Erratum: charm quark contribution to $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ at next-to-next-to-leading order

Andrzej J. Buras,^a Martin Gorbahn,^{b,c} Ulrich Haisch^{d,e} and Ulrich Nierste^{c,d}

^a*Physik Department, Technische Universität München,
D-85748 Garching, Germany*

^b*IPPP, Physics Department, University of Durham,
DH1 3LE, Durham, U.K.*

^c*Institut für Theoretische Teilchenphysik, Universität Karlsruhe,
D-76128 Karlsruhe, Germany*

^d*Theoretical Physics Department, Fermilab,
Batavia, IL 60510, U.S.A.*

^e*Institut für Theoretische Physik, Universität Zürich,
CH-8057 Zürich, Switzerland*

E-mail: andrzej.buras@tum.de, mgorbahn@particle.uni-karlsruhe.de,
uhaisch@physik.unizh.ch, nierste@particle.uni-karlsruhe.de

ERRATUM TO: [JHEP11\(2006\)002](#)

ABSTRACT: We correct the treatment of anomalous triangle diagrams occurring in the effective theory in which the heavy top quark is integrated out. To this end we determine the initial conditions and anomalous dimensions of the operator describing the Z -mediated coupling of neutrinos to quarks and further rectify the bilocal renormalization group evolution. Our changes affect the charm-quark contribution $P_c(X)$ at the next-to-leading and next-to-next-to-leading orders, but are numerically negligible as they amount to relative shifts below a permille.

1 Z penguin

Contrary to the statements in section 6.1 the diagrams in figure 9 vanish for $m_t \rightarrow \infty$. As a consequence, the Wilson coefficient of the Chern-Simons operator Q_{CS} also vanishes and the anomalous diagrams will not drop out in a next-to-leading (NLO) and next-to-next-to-leading order (NNLO) calculation of the charm-quark contribution to $K \rightarrow \pi^+ \nu \bar{\nu}$. There is no more reason to include Q_{CS} in the definition of Q_Z in eq. (5.2). Further the

Wilson coefficients of Q_V^q and Q_A^q are different, so that the definition of Q_Z is not helpful for the description of the anomalous effects. To maintain the canonical normalization of the coefficients we define

$$Q_Z^q = -I_q^3 Q_A^q \quad (1.1)$$

recalling that $I_q^3 = \pm 1/2$ denotes the third component of the weak isospin. Since complete isospin doublets do not contribute to the anomalous triangle diagram, it is sufficient to consider diagrams with a bottom-quark loop stemming from matrix elements involving Q_Z^b . A useful operator basis is $(Q_Z^b, Q_Z^{\text{sg}}, Q_Z^{\text{ns}})$ with the flavor-singlet and flavor-non-singlet operators

$$\begin{aligned} Q_Z^{\text{sg}} &= Q_Z^s + Q_Z^d - Q_Z^c - Q_Z^u \\ &= \frac{1}{2} \sum_{\ell=e,\mu,\tau} [\bar{s}\gamma_\mu\gamma_5 s + \bar{d}\gamma_\mu\gamma_5 d + \bar{c}\gamma_\mu\gamma_5 c + \bar{u}\gamma_\mu\gamma_5 u] \bar{\nu}_{\ell L}\gamma^\mu\nu_{\ell L}, \\ Q_Z^{\text{ns}} &= Q_Z^s + Q_Z^d + Q_Z^c + Q_Z^u \\ &= \frac{1}{2} \sum_{\ell=e,\mu,\tau} [\bar{s}\gamma_\mu\gamma_5 s + \bar{d}\gamma_\mu\gamma_5 d - \bar{c}\gamma_\mu\gamma_5 c - \bar{u}\gamma_\mu\gamma_5 u] \bar{\nu}_{\ell L}\gamma^\mu\nu_{\ell L}. \end{aligned} \quad (1.2)$$

The operator Q_Z^{ns} does not participate in the renormalization group (RG) evolution and its coefficient satisfies $C_Z^{\text{ns}}(\mu) = 1$ at all scales $\mu_c \leq \mu \leq \mu_w$. The coefficients C_Z^b and C_Z^{sg} have non-trivial initial conditions at the scale μ_w , at which the top quark is integrated out, and further run for $\mu_b \leq \mu \leq \mu_w$, with their RG evolution determined by the anomalous dimension γ_A calculated from the renormalization factor Z_{AA} defined in eqs. (6.9). In our paper we consider different renormalization schemes for γ_5 : in Q_Z^b we use the 't Hooft-Veltman (HV) scheme, which amounts to the replacement of $\bar{b}\gamma_\mu\gamma_5 b$ by $-(i/3!) \epsilon^{\mu_1\mu_2\mu_3\mu} \bar{b}\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3} b$ [17]. The diagrams for the NLO and NNLO anomalous dimension describe the mixing of Q_Z^b into Q_Z^{sg} and the diagrams of the bilocal mixing discussed in section 3 involve Q_Z^{sg} with open quark lines only, so that the definition of the naive dimensional regularization (NDR) scheme is unambiguous here. We perform these calculation in two schemes, choosing either HV or NDR for the open quark line. At NNLO, the initial condition of the Wilson coefficient C_Z^q is the same in HV and NDR:

$$C_Z^q(\mu_w) = 1 + \left(\frac{\alpha_s(\mu_w)}{4\pi} \right)^2 C_Z^{q(2)}(\mu_w), \quad \text{with} \quad C_Z^{q(2)}(\mu_w) = 2I_q^3 6C_F \ln \frac{m_t^2}{\mu_w^2}, \quad (1.3)$$

and $C_F = 4/3$. It follows that

$$C_Z^{\text{sg}}(\mu_w) = \left(\frac{\alpha_s(\mu_w)}{4\pi} \right)^2 C_Z^{b(2)}(\mu_w). \quad (1.4)$$

The results in eqs. (1.3) and (1.4) assume that the same scheme for γ_5 is used in the full standard model (SM) and the effective five-quark theory. $C_Z^q(\mu_w)$ is calculated from the two-loop diagram obtained by attaching a quark line to the two gluons to the left diagram in figure 9 and a second diagram with reversed orientation of the quark loop. The finite renormalization constant $Z_{AA}^{(2,0)}$ defined in eq. (6.12) is taken into account here. The leading

order (LO) 2×2 anomalous dimension matrix (ADM) $\hat{\gamma}_Z^{(0)}$ for (Q_Z^b, Q_Z^{sg}) vanishes. For the NLO and NNLO ADM we find

$$\hat{\gamma}_Z^{(n)} = \begin{pmatrix} \gamma_A^{(n)} & \gamma_A^{(n)} \\ 4\gamma_A^{(n)} & 4\gamma_A^{(n)} \end{pmatrix}, \quad n = 1, 2, \quad (1.5)$$

in the HV scheme, confirming the anomalous dimensions of the axial current calculated in [17]:

$$\gamma_A^{(1)} = 12C_F, \quad \gamma_A^{(2)} = \frac{284}{3}C_FC_A - \frac{8}{3}C_Ff - 36C_F^2. \quad (1.6)$$

Here $C_A = 3$ and f denotes the number of active quark flavors. A priori the element $(\hat{\gamma}_Z^{(n)})_{12}$ may depend on the choice of the scheme for Q_Z^{sg} : the HV scheme involves a finite one-loop counterterm [17] to remove a spurious anomaly from diagrams with an open quark line. This finite counterterm is absent in the NDR scheme, which instead involves a finite counterterm to the evanescent operator E_A^q defined in eq. (6.10). We find that the analytical results for $(\hat{\gamma}_Z^{(1)})_{12}$ and $(\hat{\gamma}_Z^{(2)})_{12}$ are the same in both schemes.¹

The Wilson coefficients $\vec{C}_Z = (C_Z^b, C_Z^{\text{sg}})^T$ for $\mu_b \leq \mu < \mu_W$ are obtained in the usual way by solving

$$\mu \frac{d}{d\mu} \vec{C}(\mu) = \hat{\gamma}_Z^T \vec{C}(\mu). \quad (1.7)$$

At the scale μ_b the bottom quark is integrated out and we have to match the five-flavor and four-flavor theories. The corresponding hamiltonians are

$$\begin{aligned} \mathcal{H}_{\text{eff}}^{f=5,Z} &= \frac{\pi\alpha}{M_W^2 \sin^2 \theta_W} \left[C_Z^b Q_Z^b + C_Z^{\text{sg}} Q_Z^{\text{sg}} + C_Z^{5,\text{sg}} Q_Z^{\text{sg}} \right], \\ \mathcal{H}_{\text{eff}}^{f=4,Z} &= \frac{\pi\alpha}{M_W^2 \sin^2 \theta_W} \left[C_Z^{\text{sg}} Q_Z^{\text{sg}} + C_Z^{4,\text{sg}} Q_Z^{\text{sg}} \right] \end{aligned}$$

with $C_Z^{f,\text{sg}} = 1$ in both theories. From the two-loop triangle diagrams with bottom quark one finds:

$$\begin{aligned} C_Z^{4,\text{sg}}(\mu_b) &= C_Z^{5,\text{sg}}(\mu_b) + C_Z^b(\mu_b) \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^2 6C_F \ln \frac{m_b^2}{\mu_b^2} \\ &= C_Z^{5,\text{sg}}(\mu_b) + \left(\frac{\alpha_s(\mu_b)}{4\pi} \right)^2 6C_F \ln \frac{m_b^2}{\mu_b^2} + \mathcal{O}(\alpha_s^4). \end{aligned} \quad (1.8)$$

Inserting eqs. (1.3) to (1.5) into the NNLO RG evolution equation [39] entails:

$$\begin{aligned} C_Z^{5,\text{sg}}(\mu_b) &= \frac{\gamma_A^{(1)}}{2\beta_0} \frac{\alpha_s(\mu_W) - \alpha_s(\mu_b)}{4\pi} + \frac{5}{8} \frac{(\gamma_A^{(1)})^2}{\beta_0^2} \left(\frac{\alpha_s(\mu_W) - \alpha_s(\mu_b)}{4\pi} \right)^2 \\ &\quad - \frac{\beta_0 \gamma_A^{(2)} - \beta_1 \gamma_A^{(1)}}{4\beta_0^2} \frac{\alpha_s^2(\mu_W) - \alpha_s^2(\mu_b)}{(4\pi)^2} + \left(\frac{\alpha_s(\mu_W)}{4\pi} \right)^2 C_Z^{b(2)}(\mu_W), \end{aligned} \quad (1.9)$$

$$C_Z^{5,b}(\mu_b) = 1 + C_Z^{5,\text{sg}}(\mu_b).$$

Numerically, we find

$$C_Z^{\text{sg}}(m_b) = -0.011, \quad (1.10)$$

with negligible variation when μ_W is changed between M_W and m_t or if m_b , m_t , or $\alpha_s(M_Z)$ are varied within their experimental errors.

¹The elements in the second row of $\hat{\gamma}_Z^{(n)}$ involve an insertion of Q_Z^{sg} into a closed quark loop and are hence not properly defined in the NDR scheme. This ambiguity is irrelevant for a NNLO analysis of $P_c(X)$.

Figure 1. $\Delta P_c(X)$ as a function of the renormalization scale μ_c . The dashed (solid) line represents the NLO (NNLO) result.

2 Bilocal evolution

In order to determine the charm-quark contribution of $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ we need the bilocal mixings of $(C_Z^c, C_\pm)$ and $(C_Z^b, C_\pm)$ into the coefficient C_ν . The relevant three-loop anomalous dimension

$$\gamma_{\pm, \nu}^{A\Delta(2)} = 6Z_{\pm A, \nu}^{\Delta(3,1)} - 4Z_{AE}^{(2,1)} Z_{\pm E, \nu}^{P(1,0)} - 4Z_{AA}^{\Delta(2,0)} Z_{\pm A, \nu}^{P(1,1)} \quad (2.1)$$

can be inferred from eqs. (6.11), (6.12), (6.19) and (6.20). In the NDR scheme we obtain:

$$\gamma_{\pm, \nu}^{A\Delta(2)} = 12C_F(1 \pm 5C_A). \quad (2.2)$$

We write the Wilson coefficient for the Z -penguin contribution to C_ν as $C_\nu^P + C_\nu^\Delta$. Here C_ν^P is the piece coming from the non-anomalous diagrams obtained by solving the RG equation in (6.4), with the solution quoted in eq. (6.43). C_ν^Δ is the new contribution from the triangle diagrams. The RG equation read

$$\mu \frac{d}{d\mu} [C_\nu^P(\mu) + C_\nu^\Delta(\mu)] = \gamma_\nu [C_\nu^P(\mu) + C_\nu^\Delta(\mu)] + 4 \sum_{\substack{i=\pm \\ j=b, \text{nsg}, \text{sg}}} \gamma_{ij, \nu} C_i(\mu) C_Z^j(\mu). \quad (2.3)$$

From the definition of $Q_{b, \text{nsg}, \text{sg}}$ we find

$$\gamma_{\pm b, \nu} = \frac{1}{2} \gamma_{\pm, \nu}^{A\Delta}, \quad \gamma_{\pm \text{nsg}, \nu} = \gamma_{\pm, \nu}^P = -\frac{1}{2} \gamma_{\pm, \nu}^A, \quad \gamma_{\pm \text{sg}, \nu} = \frac{1}{2} \gamma_{\pm, \nu}^A + \frac{1}{2} \gamma_{\pm, \nu}^{A\Delta}. \quad (2.4)$$

The expressions for $\gamma_{\pm, \nu}^A$, calculated from the non-anomalous diagrams, are listed in eq. (6.8). Subtracting the RGE for $C_\nu^P(\mu)$ in eq. (6.4) from (2.4) we find:

$$\mu \frac{d}{d\mu} C_\nu^\Delta(\mu) = \gamma_\nu C_\nu^\Delta(\mu) + 2 \sum_{i=\pm} \left[\gamma_{i, \nu}^{A\Delta} C_i(\mu) C_Z^b(\mu) + (\gamma_{i, \nu}^{A\Delta} + \gamma_{i, \nu}^A) C_i(\mu) C_Z^{\text{sg}}(\mu) \right]. \quad (2.5)$$

This RG equation splits into two linear differential equations which are easily solved, as discussed for example in [65] for an analogous case.

To NNLO only the one-loop matrix elements involving a double operator insertion $(Q_\pm^q, Q_Z^{\text{sg}})$ gives a non-trivial contribution to the matching at μ_c . We write the contribution to the tree-level matrix element $\langle Q_\nu \rangle^{(0)}$ in the following way

$$\langle Q_\pm^{\text{sg}}(\mu_c) \rangle = \frac{\alpha_s(\mu_c)}{4\pi} r_\pm^{\text{sg}(1)}(\mu_c) \langle Q_\nu \rangle^{(0)} = -\frac{\alpha_s(\mu_c)}{4\pi} r_\pm^{P(1)}(\mu_c) \langle Q_\nu \rangle^{(0)}, \quad (2.6)$$

where $r_{\pm}^{P(1)}(\mu_c)$ codifies the one-loop corrections detailed in eqs. (6.40) and (6.41) and $Q_{\pm}^{\text{sg}}$ is defined analogously to $Q_{\pm}^P$ in eq. (3.1).

We have now collected all ingredients necessary to determine the missing contribution $\Delta P_c(X)$ to the charm-quark contribution in $K^+ \rightarrow \pi^+ \nu \bar{\nu}$, represented by the parameter $P_c(X)$ defined in eq. (2.9) of the paper. For the NLO and NNLO coefficient in the perturbative expansion

$$\Delta P_c(X) = \Delta P_c^{(1)}(X) + \frac{\alpha_s(\mu_c)}{4\pi} \Delta P_c^{(2)}(X), \quad (2.7)$$

we plot the μ_c -dependence in figure 1. The charm-quark corrections due to anomalous triangle graphs lead to shifts in $P_c(X)$ in the ballpark of -0.0001 with the exact value depending on the renormalization scale. Recalling that $P_c(X) = 0.369$ ($P_c(X) = 0.375$) at NLO (NNLO) the corresponding relative shift amounts to -0.05% (-0.01%). One can understand the smallness of the NLO contribution to $\Delta P_c(X)$ by inspecting the logarithmic expansion of our result: the NLO term starts only at $\mathcal{O}(\alpha_s^2 \ln(\mu_W^2/\mu_b^2) \ln(\mu_W^2/\mu_c^2))$, which is of the same order in α_s as the NNLO correction containing only one large logarithm. Expanding $\Delta P_c(X)$ in α_s we find explicitly

$$\Delta P_c(X) = \frac{1}{\lambda^4} \frac{m_c^2}{M_W^2} \left(\frac{\alpha_s}{4\pi} \right)^2 \left\{ \left(\ln \frac{\mu_W^2}{\mu_b^2} - 2 \ln \frac{\mu_W^2}{\mu_c^2} + 1 \right) \ln \frac{\mu_W^2}{\mu_b^2} \right\} + \mathcal{O} \left(\alpha_s^2 \ln^0 \frac{\mu_W^2}{\mu_b^2}, \alpha_s^3 \right), \quad (2.8)$$

where the α_s^2 suppression is accompanied by terms with small numerical coefficients of opposite sign. Notice that the exact result $\Delta P_c(X)$ is smaller by two orders of magnitude than the total NNLO uncertainty in $P_c(X)$ of ± 0.015 . In view of these numbers it is clear that our erroneous treatment of anomalous triangle diagrams has no phenomenological impact.

Acknowledgments

We are grateful to Konstantin Chetyrkin and Mikolaj Misiak for pointing our mistake out to us. We also thank Johann Kühn for bringing the article [66], in which the large logarithm occuring from the anomalous triangle diagrams has been discovered, to our attention. We further thank Mikolaj Misiak and Gerhard Buchalla for a thorough proofreading and comments on this erratum.

References

- [65] J. Brod and M. Gorbahn, ϵ_K at next-to-next-to-leading order: the charm-top-quark contribution, *Phys. Rev. D* **82** (2010) 094026 [[arXiv:1007.0684](#)] [[INSPIRE](#)].
- [66] J.C. Collins, F. Wilczek and A. Zee, Low-energy manifestations of heavy particles: application to the neutral current, *Phys. Rev. D* **18** (1978) 242 [[INSPIRE](#)].