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Noncommutative Chern-Simons gauge and gravity theories and their geometric Seiberg-Witten map

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Abstract: We use a geometric generalization of the Seiberg-Witten map between noncommutative and commutative gauge theories to find the expansion of noncommutative Chern-Simons (CS) theory in any odd dimension D and at first order in the noncommutativity parameter θ. This expansion extends the classical CS theory with higher powers of the curvatures and their derivatives.

A simple explanation of the equality between noncommutative and commutative CS actions in $D = 1$ and $D = 3$ is obtained. The θ dependent terms are present for $D \geq 5$ and give a higher derivative theory on commutative space reducing to classical CS theory for $\theta \to 0$. These terms depend on the field strength and not on the bare gauge potential. In particular, as for the Dirac-Born-Infeld action, these terms vanish in the slowly varying field strength approximation: in this case noncommutative and commutative CS actions coincide in any dimension.

The Seiberg-Witten map on the $D = 5$ noncommutative CS theory is explored in more detail, and we give its second order θ-expansion for any gauge group. The example of extended $D = 5$ CS gravity, where the gauge group is $\mathrm{SU}(2,2)$, is treated explicitly.

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Contents

1 Introduction and summary

Noncommutative (NC) actions can be expanded order by order in the noncommutativity parameter θ and be interpreted as effective actions on commutative spacetimes, the noncommutativity leading to extra interaction terms, possibly capturing some quantum spacetime effect. These actions involve higher derivatives in the field strengths. For gauge theory actions one can expand not only the $\star$-products but also the noncommutative fields in terms of the commutative ones using the Seiberg-Witten (SW) map [1]. This allows to define NC gauge theories with any simple gauge group in arbitrary representations [2, 3]. In the literature deformations have been studied mainly at first order in θ.

In [4–9], extensions of Yang-Mills and gravity theories have been obtained at second order in the noncommutativity parameter θ starting from NC actions. The second order expansion is needed in the case of $D = 4$ gravity theories because the first order θ-correction vanishes.[1]

Some gauge theory actions have the remarkable property of being invariant under the SW map. This is notably the case for the CS action in 3 dimensions [13] and, if we consider slowly varying field strengths, for the Dirac-Born-Infeld theory in any dimension [1,

[1] A complementary route, named θ-exact approach, is to expand the NC actions in power series of the gauge potential while keeping all orders in θ, see [10, 11] for expansions up to second order in the gauge potential and quantum field theories applications and [12] for expansions up to third order.

14]. Noncommutative CS actions can be studied in any (odd) dimension [15–18]. In [18] noncommutativity is given by a Drinfeld twist defined by a set of commuting vector fields (so-called abelian Drinfeld twist), this noncommutativity including as a special case Moyal-Groenewold noncommutativity.

In this paper we apply the geometric Seiberg-Witten map [5] (i.e. the geometric generalization of the SW map that applies to Drinfeld twist noncommutativity) to the noncommutative Chern-Simons actions in any odd dimension studied in [18]. We obtain the correction to the classical action up to first order in the noncommutativity parameter θ. The correction is expressed in terms of the curvature $R = d\Omega - \Omega \wedge \Omega$, of its contraction along the vector fields determining the noncommutativity and of its covariant derivative. These terms are covariant under gauge transformations and therefore the correction is truly gauge invariant (not just up to boundary terms). The construction of these extended commutative CS actions obtained by adding correction terms order by order in θ applies to any gauge group G. For slowly varying field strength we show that these correction terms vanish, so that, as for Dirac-Born-Infeld theory, also noncommutative and commutative CS theories coincide in any dimension in this approximation.

The variation of the NC Chern-Simons form under SW map has an intriguing structure. We find that the SW map relates the NC topological terms $Tr(R^n)$ and $Tr(R^{n+1})$ and therefore relates the NC CS forms in D and in $D+2$ dimensions. In fact D-dimensional CS forms are mapped into double contractions of the $(D+2)$-dimensional CS forms, plus contractions of $(D+1)$-forms, plus extra terms that are covariant under gauge transformations. Since in D dimensions $(D+2)$- and $(D+1)$-forms vanish, only the extra covariant terms are relevant in computing the SW map of the CS action. In $D = 1$ and $D = 3$ the extra covariant terms are absent, which easily explains why in these cases the SW map is trivial, as first observed by [13]. Our results confirm those of [19], where, using a different approach based on operator valued fields, a (generalized) NC CS action defined only in terms of covariant derivatives was shown to be nontrivial under the SW map in dimension $D > 3$. We sharpen the findings in [19] by explicitly computing and analyzing the extra terms in $D > 3$. Moreover we are not constrained to consider Moyal-Groenewold noncommutativity, and our commutative limit, never involving the inverse of $\theta^{\mu\nu}$, is well under control for $\theta \to 0$.

The first nontrivial θ dependence occurs in $D = 5$ NC CS theory. For this case we compute also the second order expansion in θ.

Next we specialize to NC CS gravity [18] where the gauge group is $G = \mathrm{SU}(2,2)$, and explicitly compute its expansion to first order in θ in terms of component fields.

In this paper we focus on local properties of CS forms: in particular the connection Ω is always globally defined, and the underlying principal G-bundle is trivial.

It would be interesting to extend our analysis to the $D = 5$ noncommutative Chern-Simons supergravity theory constructed in [18], invariant under the local action of the $\star$-supergroup $\mathrm{U}(2,2|N)$ that includes N supersymmetries. In this case the SW map relates $\star$-supersymmetry to ordinary supersymmetry, so that the θ-correction terms of the SW expansion are separately invariant under ordinary supersymmetry. The result is an extended $D = 5$ CS supergravity with (locally) supersymmetric higher order terms. This work is in progress.

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The paper is organized as follows. In section 2 we recall a few facts on Chern-Simons forms and their noncommutative versions. In section 3 we recall the geometric generalization of the Seiberg-Witten map. In section 4 we compute the SW variation of the NC topological term $Tr(R^n)$, of the CS forms and of the CS actions. In section 5 we apply these results to $D = 5$ CS and present the first and second order corrections in θ; finally we consider the case of $D = 5$ CS gravity. In appendix A we give the derivation of the SW variation of the noncommutative topological term $Tr(R^n)$. In appendix B we collect useful identities, and appendix C contains a summary of $D = 5$ gamma matrix properties.

2 Chern-Simons forms and their noncommutative versions

Commutative CS forms. The CS Lagrangian in $(2n-1)$-dimensions is a $(2n-1)$-form given in terms of the G-gauge connection Ω and its exterior derivative $d\Omega$, or equivalently its curvature 2-form $R = d\Omega - \Omega \wedge \Omega$, by the following expressions (see e.g. [20, 21]):

$$L_{\rm CS}^{(2n-1)} = n \int_0^1 Tr[\Omega(td\Omega - t^2\Omega^2)^{n-1}]dt = n \int_0^1 t^{n-1} Tr[\Omega(R + (1-t)\Omega^2)^{n-1}]dt \tag{2.1}$$

where we have omitted writing explicitly the wedge product. For example:

$$L_{\rm CS}^{(1)} = Tr[\Omega] \tag{2.2}$$

$$L_{\rm CS}^{(3)} = Tr\left[R\Omega + \frac{1}{3}\Omega^3\right] \tag{2.3}$$

$$L_{\rm CS}^{(5)} = Tr\left[R^2\Omega + \frac{1}{2}R\Omega^3 + \frac{1}{10}\Omega^5\right] \tag{2.4}$$

$$L_{\rm CS}^{(7)} = Tr\left[R^3\Omega + \frac{2}{5}R^2\Omega^3 + \frac{1}{5}R\Omega^2 R\Omega + \frac{1}{5}R\Omega^5 + \frac{1}{35}\Omega^7\right]. \tag{2.5}$$

These expressions are obtained by solving the condition

$$dL_{\rm CS}^{(2n-1)} = Tr(R^n)\ . \tag{2.6}$$

The CS form $L_{\rm CS}^{(2n-1)}$ contains (exterior products of) the Lie(G)-valued gauge potential one-form Ω and its exterior derivative. The trace Tr is taken on some representation of the Lie algebra Lie(G).[2]

Because of (2.6), the CS action on the boundary ∂M of a manifold M is related to a topological action in $2n$ dimensions via Stokes theorem:

$$\int_{\partial M} L_{\rm CS}^{(2n-1)} = \int_M Tr(R^n)\ . \tag{2.7}$$

Infinitesimal gauge transformations are defined by

$$\delta_\varepsilon \Omega = d\varepsilon - \Omega\varepsilon + \varepsilon\Omega, \qquad \Rightarrow \qquad \delta_\varepsilon R = -R\varepsilon + \varepsilon R \tag{2.8}$$

[2] More generally Tr can be any multilinear function of the Lie algebra, invariant under cyclic permutations. In this paper Tr stands for the usual matrix trace.

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so that $Tr(R^n)$ is manifestly gauge invariant. Therefore also the CS action is gauge invariant under infinitesimal gauge transformations.[3]

Considering $L_{\rm CS}^{(2n-1)}$ as a function of Ω and R, a convenient formula for its gauge variation is (see for example ref. [18])

$$\delta_\varepsilon L_{\rm CS}^{(2n-1)} = d(j_\varepsilon L_{\rm CS}^{(2n-1)}) \tag{2.9}$$

where j_ε is a contraction acting selectively on Ω, i.e.

$$j_\varepsilon \Omega = \varepsilon, \qquad j_\varepsilon R = 0 \tag{2.10}$$

with the graded Leibniz rule $j_\varepsilon(\Omega\Omega) = j_\varepsilon(\Omega)\Omega - \Omega j_\varepsilon(\Omega) = \varepsilon\Omega - \Omega\varepsilon$ etc. Considering instead $L_{\rm CS}^{(2n-1)}$ as a function of Ω and $d\Omega$, formula (2.9) holds with the rules $j_\varepsilon\Omega = \varepsilon$ and $j_\varepsilon d\Omega = \varepsilon\Omega - \Omega\varepsilon$.

$\star$-Exterior products from abelian Drinfeld twists. The preceding discussion is based on algebraic manipulations, and relies on the (graded) cyclicity of Tr. As such, it can be exported immediately to the noncommutative setting, provided we ensure that cyclicity holds. The noncommutativity we consider here is controlled by an abelian twist, and amounts to a deformation of the exterior product:

$$\begin{aligned}\tau \wedge_\star \tau' &\equiv \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{2}\right)^n \theta^{A_1B_1}\cdots\theta^{A_nB_n}(\ell_{A_1}\cdots\ell_{A_n}\tau)\wedge(\ell_{B_1}\cdots\ell_{B_n}\tau') \\ &= \tau\wedge\tau' + \frac{i}{2}\theta^{AB}(\ell_A\tau)\wedge(\ell_B\tau') + \frac{1}{2!}\left(\frac{i}{2}\right)^2\theta^{A_1B_1}\theta^{A_2B_2}(\ell_{A_1}\ell_{A_2}\tau)\wedge(\ell_{B_1}\ell_{B_2}\tau') + \cdots\end{aligned} \tag{2.11}$$

where θ^{AB} is a constant antisymmetric matrix, and ℓ_A are Lie derivatives along commuting vector fields X_A. The product is associative due to $[X_A, X_B] = 0$ ($\Rightarrow [\ell_A, \ell_B] = 0$). If the vector fields X_A are chosen to coincide with the partial derivatives ∂_μ, and if τ, τ' are 0-forms, then $\tau\star\tau'$ reduces to the well-known Moyal-Groenewold product [22, 23]. A short review on twisted differential geometry can be found for example in [24].

The deformed exterior product differs from the undeformed one by a total Lie derivative, indeed since $[\ell_A, \ell_B] = 0$ we can write

$$\begin{aligned}\tau\wedge_\star\tau' &= \tau\wedge\tau' + \ell_{A_1}\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n\theta^{A_1B_1}\cdots\theta^{A_nB_n}(\ell_{A_2}\cdots\ell_{A_n}\tau)\wedge(\ell_{B_1}\ell_{B_2}\cdots\ell_{B_n}\tau') \\ &= \tau\wedge\tau' + \ell_{A_1}Q^{A_1}\end{aligned} \tag{2.12}$$

where for brevity we have renamed the summation Q^{A_1}. In particular when $\tau\wedge\tau'$ is a top form we have

$$\int \tau\wedge_\star\tau' = \int\tau\wedge\tau' \tag{2.13}$$

[3]Notice that this argument does not work for finite gauge transformations because not all finite gauge transformations on the boundary ∂M are induced by finite gauge transformations in the bulk M. In general under finite gauge transformations the CS form changes by a locally exact form, related to a winding number. Hence only the equations of motion are invariant under finite transformations.

for suitable boundary conditions; indeed $\int \ell_{A_1} Q^{A_1} = \int (i_{A_1} d + d i_{A_1}) Q^{A_1} = 0$ because $dQ^{A_1} = 0$ since Q^{A_1} is a top form, and $\int d i_{A_1} Q^{A_1} = 0$ if we integrate on a manifold without boundary or if the forms τ and τ' have suitable boundary conditions. The equality (2.13) implies that the integral of $\star$-wedge products of homogeneous forms has the usual graded cyclic property $\int \tau \wedge_\star \tau' = (-1)^{\deg(\tau)\deg(\tau')} \int \tau' \wedge_\star \tau$. Notice however that in general $\int \tau \wedge_\star \tau' \wedge_\star \tau'' \neq \int \tau \wedge \tau' \wedge \tau''$.

If we now consider homogeneous forms $\mathcal{T}, \mathcal{T}'$ that are Lie algebra valued, the trace of the $\wedge_\star$-product of forms is still graded cyclic up to total Lie derivative terms:

$$Tr(\mathcal{T} \wedge_\star \mathcal{T}') = (-1)^{\deg(\mathcal{T})\deg(\mathcal{T}')}\, Tr(\mathcal{T}' \wedge_\star \mathcal{T}) + \ell_A Q^A \; ; \tag{2.14}$$

and for suitable boundary conditions the integral of the trace has the graded cyclic property.

Noncommutative CS forms. We define noncommutative Chern-Simons actions by replacing $\wedge$-products with $\wedge_\star$-products in the commutative Chern-Simons action. This procedure is unique if we integrate over manifolds without boundary or if the fields are properly behaving at the boundary; it is not unique for CS forms because of the cyclic ordering ambiguities (2.14), that are however irrelevant in the present paper. We denote by $L^{(2n-1)}_{\mathrm{CS}^*}$ any one of the NC generalizations of the CS form $L^{(2n-1)}_{\mathrm{CS}}$.

The check that the exterior derivative of the commutative CS form $L^{(2n-1)}_{\mathrm{CS}}$ gives $Tr(R^n)$ is algebraic, and relies only on the Leibniz rule property of the exterior derivative and on the graded cyclicity of the trace. Since the exterior derivative satisfies the Leibniz rule also in the noncommutative case, and the graded cyclicity of the trace holds up to total Lie derivatives, we can conclude that the noncommutative Chern-Simons form satisfies the relation

$$dL^{(2n-1)}_{\mathrm{CS}^*} = Tr(R^{\wedge_\star n}) + \ell_C Q^{(2n)\,C} \tag{2.15}$$

where $Q^{(2n)C}$ is due to cyclic reorderings and is a sum of wedge products of Lie derivatives of connections and of their exterior derivatives.

We note that $Q^{(2n)C}$ is local in the noncommutative connection, in the sense that expanding the $\wedge_\star$-product, for any finite order in θ there is a finite number of Lie or exterior derivatives.

In the noncommutative case the gauge group G usually has to be extended, because $\star$-commutators in general do not close in the original Lie algebra Lie(G). For example

$$\begin{aligned} \Omega \wedge_\star \Omega &= \Omega^a \wedge_\star \Omega^b\, T^a T^b \\ &= \frac{1}{2}(\Omega^a \wedge_\star \Omega^b - \Omega^b \wedge_\star \Omega^a)[T^a, T^b] + \frac{1}{2}(\Omega^a \wedge_\star \Omega^b + \Omega^b \wedge_\star \Omega^a)\{T^a, T^b\}\,, \end{aligned} \tag{2.16}$$

with the second term nonvanishing because the $\wedge_\star$-product is not antisymmetric. We therefore consider Lie algebras with representations T^a that close under the usual matrix product (i.e. under commutators and anticommutators). Note however that this restriction can be lifted when using the Seiberg-Witten map (see section 5).

It is easy to prove the invariance of the noncommutative Chern-Simons action under infinitesimal $\star$-gauge transformations defined by:

$$\delta^\star_\varepsilon \Omega = d\varepsilon - \Omega \star \varepsilon + \varepsilon \star \Omega, \qquad \Rightarrow \qquad \delta^\star_\varepsilon R = -R \star \varepsilon + \varepsilon \star R\,. \tag{2.17}$$

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Indeed

$$\delta^\star_\varepsilon \int L^{(2n-1)}_{\text{CS}^*} = \int d(j_\varepsilon L^{(2n-1)}_{\text{CS}^*}) = 0 \tag{2.18}$$

for suitable boundary conditions. This is so because in the $\star$-deformed case the variation formulae (2.9), (2.10) still hold true under integration.

For example the $D = 5$ $\star$-Chern-Simons action reads

$$\int L^{(5)}_{\text{CS}^*} = \int Tr\left[R \wedge_\star R \wedge_\star \Omega + \frac{1}{2} R \wedge_\star \Omega \wedge_\star \Omega \wedge_\star \Omega + \frac{1}{10} \Omega \wedge_\star \Omega \wedge_\star \Omega \wedge_\star \Omega \wedge_\star \Omega\right]. \tag{2.19}$$

and is invariant under the $\star$-gauge variations (2.17).

3 The Seiberg-Witten map

In the framework of Moyal deformed gauge theories, Seiberg and Witten showed how to relate noncommutative fields (that transform under deformed gauge transformations) to ordinary fields, called also classical fields, transforming with the usual gauge variation laws. The Seiberg-Witten map expresses the NC fields as functions of the ordinary fields in such a way that usual gauge variations on the latter induce $\star$-gauge variations on the former. The map is nonlinear, and is determined order by order in the noncommutativity parameter θ.

Under this map, a NC action can be re-expressed in terms of classical fields. The result is invariant under usual gauge variations (since the NC action is invariant under $\star$-gauge variations), and can be written as the classical action plus higher order θ corrections, each of which is separately gauge invariant under usual gauge variations (because usual gauge variations do not involve θ). This map provides therefore an interesting mechanism to generate extensions of usual commutative actions, with interaction terms that depend on θ (for gravity actions see [5–9]).

Denoting by $\widehat{\Omega}$ the NC gauge field, and by $\widehat{\varepsilon}$ the NC gauge parameter, the Seiberg-Witten map relates $\widehat{\Omega}$ to the ordinary Ω, and $\widehat{\varepsilon}$ to the ordinary ε and to Ω so as to satisfy:

$$\widehat{\Omega}(\Omega) + \widehat{\delta}_{\widehat{\varepsilon}} \widehat{\Omega}(\Omega) = \widehat{\Omega}(\Omega + \delta_\varepsilon \Omega) \tag{3.1}$$

with

$$\delta_\varepsilon \Omega = d\varepsilon + \varepsilon\, \Omega - \Omega \varepsilon\,, \tag{3.2}$$

$$\widehat{\delta}_{\widehat{\varepsilon}} \widehat{\Omega} = d\widehat{\varepsilon} + \widehat{\varepsilon} \star \widehat{\Omega} - \widehat{\Omega} \star \widehat{\varepsilon}\,. \tag{3.3}$$

Thus the dependence of the NC gauge field on the ordinary gauge field is determined by requiring that ordinary gauge variations of Ω inside $\widehat{\Omega}(\Omega)$ produce the noncommutative gauge variation of $\widehat{\Omega}$.

The condition (3.1) is satisfied if the following differential equations in the noncommutativity parameter θ^{AB} hold [1, 5]:

$$\delta_\theta \widehat{\Omega} \equiv \delta\theta^{AB} \frac{\partial}{\partial \theta^{AB}} \widehat{\Omega} = \frac{i}{4} \delta\theta^{AB} \{\widehat{\Omega}_A, \ell_B \widehat{\Omega} + \widehat{R}_B\}_\star\,, \tag{3.4}$$

$$\delta_\theta \widehat{\varepsilon} \equiv \delta\theta^{AB} \frac{\partial}{\partial \theta^{AB}} \widehat{\varepsilon} = \frac{i}{4} \delta\theta^{AB} \{\widehat{\Omega}_A, \ell_B \widehat{\varepsilon}\}_\star\,, \tag{3.5}$$

where:

- $\widehat{\Omega}_A$, $\widehat{R}_A$ are defined as the contraction i_A along the tangent vector X_A of the exterior forms $\widehat{\Omega}$, $\widehat{R}$, i.e. $\widehat{\Omega}_A \equiv i_A\widehat{\Omega}$, $\widehat{R}_A \equiv i_A\widehat{R}$.

- The bracket $\{\,,\;\}_\star$ is the usual $\star$-anticommutator, for example $\{\Omega_A, R_B\}_\star = \Omega_A \star R_B + R_B \star \Omega_A$.

The differential equations (3.4)–(3.5) hold for any abelian twist defined by arbitrary commuting vector fields X_A [5]. They reduce to the usual Seiberg-Witten differential equations [1] in the case of a Moyal-Groenewold twist, i.e. when $X_A \to \partial_\mu$.

We can solve these differential equations order by order in θ by expanding $\widehat{\Omega}$ and $\widehat{\varepsilon}$ in power series of θ, so that (the factor $\frac{i}{2}$ is inserted for ease of later notation) $\widehat{\Omega} = \Omega + \frac{i}{2}\theta^{AB}\Omega'_{AB} - \frac{1}{8}\theta^{AB}\theta^{EF}\Omega''_{AB\,EF} + \ldots$ where $\frac{i}{2}\Omega'_{AB} = \frac{\partial}{\partial\theta^{AB}}\widehat{\Omega}|_{\theta=0}$ etc., and similarly for $\widehat{\varepsilon}$. For example up to first order in θ from (3.4) and (3.5) we immediately find

$$\widehat{\Omega} = \Omega + \frac{i}{4}\theta^{AB}\{\Omega_A, \ell_B\Omega + R_B\} + \mathcal{O}(\theta^2)\,, \tag{3.6}$$

$$\widehat{\varepsilon} = \varepsilon + \frac{i}{4}\theta^{AB}\{\Omega_A, \ell_B\varepsilon\} + \mathcal{O}(\theta^2)\ . \tag{3.7}$$

Recursive formulas were found in [25] for the Moyal-Weyl product, and generalized for the geometric SW map in [5]. Typically $\widehat{\Omega}$ is a power series in θ of sums of products of commutative connections, also contracted and differentiated (e.g. $\Omega_A, i_A d\Omega, \ell_A\ell_B\Omega$, etc.). Again we say that $\widehat{\Omega}$ is local in the commutative connection because for every power of θ only a finite number of exterior derivatives appears. It follows that in this framework noncommutative Lagrangians are power series in θ of commutative Lagrangians that are local in the connection Ω.

4 The SW variation of NC Chern-Simons forms

In the following we omit the hat denoting noncommutative fields, the $\star$ and $\wedge_\star$ products, and simply write $\{\,,\;\}$, $[\,,\;]$ for the $\star$-anticommutator and the $\star$-commutator $\{\,,\;\}_\star$, $[\,,\;]_\star$.

SW variation of $\boldsymbol{Tr(R^n)}$. An expression equivalent to (3.4) for the SW variation of the connection 1-form is

$$\delta_\theta\Omega = \frac{i}{4}\delta\theta^{AB}\{\Omega_A, \mathbb{L}_B\Omega - d\Omega_B\}\ . \tag{4.1}$$

The "fat" Lie derivative $\mathbb{L}_B$ is defined by $\mathbb{L}_B \equiv \ell_B + L_B$ where L_B is the covariant Lie derivative along the tangent vector X_B; it acts on every field P as

$$L_B P = \ell_B P - [\Omega, P].$$

In fact the covariant Lie derivative L_B can be written in Cartan form:

$$L_B = i_B D + D i_B\,, \tag{4.2}$$

where D is the covariant derivative: $DP = dP - [\Omega, P]$ for P even form, $DP = dP - \{\Omega, P\}$ for P odd form. In particular $DR = 0$ (Bianchi identity) follows from the definition of R. Moreover $L_A R = i_A DR + D i_A R = DR_A$.

The SW variation of the connection implies the following variation for the curvature 2-form $R = d\Omega - \Omega \wedge \Omega$ (an easy derivation uses equation (A.4) with $P = Q = \Omega$),

$$\delta_\theta R = \frac{i}{4}\delta\theta^{AB}(\{\Omega_A, \mathbb{L}_B R\} - [R_A, R_B]) \ . \tag{4.3}$$

From this formula and iterated use of (A.4) the SW variation of the trace of R^n can be proven to be (see appendix A):

$$\delta_\theta Tr(R^n) = \frac{i}{2}\delta\theta^{AB} Tr\left(\frac{1}{n+1} i_B i_A R^{n+1}\right) + \frac{i}{2}\delta\theta^{AB}\Big(d\,\mathrm{U}_{AB} + \ell_C Q^C_{AB}\Big) \tag{4.4}$$

where the $(2n-1)$-form U_{AB} is given by

$$\mathrm{U}_{AB} = Tr\left(\sum_{i=2}^{n-1} R^{i-1} DR_{[A}(R^{n-i})_{B]}\right) \tag{4.5}$$

with $(R^{n-i})_B \equiv i_B(R^{n-i})$. Antisymmetrization in the indices ${}_A\ {}_B$ has weight one (i.e. ${}_{[A\ B]} = \frac{1}{2}{}_{A\ B} - \frac{1}{2}{}_{B\ A}$). The precise expression (see appendix A for details) of the $2n$-form Q^C_{AB}, local in the NC connection, will not be relevant in the following.

SW variation of $Tr(R^n)$ on a $2n$-dimensional manifold M. If the forms are defined on a $2n$-dimensional manifold M, $Tr(R^n)$ has top degree and its SW variation (4.4) simplifies since $R^{n+1} = 0$ being a $(2n+2)$- form. Moreover, writing $\ell_C = i_C d + d\, i_C$ and observing that $dQ^C_{AB} = 0$ because it is a $(2n+1)$-form, we obtain the SW variation of the top form $Tr(R^n)$:

$$\delta_\theta Tr(R^n) = \frac{i}{2}\delta\theta^{AB} d\big(\,\mathrm{U}_{AB} + i_C Q^C_{AB}\big). \tag{4.6}$$

Let's comment on the nontrivial information in this formula. The exactness of $\delta_\theta Tr(R^n)$ is a trivial consequence of considering R^n a top form. We write $\delta_\theta Tr(R^n) = d\eta$ and compare this expression with the SW variation of (2.15) that when R^n is a top form reads $\delta_\theta Tr(R^n) = d\delta_\theta L^{(2n-1)}_{\mathrm{CS}\star} - d i_C \delta_\theta Q^{(2n)\,C}$. Recalling the differential equation (3.4) we see that η is local in the NC connection (i.e., that expanding the $\wedge_\star$-product, for any finite order in θ we have a finite number of Lie or exterior derivatives in the NC connection).

The nontrivial information in (4.6) is that η_{AB}, defined by $\eta = \delta\theta^{AB}\eta_{AB}$, is given by the sum $\mathrm{U}_{AB} + i_C Q^C_{AB}$ where the second term is a contraction of a $2n$-form (local in the NC connection), and the first term is expressed only in terms of products of curvatures, their contractions and covariant derivatives, i.e., in terms of only gauge covariant fields.

SW variation of $L^{(2n-1)}_{\mathrm{CS}\star}$. The SW variation of $L^{(2n-1)}_{\mathrm{CS}\star}$ can be inferred from eq. (2.15): $dL^{(2n-1)}_{\mathrm{CS}\star} = Tr(R^n) + \ell_C Q^{(2n)\,C}$, where the Lie derivative term on the right hand side comes from cyclic reorderings (in the commutative limit $Q^{(2n)C} = 0$ since the trace in that case

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is cyclic). Using this relation in (4.4) yields the SW variation of $dL^{(2n-1)}_{\mathrm{CS}^\star}$ in a manifold of arbitrary dimension,

$$\delta_\theta d\, L^{(2n-1)}_{\mathrm{CS}^\star} = \frac{i}{2}\delta\theta^{AB} d\left(\frac{1}{n+1} i_B i_A (L^{(2n+1)}_{\mathrm{CS}}) + \mathrm{U}_{AB}\right) + \frac{i}{2}\delta\theta^{AB}\left(\frac{-1}{n+1} i_B i_A \ell_C Q^{(2n+2)\,C} + \ell_C Q^C_{AB}\right) + \ell_C(\delta_\theta Q^{(2n)\,C}) \,. \tag{4.7}$$

For forms living in a $2n$-dimensional manifold M, this becomes

$$\delta_\theta d\, L^{(2n-1)}_{\mathrm{CS}^\star} = \frac{i}{2}\delta\theta^{AB} d\big(\mathrm{U}_{AB} + i_C Q^C_{AB}\big) + d i_C(\delta_\theta Q^{(2n)\,C})$$

where we used the identity $\ell_C = i_C d + d\, i_C$ and the vanishing of forms of degree higher than $2n$. Equivalently on M we have

$$\delta_\theta L^{(2n-1)}_{\mathrm{CS}^\star} = \frac{i}{2}\delta\theta^{AB}\Big(\mathrm{U}_{AB} + i_C Q^C_{AB}\Big) + i_C(\delta_\theta Q^{(2n)\,C}) + d\varphi$$

for some $(2n-1)$-form φ written in terms of the connection, of exterior derivatives and of contraction operators along the noncommutative directions.[4]

We now consider a $(2n-1)$-dimensional submanifold N of M and choose commuting vector fields $\{X_A\}$ on M that restrict to vector fields on N. In this case $L^{(2n-1)}_{\mathrm{CS}^\star}$ is a top form on N, and $Q^C_{AB} = \delta_\theta Q^{(2n)\,C} = 0$ being $2n$-forms on the $(2n-1)$-dimensional manifold N. The SW variation of the CS action on a manifold N with no boundary or with fields that have appropriate boundary conditions is therefore

$$\begin{aligned}\delta_\theta \int L^{(2n-1)}_{\mathrm{CS}^\star} &= \frac{i}{2}\delta\theta^{AB}\int \mathrm{U}_{AB} \\ &= \frac{i}{2}\delta\theta^{AB}\int Tr\left(\sum_{i=2}^{n-1} R^{i-1} D R_A (R^{n-i})_B\right) \\ &= \frac{i}{2}\delta\theta^{AB}\int Tr\left(R D R_A \sum_{k=0}^{n-3}(k+1) R^{n-3-k} R_B R^k\right)\end{aligned} \tag{4.8}$$

where in the last equality we have evaluated the contraction operator i_B on (R^{n-i}), integrated by parts and cyclically reordered the terms in the sum.

This variation is zero for $n = 1, 2$. The first non vanishing SW variation of a Chern-Simons action occurs for $n = 3$. In particular in three dimensions the SW expansion of the noncommutative Chern-Simons action equals the commutative Chern-Simons action; this result, for Moyal-Groenewold noncommutativity, was obtained in [13].[5]

In higher dimensions the variation is nonvanishing, and is expressed in terms of the gauge covariant quantities R, R_A and their covariant derivatives.

[4] The local structure of φ follows observing that the SW map is local in the sense discussed at the end of section 3.

[5] We mention that the solution to (3.1) is not unique. For example if $\hat{A}$ is a solution, any finite noncommutative gauge transformation of $\hat{A}$ gives another solution. Another source of ambiguities is related to field redefinitions of the gauge potential. Use of a nonstandard solution to SW map may lead to different results, see [26] where a nontrivial second order in θ expansion of the $D = 3$ CS action is constructed via a nonstandard solution to SW map.

Slowly varying fields and invariance of NC CS action under SW map. In [1] (section 4.1) it is shown that for slowly varying field strength the noncommutative and commutative Dirac-Born-Infeld actions coincide (up to a redefinition of the coupling constant and of the metric). In our geometric framework, where the noncommutativity is given by the vector fields $\{X_A\}$, we can consider field strengths that are slowly varying just along the noncommutative directions. The gauge covariant formulation of the slowly varying field strength condition is $L_A R \sim 0$. In this case the noncommutative and commutative CS actions coincide. Indeed $DR_A = i_A DR + Di_A R = L_A R \sim 0$, and hence $\mathrm{U}_{AB} \sim 0$ (cf. (4.8)).

This result holds in particular in the U(1) case where the slowly varying field strenght condition on commutative spacetime reads $\ell_A R^{\text{commutative}} \sim 0$. For nondegenerate Moyal-Groenewold noncommutativity this is equivalent to $\partial_\mu R^{\text{commutative}}_{\nu\sigma} \sim 0$ that is the condition considered in [1].

5 Extended CS actions from NC CS actions

Consider the Taylor series expansion of a NC CS action in powers of θ (the θ dependence is due to the $\star$-product and to the SW map),

$$\int L^{(2n-1)}_{\mathrm{CS}^\star} = \int L^{(2n-1)}_{\mathrm{CS}} + \frac{i}{2}\theta^{AB}\int L^{(2n-1)\,\prime}_{\mathrm{CS}\;AB} - \frac{1}{8}\theta^{AB}\theta^{EF}\int L^{(2n-1)\,\prime\prime}_{\mathrm{CS}\;ABEF} + \mathcal{O}(\theta^3) \tag{5.1}$$

where $\int L^{(2n-1)}_{\mathrm{CS}} = \int L^{(2n-1)}_{\mathrm{CS}^\star}|_{\theta=0}$, $\frac{i}{2}\int L^{(2n-1)\,\prime}_{\mathrm{CS}\;AB} = \frac{\partial}{\partial\theta^{AB}}\int L^{(2n-1)}_{\mathrm{CS}^\star}|_{\theta=0}$, etc.. The right hand side is a higher derivative action on commutative spacetime. It is an extension, with θ corrections, of the commutative CS action $\int L^{(2n-1)}_{\mathrm{CS}}$. The result of the previous section gives the first order θ-correction to the commutative CS theory, so that the action of the extended CS theory reads

$$\int L^{(2n-1)}_{\mathrm{CS}} + \frac{i}{2}\theta^{AB}\int Tr\left(RDR_A\sum_{k=0}^{n-3}(k+1)R^{n-3-k}R_BR^k\right) . \tag{5.2}$$

We notice that this action is well defined for any gauge group G, and that it has the same (off shell) degrees of freedom as the usual CS action. Like in modified gravity theories the θ correction is just a further interaction term among the fields.

Note. In section 2 we had to consider NC CS actions with fields valued in a Lie algebra representation T^a closed under the matrix product rather than under commutators (recall (2.16)). This in general is a severe restriction on the gauge group G (typically requiring $G = \mathrm{U}(N)$). Here however, the SW map relates the noncomutative fields corresponding to products of generators $T^aT^b\ldots$ to the classical gauge fields of any gauge group G [2, 3]. Thus the SW map allows to define NC CS actions for any gauge group.

5.1 $D = 5$ CS form to second order in θ

To evaluate the second order variation of $\int L^{(5)}_{\mathrm{CS}^\star}$,

$$\delta_\theta\delta_\theta\int L^{(5)}_{\mathrm{CS}^\star} = \frac{i}{2}\delta_\theta\delta\theta^{AB}\int RDR_AR_B \tag{5.3}$$

we need the SW variation of R_B and DR_A. The first one is easily obtained by applying the contraction operator i_B to the SW variation of the curvature 2-form R, eq. (4.3). The second one is obtained by summing the SW variation of dR_A to the SW variation of $\{\Omega, R_A\}$, that is evaluated using (4.1) and (A.4). The result is

$$\begin{aligned}
\delta_\theta R_B &= \frac{i}{4}\theta^{CD}\Big(\{\Omega_C, \mathbb{L}_D R_B\} - 2\{R_{CB}, R_D\}\Big)\,, \\
\delta_\theta DR_A &= \frac{i}{4}\theta^{CD}\Big(\{\Omega_C, \mathbb{L}_D DR_A\} + 2\{DR_C, R_{DA}\} - L_A[R_C, R_D]\Big)\,.
\end{aligned} \tag{5.4}$$

Next with the help of (A.4) we compute $\delta_\theta DR_A R_B$ and finally using (B.3) we obtain[6]

$$\begin{aligned}
\delta_\theta\delta_\theta \int L^{(5)}_{\text{CS}\star} &= \frac{i}{2}\delta_\theta\delta\theta^{AB}\int Tr(R\,DR_A R_B) &(5.5)\\
&= -\frac{1}{4}\delta\theta^{AB}\delta\theta^{CD}\int Tr\Big(DR_A\Big(\{R_B R, R_{CD}\} + \{R_B, R_{CD}R\} + 2\{R_{BC}, R_D\}R + \\
&\quad +2\{R_{BC}, R_D R\} - 2[R_B, R_C R_D] + 2[R_{BC}, [R, R_D]] - 2i_D(DR_B)DR_C\Big)\Big).
\end{aligned}$$

The expansion at second order in power series of θ of the $D = 5$ noncommutative CS action (2.19) is then given by

$$\int L^{(5)}_{\text{CS}\star} = \int L^{(5)}_{\text{CS}} + \frac{i}{2}\theta^{AB}\int Tr(RDR_A R_B) - \frac{1}{8}\theta^{AB}\theta^{CD}\int L^{(5)\,\prime\prime}_{\text{CS}\;ABCD} + \mathcal{O}(\theta^3) \tag{5.6}$$

where $L^{(5)\,\prime\prime}_{\text{CS}\;ABCD}$ is the integrand in (5.5).

5.2 Extended $D = 5$ CS gravity to first order in θ

CS gravities and supergravities [27–31] present interesting alternatives to standard (super)gravities in odd dimensions. Indeed CS gravities are a particular case of Lovelock gravities [32], with at most second order equations for the metric. Moreover the gauge (super)group contains the anti-de Sitter (super)algebra, so that the theory is translation invariant and does not have dimensionful coupling constants. One can use group contraction to recover the (super)Poincaré algebra, but retrieving the Einstein-Hilbert term in this limit is problematic. There are however techniques (S-expansion method [33]) to recover Poincaré gravity from CS gravity with a particular "expanded" gauge algebra.

We study here the example of $D = 5$ Chern-Simons AdS pure gravity. The commutative $\mathrm{SU}(2,2)$ connection and curvature are given by

$$\Omega = \frac{1}{4}\omega^{ab}\gamma_{ab} - \frac{i}{2}V^a\gamma_a, \qquad R = \frac{1}{4}R^{ab}\gamma_{ab} - \frac{i}{2}R^a\gamma_a \tag{5.7}$$

with

$$R^{ab} = d\omega^{ab} - \omega^{ac}\omega_c^{\ b} + V^aV^b, \qquad R^a = dV^a - \omega^a_{\ c}V^c \tag{5.8}$$

[6] The last two terms are obtained from the term $-2DR_A(L_D R_B)L_C R$ by use of the Cartan identity $L_D = i_D D + D i_D$, integrating by parts the exterior covariant derivative, observing that $DDR_A = -[R, R_A]$ and renaming indices.

all indices $a, b, \ldots$ running on five values. The $D = 5$ gamma matrices γ_a, together with their commutators $\gamma_{ab} \equiv \frac{1}{2}[\gamma_a, \gamma_b]$, close on the $D = 5$ AdS algebra $\mathrm{SU}(2,2) \approx \mathrm{SO}(2,4)$. The $\mathrm{SU}(2,2)$ connection contains both the vielbein V^a and the spin connection ω^{ab}, and correspondingly the $\mathrm{SU}(2,2)$ curvature contains both the AdS curvature R^{ab} and the torsion R^a. After applying the SW map to the $D = 5$ noncommutative CS action (2.19), and using the expression for the first order correction in (5.6), we obtain the extended $D = 5$ AdS gravity action:

$$\begin{aligned}\int L^{(5)}_{\mathrm{CS}\star} = &\int \frac{1}{8}\epsilon_{abcde}\left(R^{ab}R^{cd}V^e + \frac{2}{3}R^{ab}V^cV^dV^e + \frac{1}{5}V^aV^bV^cV^dV^e\right) + \\ &+\frac{1}{2}\theta^{AB}\Big(R^{ab}DR^{ac}_AR^{bc}_B + 2R^{ab}V^aR^c_AR^{bc}_B + R^{ab}DR^a_AR^b_B + \\ &+R^{ab}R^{ac}_AV^cR^b_B + R^aD(R^{ab}_AR^b_B) + 2R^aV^{[a}R^{b]}_AR^b_B + R^aR^{bc}_AV^cR^{ab}_B\Big) + O(\theta^2),\end{aligned} \tag{5.9}$$

where D is the $\mathrm{SO}(1,4)$ Lorentz covariant derivative (with connection ω^{ab}).

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A The SW variation of $Tr(R^n)$

We first recall some formulas for the variation of a $\wedge_\star$-product of fields [6]. We omit writing explicitly star products.

Lemma 1. *Let P, Q be arbitrary exterior forms. Then*

$$\{\Omega_{[A}, \mathbb{L}_{B]}P\}Q + P\{\Omega_{[A}, \mathbb{L}_{B]}Q\} + 2\ell_{[A}P\ell_{B]}Q = \{\Omega_{[A}, \mathbb{L}_{B]}(PQ)\} + 2L_{[A}PL_{B]}Q\,, \tag{A.1}$$

where we recall that the bracket $_{[A\,B]}$ denotes antisymmetrization of the indices A and B with weight 1, so that for example $\widehat{\Omega}_{[A}\mathbb{L}_{B]} = \frac{1}{2}(\widehat{\Omega}_A\mathbb{L}_B - \widehat{\Omega}_B\mathbb{L}_A)$.

The proof is by a straightforward calculation.

Lemma 2. *Let P, Q be arbitrary exterior forms and $P'_{[A\,B]}, Q'_{[A\,B]}$ be defined by their variations via the equations*

$$\delta_\theta P = \frac{i}{4}\delta\theta^{AB}\left(\{\Omega_A, \mathbb{L}_BP\} + P'_{[A\,B]}\right), \tag{A.2}$$

$$\delta_\theta Q = \frac{i}{4}\delta\theta^{AB}\left(\{\Omega_A, \mathbb{L}_BQ\} + Q'_{[A\,B]}\right). \tag{A.3}$$

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Then

$$\delta_\theta(PQ) = \frac{i}{4}\delta\theta^{AB}\Big(\{\Omega_A, \mathbb{L}_B(PQ)\} + 2L_A P L_B Q + P'_{[A\,B]}Q + PQ'_{[A\,B]}\Big)\ . \tag{A.4}$$

This result easily follows from the previous lemma and from the $\wedge_\star$-product variation $P \wedge_{\star_{\theta+\delta\theta}} Q = P \wedge_{\star_\theta} Q + \frac{i}{2}\delta\theta^{AB}\ell_A P \wedge_{\star_\theta} \ell_B Q$.

We can now apply formula (A.4) to $\delta_\theta R^n$ written as $\delta_\theta(RR^{n-1})$. Recalling the SW variation of R given in (4.3), and defining $(R^{n-1})'_{[AB]}$ from

$$\delta_\theta R^{n-1} = \frac{i}{4}\delta\theta^{AB}\Big(\{\Omega_A, \mathbb{L}_B R^{n-1}\} + (R^{n-1})'_{[AB]}\Big) \tag{A.5}$$

one finds

$$\delta_\theta R^n = \frac{i}{4}\delta\theta^{AB}\Big(\{\Omega_A, \mathbb{L}_B R^n\} + 2L_A R L_B R^{n-1} - 2R_A R_B R^{n-1} + R(R^{n-1})'_{[AB]}\Big)\ . \tag{A.6}$$

Comparison with $\delta_\theta R^n = \frac{i}{4}\delta\theta^{AB}\Big(\{\Omega_A, \mathbb{L}_B R^n\} + (R^n)'_{[AB]}\Big)$ leads to the recursive relation

$$\begin{aligned}(R^n)'_{[AB]} &= 2L_{[A} R L_{B]} R^{n-1} - 2R_{[A}R_{B]}R^{n-1} + R(R^{n-1})'_{[AB]}\\ &= 2DR_{[A}D(R^{n-1})_{B]} - 2R_{[A}R_{B]}R^{n-1} + R(R^{n-1})'_{[AB]}\end{aligned} \tag{A.7}$$

with initial condition $R'_{[AB]} = -[R_A, R_B] = -2R_{[A}R_{B]}$. This recursive relation is easily seen to be solved by

$$(R^n)'_{[AB]} = 2\sum_{i=1}^{n-1} R^{i-1}DR_{[A}D(R^{n-i})_{B]} - 2\sum_{i=1}^{n} R^{i-1}R_{[A}R_{B]}R^{n-i}\ . \tag{A.8}$$

Using this expression, the Leibniz rule for $\mathbb{L}_B$ and the identity $\mathbb{L}_B\Omega_A = R_{BA}$, we can rewrite the SW variation of $Tr(R^n)$ as

$$\begin{aligned}\delta_\theta Tr(R^n) &= \frac{i}{4}\delta\theta^{AB} Tr\Big(\mathbb{L}_B\{\Omega_A, R^n\} + \{R_{AB}, R^n\} - 2\sum_{i=1}^{n} R^{i-1}R_A R_B R^{n-i} +\\ &\quad + 2\sum_{i=1}^{n-1} R^{i-1}DR_A D(R^{n-i})_B\Big)\\ &= \frac{i}{2}\delta\theta^{AB} Tr\Bigg(\ell_B\{\Omega_A, R^n\} + R_{AB}R^n - nR_A R_B R^{n-1} + \ell_C \check{Q}^C_{AB} +\\ &\quad + \sum_{i=1}^{n-1} R^i R_A (R^{n-i})_B - \sum_{i=1}^{n-1} R^{i-1}R_A R(R^{n-i})_B + D\sum_{i=1}^{n-1} R^{i-1}DR_A(R^{n-i})_B\Bigg)\end{aligned} \tag{A.9}$$

where in the third line we have used cyclic reorderings to simplify the first line; the effect of these reorderings is the addition of a total Lie derivative term $\ell_C \check{Q}^C_{AB}$ that can be explicitly computed. The last line is the rewriting of the second line using the Leibniz rule for D: $R^{i-1}DR_A D(R^{n-i})_B = D\big(R^{i-1}DR_A(R^{n-i})_B\big) - R^{i-1}DDR_A(R^{n-i})_B$ and $DDR_A = -RR_A + R_A R$.

We next use the Leibniz rule for the contraction operator in the form $R_A(R^{n-i})_B = R_A R_B R^{n-i-1} + R_A R(R^{n-i-1})_B$ and then cyclic reorder the first and second terms in the last line: they drastically simplify to just two summands (up to a total Lie derivative absorbed in the term $\ell_C \check{Q}^C_{AB}$), and we obtain

$$\begin{aligned}
\delta_\theta Tr(R^n) &= \frac{i}{2}\delta\theta^{AB} Tr\Bigg(\ell_B\{\Omega_A, R^n\} + R_{AB}R^n - R_A R_B R^{n-1} - R_A R(R^{n-1})_B + \\
&\quad + D\sum_{i=1}^{n-1} R^{i-1} D R_A (R^{n-i})_B + \ell_C \check{Q}^C_{AB}\Bigg) \\
&= \frac{i}{2}\delta\theta^{AB} Tr\Big(\ell_B\{\Omega_A, R^n\} + \frac{1}{n+1} i_B i_A R^{n+1} + \ell_C \check{Q}^C_{AB}\Big) + \\
&\quad + \frac{i}{2}\delta\theta^{AB}\, d\, Tr\left(\sum_{i=2}^{n-1} R^{i-1} D R_A (R^{n-i})_B\right). \qquad \text{(A.10)}
\end{aligned}$$

To derive the last equality we observe that up to cyclic reorderings (absorbed in the $\ell_C \check{Q}^C_{AB}$ term):

- $Tr(i_B i_A R^{n+1}) = (n+1) Tr[R_{AB}R^n - R_A(R^n)_B] = (n+1) Tr[R_{AB}R^n - R_A R_B R^{n-1} - R_A R(R^{n-1})_B]$,
- the covariant derivative can be replaced by the exterior derivative,
- the first term in the sum $\delta\theta^{AB} d\, Tr\big(\sum_{i=1}^{n-1} R^{i-1} D R_A (R^{n-i})_B\big)$, i.e. $\delta\theta^{AB} d\, Tr\big(D R_A (R^{n-1})_B\big)$, vanishes.[7]

In conclusion the SW variation of $Tr(R^n)$ is given by

$$\begin{aligned}
\delta_\theta Tr(R^n) &= \frac{i}{2}\delta\theta^{AB} Tr\left(\frac{1}{n+1} i_B i_A R^{n+1}\right) + \frac{i}{2}\delta\theta^{AB}\, d\, Tr\left(\sum_{i=2}^{n-1} R^{i-1} D R_A (R^{n-i})_B\right) \\
&\quad + \frac{i}{2}\delta\theta^{AB} \ell_C Q^C_{AB} \qquad \text{(A.11)}
\end{aligned}$$

where the sum $Tr(\ell_B\{\Omega_A, R^n\} + \ell_C \check{Q}^C_{AB})$ has been renamed $\ell_C Q^C_{AB}$.

B Useful identities

Cartan formulae. The usual Cartan calculus formulae simplify if we consider commuting vector fields X_A, and read

$$\begin{aligned}
&\ell_A = i_A d + d i_A\,, && L_A = i_A D + D i_A && \\
&[\ell_A, \ell_B] = 0\,, && [L_A, L_B] = i_A i_B R && \\
&[\ell_A, i_B] = 0\,, && [L_A, i_B] = 0 && \\
&i_A i_B + i_B i_A = 0\,, && d \circ d = 0\,, && D \circ D = -R\,.
\end{aligned}$$

[7]One proves that up to cyclic reorderings $Tr\big(DR_A(R^{n-1})_B\big)$ is a total derivative, and therefore its exterior derivative vanishes (since $d^2 = 0$). Indeed $Tr\big(DR_{[A}(R^m)_{B]}\big) = Tr\big(\sum_{j=0}^{m-1} DR_{[A}R^j R_{B]} R^{m-j-1}\big)$ and the terms in this sum combine in pairs to give total derivatives (for m odd the central term is by itself a total derivative). For example up to cyclic reorderings $Tr\big(DR_{[A}R^j R_{B]}R^{m-j-1} + DR_{[A}R^{m-j-1}R_{B]}R^j\big) = Tr\big(DR_{[A}R^j R_{B]}R^{m-j-1} + R_{[B}R^j DR_{A]}R^{m-j-1}\big) = Tr\big(D(R_{[A}R^j R_{B]}R^{m-j-1})\big) = dTr\big(R_{[A}R^j R_{B]}R^{m-j-1}\big)$.

Other useful identities are (cf. also [6]):

$$\theta^{AB} L_A L_B P = -\frac{1}{2}\theta^{AB}[R_{AB}, P] \tag{B.1}$$

$$\theta^{AB}\mathbb{L}_A \Omega_B = \theta^{AB} R_{AB} \tag{B.2}$$

$$\theta^{AB}\int Tr\Big(\{\Omega_A, \mathbb{L}_B(PQ)\} + 2L_A P\, L_B Q\Big) = \theta^{AB}\int Tr\Big(\{R_{AB}, P\}Q\Big) \tag{B.3}$$

where $L_A P = \ell_A P - [\Omega_A, P]$, $\mathbb{L}_A \equiv \ell_A + L_A$ and $R_A \equiv i_A R$, $R_{AB} \equiv i_B i_A R$.

C Gamma matrices in $D = 5$

We summarize in this appendix our gamma matrix conventions in $D = 5$.

$$\eta_{ab} = (1,-1,-1,-1,-1), \qquad \{\gamma_a, \gamma_b\} = 2\eta_{ab}, \qquad [\gamma_a, \gamma_b] = 2\gamma_{ab}, \tag{C.1}$$

$$\gamma_0\gamma_1\gamma_2\gamma_3\gamma_4 = -1, \qquad \varepsilon_{01234} = \varepsilon^{01234} = 1, \tag{C.2}$$

$$\gamma_a^\dagger = \gamma_0\gamma_a\gamma_0, \tag{C.3}$$

$$\gamma_a^T = C\gamma_a C^{-1}, \qquad C^2 = -1, \qquad C^\dagger = C^T = -C. \tag{C.4}$$

C.1 Gamma identities

$$\gamma_a\gamma_b = \gamma_{ab} + \eta_{ab} \tag{C.5}$$

$$\gamma_{abc} = \frac{1}{2}\epsilon_{abcde}\gamma^{de} \tag{C.6}$$

$$\gamma_{abcd} = -\epsilon_{abcde}\gamma^e \tag{C.7}$$

$$\gamma_{ab}\gamma_c = \eta_{bc}\gamma_a - \eta_{ac}\gamma_b + \frac{1}{2}\epsilon_{abcde}\gamma^{de} \tag{C.8}$$

$$\gamma_c\gamma_{ab} = \eta_{ac}\gamma_b - \eta_{bc}\gamma_a + \frac{1}{2}\epsilon_{abcde}\gamma^{de} \tag{C.9}$$

$$\gamma^{ab}\gamma_{cd} = -\varepsilon^{ab}{}_{cde}\gamma^e - 4\delta^{[a}_{[c}\gamma^{b]}{}_{d]} - 2\delta^{ab}_{cd} \tag{C.10}$$

where $\delta^{ab}_{cd} \equiv \frac{1}{2}(\delta^a_c\delta^b_d - \delta^b_c\delta^a_d)$, and indices antisymmetrization in square brackets has total weight 1.

References

[1] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [hep-th/9908142] [INSPIRE].

[2] B. Jurčo, S. Schraml, P. Schupp and J. Wess, *Enveloping algebra valued gauge transformations for nonAbelian gauge groups on noncommutative spaces*, *Eur. Phys. J.* **C 17** (2000) 521 [hep-th/0006246] [INSPIRE].

[3] B. Jurčo, L. Möller, S. Schraml, P. Schupp and J. Wess, *Construction of nonAbelian gauge theories on noncommutative spaces*, *Eur. Phys. J.* **C 21** (2001) 383 [hep-th/0104153] [INSPIRE].

[4] L. Möller, *Second order of the expansions of action functionals of the noncommutative standard model*, *JHEP* **10** (2004) 063 [hep-th/0409085] [INSPIRE].

[5] P. Aschieri and L. Castellani, *Noncommutative gravity coupled to fermions: second order expansion via Seiberg-Witten map*, *JHEP* **07** (2012) 184 [arXiv:1111.4822] [INSPIRE].

[6] P. Aschieri, L. Castellani and M. Dimitrijević, *Noncommutative gravity at second order via Seiberg-Witten map*, *Phys. Rev.* **D 87** (2013) 024017 [arXiv:1207.4346] [INSPIRE].

[7] M. Dimitrijević, V. Radovanović and H. Stefancic, *AdS-inspired noncommutative gravity on the Moyal plane*, *Phys. Rev.* **D 86** (2012) 105041 [arXiv:1207.4675] [INSPIRE].

[8] E. Di Grezia, G. Esposito, M. Figliolia and P. Vitale, *The Seiberg-Witten map for non-commutative pure gravity and vacuum Maxwell theory*, *Int. J. Geom. Meth. Mod. Phys.* **10** (2013) 1350023 [arXiv:1209.1331] [INSPIRE].

[9] M. Dimitrijević and V. Radovanović, *Noncommutative* SO(2, 3) *gauge theory and noncommutative gravity*, *Phys. Rev.* **D 89** (2014) 125021 [arXiv:1404.4213] [INSPIRE].

[10] P. Schupp and J. You, *UV/IR mixing in noncommutative QED defined by Seiberg-Witten map*, *JHEP* **08** (2008) 107 [arXiv:0807.4886] [INSPIRE].

[11] R. Horvat, A. Ilakovac, J. Trampetic and J. You, *Self-energies on deformed spacetimes*, *JHEP* **11** (2013) 071 [arXiv:1306.1239] [INSPIRE].

[12] C.P. Martin, *Computing the θ-exact Seiberg-Witten map for arbitrary gauge groups*, *Phys. Rev.* **D 86** (2012) 065010 [arXiv:1206.2814] [INSPIRE].

[13] N.E. Grandi and G.A. Silva, *Chern-Simons action in noncommutative space*, *Phys. Lett.* **B 507** (2001) 345 [hep-th/0010113] [INSPIRE].

[14] B. Jurčo, P. Schupp and J. Wess, *NonAbelian noncommutative gauge theory via noncommutative extra dimensions*, *Nucl. Phys.* **B 604** (2001) 148 [hep-th/0102129] [INSPIRE].

[15] A.H. Chamseddine and J. Fröhlich, *The Chern-Simons action in noncommutative geometry*, *J. Math. Phys.* **35** (1994) 5195 [hep-th/9406013] [INSPIRE].

[16] A.P. Polychronakos, *Noncommutative Chern-Simons terms and the noncommutative vacuum*, *JHEP* **11** (2000) 008 [hep-th/0010264] [INSPIRE].

[17] S. Cacciatori and L. Martucci, *Noncommutative AdS supergravity in three-dimensions*, *Phys. Lett.* **B 542** (2002) 268 [hep-th/0204152] [INSPIRE].

[18] L. Castellani, *Chern-Simons supergravities, with a twist*, *JHEP* **07** (2013) 133 [arXiv:1305.1566] [INSPIRE].

[19] A.P. Polychronakos, *Seiberg-Witten map and topology*, *Annals Phys.* **301** (2002) 174 [hep-th/0206013] [INSPIRE].

[20] M. Nakahara, *Geometry, Topology and Physics*, Taylor and Francis (2003).

[21] T. Eguchi, P.B. Gilkey and A.J. Hanson, *Gravitation, Gauge Theories and Differential Geometry*, *Phys. Rept.* **66** (1980) 213 [INSPIRE].

[22] J.E. Moyal, *Quantum mechanics as a statistical theory*, *Proc. Cambridge Phil. Soc.* **45** (1949) 99 [INSPIRE].

[23] H.J. Groenewold, *On the Principles of elementary quantum mechanics*, *Physica* **12** (1946) 405 [INSPIRE].

[24] P. Aschieri and L. Castellani, *Noncommutative $D = 4$ gravity coupled to fermions*, *JHEP* **06** (2009) 086 [arXiv:0902.3817] [INSPIRE].

[25] K. Ulker and B. Yapiskan, *Seiberg-Witten maps to all orders*, *Phys. Rev.* **D 77** (2008) 065006 [arXiv:0712.0506] [INSPIRE].

[26] L.C.Q. Vilar, O.S. Ventura, R.L. P.G. Amaral, V.E.R. Lemes and L.O. Buffon, *On The Complete Seiberg-Witten Map For Theories With Topological Terms*, *JHEP* **04** (2007) 018 [hep-th/0612287] [INSPIRE].

[27] A.H. Chamseddine, *Topological Gauge Theory of Gravity in Five-dimensions and All Odd Dimensions*, *Phys. Lett.* **B 233** (1989) 291 [INSPIRE].

[28] A.H. Chamseddine, *Topological gravity and supergravity in various dimensions*, *Nucl. Phys.* **B 346** (1990) 213 [INSPIRE].

[29] R. Troncoso and J. Zanelli, *Gauge supergravities for all odd dimensions*, *Int. J. Theor. Phys.* **38** (1999) 1181 [hep-th/9807029] [INSPIRE].

[30] J. Zanelli, *Lecture notes on Chern-Simons (super-)gravities. Second edition (February 2008)*, hep-th/0502193 [INSPIRE].

[31] J. Zanelli, *Chern-Simons Forms in Gravitation Theories*, *Class. Quant. Grav.* **29** (2012) 133001 [arXiv:1208.3353] [INSPIRE].

[32] D. Lovelock, *The Einstein tensor and its generalizations*, *J. Math. Phys.* **12** (1971) 498 [INSPIRE].

[33] F. Izaurieta, E. Rodriguez, P. Minning, P. Salgado and A. Perez, *Standard General Relativity from Chern-Simons Gravity*, *Phys. Lett.* **B 678** (2009) 213 [arXiv:0905.2187] [INSPIRE].

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