E.1):

$$\det \hat{G}(b, \tilde{\beta}) = P(b, \tilde{\beta})^2 = \left[(b_3^2 + b_-^2)(\tilde{\beta}^2 - 1) - \tilde{\beta}^2(\tilde{\beta}^2 - 1 + b_+^2)\right]^2 = \left[(\tilde{\beta}^2 - 1)(\rho^2 - \tilde{\beta}^2) - \tilde{\beta}^2 b_+^2\right]^2,$$

where P is the polynomial given in (4.10). Using (4.1), (F.10) and (F.6), we find that $\tilde{\beta}$ can be expressed as follows as a function of α_1 on the locus $\mathcal{G}$:

$$\begin{aligned} \tilde{\beta}^2 = &\frac{1}{4}\left(1 + \rho^2 - b_+^2 + 2b_- + \frac{2b_3^2}{1 + b_- - b_+}\right) + \langle U_2, V_3\rangle - \langle U_2, W\rangle + ||U_2||^2 - \frac{b_3\langle U_2, V_2\rangle}{1 + b_- - b_+} \\ &+ \alpha_1\left[\frac{1}{2}\left(-1 + 2\beta^2 - \rho^2 + b_+^2\right) + \langle U_2, V_3\rangle + \langle U_2, W\rangle + \frac{b_3}{1 + b_- - b_+}||U_2||^2\right] \\ &+ \frac{\alpha_1^2(-1 + b_- - b_+)[\rho^2 - (1 - b_+)^2]}{1 + b_- - b_+}. \end{aligned} \tag{F.16}$$

On the locus $\mathcal{G}_0$, we have $\zeta = \lambda\hat{\zeta}$, where $\hat{\zeta} \stackrel{\text{def.}}{=} \frac{\zeta}{\lambda}$ is a unit norm spinor of negative chirality defined on $\mathcal{G}_0$ and which is orthonormal to ξ_1^- and ξ_2^- at every point of $\mathcal{G}_0$. On this locus, we can write $U_i = \lambda\hat{U}_i$, with:

$$\hat{U}_i \stackrel{\text{def.}}{=} \mathscr{B}(\hat{\zeta}, \gamma_a\xi_i^+)e^a \in \Omega^1(\mathcal{G}_0) \quad (i = 1, 2). \tag{F.17}$$

Substituting this into (F.10), we find an expression for the Gram matrix $\hat{\tilde{G}}$ as a function of α_1, α_2 and λ, where α_2 and λ can be expressed as functions of α_1 using the previous proposition. Thus $\hat{\tilde{G}}(\alpha_1)$ must equal the matrix $\hat{G}(b, \tilde{\beta}(\alpha_1))$ of (F.15) (where $\tilde{\beta}(\alpha_1)$ is given by (F.16)) for any $\alpha_1 \in [-1, 1]$. Expanding both of these matrices to order two in α_1, we find three linear systems in the quantities $\langle U_i, V_+\rangle, \langle U_i, V_-\rangle, \langle U_i, V_3\rangle$ and $\langle U_i, W\rangle$, which can be shown to be equivalent[14] with (F.13). Using (F.13), relation (F.16) simplifies to (F.14). Substituting (F.13) into $\hat{\tilde{G}}$, we find that $\hat{\tilde{G}}$ equals the matrix $\hat{G}(b, \tilde{\beta})$ of (F.15), where $\tilde{\beta}$ is given by (F.14). It follows that there are no further constrains on U_1 and U_2 and hence that equality of $\hat{\tilde{G}}(\alpha_1)$ and (F.15) is equivalent with relations (F.13) and (F.14) on the locus $\mathcal{G}_0$. These relation also hold on $\mathcal{G} \setminus \mathcal{G}_0$ since U_1, U_2 and λ vanish on that locus. □

Since U_1 depends continuously on α_1, relation (F.14) shows that:

$$\tilde{\beta}^2 = t(B, \alpha_1)$$

[14] At this step we used `Mathematica`®, which we acknowledge here.

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where $B = (b, \beta) \in \mathfrak{P}$ and $t : \mathfrak{P} \times [-1, 1] \to \mathbb{R}$ is a continuous function. Since $\tilde{\beta}$ is the function associated by relation (4.3) to the system of everywhere orthonormal spinors $(\tilde{\xi}_1, \tilde{\xi}_2)$, we know that $\tilde{\beta}(p)$ must belong to the interval $J(p) = J(b(p))$ for any value of $\alpha_1(p)$, where $J(b)$ was defined in (4.13). Hence the image of the function $t_B : [-1, 1] \to \mathbb{R}$ defined through:

$$t_B(\alpha_1) \stackrel{\text{def.}}{=} t(B, \alpha_1) \quad (B \in \mathfrak{P})$$

is contained in the interval $[f_-(b), f_+(b)]$. On the sub-locus of $\mathcal{G} \setminus \mathcal{G}_0$ where $\alpha_1 = \pm 1$, we have $\zeta = 0$ and $U_1 = 0$, hence (F.14) gives:

$$t(B, +1) = \beta^2 \,, \quad t(B, -1) = -\beta^2 + 1 + \rho^2 - b_+^2 \tag{F.18}$$

while on the locus $\mathcal{G}_0$, relation (F.14) gives:

$$t(B, \alpha_1) = \alpha_1 \beta^2 + \frac{1 - \alpha_1}{2}(1 + \rho^2 - b_+^2) - \lambda_M \sqrt{1 - \alpha_1^2} \langle \hat{U}_1, V_2 \rangle \quad (\alpha_1 \in (-1, 1)) \,, \tag{F.19}$$

which shows that t is differentiable on $\mathfrak{P} \times (-1, 1)$.

Proposition. Let $B \in \partial\mathfrak{P}$. Then the image of t_B equals the interval $[f_-(b), f_+(b)]$ and hence the image of the function $\sqrt{t_B}$ equals the interval $J(b)$ defined in (4.13).

Proof. The condition $B \in \partial\mathfrak{P}$ means that $\beta = \sqrt{f_\pm(b)}$, where the functions $f_\pm(b) = f_\pm(b_+, \rho)$ were defined in (4.11). Then $t_B(+1) = \beta^2 = f_\pm(b)$ while $t_B(-1) = 1 + \rho^2 - b_+^2 - f_\pm(b) = f_\mp(b)$, where we used (4.11). Thus:

$$t_B(+1) = f_\pm(p) \quad \text{and} \quad t_B(-1) = f_\mp(p) \,. \tag{F.20}$$

Since t_B is continuous, its image (which is contained in $[f_-(b), f_+(b)]$) is an interval which must contain the two values (F.20) and hence must equal $[f_-(b), f_+(b)]$. □

F.2 Explicit spinor deformations which break the stabilizer from SU(3) to SU(2)

Let $\tilde{B} = (b, \tilde{\beta}) : M \to \mathfrak{P}$ be the function (4.5) defined by the system of spinors $(\tilde{\xi}_1, \tilde{\xi}_2)$ and let $\tilde{\mathcal{D}}_0 \stackrel{\text{def.}}{=} \ker \tilde{V}_1 \cap \ker \tilde{V}_2 \cap \ker \tilde{V}_3 \cap \ker \tilde{W}$.

Proposition. Let $p \in \mathcal{G}$ be such that $B(p) = (b(p), \beta(p))$ belongs to $\partial\mathfrak{P}$ and let β_0 be any point in the interior of the interval $J(b)$. Then we can find a deformation $(\xi_1, \xi_2) \to (\tilde{\xi}_1, \tilde{\xi}_2)$ such that $(\tilde{\xi}_1, \tilde{\xi}_2)$ is a system of everywhere-orthonormal Majorana spinors on M and such that $\tilde{B} = (b, \tilde{\beta})$ with $\tilde{\beta}(p) = \beta_0$.

Proof. Follows immediately from the results of the previous subsection. □

Remark. Together with the results of subsection 3.5, the proposition implies that, for every value $B_0 \in \mathfrak{P}$ and every point $p \in M$, there exists a pair of everywhere-orthonormal Majorana spinors (ξ_1, ξ_2) on M whose function B satisfies $B(p) = B_0$. In particular, all points of $\mathfrak{P}$ can be realized by some two-dimensional and $\mathscr{B}$-compatible locally-nondegenerate subspace $\mathcal{K} \subset \Gamma(M, S)$.

Corollary. Let $p \in \mathcal{G}$ be such that $H_p \simeq \mathrm{SU}(3)$. Then $\dim \mathcal{D}_0(p) = 6$ and $B(p) \in \mathrm{Int}\mathfrak{I} \sqcup \mathrm{Int}\mathfrak{D} \sqcup \mathfrak{A} \sqcup \mathrm{Int}\mathfrak{C}^+ \sqcup \mathrm{Int}\mathfrak{C}^- \subset \mathfrak{P}$. Moreover, we can find a deformation $(\xi_1, \xi_2) \to (\tilde{\xi}_1, \tilde{\xi}_2)$ (given explicitly in the previous subsection) such that $(\tilde{\xi}_1, \tilde{\xi}_2)$ is a system of everywhere-orthonormal Majorana spinors on M and such that:

- $\dim \tilde{\mathcal{D}}_0(p) = 4$
- The stabilizer $\tilde{H}_p$ of $(\tilde{\xi}_1(p), \tilde{\xi}_2(p))$ inside $\mathrm{Spin}(T_pM, g_p) \simeq \mathrm{Spin}(8)$ is isomorphic with $\mathrm{SU}(2)$.

Proof. For $p \in \mathcal{G}$ such that $B(p) \in \partial\mathfrak{P}$, we have $\mathrm{rk}\hat{G}(B(p)) = 2$ and (2.18) implies that the 1-forms $V_1(p), V_2(p), V_3(p)$ and $W(p)$ are stabilized by a subgroup containing $\mathrm{SU}(3)$. Since $\dim \mathcal{D}_0(p) \in \{4, 6\}$ for $p \in \mathcal{G}$ (see appendix E) and since $\mathrm{SU}(3)$ does not embed into $\mathrm{SO}(4)$, we must have $\dim \mathcal{D}_0(p) = 6$ and the common stabilizer of the one-forms must equal $\mathrm{SO}(6)$. In particular, the space spanned by $V_1(p), V_2(p), V_3(p)$ and $W(p)$ inside T_p^*M has dimension two. Since $\dim \mathcal{D}_0(p) = 6$, the results of appendix E, imply that the point $B(p)$ belongs to the subset $\mathrm{Int}\mathfrak{I} \sqcup \mathrm{Int}\mathfrak{D} \sqcup \mathfrak{A} \sqcup \mathrm{Int}\mathfrak{C}^+ \sqcup \mathrm{Int}\mathfrak{C}^-$ of the frontier $\partial\mathfrak{P}$. Let $(\tilde{\xi}_1, \tilde{\xi}_2)$ be chosen as in the previous proposition. Then we have $\tilde{B}(p) \in \mathrm{Int}\mathfrak{P}$ and hence $\mathrm{rk}\hat{\tilde{G}}(p) = \mathrm{rk}\hat{G}(b(p), \tilde{\beta}(p)) = 4$ by the results of appendix E. Thus the 1-forms $\tilde{V}_1(p), \tilde{V}_2(p), \tilde{V}_3(p)$ and $\tilde{W}(p)$ are linearly independent at p and we have $\dim \tilde{\mathcal{D}}_0(p) = 4$. Moreover, the spinor $\zeta(p)$ of the previous subsection is non-zero and hence $\tilde{H}_p$ is isomorphic with $\mathrm{SU}(2)$ (see subsection 3.5). $\square$

Remark. The orthogonal complement of $K_-(p)$ inside S_p^- equals the space $\Xi^-(p)$ considered in the proof of point 4 of the Proposition of subsection 3.5, a space which carries the fundamental representation of the group $H_p'' \stackrel{\text{def.}}{=} \mathrm{Stab}_{\mathrm{Spin}(T_pM, g_p)}(\xi_1^+(p), \xi_2^+(p), \xi_2^-(p)) \simeq \mathrm{SU}(3)$. The fact that the deformed spinor $\tilde{\xi}_1^-(p)$ has non-zero projection $\zeta(p)$ on the space $\Xi^-(p)$ is responsible for breaking the stabilizer group at p from $\mathrm{SU}(3)$ to $\mathrm{SU}(2)$.

G The non-generic assumption made in [26]

Let $\pi_1 : \hat{M} \to M$ and $\pi_2 : \hat{M} \to S^1$ denote the projections on the first and second factor of the direct product $\hat{M} = M \times S^1$ (which, as in [26], we endow with the direct product metric). Let $\theta \in \Omega^1(\hat{M})$ be the π_2-pullback of the canonical normalized 1-form of S^1 (notice that θ is the normalized Killing form on $\hat{M}$ corresponding to the symmetry given by rotations along the circle). Loc. cit. uses three one-forms[15] $\hat{V}_+, \hat{V}_-, \hat{V}_3 \in \Omega^1(\hat{M})$ defined on the 9-manifold $\hat{M}$ which are invariant under S^1-rotations and hence are given by:

$$\hat{V}_r = \pi_1^*(V_r) + (b_r \circ \pi_1)\theta\,, \quad \forall r \in \{+, -, 3\} \tag{G.1}$$

where $V_r \in \Omega^1(M)$ and $b_r \in \mathcal{C}^\infty(M, \mathbb{R})$. The quantities V_r, b_r turn out to coincide with the 0-forms and 1-forms given in (2.2) and (2.4). Indeed, it is easy to see that the algebraic constraints for (G.1) given in equations [26, eq. (2.15)] are equivalent with the system (4.2)

[15]The one-forms used by [26] on $\hat{M}$ are denoted there by V_+, V_- and V_3. The relation with our notation is $\hat{V}_\pm^{\text{here}} = \frac{1}{2} V_\pm^{\text{there}}$ and $\hat{V}_3^{\text{here}} = V_3^{\text{there}}$, cf. [26, eq. (2.26)].

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for V_r if one takes into account relation [26, eq. (2.26)]. Since θ and $\pi_1^*(V_r)$ are orthogonal at every point of $\hat{M}$, relations (G.1) give:

$$\langle \theta, \hat{V}_r \rangle = b_r \circ \pi_1 \,. \tag{G.2}$$

Loc cit. makes intensive use of the assumption (cf. [26, eq. (3.9)]) that the following relation holds on $\hat{M}$:

$$\theta = \frac{2}{1+\hat{\alpha}} \langle \theta, \hat{V}_+ \rangle \hat{V}_+ + \frac{2}{1-\hat{\alpha}} \langle \theta, \hat{V}_- \rangle \hat{V}_- + \frac{2}{1-\hat{\alpha}} \langle \theta, \hat{V}_3 \rangle \hat{V}_3 \,, \tag{G.3}$$

where[16] $\hat{\alpha} \in \mathcal{C}^\infty(\hat{M}, \mathbb{R})$ is a function independent of the S^1 coordinate, hence $\hat{\alpha} = \alpha \circ \pi_1$ for any $\alpha \in \mathcal{C}^\infty(M, \mathbb{R})$. To arrive at (G.3), we used the fact that $\hat{V}_\pm^{\rm here} = \frac{1}{2} V_\pm^{\rm there}$ and $\hat{V}_3^{\rm here} = V_3^{\rm there}$. Comparing with (4.2), it is not hard to check that $\alpha = 1 - 2\beta^2$, where β was defined in (4.3). Equations (G.2) give:

$$\frac{2}{1 \pm \hat{\alpha}} \langle \theta, \hat{V}_\pm \rangle = a_\pm \circ \pi_1 \,, \quad \frac{2}{1 - \hat{\alpha}} \langle \theta, \hat{V}_3 \rangle = a_3 \circ \pi_1 \,,$$

where $a_{\pm,3} \in \mathcal{C}^\infty(M, \mathbb{R})$ are given by:

$$a_\pm \stackrel{\rm def.}{=} \frac{2b_\pm}{1 \pm \alpha} \,, \quad a_3 \stackrel{\rm def.}{=} \frac{2b_3}{1-\alpha} \,.$$

Hence (G.3) takes the form:

$$\theta = (a_+ \circ \pi_1) \hat{V}_+ + (a_- \circ \pi_1) \hat{V}_- + (a_3 \circ \pi_1) \hat{V}_3 \,. \tag{G.4}$$

Since θ and $\pi_1^*(V_r)$ are orthogonal at every point of $\hat{M}$, substituting (G.1) into (G.4) and projecting onto $\pi_1^*(T^*M)$ gives:

$$a_+ V_+ + a_- V_- + a_3 V_3 = 0 \,. \tag{G.5}$$

Hence equation [26, eq. (3.9)] requires that V_+, V_- and V_3 be linearly dependent at every point of M, a requirement which cannot be satisfied in the generic case. In the non-generic case when (G.5) holds, we have $\mathrm{rk}\mathcal{D} \geq 6$ on M and hence the SU(2) locus $\mathcal{U}$ of M must be empty (see table 5).

The fact that the SU(2) locus $\mathcal{U}$ is not generally empty follows from the results of subsection 3.5 (which gives a proof of this fact directly in terms of spinors), from the results of appendix E (which shows that the 1-forms $V_1(p), V_2(p), V_3(p)$ and $W(p)$ are linearly independent in the generic case) and also from the results of appendix F, which gives an explicit construction of a family of spinor deformations which can be used to break the stabilizer group H_p from SU(3) to SU(2). The condition $\mathcal{U} = \emptyset$ is a very strong restriction since the locus $\mathcal{U}$ is *open* in M. This condition amounts to vanishing of the spinor projection $\zeta(p)$ arising in the proof of point 4 of the Proposition of subsection 3.5 for *every* point p of M; it is also equivalent with the condition that the image of the map

[16] The function $\hat{\alpha}$ is denoted by α in [26].

B defined in (4.5) is contained in the *frontier* $\partial\mathfrak{P}$ of the four-dimensional semi-algebraic body $\mathfrak{P}$, rather that in the body $\mathfrak{P}$ itself.

We also note that the cosmooth generalized distribution $\hat{\mathcal{D}} \stackrel{\text{def.}}{=} \ker \hat{V}_+ \cap \ker \hat{V}_- \cap \ker \hat{V}_3$ defined on $\hat{M}$ may have transverse or non-transverse intersection with the distribution $\pi_1^*(TM)$. This is one reason why one cannot conclude (as [26] does) that the stabilizer stratification of M would be "directly inherited" from that of $\hat{M}$. As we show in a different publication, the relation between the stabilizer stratifications of M and $\hat{M}$ is in fact rather involved, in particular due to the non-transversality issue mentioned above.

Remark. Loc cit. gives an argument (see the discussion there introducing equation [26, (3.9)]) according to which (G.3) would always have to hold. That argument relies on confusing θ (a one-form which exists on $\hat{M}$ by the definition of $\hat{M} \stackrel{\text{def.}}{=} M \times S^1$ and therefore is not a spinor bilinear) with a combination of one-forms constructed from the canonical lifts to $\hat{M}$ of the supersymmetry generators $\xi_1, \xi_2 \in \Gamma(M, S)$. It is further based on the assumption that θ would induce, in certain cases, a *nowhere-vanishing* vector field/one-form on M. However, the projection of θ on the bundle $\pi_1^*(T^*M) \subset T^*\hat{M}$ always vanishes, hence that projection can never define a non-vanishing one-form on M and thus it can never give a non-trivial singlet for the structure group of M. For these reasons, the argument given in loc. cit. cannot be used to conclude that θ would always have to be a linear combination of $\hat{V}_\pm$ and $\hat{V}_3$.

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