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$N = 2$ supersymmetric QCD and elliptic potentials

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Abstract: We investigate the relation between the four dimensional $N = 2$ SU(2) super Yang-Mills theory with four fundamental flavors and the quantum mechanics model with Treibich-Verdier potential described by the Heun equation in the elliptic form. We study the precise correspondence of quantities in the gauge theory and the quantum mechanics model. An iterative method is used to obtain the asymptotic expansion of the spectrum for the Schrödinger operator, we are able to fix the precise relation between the energy spectrum and the instanton partition function of the gauge theory. We also study asymptotic expansions for the spectrum which correspond to the strong coupling regions of the Seiberg-Witten theory.

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Contents

1 Introduction

Recently we have witnessed some surprising connection between supersymmetric gauge theories, conformal field theories and integrable theories. Among the many implications of physical and mathematical interests, we gain some new understanding about the nonperturbative dynamics of quantum gauge theories. The four dimensional $N = 2$ super Yang-Mills theory provides a major playground for these connections since the development of the Seiberg-Witten theory [1, 2]. Its connections to string theory, classical integrable system were among the major concerns for physics interest [3–5]. The instanton counting [6] provides an essential quantum field theory explanation for the Seiberg-Witten solution, and as recently revealed it also provides a quantum generalization for the corresponding integrable system [7]. The relation of 4D gauge theory/quantum integrable model is part of the program recently outlined by Nekrasov and Shatashvili, relating the vacuum space of various gauge theories to the Bethe states of some quantum integrable systems [8, 9].

In some very simple cases, the integrable system reduces to some simple quantum mechanical problem. Apply the conventional methods of quantum physics, we can compute the quantum corrections, obtain the spectral property of the quantum model.

Through this kind of quantization we gain a deformed version of the Seiberg-Witten theory, therefore in this context the deformation in the Nekrasov partition function has a clear physical meaning.

The relation between gauge theories and integrable systems has various interesting implications, it reveals some structures not explicitly manifested in the original formulation of the two kinds of theories. However, for the moment our understanding of their relation is based on some sporadic examples we have studied, there is not a dictionary allowing us to build the precise relation between a given gauge theory and an integrable system, or vice versa. Therefore a closer study of some particular examples is still worthwhile to better understand how the correspondence works.

In this paper we investigate one example of the gauge theory/integrable model correspondence, namely the $N = 2$ SU(2) gauge theory with matters and their relation to the spectral theory of some integrable elliptic potentials or their trigonometric limits. Our goal of the investigation is to find the precise relation between certain quantities on the two sides, especially to match the infrared dynamics of the gauge theory and the eigenvalue spectrum of the elliptic potential.

The typical example is the SU(2) $N_f = 4$ QCD and the associated BC_1 Calogero-Inozemtsev (CI) model, i.e. the quantum mechanical model described by the Heun differential equation [10, 11]. The direct connection between the two theories was not noticed during the study of classical integrability of Seiberg-Witten theory [12], recently it appears to be clear partly due to the relation between gauge theories and CFT which provides another perspective to the gauge theory/integable model subject [13]. The Heun equation in the elliptic form describes a quantum particle in the Treibich-Verdier(TV) elliptic potential [14], which is an interesting topic in its own right. The potential, with its first appearance in Darboux's work [15], is a notable object in modern study of dynamical system. It belongs to the so called integrable finite-gap potential, has close relation to KdV soliton theory and algebraic-integrability theory [16]. We hope the study about its relation to SYM theory would enrich the subject.

The quantum mechanical model, as we call it, is not Hermitian, however it fairly makes sense in the context of algebraic integrable system. The methods of the Hermitian quantum mechanics apply as well, actually the tools of complex analysis are very useful, as we show in the subsequent sections. We use "quantum correction" to refer to the ϵ-corrections to both the quantum CI model and the gauge theory.

The plan of the paper is the following.

The section 2 is devoted to some necessary background. We briefly explain how the SU(2) $N_f = 4$ gauge theory is related to the Heun equation through the AGT correspondence, hence we can identify the relation between the mass parameters of gauge theory and the coupling strength of the TV potential, given in (2.8). The couplings take the form of finite-gap potential. We explain an exact WKB method for some linear spectrum problems. For the Heun equation, a shift of the parameters, manifested as a relation between the WKB expansion of two functions Θ and Ξ in (2.19), is useful in our discussion.

In the section 3 we first apply the exact WKB quantization method to obtain the spectrum of the TV quantum model for the case when the kinetic energy is very large compared to the potential. The contour integral of the WKB perturbation is carried out by applying an iterative method, we obtain the perturbative expansion for the integral for general mass parameters. We then relate the perturbative spectrum of the TV potential and the low energy solution of the super QCD theory. The relation between the energy eigenvalue and the deformed prepotential of gauge theory is given in (3.6), by matching low order perturbations on the two sides. A consistent shift argument allows us to extend the match to higher order perturbations, therefore predict the full asymptotic eigenvalue for the Heun equation from the Nekrasov partition function of the super QCD, given in (3.11).

In the section 4 we study another perturbative expansion for the spectrum where the kinetic energy is small compared to the potential, given in (4.4). It corresponds to the strong coupling expansion of the gauge theory, the procedure of determining expansion point and performing the WKB contour integral follows the spirit of the Seiberg-Witten theory.

In the section 5 we discuss various limit cases where few other SU(2) gauge theory models are recovered.

In the appendix A we give the Heun equation in different forms, in accordance with the convention we use, they are suitable to explain different aspects about its relation to the gauge theory and the quantum mechanical model. In the appendix B we explain the iterative method to solve the polynomial equation $P_4(z) = 0$ in order to carry out the elliptic integral. In the appendix C we determine the locations of the stationary points for the elliptic potential, they are in one-to-one correspondence with the singularities in the gauge theory moduli space.

2 SU(2) $N_f = 4$ super QCD and spectrum of elliptic potential

2.1 Schrödinger equation from AGT correspondence

There is not an obvious way to derive the Heun equation from any aspects of the gauge theory, it appears through a recently discovered relation between $N = 2$ SU(2) gauge theory and Liouville CFT, the Alday-Gaiotto-Tachikawa (AGT) correspondence [13]. The AGT states a relation between Nekrasov partition function of $N = 2$ gauge theory and the chiral part of the conformal block of Liouville CFT, both associated to certain punctured Riemann surfaces. For SU(2) $N_f = 4$ supersymmetric QCD, the relevant instanton partition function for gauge theory in the Ω background (ϵ_1, ϵ_2) is related to the 4-point conformal block on the sphere.

In CFT theory, the degenerate operators satisfy constraint conditions of Virasoro generators. Inserting a degenerate operator in the correlator results in certain differential equation [17]. Sometimes these differential equations are very useful, for example, along this line in the minimal models of CFT the 4-point correlator including one degenerate operator can be solved through the hypergeometric equation [17, 18]. In the AGT context, we can do the similar thing for Liouville CFT, and the corresponding gauge theory is also affected. It is argued that the CFT correlator with an additional degenerate operator inserted is related to the $N = 2$ gauge theory with surface operator [19], and the resulting conformal block/Nekrasov partition function in the Nekrasov-Shatashvili (NS) limit is related to the eigenfunction of the corresponding quantum integrable system [20].

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In the case of 4-point correlator with an extra degenerate operator constrained by level two Virasoro operators, the procedure results in a second order partial differential equation for the 5-point conformal block [17]. In the NS limit $\epsilon_1 = \epsilon, \epsilon_2 = 0$, which is relevant for the quantum integrability, the equation becomes the normal form of the Heun equation [21],

$$\left(-\epsilon^2\partial_z^2 + \mathrm{U}(z, m, \epsilon, \Theta)\right)\Phi(z) = 0, \tag{2.1}$$

where

$$\begin{aligned}\mathrm{U}(z) = & \frac{\tilde{m}_1^2 - \frac{\epsilon^2}{4}}{z^2} + \frac{m_1(m_1-\epsilon)}{(z-q)^2} + \frac{m_0(m_0-\epsilon)}{(z-1)^2} \\ & - \frac{m_0(m_0-\epsilon) + m_1(m_1-\epsilon) - \tilde{m}_0^2 + \tilde{m}_1^2}{z(z-1)} - \frac{(1-q)\Theta}{z(z-q)(z-1)}.\end{aligned} \tag{2.2}$$

In this equation z is the position of the degenerate operator and it takes complex value, and Θ is the accessory parameter. All the parameters are related to gauge theory quantities through the AGT correspondence. $m_0, \tilde{m}_0, m_1, \tilde{m}_1$ are parameters determining the conformal weight of the four primary operators of CFT, they are related to the physical mass of the four flavors μ_i in gauge theory as[1]

$$\begin{aligned} m_0 &= \frac{1}{2}(\mu_1 + \mu_2), \qquad \tilde{m}_0 = \frac{1}{2}(\mu_1 - \mu_2), \\ m_1 &= \frac{1}{2}(\mu_3 + \mu_4), \qquad \tilde{m}_1 = \frac{1}{2}(\mu_3 - \mu_4). \end{aligned} \tag{2.3}$$

The singularity parameter q is the cross ratio of the position of the four non-degenerate primary operators in CFT, identical to the UV coupling of the super QCD theory.

We can further rewrite the equation (2.1) in the elliptic form, see e.g. [11, 22, 23]. Define the new coordinate x through

$$z = \frac{\wp(x,p) - e_2(p)}{e_1(p) - e_2(p)}, \tag{2.4}$$

and the new function $W(x)$ by[2]

$$\Phi(z) = (\wp(x) - e_1)^{-\frac{m_0}{\epsilon}+\frac{1}{4}}(\wp(x) - e_2)^{-\frac{\tilde{m}_1}{\epsilon}-\frac{1}{4}}(\wp(x) - e_3)^{-\frac{m_1}{\epsilon}+\frac{1}{4}}W(x), \tag{2.5}$$

where $\wp(x)$ is the double periodic Weierstrass elliptic function. The nome $p = \exp\left(2\pi i\frac{\omega_2}{\omega_1}\right)$, where $2\omega_1, 2\omega_2$ are the periods of $\wp(x)$. p is related to the instanton counting parameter q in (2.2) by a relation

$$q = \frac{\theta_2^4(p)}{\theta_3^4(p)}. \tag{2.6}$$

[1] In the AGT paper [13] the flavors μ_1, μ_2 are in the antifundamental representation and μ_3, μ_4 are in the fundamental representation. In the Nekrasov partition function, the fundamental matter of mass μ contributes the same factor as the antifundamental matter with mass $\epsilon_1 + \epsilon_2 - \mu$ contributes. Therefore the parameters in AGT paper [13], in the NS limit, are related to ours by $m_0 = \epsilon - m_0^{\text{agt}}, \tilde{m}_0 = -\tilde{m}_0^{\text{agt}}, m_1 = m_1^{\text{agt}}, \tilde{m}_1 = \tilde{m}_1^{\text{agt}}$. But note that they appear in the equation as $m_0(m_0-\epsilon)$ and $\tilde{m}_0^2 - \frac{\epsilon^2}{4}$ unaffected.

[2] The identification of mass parameters to the γ, η, λ in the equation is not unique, see the appendix A. The coefficients for the equation in the elliptic form depends on the identification. We choose the first column identification in (A.5).

Define $\omega_0 = 0, \omega_3 = \omega_1 + \omega_2$, we have $e_{1,2,3} = \wp(\omega_{1,2,3})$. Then we get the following form of the equation,

$$\frac{d^2}{dx^2}W + \left(E - \sum_{i=0}^{3} b_i \wp(x + \omega_i)\right) W = 0. \tag{2.7}$$

It is an eigenvalue problem for the Schrödinger operator, the multi-component elliptic potential is the Treibich-Verdier potential [14].

Here we should emphasize a difference between the Lamé potential and the TV potential, concerned about the nome of the elliptic function. For the Lamé potential which is related to the $N = 2^*$ gauge theory, the nome of the function $\wp(x; \omega_1, \omega_2)$ is identified with the instanton counting parameter q, which is also the modulus of the torus in the M5-brane construction [24]. However, for the TV potential the nome of the elliptic function is p, it is related to the instanton counting parameter q by the relation (2.6). In the brane construction q is the cross ratio of the base punctured sphere while p is the modulus of the covering elliptic curve [24].

The Heun equation is equivalent to the quantum BC_1 Calogero-Inozemtsev model which is the simplest case of integrable models of Calogero type associated to the BC_N Lie algebra. The coupling strengths are related to the gauge theory mass parameters,

$$\begin{aligned} b_0 &= \left(\frac{2\tilde{m}_0}{\epsilon} - \frac{1}{2}\right)\left(\frac{2\tilde{m}_0}{\epsilon} + \frac{1}{2}\right), \qquad & b_1 &= \left(\frac{2m_0}{\epsilon} - \frac{3}{2}\right)\left(\frac{2m_0}{\epsilon} - \frac{1}{2}\right), \\ b_2 &= \left(\frac{2\tilde{m}_1}{\epsilon} - \frac{1}{2}\right)\left(\frac{2\tilde{m}_1}{\epsilon} + \frac{1}{2}\right), \qquad & b_3 &= \left(\frac{2m_1}{\epsilon} - \frac{3}{2}\right)\left(\frac{2m_1}{\epsilon} - \frac{1}{2}\right). \end{aligned} \tag{2.8}$$

The eigenvalue E is related to the accessory parameter by

$$E = 4(e_1 - e_3)\frac{\Theta}{\epsilon^2} + f\left(\frac{m_0}{\epsilon}, \frac{m_1}{\epsilon}, \frac{\tilde{m}_0}{\epsilon}, \frac{\tilde{m}_1}{\epsilon}, e_{1,2,3}\right), \tag{2.9}$$

where the precise form of the function f can be recovered from (A.6), (A.13) in appendix A, and the modulus of e_i is p. Therefore we also call Θ as energy eigenvalue although the normal form of the Heun equation does not display it as eigenvalue. Θ is a function of $m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2, q, \epsilon$, and the quantum number we call it ν, the precise relation can be computed by the WKB method and related to gauge theory partition function.

Note that the map (2.4) is not one-to-one, in the complex plane every fundamental region of the x-plane is mapped to the whole complex plane of z. The points of $x = \omega_1, \omega_2, \omega_3, 0$, module periods, are mapped to $z = 1, 0, q, \infty$, respectively.

Therefore we get the direct relation between the supersymmetric QCD theory and the quantum model of Treibich-Verdier potential, we will analyse the asymptotic spectrum of the potential and its relation to the duality of gauge theory. Actually, the Heun equation satisfied by the 5-point conformal block with one degenerate operator was already discovered for a different purpose, in the Weierstrass form [25]. This equation can be derived directly following the original paper of Belavin, Polyakov and Zamolodchikov [17], and has been used in many papers in different forms, there are few recent study relevant to the Heun equation in the context of relations among conformal field theory, integrable theory and matrix models [22, 23, 26–29].

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2.2 Integrable finite-gap potentials

The following linear spectrum problem for the potential $u(x)$,

$$-\psi'' + u(x)\psi = E\psi\,, \tag{2.10}$$

is an outstanding source of knowledge for subjects from classical functional analysis to quantum physics and integrable theory. In real analysis, a particular interesting case is when the potential is periodic, $u(x) = u(x+T)$, then the solution of the equation takes the form of Floquet-Bloch wave function,

$$\psi(x+T) = e^{i\nu T}\psi(x)\,. \tag{2.11}$$

For every quantum number ν the spectrum has a solution $E(\nu)$, however for stable solutions the reverse function $\nu = \nu(E)$ must be real, i.e. $|\cos\nu T| \leqslant 1$, this condition restricts the allowed energy inside some bands in the line of E, with the boundaries determined by the periodic and anti-periodic solution $E(\nu_*)$, with ν_* determined by $e^{i\nu_* T} = \pm 1$. Other regions are the forbidden zones where solutions are not stable. As one dimensional model for electrons in a lattice, the gap structure is responsible for the electroconductivity of materials.

Generally, the width of energy gaps decreases as the energy increases, it is possible somewhere in the spectrum the gap disappears. If moreover the spectrum is bounded from below, then the number of forbidden zones is a finite number, and the potential is called *finite-gap potential*. The finite-gap potentials are studied in the context of algebraic integrability theory where analytical tools of Riemann surface are available [16]. The Hamiltonian system of the KdV hierarchy generates a family of elliptic finite-gap potentials by the fact that (quasi)periodic solutions of the stationary higher KdV equations are finite-gap potentials [30, 31]. The g-gap potentials can be written in term of the Riemann Theta function on the genus g surface [32], and they are isospectral under the KdV flows.

Interestingly, the elliptic potentials related to 4D SU(2) SYM theories are among the most studied finite-gap potentials. The Lamé potential, acquiring the name from the Lamé equation,

$$u(x) = g(g+1)\wp(x)\,, \tag{2.12}$$

is a finite-gap potential with g forbidden zones in the spectrum when g is an integer, a result due to E.L. Ince. The Lamé equation is related to the mass deformed SU(2) $N = 4$ Yang-Mills theory($N = 2^*$ theory), we have analysed their relation in a previous work [33]. The Treibich-Verdier potential is a generalization of the Lamé potential, it is also a finite-gap potential when the coupling coefficients b_i take the following form,

$$u(x) = \sum_{i=0}^{3} g_i(g_i+1)\wp(x+\omega_i)\,, \tag{2.13}$$

and g_i are integers [14]. When the couplings are identified with the mass parameters of the supersymmetric QCD, the coupling constants b_i in (2.8) indeed can be written in the form above, but in general the corresponding g_i are not integers. Of course, in the gauge theory there is not a reason to demand g_i to be integers. If we were satisfied by the relation of gauge theory and the spectral solution of Schrödinger operator then it is not necessary to impose the integer condition for g_i. However, if the masses take special values that g_i are integers then we can further relate the ϵ_1-deformed gauge theory to the classical elliptic solution of KdV theory. Moreover, the full deformed gauge theory/quantum CFT with nonzero ϵ_1, ϵ_2 provides a deformation for the classical KdV solution, as demonstrated for the Lamé potential in [34]. In this paper we do not focus on this point, but similar arguments can be made for the Treibich-Verdier potential.

There are few ways to reduce the TV potential to the Lamé potential.

- The obvious one is by turning any three parameters among b_0, b_1, b_2, b_3 to zero.
- We can also turn them to $b_0 = b_1 = b_2 = b_3 = b$ and use the duplication formula for the elliptic function,

$$4\wp(2x) = \wp(x) + \wp(x+\omega_1) + \wp(x+\omega_2) + \wp(x+\omega_3)\,. \tag{2.14}$$

- The less obvious case is when b_i take special value, including: ① $b_0 = b_1 \neq 0, b_2 = b_3 = 0$ or $b_0 = b_1 = 0, b_2 = b_3 \neq 0$; ② $b_0 = b_2 \neq 0, b_1 = b_3 = 0$ or $b_0 = b_2 = 0, b_1 = b_3 \neq 0$. We can reduce the potential in this class to the Lamé potential because we have the following relation by the Landen transformation,

$$\begin{aligned}
\wp(x;\omega_1,2\omega_2) &= \wp(x;2\omega_1,2\omega_2) + \wp(x+\omega_1;2\omega_1,2\omega_2) - \wp(\omega_1;2\omega_1,2\omega_2)\,,\\
\wp(x;2\omega_1,\omega_2) &= \wp(x;2\omega_1,2\omega_2) + \wp(x+\omega_2;2\omega_1,2\omega_2) - \wp(\omega_2;2\omega_1,2\omega_2)\,,
\end{aligned} \tag{2.15}$$

  we keep the dependence on periods explicit to emphasize the change of the parameter $\tau = \omega_2/\omega_1$. However, other cases of $b_0 = b_3 \neq 0, b_1 = b_2 = 0$ or $b_0 = b_3 = 0, b_1 = b_2 \neq 0$ are not allowed. A further scaling limit reduces the Lamé equation the Mathieu equation which is related to the $N = 2$ SU(2) pure Yang-Mills theory. The spectrum of the Mathieu potential $u(x) = \cos x$ contains infinite many gaps.

At this point it is also worth to mention that the BC_1 CI model/TV potential is related to the sixth Painlevé equation. The sixth Painlevé equation can be written in the form of a time dependent Hamiltonian system whose potential is the TV potential [35]. Accordingly, the sixth Painlevé equation is a non-autonomous version of the classical BC_1 CI model, and in fact the relation to classical Hamiltonian system can be generalised to all other Painlevé equations [36].

2.3 The linear spectrum problem and exact WKB method

For a linear spectrum problem, a standard method to obtain the perturbative solution is the WKB method when there is an expansion parameter ϵ. For a potential with turning points x_1, x_2, the quantization condition for the exact wave function is

$$\frac{1}{\pi}\int_{x_1}^{x_2}\sum_{n=0}^{\infty}\epsilon^{n-1}p_n(E,u(x))dx = \nu\,. \tag{2.16}$$

Depending on the boundary condition there may be a requirement about the value of ν, for the periodic potential there is no other requirement other than the gap condition discussed previously. It gives the relation between the eigenvalue and the quantum number ν and other parameters. However, for a general potential on the real axis the higher WKB components p_n contain nonintegrable singularities at the turning points. Some efforts were devoted to overcome this difficult to achieve the exact WKB quantization method [37, 38]. Two useful tricks are introduced:

(1.) We should extend the integrals to the complex plane, with branch cuts determined by the equation itself. So the quantization condition becomes a contour integral along the branch cut.

(2.) In the contour integral, we can trade the higher order integrands p_n by derivatives of less singular functions with respect to parameters (energy, couplings). The integral and derivative operations are commutative. Therefore the integration can be performed for the less singular integrands whose singularities become integrable.

This kind of exact WKB method works effectively for the trigonometric potential $u(x) = \cos x$, as demonstrated by previous works [39]. The branch cut plane is the Seiberg-Witten curve in gauge theory, the contour integral of the WKB component p_0 is directly related to the contour integral of the Seiberg-Witten form. The higher order WKB corrections are identified with the Omega background deformation of the gauge theory in the NS limit [7].

The exact WKB method has been used to investigate the Lamé potential which is related to $N = 2^*$ gauge theory model [33]. We learned that for a quantum particle moving in an elliptic potential $u(x) = g(g + 1)\wp(x)$, at every stationary point of the potential (module periods) there is an asymptotic expansion for the eigenvalue and eigenfunction. In fact the stationary points of the potential are related to the weak and strong coupling regions of the gauge theory where asymptotic expansions for the effective action is available and they are related by electric-magnetic duality. This paper is a natural continuation of our previous work, here we will analyse the eigenvalue expansion of the TV elliptic potential, expanded as the WKB series.

Application to the Heun equation. We start the WKB analysis from the normal form the Heun equation (2.1), because the integral turns out to be simpler than other forms of the equation. In some earlier works, e.g. [21, 40] the leading order WKB computation is carried out for simple cases like taking equal mass limit or massless limit. We extend their analysis to generic mass value, and also determine other asymptotic expansions which correspond to strong coupling singularities of gauge theory. However, the potential $\mathrm{U}(z, m, \epsilon, \Xi)$ involving the Plank constant ϵ makes the WKB analysis more complicated. There is a simple way to take into account of this effect. We do not need to work on the Schrödinger equation (2.1), instead we start from the Schrödinger equation with the following potential $V(z, m, \Xi)$,

$$\left(-\epsilon^2\partial_z^2 + V(z, m, \Xi)\right)\Psi(z) = 0\,, \tag{2.17}$$

where

$$V(z)=\frac{\tilde{m}_1^2}{z^2}+\frac{m_1^2}{(z-q)^2}+\frac{m_0^2}{(z-1)^2}-\frac{m_0^2+m_1^2-\tilde{m}_0^2+\tilde{m}_1^2}{z(z-1)}-\frac{(1-q)\Xi}{z(z-q)(z-1)}\,. \tag{2.18}$$

The masses $m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2$ are free parameters, they appear in the eigenvalue Ξ and the corresponding eigenfunction. We can shift them by any quantity $\delta m_0^2, \delta m_1^2, \delta\tilde{m}_0^2, \delta\tilde{m}_1^2$, then the eigenvalue and eigenfunction with the new parameters $m_0^2+\delta m_0^2, m_1^2+\delta m_1^2, \tilde{m}_0^2+\delta\tilde{m}_0^2, \tilde{m}_1^2+\delta\tilde{m}_1^2$ still satisfy the Schrödinger equation (2.17). This is of course true when the shifts take the special values: $\delta m_0^2=-\epsilon m_0, \delta m_1^2=-\epsilon m_1, \delta\tilde{m}_0^2=-\epsilon^2/4, \delta\tilde{m}_1^2=-\epsilon^2/4$. This fact is already clear in our study of $N=2^*$ gauge theory/Lamé potential where the adjoint mass appears in the shifted form as $m(m-\epsilon)$ [33].

Therefore, we can solve the eigenvalue problem of the Schrödinger equation (2.17) with potential (2.18), we get the eigenvalue as function of a quantum number ν and all other parameters, $\Xi\left(\nu, m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2, q, \epsilon\right)$. Then the eigenvalue Θ of equation (2.1) with the potential (2.2) takes the same functional form as Ξ but with the mass parameters shifted,

$$\Theta\left(\nu, m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2, q, \epsilon\right)=\Xi\left(\nu, m_0(m_0-\epsilon), m_1(m_1-\epsilon), \tilde{m}_0^2-\frac{\epsilon^2}{4}, \tilde{m}_1^2-\frac{\epsilon^2}{4}, q, \epsilon\right). \tag{2.19}$$

This fact not only simplify the computation, we also use this fact to fix the precise relation between the spectrum expansion and the gauge theory partition function, as we explain later. We emphasize it is the equation (2.1) directly related to gauge theory and CFT.

The WKB form of the wave function is expanded by the Plank constant ϵ,

$$\Psi(z)=\exp i\int^z dx\Big(\frac{p_0(x)}{\epsilon}+p_1(x)+\epsilon p_2(x)+\cdots\Big)\,. \tag{2.20}$$

The Schrödinger equation gives $p_n(z)$ order by order,

$$p_0(z)=i\sqrt{V(z)}, \quad p_1(z)=\frac{i}{2}(\ln p_0)', \quad p_2(z)=\frac{-p_1^2+ip_1'}{2p_0}, \quad \cdots\,. \tag{2.21}$$

The potential contains four turning points determined by $p_0(z)=0$, therefore in the complex plane there are two branch cuts and the curve is a torus.

The monodromy of the wave function along the contour α or β of the curve can be computed by

$$\nu=\frac{1}{2\pi\epsilon}\oint_{\alpha,\beta}p(z)dz\,. \tag{2.22}$$

ν can be expanded in accordance with $p(z)$, $\nu=\epsilon^{-1}\nu_0+\epsilon\nu_2+\epsilon^3\nu_4+\cdots$, and we have $\nu_n=(2\pi)^{-1}\oint p_n(z)dz$.

First, let us work out the leading order monodromy, it is

$$\nu_0=\frac{1}{2\pi}\oint_{\alpha,\beta}p_0(z)dz=\oint_{\alpha,\beta}\frac{i}{2\pi}\frac{\sqrt{P_4(z)}}{z(z-q)(z-1)}dz\,, \tag{2.23}$$

where $P_4(z)$ is a quartic polynomial of z whose explicit form is in appendix B. As shown in [21, 40] it is simpler to compute the contour integral for $\partial_\Xi\nu_0$ first because

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$\partial_\Xi P_4(z) = -(1-q)z(z-q)(z-1)$, therefore we have

$$\frac{\partial \nu_0}{\partial \Xi} = \frac{1-q}{4\pi i} \oint_{\alpha,\beta} \frac{dz}{\sqrt{P_4(z)}} \,. \tag{2.24}$$

As $P_4(z)$ is a quartic polynomial, the integral is complete elliptic integral. The result depends on the four roots of the equation $P_4(z) = 0$. Suppose we have four roots $z_i, i = 1,2,3,4$, we can factorize $P_4(z)$ as $P_4(z) = \tilde{m}_0^2(z-z_1)(z-z_2)(z-z_3)(z-z_4)$, where the factor $\tilde{m}_0^2$ is the coefficient of z^4. Then the integrals are

$$\begin{aligned} \int_{z_1}^{z_2} \frac{dz}{\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}} &= \frac{2i}{\sqrt{(z_1-z_3)(z_2-z_4)}} K\left(k^2\right), \\ \int_{z_2}^{z_3} \frac{dz}{\sqrt{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}} &= \frac{-2}{\sqrt{(z_1-z_3)(z_2-z_4)}} K\left(k^{'2}\right), \end{aligned} \tag{2.25}$$

for contour α and β respectively. $K(k^2)$ is the complete elliptic integrals of the first kind, k is the modulus given by the cross ratio of the four roots, and $k^{'}$ is the complementary modulus.

$$k^2 = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)}, \qquad k^{'2} = 1-k^2 = \frac{(z_2-z_3)(z_1-z_4)}{(z_1-z_3)(z_2-z_4)} \,. \tag{2.26}$$

The modulus k^2 and q are different quantities, k^2(or $k^{'2}$) describes the IR coupling of gauge theory while q describes the UV coupling of gauge theory.

In order to get an asymptotic expansion, we need the four roots with a hierarchical structure that makes either k^2 or $k^{'2}$ small. This happens when two of the roots collide while other two remain at finite distance, for the case of generic parameters, there are six possible ways. This can be achieved by turning the parameters in the polynomial $P_4(z)$, in $N = 2$ gauge theory this is controlled by moving in the moduli space. In the following, we will discuss the asymptotic expansions for the spectrum of TV potential, which are related to massless particles of different U(1) charge in the gauge theory context, therefore we use the gauge theory terminology referring them as "electric/magnetic/dyonic" expansions.

The first order perturbation given by the contour integral $\oint p_1(z)dz$ does not contribute because the integrand is a total derivative. Actually, all odd order contour integral are zero because the integrands $p_{2i+1}(z)$ are all total derivatives. Higher order contour integral $\oint p_{2i}(z)dz$ can be generated from the leading order monodromy, by the action of certain differential operators with respect to the energy and mass parameters, as have demonstrated for other simpler potentials [33, 39, 41]. However, for the TV potential we have more mass parameters hence the higher order differential operators are harder to compute, so we restrict to the leading order. But concerned about the higher order ϵ-expansion, there is a consistency condition allowing us to obtain them from the gauge theory side, see the next section. Moreover, another different method using the KdV Hamiltonians is recently developed in our subsequent paper [34] and we briefly discuss in the Conclusion.

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3 Perturbative spectrum of Treibich-Verdier potential

3.1 The expansion for large Ξ

We start from the expansion region that corresponds to the electric expansion of gauge theory where the effective coupling is weak. The first input from the gauge theory is the identification of parameter Θ, or Ξ after the masses shifted, with the moduli space which is a large quantity in the electric region, $\Xi \gg m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2$. Then the contour integral is along the α-cycle where roots $z_1 \sim 0, z_2 \sim q$ are close with each other. The leading order monodromy ν_0 is identified with the v.e.v of the scalar field in the undeformed gauge theory a_0. When gauge theory is deformed, the v.e.v of the adjoint scalar is also deformed as $a = a_0 + \epsilon^2 a_2 + \epsilon^4 a_4 + \cdots$. We identify ν with a by $\nu = \frac{a}{\epsilon}$, i.e. we have $\nu_n = a_n$.

Let us compute the leading order WKB integral for $\nu_0 = a_0$. For large Ξ, the equation $P_4(z) = 0$ can be iteratively solved order by order in the Ξ expansion if we correctly choose the leading order solution. For details of the method see appendix B. The result is

$$\begin{aligned}
z_1 &= \frac{q\tilde{m}_1^2}{(1-q)\Xi} + \frac{f_1^2}{\Xi^2} + \frac{f_1^3}{\Xi^3} + \cdots\,, \\
z_2 &= q - \frac{qm_1^2}{\Xi} + \frac{f_2^2}{\Xi^2} + \frac{f_2^3}{\Xi^3} + \cdots\,, \\
z_3 &= 1 + \frac{m_0^2}{\Xi} + \frac{f_3^2}{\Xi^2} + \frac{f_3^3}{\Xi^3} + \cdots\,, \\
z_4 &= \frac{(1-q)\Xi}{\tilde{m}_0^2} + f_4^0 + \frac{f_4^1}{\Xi} + \frac{f_4^2}{\Xi^2} + \cdots\,.
\end{aligned} \tag{3.1}$$

The coefficients f_i^j with the subscript denotes roots $i = 1, 2, 3, 4$, the superscript denotes the (minus of) power of Ξ. We present the first few f_i^j in appendix B. The roots have the right hierarchical pattern, $z_1 \sim 0 \ll z_2 \sim q \ll z_3 \sim 1 \ll z_4 \sim \infty$, then the modulus k is

$$k^2 = q - \frac{m_0^2 + \tilde{m}_0^2 + m_1^2 + \tilde{m}_1^2}{\Xi} q + \mathcal{O}\left(\Xi^{-2} q\right) + \mathcal{O}\left(q^2\right). \tag{3.2}$$

It remains small because q is small. Then the $\partial_\Xi \nu_0$ and its Ξ expansion is given by

$$\begin{aligned}
\frac{\partial a_0}{\partial \Xi} &= \frac{(1-q)}{\pi \tilde{m}_0} \frac{K(k^2)}{\sqrt{(z_1 - z_3)(z_2 - z_4)}} \\
&= \frac{1}{2} \Xi^{-\frac{1}{2}} \left(h_0 + h_1 \Xi^{-1} + h_2 \Xi^{-2} + \cdots\right),
\end{aligned} \tag{3.3}$$

where h_i are functions of $m_0, m_1, \tilde{m}_0, \tilde{m}_1, q$. Integrate Ξ, and reverse the series, we get

$$\Xi = \frac{a_0^2}{h_0^2}\left[1 + 2h_0 h_1 a_0^{-2} + \left(\frac{2}{3} h_0^3 h_2 - h_0^2 h_1^2\right) a_0^{-4} + \left(\frac{2}{5} h_0^5 h_3 - 2h_0^4 h_1 h_2 + 2h_0^3 h_1^3\right) a_0^{-6} + \cdots\right]. \tag{3.4}$$

Substituting the expressions of h_i, we have

$$\Xi = a_0^2 - m_1^2 - \tilde{m}_1^2 + \frac{\left(a_0^2 + m_0^2 - \tilde{m}_0^2\right)\left(a_0^2 + m_1^2 - \tilde{m}_1^2\right)}{2a_0^2} q + \mathcal{O}\left(q^2\right). \tag{3.5}$$

The first non-zero quantum correction to ν is $\nu_2 = a_2$. Following the method of earlier work [39], it would be given by a differential operator acting on the leading order result. We do not proceed here because the coefficients are very lengthy if the masses are of generic value. Including all quantum effects, the monodromy can be expanded as

$\epsilon^{-1}a = \epsilon^{-1}a_0 + \epsilon a_2 + \epsilon^3 a_4 + \cdots$, a_n are function of Ξ. The inverse gives the expansion of Ξ in the form $\Xi = \Xi_0 + \epsilon^2\Xi_2 + \epsilon^4\Xi_4 + \cdots$. with Ξ_{2n} functions of a, instead of a_0 [41]. In this form we can relate the function $\Xi(a, m, q, \epsilon)$ to the Nekrasov partition function of the gauge theory, through the Matone's relation [42, 43]. In the following section, we will show how to match the WKB expansion with the gauge theory result without performing higher order WKB computation.

3.2 Match with the instanton partition function

According to the gauge theory/integrable model relation, the moduli parameter of gauge theory is proportional to the energy eigenvalue of integrable model, the v.e.v of adjoint scalar is identified with the momentum of quasi-particles. In order to establish the precise relation between the two models, we need to compute the expansions on the two sides, at least for the first few orders. For $N = 2$ gauge theory, the leading order solution can be computed from the Seiberg-Witten curve, but this mechanism is unable to get information about quantum corrections.

There is a particular region in the moduli space where the gauge theory is formulated by a Lagrangian and the coupling is weak, therefore the theory is in good control by QFT method. The exponentially suppressed instanton contribution is given by a counting algorithm [6], incorporating the ϵ-deformation by the Ω background, the ϵ-corrected effective action obtained is directly related to the quantum spectrum of the model. The effective action contains the perturbative part and the instanton part [6, 44]. The perturbative part of the SU(2) theory with four flavors is $Z^{\text{pert}} = \exp -\frac{1}{\epsilon_1\epsilon_2}\left(a^2 \ln q + \cdots\right)$, where we omit terms that do not depend on q, because only the first term play a role in the identification (3.6). The instanton contribution $Z^{\text{inst}} = 1 + Z_1 q + Z_2 q^2 + \cdots$ can be computed from the Nekrasov partition function [6]. The deformed prepotential is obtained from the instanton partition function by $\mathcal{F} = -\epsilon_1\epsilon_2 \ln(Z^{\text{pert}} Z^{\text{inst}})$, we take the NS limit $\epsilon_1 = \epsilon, \epsilon_2 = 0$, then we can expand the deformed prepotential as $\mathcal{F} = \mathcal{F}_{(0)} + \epsilon\mathcal{F}_{(1)} + \cdots$ where at each order $\mathcal{F}_{(n)}$ contains the perturbative part and the instanton part. As emphasized in AGT paper [13], the Nekrasov partition function actually computes the instanton action of U(2) gauge theory, therefore an U(1) factor need to be subtracted to get result of SU(2) gauge theory. We use a form of the instanton computing formula convenient for program treatment [45, 46].

It is the function Θ associated to the Heun equation (2.1) directly related to the gauge theory, therefore we need to shift the masses parameters of the function Ξ for the equation (2.17) obtained in the previous section to get Θ, then relate it to gauge theory. The relation between the function Θ and the deformed prepotential of U(2) $N_f = 4$ supersymmetric QCD in the limit $\epsilon_1 = \epsilon, \epsilon_2 \to 0$ can be fixed,

$$\Theta\big(a, m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2, q, \epsilon\big) + m_1(m_1-\epsilon) + \tilde{m}_1^2 + \frac{2q}{1-q}(m_0-\epsilon)(m_1-\epsilon) = q\frac{\partial}{\partial q}\mathcal{F}(a, \mu_i, q, \epsilon), \quad (3.6)$$

where on the left hand side the mass parameters are $m_0, m_1, \tilde{m}_0, \tilde{m}_1$, on the right hand side the mass parameters are $\mu_1, \mu_2, \mu_3, \mu_4$, they are related through (2.3). This relation, often not written in the form above, has been checked in equal mass case [21], used in discussion of relation to classical conformal block [22, 27, 29]. In this paper we carry out a direct spectral analysis for Θ when masses take generic value, find its three asymptotic expansions in the parameter space (relation (3.6) is one of them), and check these facts against the gauge theory.

Let us explain how it is determined from the process of obtaining the equation (2.1) from $\epsilon_2 \to 0$ limit of the BPZ [17] equation of Liouville CFT, see e.g. [19, 21–23, 25, 27, 29]. The null decoupling equation is a partial differential equation, the chiral part of the integrand of the 5-point correlation function is interpreted as the wave function. In the classical limit $\epsilon_2 \to 0$, the null operator decouples because it is a light operator, and the equation becomes an ordinary differential equation. Then Θ is the accessory parameter of the Fushian equation, given by the classical limit of chiral integrand of 4-point function of heavy operators [47]. The 4-point conformal block is further related to the Nekrasov instanton action by the AGT relation [13]. Taking account all these relations, we can fix the relation (3.6). In particular, the $-a^2+m_1(m_1-\epsilon)+\tilde{m}_1^2$ part comes from $\Delta_{\text{int}}-\Delta_3-\Delta_4$, as explained in [21], the term $\frac{2q}{1-q}(m_0-\epsilon)(m_1-\epsilon)$ comes from the U(1) factor in AGT relation because in (3.6) we use deformed prepotential of U(2) gauge theory (already set $a_1 = -a_2 = a$). We have confirmed this relation to higher order ϵ expansion by a different method of computing Θ from the KdV Hamiltonians [34].

From this relation we can derive the instanton part of the prepotential of the gauge theory $\mathcal{F}^{\text{inst}}_{(0)}$ and $\mathcal{F}^{\text{inst}}_{(1)}$ from the leading order result of Ξ in (3.5), and the relation (2.19) and (3.6). The first few order results, now written in terms of the physical mass μ_i used in gauge theory, are

$$\begin{aligned}\mathcal{F}^{\text{inst}}_{(0)} = \frac{1}{2a^2}&\left(a^4 + a^2\sum_{i<j}\mu_i\mu_j + \prod_i \mu_i\right) q\\ &+\frac{1}{64a^6}\left[13a^8 + a^6\left(\sum_i \mu_i^2 + 16\sum_{i<j}\mu_i\mu_j\right) + a^4\left(\sum_{i<j}\mu_i^2\mu_j^2 + 16\prod_i \mu_i\right)\right.\\ &\left.-3a^2\sum_{i<j<k}\mu_i^2\mu_j^2\mu_k^2 + 5\prod_i\mu_i^2\right] q^2 + \mathcal{O}\left(q^3\right),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\mathcal{F}^{\text{inst}}_{(1)} = -\frac{1}{4a^2}&\left(5a^2\sum_i\mu_i + \sum_{i<j<k}\mu_i\mu_j\mu_k\right) q\\ &-\frac{1}{64a^6}\left[41a^6\sum_i\mu_i + a^4\left(\sum_{i<j}\mu_i^2\mu_j + \sum_{i<j}\mu_i\mu_j^2 + 8\sum_{i<j<k}\mu_i\mu_j\mu_k\right)\right.\\ &\left.-3a^2\sum_{i<j<k}\left(\mu_i^2\mu_j^2\mu_k + \mu_i^2\mu_j\mu_k^2 + \mu_i\mu_j^2\mu_k^2\right) + 5\prod_l\mu_l\sum_{i<j<k}\mu_i\mu_j\mu_k\right] q^2\\ &+\mathcal{O}\left(q^3\right),\end{aligned} \tag{3.8}$$

where the indices $i, j, k, l \in \{1,2,3,4\}$. The $\mathcal{F}_{(0)}$ is the Seiberg-Witten solution. They precisely agree with the results of Nekrasov instanton partition function for U(2) $N_f = 4$ theory with generic masses, hence confirm the relation (3.6) for low order of ϵ-expansion.

In order to validate the relation (3.6), we should work out few higher order WKB analysis of the Heun equation. In principle this can be done. However, even without the higher order WKB results there is a nontrivial consistency condition which allows us to proceed further. The functions Ξ and Θ, expanded as series of ϵ, have different forms. Because in the equation (2.17) the potential $V(z)$ does not contain ϵ, the Schrödinger operator is invariant under the change $\epsilon \to -\epsilon$, therefore the spectrum function Ξ contains only even order of ϵ. Indeed, for a potential independent of ϵ the contour integrals for odd order WKB component p_{2n+1} always vanish. Therefore we have

$$\Xi = \sum_n \epsilon^{2n} \xi_{2n}\left(a, m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2, q\right), \qquad n \geqslant 0 . \tag{3.9}$$

But in the equation (2.1) the potential $\mathrm{U}(z,\epsilon)$ involves ϵ, the integrals of p_{2n+1} are nonzero, the function Θ should contain all order of ϵ,

$$\Theta = \sum_n \epsilon^{n} \theta_{n}\left(a, m_0^2, m_1^2, \tilde{m}_0^2, \tilde{m}_1^2, q\right), \qquad n \geqslant 0 .$$

As discussed previously, if we shift the arguments of masses in the function Θ in a proper way, then formally we can write it as a series with only even power of ϵ,

$$\Theta = \sum_n \epsilon^{2n} \widetilde{\theta}_{2n}\left(a, m_0(m_0-\epsilon), m_1(m_1-\epsilon), \tilde{m}_0^2 - \frac{\epsilon^2}{4}, \tilde{m}_1^2 - \frac{\epsilon^2}{4}, q\right), \qquad n \geqslant 0 , \tag{3.10}$$

now with $\widetilde{\theta}_{2n+1}$ vanish, and the new functions $\widetilde{\theta}_{2n}$ take the same functional form as ξ_{2n}, merely with arguments shifted. Notice that if we suppose the relation (3.6) correct, then the function Θ can be derived from the deformed prepotential, indeed it contains terms of all order of ϵ. It is a nontrivial requirement that we can rearrange the expansion in such a way where all odd order ϵ terms vanish by shifting the mass parameters. Especially, from this formula it is clear θ_1 entirely comes from θ_0 by the shift of m_0^2 and m_1^2, therefore $\theta_1 = \left(-m_0 \frac{\partial}{\partial m_0^2} - m_1 \frac{\partial}{\partial m_1^2}\right)\theta_0 = -\frac{1}{2}\left(\frac{\partial}{\partial m_0} + \frac{\partial}{\partial m_1}\right)\theta_0$. This is consistent with the ϵ-order result in [21] by explicitly examining the first two order WKB perturbation for the equation (2.1).

The rearrangement indeed works. As the instanton counting can be easily computed in a programmed way, therefore we can use the relation (3.6) to predict the higher WKB expansion for the function Θ. Then we shift the masses as explained and get the expansion (3.10), from the functional form of $\widetilde{\theta}_{2n}$ we obtain the function ξ_{2n}. For example, the first few order of ϵ-expansion for Ξ are

$$\begin{aligned}\Xi = {} & a^2 - m_1^2 - \tilde{m}_1^2 + \frac{\left(a^2 + m_0^2 - \tilde{m}_0^2\right)\left(a^2 + m_1^2 - \tilde{m}_1^2\right)}{2a^2} q + \mathcal{O}\left(q^2\right) \\ & + \epsilon^2 \left(-\frac{1}{4} + \frac{-a^4 + \left(m_0^2 - \tilde{m}_0^2\right)\left(m_1^2 - \tilde{m}_1^2\right)}{8a^4} q + \mathcal{O}\left(q^2\right)\right) \\ & + \epsilon^4 \left(\frac{\left(m_0^2 - \tilde{m}_0^2\right)\left(m_1^2 - \tilde{m}_1^2\right)}{32a^6} q + \mathcal{O}\left(q^2\right)\right) + \mathcal{O}\left(\epsilon^6\right) .\end{aligned} \tag{3.11}$$

However this does not mean we can shift the masses in the same way to make the odd order $\mathcal{F}_{(2n+1)}$ vanish, because in the relation (3.6) the term $(m_0 - \epsilon)(m_1 - \epsilon)$ is not in the shifted form, it plays a special role in the rearrangement.

4 Perturbative spectrum for small Δ

According to the duality of the gauge theory, there are other asymptotic expansions given by contour integrals along the dual cycles, such as β and $\alpha + \beta$. Stated in terms of the monodromy of the wave function in (2.22), the contour integral along the β cycle gives the magnetic dual description. The dual of a is denoted as a_D, deformed as $a_D = a_{D0} + \epsilon^2 a_{D2} + \epsilon^4 a_{D4} + \cdots$, now we can identify $\nu_n = a_{Dn}$. As we actually only work out the leading order, we omit the subscript. In order to determine the location for other expansions in the strong coupling regions in the moduli space, we need to find finite value solutions for the six degree equation $\mathcal{D}(P_4)(\Xi) = 0$. In the case of generic mass, the solutions are complicated, however, asymptotic solutions can be found by the iterative method as explained in appendix B.

Magnetic expansion. One of the solutions takes the form which corresponds to the dual magnetic description of the gauge theory dynamics,

$$\Xi_{\text{mag}} = -\left(m_1^2 + \tilde{m}_1^2\right) + g_n q^{n/2}, \qquad n = 1, 2, 3, 4, \cdots . \tag{4.1}$$

The first few g_n are

$$\begin{aligned} g_1 &= 2\left(\left(m_0^2 - \tilde{m}_0^2\right)\left(m_1^2 - \tilde{m}_1^2\right)\right)^{1/2}, \\ g_2 &= -\frac{m_1^4\tilde{m}_0^2 + m_0^4\tilde{m}_1^2 - 2m_0^2\tilde{m}_0^2\tilde{m}_1^2 - 2m_1^2\tilde{m}_0^2\tilde{m}_1^2 + \tilde{m}_0^4\tilde{m}_1^2 + \tilde{m}_0^2\tilde{m}_1^4}{\left(m_0^2 - \tilde{m}_0^2\right)\left(m_1^2 - \tilde{m}_1^2\right)}, \\ &\cdots . \end{aligned} \tag{4.2}$$

In the mass decoupling limit only the term $\mathcal{O}\left(q^{1/2}\right)$ survives, it is symmetric w.r.t. the masses, therefore the shifted coordinate $\widetilde{\widetilde{\Xi}}_{\text{mag}}$, defined in appendix B, is finite under all steps of the successive decoupling limits.

Then we set $\Xi = \Xi_{\text{mag}} + \Delta$, with Δ a small quantity compare to $q^{1/2}\left(m_0^2 - \tilde{m}_0^2\right)^{1/2}\left(m_1^2 - \tilde{m}_1^2\right)^{1/2} = q^{1/2}(\mu_1\mu_2\mu_3\mu_4)^{1/2}$. Substitute $\Xi = \Xi_{\text{mag}} + \Delta$ into $P_4(z)$, the polynomial now involves the small quantity Δ, then we can continue to iteratively solve the equation $P_4(z) = 0$. The first few order of $z_i, i = 1, 2, 3, 4$, are presented in appendix B. The roots indeed give us the small complementary modulus as

$$k'^2 = \frac{2\Delta^{1/2}}{\left(m_0^2 - \tilde{m}_0^2\right)^{1/4}\left(m_1^2 - \tilde{m}_1^2\right)^{1/4} q^{1/4}} - \frac{2\Delta}{\left(m_0^2 - \tilde{m}_0^2\right)^{1/2}\left(m_1^2 - \tilde{m}_1^2\right)^{1/2} q^{1/2}} + \cdots \ll 1 . \tag{4.3}$$

In k'^2 we only present the leading order terms that survive in the full mass decoupling limit which scale as $(q\mu_1\mu_2\mu_3\mu_4)^{-n/4}$.

Substitute these data to the integral (2.24), we finally get the reverse expansion which would be compact if written in terms of physical masses, it begins as

$$\Delta = \Bigg(-2(\mu_1\mu_2\mu_3\mu_4)^{1/4}q^{1/4} + \frac{\sum_{i<j<k}\mu_i^2\mu_j^2\mu_k^2}{2(\mu_1\mu_2\mu_3\mu_4)^{5/4}}q^{3/4} \\ + \frac{9\sum_{i<j<k}\mu_i^4\mu_j^4\mu_k^4 - 10(\mu_1\mu_2\mu_3\mu_4)^2\sum_{i<j}\mu_i^2\mu_j^2 - 40(\mu_1\mu_2\mu_3\mu_4)^3}{32(\mu_1\mu_2\mu_3\mu_4)^{11/4}}q^{5/4} + \mathcal{O}\left(q^{7/4}\right)\Bigg)\hat{a}_D \\ + \Bigg(\frac{1}{8} - \frac{3\sum_{i<j<k}\mu_i^2\mu_j^2\mu_k^2}{16(\mu_1\mu_2\mu_3\mu_4)^{3/2}}q^{1/2} \\ - \frac{39\sum_{i<j<k}\mu_i^4\mu_j^4\mu_k^4 - 30(\mu_1\mu_2\mu_3\mu_4)^2\sum_{i<j}\mu_i^2\mu_j^2 - 8(\mu_1\mu_2\mu_3\mu_4)^3}{128(\mu_1\mu_2\mu_3\mu_4)^3}q + \mathcal{O}\left(q^{3/2}\right)\Bigg)\hat{a}_D^2 \\ + \mathcal{O}\left(\hat{a}_D^3\right), \tag{4.4}$$

where we have defined $\hat{a}_D = ia_D$, as in [41]. The masses appear in a symmetric way as expected. From the leading order expansion obtained above, which is the first order in $\Xi = \Xi_0 + \epsilon^2\Xi_2 + \cdots$ if we recover the abbreviated subscript, we can derive the first order quantum correction to $\Theta = \Theta_0 + \epsilon\Theta_1 + \cdots$ which comes from the shift of m_0^2, m_1^2,

$$\Theta_0 = \Xi_0, \qquad \Theta_1 = -\frac{1}{2}\left(\frac{\partial}{\partial m_0} + \frac{\partial}{\partial m_1}\right)\Theta_0 = -\sum_{i=1}^{4}\frac{\partial}{\partial \mu_i}\Theta_0\,, \tag{4.5}$$

where $\Theta_0 = \Theta_0(\mu_i, \hat{a}_D, q)$ for the dual expansion.

Dyonic expansion. The dyonic expansion comes from a rotation of the phase of q by 2π. From the relation $q = \exp(2\pi i\tau)$ with $\tau = \frac{4\pi i}{g_{uv}^2} + \frac{\theta}{2\pi}$, the rotation $q \to e^{2\pi i}q$ induces the shift of the theta angle by 2π, this would shift the electric charge of a magnetic particle in the low energy gauge theory according to the Witten effect [48], resulting a dyon of charge (1,1).

5 Various limit cases

We provide few limit cases, where we can recover few other gauge theory models and confirm their relations.

(I). Equal masses limit. There are few cases the polynomial $P_4(z)$ degenerates to lower order, it is easy to solve $P_4(z) = 0$ and the results can be presented in a more compact form. We can take all masses equal $\mu_i = m$, then we have $\tilde{m}_0 = 0, \tilde{m}_1 = 0$. It is also simple to study the case for $\mu_1 = \mu_2, \mu_3 \neq \mu_4$ where we have $\tilde{m}_0 = 0$, and the case for $\mu_1 \neq \mu_2, \mu_3 = \mu_4$ where we have $\tilde{m}_1 = 0$.

(II). Massless limit. We can turn some flavors massless, while keep other flavors massive. This is allowed in the electric expansion, but not allowed in the dual expansions because this violates our assumption $\Delta \ll q^{1/2}(\mu_1\mu_2\mu_3\mu_4)^{1/2}$. A particular interesting case is the full massless limit, $m_0 = m_1 = \tilde{m}_0 = \tilde{m}_1 = 0$, where the gauge theory becomes conformal. Let us look at the Seiberg-Witten solution,

$$a_0 = -\frac{\sqrt{(1-q)\Xi}}{2\pi}\oint_\alpha \frac{dz}{\sqrt{z(z-q)(z-1)}} = -\frac{2K(q)}{\pi}\sqrt{(1-q)\Xi}\,. \tag{5.1}$$

From $\Xi = q\frac{\partial \mathcal{F}}{\partial q}$ we get the prepotential, therefore the instanton corrected effective coupling is

$$4\pi i\tau_{ir} = \frac{\partial^2 \mathcal{F}}{\partial a_0^2} = \frac{\pi^2}{2}\int \frac{dq}{q(1-q)K^2(q)} = -2\pi\frac{K(1-q)}{K(q)} \,. \tag{5.2}$$

If we set $q = \exp(2\pi i\tau_{uv}), p = \exp(4\pi i\tau_{ir})$, then for weak coupling $|q| \ll 1$ we have the relation

$$2\pi i\tau_{ir} = 2\pi i\tau_{uv} - 4\ln 2 + \frac{1}{2}q + \frac{13}{64}q^2 + \frac{23}{192}q^3 + \frac{2701}{32768}q^4 + \cdots \,, \tag{5.3}$$

or the inverse relation

$$q = \frac{\theta_2^4(p)}{\theta_3^4(p)} \,, \tag{5.4}$$

which is exactly the relation obtained earlier in another context [49], discussed later by e.g. [13, 40]. It is the same relation of (2.6). The duality of the massless gauge theory is encoded in the modular transformation of the Theta functions.

(III). Mass decoupling limit. In the infinite mass limit, by turning the UV coupling properly, the resulting theory is a gauge theory with less flavors. In both electric and magnetic expansions, in the final results the mass parameters always appear in a symmetric way. This feature makes the mass decoupling procedure straightforward, we can decouple any one, any number, of the four flavors.

For example, if we keep $\mu_{1,2,3}$ finite and turn $\mu_4 \to \infty, q \to 0$ while make $q\mu_4 = \Lambda_3$ finite, we get the $N = 2$ gauge theory with $N_f = 3$. By turning $\mu_3, \mu_4 \to \infty, q \to 0$ with $q\mu_3\mu_4 = \Lambda_2^2$ we get the $N = 2$ gauge theory with $N_f = 2$. And $\mu_2, \mu_3, \mu_4 \to \infty, q \to 0$ with $q\mu_2\mu_3\mu_4 = \Lambda_1^3$ gives the $N = 2$ gauge theory with $N_f = 1$. Finally we get the $N = 2$ pure Yang-Mills theory by decoupling all flavors as $\mu_{1,2,3,4} \to \infty, q \to 0$ with $q\mu_1\mu_2\mu_3\mu_4 = \Lambda^4$. We can compare the decoupling results of formula (3.7) to some previous results [50].

We can also do the decoupling limit for the magnetic(and dyonic) expansion. For example, we first set all masses equal $\mu_i = m$, then turn $m \to \infty, q \to 0$ and keep $q^{1/4}m = \Lambda$ finite. In the formula (4.4), only the following terms survive in the limit,

$$\Delta_{N_f=0} = -2\hat{a}_D\Lambda + \frac{1}{2^3}\hat{a}_D^2 + \frac{1}{2^7}\frac{\hat{a}_D^3}{\Lambda} + \frac{5}{2^{12}}\frac{\hat{a}_D^4}{\Lambda^2} + \frac{33}{2^{17}}\frac{\hat{a}_D^5}{\Lambda^3} + \frac{63}{2^{20}}\frac{\hat{a}_D^6}{\Lambda^4} + \cdots \,. \tag{5.5}$$

In agreement with the result of pure gauge theory [41], if we scale $\hat{a}_D \to 2\hat{a}_D$ because in our parameterization the singularities in the moduli space are at $\pm 2\Lambda^2$.

(IV). Limits related to $N = 2^*$ SYM. According to the AGT correspondence, the partition function of the $N = 2$ $N_f = 4$ QCD is related to the 4-point conformal block on the sphere, and the partition function of the $N = 2^*$ SYM is related to the 1-point conformal block on the torus. It turns out that there is a surprising relation between the two pairs. We will show that, from the perspective of relation between gauge theory/CFT/integrable potential, these facts are in consistent with that the TV potential reduces to the Lamé potential in particular limits.

The relation on the CFT side is reflected through the fact that the 1-point correlation for a generic primary operator on the torus is related to 4-point correlator on the sphere with a special choice of conformal weight for the primary operators [25]. With a proper identification of parameters as in (2.8), in the NS limit, this choice makes $m_0 = m_1 = \tilde{m}_1 = \frac{\epsilon}{4}$ and $\tilde{m}_0$ remains free, therefore we have

$$b_0 = \left(\frac{2\tilde{m}_0}{\epsilon} - \frac{1}{2}\right)\left(\frac{2\tilde{m}_0}{\epsilon} + \frac{1}{2}\right), \qquad b_1 = b_2 = b_3 = 0\,. \tag{5.6}$$

The TV potential in this limit becomes the Lamé potential with a single $\wp(x)$ function. It implies a partial massless limit for the corresponding gauge theory, $\mu_1 = \tilde{m}_0 + \frac{\epsilon}{4}, \mu_2 = -\tilde{m}_0 + \frac{\epsilon}{4}, \mu_3 = \frac{\epsilon}{2}, \mu_4 = 0$.

Meanwhile, a relation between the Nekrasov partition functions of the $N_f = 4$ gauge theory and the $N = 2^*$ gauge theory is also recently found [51]. The choice of the flavor masses is

$$\mu_1 = \frac{1}{2}M, \quad \mu_2 = \frac{1}{2}(M + \epsilon_1), \quad \mu_3 = \frac{1}{2}(M + \epsilon_2), \quad \mu_4 = \frac{1}{2}(M + \epsilon_1 + \epsilon_2)\,. \tag{5.7}$$

In the NS limit, it implies the parameters for the elliptic potential are

$$b_0 = b_2 = 0, \qquad b_1 = b_3 = \frac{M}{\epsilon}\left(\frac{M}{\epsilon} - 1\right). \tag{5.8}$$

The TV potential in this limit is actually also the Lamé potential, because we have the relation (2.15) for the elliptic function.

6 Conclusion

In this paper we investigate the relation between the SU(2) super QCD models and the quantum mechanics models of some elliptic potentials. We carry out a detailed study of SU(2) gauge theory with four fundamental flavors and the associated spectrum of the Treibich-Verdier potential. This relation implies a few other cases of gauge theory/elliptic potential correspondence, by taking various limits on both sides. Our conclusion of the analysis is that the Coulomb branch low energy dynamics of these SU(2) super QCD theories are equivalent to the spectral problem of elliptic potentials.

We compare various aspects on both sides. For example, the Treibich-Verdier potential has six stationary points, they correspond to six singularities in the moduli of the gauge theory. We analyse the asymptotic expansions of the eigenvalue of the Schrödinger operator at these singularities, we can match one of the expansions for very large eigenvalue with the instanton action of gauge theory. An iterative method is used to factorize the polynomial $P_4(z)$ in a proper way to find the asymptotic spectrum expansions for the elliptic potential, applicable to all stationary/singularity points. This method is practically useful to obtain the dual expansion of gauge theory prepotential. We can compare various limit cases of our computation to previous literatures. The study supports the fact that the new parametrisation of the Seiberg-Witten curve, manifested as the potential $V(z)$ in (2.18), indeed captures all the ingredients of the low energy gauge theory as originally formulated [2].

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There are obvious questions related to the problem we have studied. Can we generalise the gauge theory/elliptic potential relation to general quiver gauge theories with more SU(2) groups? For general $N = 2$ gauge theory with SU(N_c) gauge group and $N_f = 2N_c$ flavor, is there a corresponding integrable model with the potential of elliptic form? Can we use this gauge theory/elliptic potential relation to find more finite-gap potentials?

Lastly, concerned about the relation to KdV theory, we recently indeed made further study where we give another method to derive deformed gauge theory prepotential in the NS limit from classical KdV Hamiltonians, and the deformed prepotential for generic value of ϵ_1, ϵ_2 is interpreted as a "quantum" deformation of the KdV Hamiltonians [34]. All the method and arguments can be directly applied to the TV potential. Especially the integration on elliptic function in (2.7) naturally gives the spectrum/prepotential in terms of quasimodular functions of p, using (2.6) we precisely recover the results of this paper including (3.7), (3.8). Working with the normal form of the Heun equation, the higher order WKB analysis for the TV potential would be technically difficult to obtain, hence we cease to work out them in this paper. The approach from KdV theory avoids this problem and higher order ϵ expression are relatively easier to derive, however the method can only applies to the asymptotic electric region [34]. Our conclusion is that the method developed here and the method from KdV theory are complementary, their consistency is a encouraging evidence for the claims we have made.

A The Heun equation

We used the convention a bit different from the NIST handbook [11] to relate the equation to gauge theory quantities, therefore in order to avoid confusion we go through some details on different forms of the Heun equation.

The Heun equation is the general second order Fuchsian linear differential equation with four regular singularities, it is an extension of the hypergeometric equation. It can be written in several different forms, the most familiar form is

$$\frac{d^2}{dz^2}w(z) + \left(\frac{\gamma}{z} + \frac{\eta}{z-1} + \frac{\lambda}{z-q}\right)\frac{d}{dz}w(z) + \frac{\alpha\beta z - Q}{z(z-1)(z-q)}w(z) = 0\,, \tag{A.1}$$

with $\alpha + \beta + 1 = \gamma + \eta + \lambda$, q is called the singularity parameter, $\alpha, \beta, \gamma, \eta, \lambda$ are called the exponent parameters, and Q is called the accessory parameter. There are four regular singularities at $0, q, 1, \infty$, the merging of regular singularities gives the Confluent Heun equations with irregular singularities.

Define a new function $\widetilde{w}(z)$ by

$$w(z) = z^{-\gamma/2}(z-1)^{-\eta/2}(z-q)^{-\lambda/2}\widetilde{w}(z)\,, \tag{A.2}$$

then we can write the Heun equation in the normal form,

$$\frac{d^2}{dz^2}\widetilde{w}(z) - \left[\frac{D}{z^2} + \frac{F}{(z-q)^2} + \frac{E}{(z-1)^2} + \frac{(1-q)B - qA}{z(z-1)} + \frac{q(1-q)(A+B)}{z(z-1)(z-q)}\right]\widetilde{w}(z) = 0. \tag{A.3}$$

with $A+B+C=0$. The parameters A,B,C,D,E,F are related to $\alpha,\beta,\gamma,\eta,\lambda$ by

$$A=\frac{2Q-q\gamma\eta-\gamma\lambda}{2q}\,,\qquad B=\frac{2Q-2\alpha\beta+\gamma\eta+\lambda\eta-q\gamma\eta}{2(1-q)}\,,$$
$$D=\frac{1}{2}\gamma(\frac{1}{2}\gamma-1)\,,\qquad E=\frac{1}{2}\eta\left(\frac{1}{2}\eta-1\right),\qquad F=\frac{1}{2}\lambda\left(\frac{1}{2}\lambda-1\right). \tag{A.4}$$

Compare with the equation (2.1) obtained from CFT/gauge theory, we can identify the parameters $\gamma,\eta,\lambda,\alpha,\beta,Q$ with $m_0,m_1,\tilde{m}_0,\tilde{m}_1,q,\epsilon$ of the SU(2) $N_f=4$ theory. The identification is not unique, we have

$$\begin{aligned}\gamma&=\frac{2\tilde{m}_1}{\epsilon}+1\,, &&\text{or} && -\frac{2\tilde{m}_1}{\epsilon}+1\,,\\ \eta&=\frac{2m_0}{\epsilon}\,, &&\text{or} && -\frac{2m_0}{\epsilon}+2\,,\\ \lambda&=\frac{2m_1}{\epsilon}\,, &&\text{or} && -\frac{2m_1}{\epsilon}+2\,,\end{aligned} \tag{A.5}$$

and two more relations,

$$\begin{aligned}\alpha\beta&=\frac{\gamma\eta+\gamma\lambda+\eta\lambda}{2}+\frac{m_0}{\epsilon}\left(\frac{m_0}{\epsilon}-1\right)+\frac{m_1}{\epsilon}\left(\frac{m_1}{\epsilon}-1\right)-\frac{\tilde{m}_0^2}{\epsilon^2}+\frac{\tilde{m}_1^2}{\epsilon^2},\\ Q&=-(1-q)\frac{\Theta}{\epsilon^2}+\frac{q\gamma\eta+\gamma\lambda}{2}+q\left[\frac{m_0}{\epsilon}\left(\frac{m_0}{\epsilon}-1\right)+\frac{m_1}{\epsilon}\left(\frac{m_1}{\epsilon}-1\right)-\frac{\tilde{m}_0^2}{\epsilon^2}+\frac{\tilde{m}_1^2}{\epsilon^2}\right].\end{aligned} \tag{A.6}$$

Terms involving masses in $\alpha\beta$ and Q do not depend on parameter identification choice of (A.5).

We can transform the equation to the elliptic form for the study of the TV potential. Define the coordinate ζ by

$$z=q\times\mathrm{sn}^2(\zeta,q)\,, \tag{A.7}$$

and define a new function $\widehat{w}(\zeta)$ by

$$w(z)=(\mathrm{sn}\zeta)^{(1-2\gamma)/2}(\mathrm{dn}\zeta)^{(1-2\eta)/2}(\mathrm{cn}\zeta)^{(1-2\lambda)/2}\widehat{w}(\zeta)\,, \tag{A.8}$$

where the Jacobi elliptic functions $\mathrm{sn}(\zeta,q)$ etc. depend on the nome q. The quarter periods of $\mathrm{sn}(\zeta,q)$ etc. are complete elliptic integral $K(q),iK'(q)$ (the same function as in (2.25), but with different arguments due to mass deformation). Then the Heun equation transforms to the form

$$\frac{d^2}{d\zeta^2}\widehat{w}(\zeta)+\left(h-b_0q\mathrm{sn}^2\zeta-b_1q\frac{\mathrm{cn}^2\zeta}{\mathrm{dn}^2\zeta}-b_2\frac{1}{\mathrm{sn}^2\zeta}-b_3\frac{\mathrm{dn}^2\zeta}{\mathrm{cn}^2\zeta}\right)\widehat{w}(\zeta)=0\,, \tag{A.9}$$

where

$$\begin{aligned}&h=-4Q+(\gamma+\lambda-1)^2+q(\gamma+\eta-1)^2, &&b_0=-4\alpha\beta+\left(\gamma+\eta+\lambda-\frac{1}{2}\right)\left(\gamma+\eta+\lambda-\frac{3}{2}\right),\\ &b_1=\left(\eta-\frac{1}{2}\right)\left(\eta-\frac{3}{2}\right),\quad b_2=\left(\gamma-\frac{1}{2}\right)\left(\gamma-\frac{3}{2}\right), &&b_3=\left(\lambda-\frac{1}{2}\right)\left(\lambda-\frac{3}{2}\right).\end{aligned} \tag{A.10}$$

This is the equation appears in Darboux's work [15]. In the limit $q\to 0$ and other quantities remain finite, the potential reduces to the Pöschl-Teller potential.

In order to further transform the equation to the Weierstrass elliptic form, we define variables x by

$$\frac{\zeta+iK'}{(e_1-e_2)^{1/2}}=x,\quad \text{i.e.}\quad \mathrm{sn}^2(\zeta,q)=\frac{\wp(x,p)-e_2(p)}{e_3(p)-e_2(p)}. \tag{A.11}$$

The half periods of $\wp(x)$ are ω_1, ω_2, we have $K = (e_1 - e_2)^{1/2}\omega_1, iK' = (e_1 - e_2)^{1/2}\omega_2$. The nome $p = \exp\left(2\pi i \frac{\omega_2}{\omega_1}\right) = \exp\left(-2\pi \frac{K'}{K}\right)$ is related to q as in (2.6). Then we transform the equation (A.9) to the following form,

$$\frac{d^2}{dx^2}\widehat{w}(x) + \left(E - \sum_{i=0}^{3} b_i \wp(x + \omega_i))\right)\widehat{w}(x) = 0\,, \tag{A.12}$$

where

$$\begin{aligned} E &= (e_1 - e_2)h + e_2 \sum_i b_i \\ &= 4(e_2 - e_1)Q - 4e_2\alpha\beta + e_1(\gamma + \lambda - 1)^2 + e_2(\eta + \lambda - 1)^2 + e_3(\gamma + \eta - 1)^2. \end{aligned} \tag{A.13}$$

The stationary points of the potential are determined by the condition $\sum_i b_i \partial_x \wp(x + \omega_i) = 0$, if rewritten in the variable z it is a polynomial equation of degree six $Q_6(z) = 0$. The six solutions are in correspondence to the six solutions for the discriminant equation $\mathcal{D}(P_4)(\Xi) = 0$, discussed in the next two appendixes.

B Iterative solution for $P_4(z) = 0$

The degenerate points of $P_4(z)$. Let us start from some basic facts about quartic polynomial. For a quartic polynomial defined in the complex domain

$$P_4(z) = az^4 + bz^3 + cz^2 + dz + e\,, \tag{B.1}$$

we can associated an elliptic curve to the polynomial, $y^2 = P_4(z)$. The shape of the curve is controlled by the modulus parameter (2.26), which is given by the cross ratio of the roots of the equation $P_4(z) = 0$.

These elliptic curves are used in the Seiberg-Witten theory to determine the dynamics of the gauge theory, when the curve degenerates the gauge theory has a weak coupling description. It happens when two of the roots collide, making either the modulus k^2 or its complementary modulus k'^2 small. k^2 and k'^2 are related to the gauge coupling and the dual gauge coupling, respectively. In this paper we need to deal with an integral that is similar to the integration of the Seiberg-Witten form, the spirit is the same: when the curve degenerate the integral can be written as an asymptotic expansion. The condition for the degeneration is given by the vanishing of the discriminant of the polynomial,

$$\begin{aligned} \mathcal{D}(P_4) = {} & b^2c^2d^2 - 4ac^3d^2 - 4b^3d^3 + 18abcd^3 - 27a^2d^4 - 4b^2c^3e + 16ac^4e \\ & + 18b^3cde - 80abc^2de - 6ab^2d^2e + 144a^2cd^2e - 27b^4e^2 + 144ab^2ce^2 \\ & - 128a^2c^2e^2 - 192a^2bde^2 + 256a^3e^3. \end{aligned} \tag{B.2}$$

Now we specific to the quartic polynomial used in our story, it is (see [21, 40])

$$\begin{aligned} P_4(z) = {} & \tilde{m}_0^2 z^4 + \left(-\Xi + q\Xi - 2q\tilde{m}_0^2 + 2qm_1^2 + m_0^2 - \tilde{m}_0^2 - m_1^2 - \tilde{m}_1^2\right) z^3 \\ & + \left(\Xi - q^2\Xi + q^2\tilde{m}_0^2 - q^2m_1^2 - 2qm_0^2 + 2q\tilde{m}_0^2 - 2qm_1^2 + 2q\tilde{m}_1^2 + m_1^2 + \tilde{m}_1^2\right) z^2 \\ & + \left(-q\Xi + q^2\Xi + q^2m_0^2 - q^2\tilde{m}_0^2 + q^2m_1^2 - q^2\tilde{m}_1^2 - 2q\tilde{m}_1^2\right) z + q^2\tilde{m}_1^2. \end{aligned} \tag{B.3}$$

The polynomial does not define the Seiberg-Witten curve of the super QCD, however, the spirit of finding the degenerate points of the associated curve to carry out the asymptotic expansion is similar. The equation $P_4(z) = 0$ has general algebraic solutions, but their closed form are too cumbersome to analyse in practice, however, here we can turn some parameters very small and asymptotic solutions can be found. Starting from a weak coupling UV theory, we have q always small. We can also turn the scalar v.e.v which controls the magnitude of Ξ, therefore we can make $\Delta = \Xi - \Xi_*$ either very large or very small, where Ξ_* is a degeneration point of the polynomial $P_4(z)$.

The asymptotic solution can be expanded by small parameters, the corresponding expansion coefficients can be solve by an iterative method, if we correctly choose the leading order solution and the subsequent expansion pattern. The iterative method works as a self-checking process, it only works well when we choose the right expansion pattern, otherwise it always ends in an obstruction.

The location of the degeneration point Ξ_* where $\mathcal{D}(P_4)(\Xi_*) = 0$, determines the nature of the asymptotic solution, therefore we need to find its value and do asymptotic expansion near this point. For generic masses, the discriminant of $P_4(z)$ is a large polynomial of degree six for Ξ. Therefore there are six singularities in the Ξ space at finite position, each of them is of weight one. Their locations are

$$\begin{aligned} \Xi_* \sim & -m_1^2 - \tilde{m}_1^2 + (m_0 \pm \tilde{m}_0)^2 + \mathcal{O}\left(m^2 q\right), \\ & \pm 2 m_1 \tilde{m}_1 + \mathcal{O}\left(m^2 q\right), \\ & - m_1^2 - \tilde{m}_1^2 \pm 2\left(m_0^2 - \tilde{m}_0^2\right)^{1/2}\left(m_1^2 - \tilde{m}_1^2\right)^{1/2} q^{1/2} + \mathcal{O}\left(m^2 q\right). \end{aligned} \tag{B.4}$$

Then the shifted coordinates defined by,

$$\widetilde{\Xi} = m_1^2 + \tilde{m}_1^2 + \frac{2q}{1-q} m_0 m_1 + \Xi\,, \tag{B.5}$$

has singularities at

$$\begin{aligned} \widetilde{\Xi}_* \sim & \left(m_0 \pm \tilde{m}_0\right)^2 + \mathcal{O}\left(m^2 q\right), \\ & \left(m_1 \pm \tilde{m}_1\right)^2 + \mathcal{O}\left(m^2 q\right), \\ & \pm 2\left(m_0^2 - \tilde{m}_0^2\right)^{1/2}\left(m_1^2 - \tilde{m}_1^2\right)^{1/2} q^{1/2} + \mathcal{O}\left(m^2 q\right). \end{aligned} \tag{B.6}$$

Note that from (3.6) we have $\widetilde{\Xi}_* \sim q\partial_q \mathcal{F}(q)$.

Note that $(m_0 \pm \tilde{m}_0), (m_1 \pm \tilde{m}_1)$ are the bare masses $\mu_1, \mu_2, \mu_3, \mu_4$, hence we conclude the first four singularities, denoted as $\Xi_{*1,2,3,4}$, are associated with the flavor μ_i which becomes massless at the singularity Ξ_{*i} of the Coulomb moduli, they are in the semiclassical region in the moduli space. The asymptotic expansion in this region is formula (3.11), when $a \sim \mu_i$ then Ξ is at the corresponding singularity. The singularity Ξ_{*i} would be pushed to infinity under the decoupling limits involving the flavor μ_i, and remain finite for other mass decoupling limits.

In the semiclassical region, Ξ is large, we should have set $\Xi = \Xi_{*1,2,3,4} + \Delta$ with Δ large, but in section 3 we treat Ξ itself as a large quantity. We can divide all the coefficients of the polynomial (B.3) by Ξ, and the degeneration at $\Xi \sim \infty$ is obvious.

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Other two singularities in (B.4) are associated with magnetic and dyonic particles, in the mass decoupling limit they survive as the corresponding strong coupling singularities for the gauge theory with less flavors.

Before we proceed to solve, it is helpful to notice the following properties the solutions should satisfy for generic cases of parameters:

(1.) When $\tilde{m}_0 \to 0$, the equation reduces to third order, it means a root is pushed to infinity.

(2.) When $\tilde{m}_1 \to 0$, the equation has a root $z = 0$.

(3.) When $m_0 \to 0$, the equation has a root $z = 1$.

(4.) When $m_1 \to 0$, the equation has a root $z = q$.

Large Ξ solution. For large Ξ, the leading order of the polynomial reduces to a cubic one, $P_4(z) = \Xi(q-1)z(z-q)(z-1) + \cdots$, therefore the leading order solution is given by the equation $z(z-q)(z-1) = 0$ which gives three roots at $0, q, 1$, the forth root z_4 is near infinity. We use small quantity $\frac{1}{\Xi}$ to control the expansion, with a little of guess, we have the large expansion of all four roots as presented in (3.1). The subleading coefficients are iteratively solved. Take z_2 as the example, if we already got the coefficient $f_2^j, 2 \le j \le n$, substitute the trial solution

$$z_2 = q + \sum_{j=1}^{n} \frac{f_2^j}{\Xi^j} + \frac{f_2^{n+1}}{\Xi^{n+1}}, \tag{B.7}$$

into the polynomial $P_4(z)$, then the leading order non-zero coefficient in the $\frac{1}{\Xi}$ expansion of $P_4(z)$ is a function of the form $c_1 + c_2 f_2^{n+1}$ with $c_{1,2}$ functions of masses and q. We can solve the equation at this order by setting $c_1 + c_2 f_2^{n+1} = 0$ therefore obtain the coefficient f_2^{n+1}. In this way we can iteratively solve all f_i^j. We give the first few of them.

$$\begin{aligned}
f_1^2 &= \frac{q\tilde{m}_1^2\left(qm_0^2 + qm_1^2 - q\tilde{m}_0^2 - \tilde{m}_1^2\right)}{(1-q)^2}, \\
f_1^3 &= -(1-q)^{-3} q\tilde{m}_1^2 \left[q^2\left(m_0^4 + 2m_0^2m_1^2 + m_1^4 - 2m_0^2\tilde{m}_0^2 - 2m_1^2\tilde{m}_0^2 + \tilde{m}_0^4 + m_0^2\tilde{m}_1^2\right)\right. \\
&\qquad \left. -3q\tilde{m}_1^2\left(m_0^2 + m_1^2 - \tilde{m}_0^2\right) + \tilde{m}_1^2\left(m_1^2 + \tilde{m}_1^2\right)\right], \\
f_2^2 &= \frac{qm_1^4(1-2q)}{1-q}, \\
f_2^3 &= -\frac{qm_1^4\left[q^2\left(4m_1^2 + \tilde{m}_0^2\right) + q\left(m_0^2 - 4m_1^2 - \tilde{m}_0^2 - \tilde{m}_1^2\right) + m_1^2 + \tilde{m}_1^2\right]}{(1-q)^2}, \\
f_3^2 &= \frac{m_0^2\left[q\left(m_1^2 - \tilde{m}_0^2 + \tilde{m}_1^2\right) + m_0^2 - m_1^2 + \tilde{m}_0^2 - \tilde{m}_1^2\right]}{1-q}, \\
f_4^0 &= \frac{q\tilde{m}_0^2 - 2qm_1^2 - m_0^2 + m_1^2 + \tilde{m}_1^2}{\tilde{m}_0^2}, \\
f_4^1 &= -\frac{q^2m_1^2 - qm_0^2 - qm_1^2 + q\tilde{m}_1^2 + m_0^2}{1-q}.
\end{aligned} \tag{B.8}$$

Note that q is automatically incorporated into the expansion in a reasonable form. These solution satisfy the properties of limits (1.)–(4.).

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Small Δ solution. Around the magnetic point, we set $\Xi = \Xi_{\text{mag}} + \Delta$, substitute this into $P_4(z)$ we get a polynomial double expanded w.r.t. to q and Δ. In this polynomial the q-expansion is infinite order, therefore the iterative solution would be more involved than the case of large Ξ. We get the following leading order solution, and the general expansion pattern,

$$
\begin{aligned}
z_1 = & -\frac{m_0^2 - \tilde{m}_0^2}{\tilde{m}_0^2} + \frac{2m_0^2 \left(m_1^2 - \tilde{m}_1^2\right)^{1/2}}{\tilde{m}_0^2 \left(m_0^2 - \tilde{m}_0^2\right)^{1/2}} q^{1/2} + \sum_{n=2}^{\infty} c_* q^{n/2} \\
& + \left(\frac{m_0^2}{\tilde{m}_0^2 \left(m_0^2 - \tilde{m}_0^2\right)} + \frac{4m_0^2 \left(m_1^2 - \tilde{m}_1^2\right)^{1/2}}{\left(m_0^2 - \tilde{m}_0^2\right)^{5/2}} q^{1/2} + \sum_{n=2}^{\infty} c_* q^{n/2} \right) \Delta \\
& + \sum_{m=2}^{\infty} \left(\sum_{n=0}^{\infty} c_* q^{n/2} \right) \Delta^m, \\
z_2 = & -\frac{\left(m_1^2 - \tilde{m}_1^2\right)^{1/2}}{\left(m_0^2 - \tilde{m}_0^2\right)^{1/2}} q^{1/2} + \frac{m_1^2 \left(m_0^2 - \tilde{m}_0^2\right)^2 - m_0^2 \left(m_1^2 - \tilde{m}_1^2\right)^2}{\left(m_0^2 - \tilde{m}_0^2\right)^2 \left(m_1^2 - \tilde{m}_1^2\right)} q + \sum_{n=3}^{\infty} c_* q^{n/2} \\
& + \left(-\frac{\left(m_1^2 - \tilde{m}_1^2\right)^{1/4}}{\left(m_0^2 - \tilde{m}_0^2\right)^{3/4}} q^{1/4} - \frac{2m_0^2 \left(m_1^2 - \tilde{m}_1^2\right)^{3/4}}{\left(m_0^2 - \tilde{m}_0^2\right)^{9/4}} q^{3/4} + \sum_{n=2}^{\infty} c_* q^{n/2+1/4} \right) \Delta^{1/2} \\
& + \sum_{m=1}^{\infty} \left(\sum_{n=0}^{\infty} c_* q^{n/2} \right) \Delta^m + \sum_{m=1}^{\infty} \left(\sum_{n=0}^{\infty} c_* q^{1/4-m/2+n/2} \right) \Delta^{m+1/2}, \\
z_3 = & -\frac{\left(m_1^2 - \tilde{m}_1^2\right)^{1/2}}{\left(m_0^2 - \tilde{m}_0^2\right)^{1/2}} q^{1/2} + \frac{m_1^2 \left(m_0^2 - \tilde{m}_0^2\right)^2 - m_0^2 \left(m_1^2 - \tilde{m}_1^2\right)^2}{\left(m_0^2 - \tilde{m}_0^2\right)^2 \left(m_1^2 - \tilde{m}_1^2\right)} q + \sum_{n=3}^{\infty} c_* q^{n/2} \\
& - \left(-\frac{\left(m_1^2 - \tilde{m}_1^2\right)^{1/4}}{\left(m_0^2 - \tilde{m}_0^2\right)^{3/4}} q^{1/4} - \frac{2m_0^2 \left(m_1^2 - \tilde{m}_1^2\right)^{3/4}}{\left(m_0^2 - \tilde{m}_0^2\right)^{9/4}} q^{3/4} + \sum_{n=2}^{\infty} c_* q^{n/2+1/4} \right) \Delta^{1/2} \\
& + \sum_{m=1}^{\infty} \left(\sum_{n=0}^{\infty} c_* q^{n/2} \right) \Delta^m - \sum_{m=1}^{\infty} \left(\sum_{n=0}^{\infty} c_* q^{1/4-m/2+n/2} \right) \Delta^{m+1/2}, \\
z_4 = & -\frac{\tilde{m}_1^2}{m_1^2 - \tilde{m}_1^2} q - \frac{2m_1^2 \tilde{m}_1^2 \left(m_0^2 - \tilde{m}_0^2\right)^{1/2}}{\left(m_1^2 - \tilde{m}_1^2\right)^{5/2}} q^{3/2} + \sum_{n=4}^{\infty} c_* q^{n/2} \\
& - \left(\frac{m_1^2 \tilde{m}_1^2}{\left(m_1^2 - \tilde{m}_1^2\right)^3} q + \frac{4m_1^2 \tilde{m}_1^2 \left(m_0^2 - \tilde{m}_0^2\right)^{1/2} \left(m_1^2 + \tilde{m}_1^2\right)}{\left(m_1^2 - \tilde{m}_1^2\right)^{9/2}} q^{3/2} + \sum_{n=4}^{\infty} c_* q^{n/2} \right) \Delta \\
& + \sum_{m=2}^{\infty} \left(\sum_{n=1}^{\infty} c_* q^n \right) \Delta^m,
\end{aligned}
\tag{B.9}
$$

where we use c_* to denote any coefficients which are functions of only mass parameters. These solution also satisfy the properties of limits (1.)–(4.). In principle the roots can be solved order by order, but the coefficients become increasingly lengthy.

C Stationary points of Treibich-Verdier potential

In this appendix we show a fact that the stationary points of the TV potential are in one-to-one correspondence with the degenerate points of the polynomial $P_4(z)$. From the equation of Darboux form (A.9), we obtain the stationary condition for the potential $\partial_\zeta V(b_i,\zeta)=0$, then using the relation of ζ and z (A.7) we get the condition $Q_6(z)=0$ which is convenient for iterative solving. Let us work with the unshifted parameters that appear in (2.17), we have

$$\begin{aligned}Q_6(z) = {} & \tilde{m}_0^2 z^6 - 2(1+q)\tilde{m}_0^2 z^5 - \left[m_0^2-\tilde{m}_0^2-q\left(m_0^2+4\tilde{m}_0^2+m_1^2-\tilde{m}_1^2\right)-q^2\left(\tilde{m}_0^2-m_1^2\right)\right]z^4 \\ & + 2q\left[\left(m_0^2-\tilde{m}_0^2-m_1^2+\tilde{m}_1^2\right)-q\left(m_0^2+\tilde{m}_0^2-m_1^2-\tilde{m}_1^2\right)\right]z^3 \\ & + q\left[m_1^2-\tilde{m}_1^2-q\left(m_0^2-\tilde{m}_0^2+m_1^2+4\tilde{m}_1^2\right)+q^2\left(m_0^2-\tilde{m}_1^2\right)\right]z^2+2q^2(1+q)\tilde{m}_1^2 z \\ & - q^3\tilde{m}_1^2. \end{aligned} \tag{C.1}$$

Applying the iterative method, we can solve the equation $Q_6(z)=0$ perturbatively. Expanded by q, they appears in pairs,

$$\begin{aligned} z_{*1,2} &= \frac{\tilde{m}_0 \pm m_0}{\tilde{m}_0} \mp \frac{m_0\left[(m_0\pm\tilde{m}_0)^2+m_1^2-\tilde{m}_1^2\right]}{2\tilde{m}_0(m_0\pm\tilde{m}_0)^2}q+\mathcal{O}\left(q^2\right), \\ z_{*3,4} &= \frac{\tilde{m}_1}{\tilde{m}_1\pm m_1}q \pm \frac{m_1\tilde{m}_1\left[m_0^2-\tilde{m}_0^2+(\tilde{m}_1\pm m_1)^2\right]}{2(m_1-\tilde{m}_1)^4}q^2+\mathcal{O}\left(q^3\right), \\ z_{*5,6} &= \pm\frac{\left(m_1^2-\tilde{m}_1^2\right)^{1/2}}{\left(m_0^2-\tilde{m}_0^2\right)^{1/2}}q^{1/2}+\frac{m_1^2\left(m_0^2-\tilde{m}_0^2\right)^2-m_0^2\left(m_1^2-\tilde{m}_1^2\right)^2}{\left(m_0^2-\tilde{m}_0^2\right)^2\left(m_1^2-\tilde{m}_1^2\right)}q\pm\mathcal{O}\left(q^{3/2}\right). \end{aligned} \tag{C.2}$$

For a stationary particle at the point z_{*i}, as it loses the kinetic energy, its energy is the potential energy,

$$h_{*i} = b_0 q\,\mathrm{sn}^2\zeta + b_1 q\frac{\mathrm{cn}^2\zeta}{\mathrm{dn}^2\zeta}+b_2\frac{1}{\mathrm{sn}^2\zeta}+b_3\frac{\mathrm{dn}^2\zeta}{\mathrm{cn}^2\zeta}\Big|_{\zeta=\zeta(z_{*i})}. \tag{C.3}$$

Then from the relation (A.6)(A.10) we can compute the energy Θ_{*i}(or Ξ_{*i}) at the stationary points z_{*i}. The six $\Xi_{*1,2,3,4,5,6}$ associated to $z_{*1,2,3,4,5,6}$ appear in pairs too,

$$\begin{aligned} \Xi_{*1,2} &= -m_1^2-\tilde{m}_1^2+(m_0\pm\tilde{m}_0)^2+\frac{m_0\left[(m_0\pm\tilde{m}_0)^2+m_1^2-\tilde{m}_1^2\right]}{m_0\pm\tilde{m}_0}q+\mathcal{O}\left(q^2\right), \\ \Xi_{*3,4} &= \pm 2m_1\tilde{m}_1+\frac{m_1\left[m_0^2-\tilde{m}_0^2+(m_1\pm\tilde{m}_1)^2\right]}{m_1\pm\tilde{m}_1}q+\mathcal{O}\left(q^2\right), \\ \Xi_{*5,6} &= -m_1^2-\tilde{m}_1^2\pm 2\left(m_0^2-\tilde{m}_0^2\right)^{1/2}\left(m_1^2-\tilde{m}_1^2\right)^{1/2}q^{1/2}+\mathcal{O}\left(m^2 q\right). \end{aligned} \tag{C.4}$$

They are the same values as in (B.4) obtained from solving $\mathcal{D}(P_4)(\Xi)=0$, and they satisfy the relation $P_4(\Xi_{*i},z_{*i})=0$. Recall that the solutions of $P_4(\Xi,z)=0$ are $z_{1,2,3,4}(\Xi)$, they depend on Ξ, when $\Xi=\Xi_{*i}$ two roots among $z_{1,2,3,4}(\Xi)$ collide. There are $6=C_4^2$ ways to collide two roots, exactly to a value z_{*i}. Therefore the six stationary points z_{*i} of the TV potential are in one-to-one correspondence with the six weak coupling asymptotic expansion points for gauge theory. So the situation is similar to the Lamé potential we have studied [33].

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