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From rarefied elliptic beta integral to parafermionic star-triangle relation

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ABSTRACT: We consider the rarefied elliptic beta integral in various limiting forms. In particular, we obtain an integral identity for parafermionic hyperbolic gamma functions which describes the star-triangle relation for parafermionic Liouville theory.

KEYWORDS: Conformal Field Models in String Theory, Supersymmetric Gauge Theory, Conformal and W Symmetry

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1 Introduction

Full understanding of the properties of non-rational $2d$ conformal field theories is one of the most important questions in string theory, quantum field theory and mathematical physics. The most important well known examples of the non-rational conformal field theory are Liouville field theory (LFT) and its various generalizations, like supersymmetric extensions, parafermionic extensions, Toda field theory, etc. Already in the seminal works [9, 36], where the three-point function in LFT was constructed, an important role of some special function Υ_b emerged. Studies of the fusion matrix [24] and boundary correlation functions [11, 25] required the use of another related function — the noncompact quantum dilogarithm S_b [10], which is called also the hyperbolic gamma function [29] (we follow the latter terminology). Both of these functions are constructed out of the Barnes double gamma function Γ_b . Study of $N = 1$ supersymmetric LFT showed that description of the three-point functions [22, 27], boundary correlation functions [12] and fusion matrix [7, 14] requires the use of supersymmetric generalizations of these functions: Υ_i , Γ_i and S_i , where $i = 0, 1$. In [4], three-point functions were studied in parafermionic LFT, which is LFT coupled with $\mathbb{Z}_N$ parafermions. It was shown there that three-point functions can be written using parafermionic generalizations of Υ_b : Υ_k , where $k = 0, \dots, N - 1$.

Analysis of the bootstrap relation and boundary three point function in the LFT carried out in [30] demonstrated that some fundamental relations between fusion matrix and three-point functions established in rational conformal field theories hold also in LFT. On the other hand, it was shown in [34] that the expressions in the above mentioned works indeed satisfy these relations due to a particular star-triangle relation [17, 35] for the hyperbolic gamma functions, which corresponds to the Faddeev-Volkov model [2] (a more complicated star-triangle relation leading to a generalization of the latter model was considered in [32]). Similar analysis of the Neveu-Schwarz sector of $N = 1$ supersymmetric LFT [23] showed

that the corresponding relations are implied by the generalization of considerations of [17] to supersymmetric hyperbolic gamma functions found in [15].

This state of affairs inspires us to think that attempts to find expressions for fusion matrix and boundary correlation functions in the parafermionic LFT inevitably will require to write parafermionic generalizations of Γ_b and S_b functions as well. In fact a parafermionic generalization of Γ_b was introduced in [23], where also some properties of this function were derived. It is also natural to assume that parafermionic generalization of S_b function should possess the star-triangle relation as well.

In this paper we would like to connect mentioned topics with the subject which developed over the last decade in higher dimensional superconformal field theories — the theory of superconformal indices [28] described in terms of the elliptic hypergeometric integrals [31]. So, the standard elliptic gamma function coincides with the superconformal index of chiral superfield of theories on $S^3 \times S^1$ space-time background. Consideration of superconformal indices of gauge theory on lens space [3] leads to a particular combination of elliptic gamma functions with different bases. It was called in [33] the rarefied elliptic gamma function due to its special product type representation, using which we introduce parafermionic hyperbolic gamma function as a particular limit. It is built from S_b functions along the same rules by which Υ_i function in [4] is built from Υ_b functions.

We show that such parafermionic hyperbolic gamma functions are related to two computable rarefied hyperbolic beta integrals, corresponding to two values of a parameter $\epsilon = 0, 1$. The one corresponding to $\epsilon = 0$ was found earlier in [13], and the second one $\epsilon = 1$ is new. Degenerating these hyperbolic beta integrals we obtain the star-triangle relation for the parafermionic LFT. For the supersymmetric case we compared obtained results with those derived earlier in [15] and found that the star-triangle relation in [15] in some cases is missing an overall sign. Thus, it appears that $4d$ superconformal indices contain a lot of important information about $2d$ systems — the $2d$ conformal field theories discussed above and integrable $2d$ lattice spin systems, for which they describe partition functions [32]. Moreover, it is known that the same hyperbolic limit of these $4d$ indices describes partition functions of $3d$ supersymmetric models on the squashed sphere S_b^3 [28]. The present work can be considered as a complement to [8], where the transition from $4d$ theories to $3d$ ones was reached by degenerating elliptic hypergeometric integrals to hyperbolic integrals, — we add to such a connection a relation to the parafermionic LFT.

We would like to add that, in view of the AGT relation between para-Liouville theory and superconformal gauge theories on $\mathbb{C}^2/\mathbb{Z}_r$, see e.g. [1, 5, 21] and references therein, it could be expected that parafermionic hyperbolic gamma functions should arise from the rarefied (or lens) elliptic gamma function.

The paper is organized in the following way. In section 2 we review the necessary formulas on elliptic gamma functions and the rarefied elliptic beta integral. In section 3 we consider parafermionic hyperbolic gamma functions. In section 4 we derived a hyperbolic beta integral and star-triangle relation for parafermionic hyperbolic gamma functions. In section 5 we consider in detail the star-triangle relation for supersymmetric case, compare it with a version of this formula obtained earlier in [15] and indicate a sign difference in them.

2 A rarefied elliptic beta integral

The standard elliptic gamma function $\Gamma(z; p, q)$ can be defined as an infinite product:

$$\Gamma(z; p, q) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - z p^j q^k}, \quad |p|, |q| < 1, \quad z \in \mathbb{C}^*. \quad (2.1)$$

The lens space elliptic gamma function is defined as a product of two standard elliptic gamma functions with different bases [3].

$$\begin{aligned} \gamma_e(z, m; p, q) &= \Gamma(z p^m; p^r, pq) \Gamma(z q^{r-m}; q^r, pq) \\ &= \prod_{j,k=0}^{\infty} \frac{1 - z^{-1} p^{-m} (pq)^{j+1} p^{r(k+1)}}{1 - z p^m (pq)^j p^{rk}} \frac{1 - z^{-1} q^m (pq)^{j+1} q^{rk}}{1 - z q^{r-m} (pq)^j q^{rk}}, \quad m \in \mathbb{Z}. \end{aligned} \quad (2.2)$$

As shown in [33], the function (2.2) can be written as a special product of the standard elliptic gamma functions with bases p^r and q^r . For $0 \leq m \leq r$ it has the form:

$$\gamma_e(z, m; p, q) = \prod_{k=0}^{m-1} \Gamma(q^{r-m} z (pq)^k; p^r, q^r) \prod_{k=0}^{r-m-1} \Gamma(p^m z (pq)^k; p^r, q^r), \quad (2.3)$$

for $m < 0$

$$\gamma_e(z, m; p, q) = \frac{\prod_{k=0}^{r-m-1} \Gamma(p^m z (pq)^k; p^r, q^r)}{\prod_{k=1}^{-m} \Gamma(q^{r-m} z (pq)^{-k}; p^r, q^r)}, \quad (2.4)$$

and for $m > r$

$$\gamma_e(z, m; p, q) = \frac{\prod_{k=0}^{m-1} \Gamma(q^{r-m} z (pq)^k; p^r, q^r)}{\prod_{k=1}^{m-r} \Gamma(p^m z (pq)^{-k}; p^r, q^r)}. \quad (2.5)$$

A convenient normalization of this function was introduced in [33]

$$\Gamma^{(r)}(z, m; p, q) = (-z)^{\frac{m(m-1)}{2}} p^{\frac{m(m-1)(m-2)}{6}} q^{-\frac{m(m-1)(m+1)}{6}} \gamma_e(z, m; p, q), \quad (2.6)$$

which yields $\Gamma^{(1)}(z, m; p, q) = \Gamma(z; p, q)$. It is this object that was called the rarefied elliptic gamma function.

Let us define a particular combination of such functions

$$\Delta_e^{(r)}(z, m; t_a, n_a | p, q) = \frac{\prod_{a=1}^6 \Gamma^{(r)}(t_a z, n_a + m + \epsilon; p, q) \Gamma^{(r)}(t_a z^{-1}, n_a - m; p, q)}{\Gamma^{(r)}(z^2, 2m + \epsilon; p, q) \Gamma^{(r)}(z^{-2}, -(2m + \epsilon); p, q)}, \quad (2.7)$$

$$\Psi_e^{(r)}(z, m; t_a, n_a | p, q) = \frac{\prod_{a=1}^6 \gamma_e(t_a z, n_a + m + \epsilon; p, q) \gamma_e(t_a z^{-1}, n_a - m; p, q)}{\gamma_e(z^2, 2m + \epsilon; p, q) \gamma_e(z^{-2}, -(2m + \epsilon); p, q)}. \quad (2.8)$$

It is shown in [33] that if parameters t_a , n_a satisfy the constraints $|t_a| < 1$ and the balancing condition

$$\prod_{a=1}^6 t_a = pq, \quad \sum_{a=1}^6 n_a = -3\epsilon, \quad \epsilon = 0, 1, \quad (2.9)$$

then one has the following integral identity

$$\kappa^{(r)} \sum_{m=0}^{r-1} \int_{\mathbb{T}} \Delta_e^{(r)}(z, m; t_a, n_a | p, q) \frac{dz}{z} = \prod_{1 \leq a < b \leq 6} \Gamma^{(r)}(t_a t_b, n_a + n_b + \epsilon; p, q), \quad (2.10)$$

where $\mathbb{T}$ is the unit circle of positive orientation and

$$\kappa^{(r)} = \frac{(p^r; p^r)_\infty (q^r; q^r)_\infty}{4\pi i}. \quad (2.11)$$

Equivalently, it can be written as

$$\begin{aligned} \kappa^{(r)} \sum_{m=0}^{r-1} \left(\frac{q}{p}\right)^{m^2+m\epsilon} \int_{\mathbb{T}} z^{-\epsilon-2m} \Psi_e^{(r)}(z, m; t_a, n_a | p, q) \frac{dz}{z} \\ = \left(\frac{q}{p}\right)^{\frac{1}{2} \sum_{a=1}^6 n_a^2} p^{\epsilon/2} q^{-3\epsilon/2} (-1)^\epsilon \prod_{a=1}^6 t_a^{-n_a} \prod_{1 \leq a < b \leq 6} \gamma_e(t_a t_b, n_a + n_b + \epsilon; p, q). \end{aligned} \quad (2.12)$$

For $\epsilon = 0$ this relation was established first by Kels [18] using a different normalization of the γ_e -functions. Note that values of the integer parameter ϵ were reduced to 0 and 1 by admissible shifts of n_a . For $r = 1$ one obtains the standard elliptic beta integral [31].

3 Parafermionic hyperbolic gamma function

The function $\Gamma(z; p, q)$ has the following limiting behaviour [29]:

$$\Gamma(e^{-2\pi v y}; e^{-2\pi v \omega_1}, e^{-2\pi v \omega_2}) \underset{v \rightarrow 0}{=} e^{-\pi(2y - \omega_1 - \omega_2)/12v\omega_1\omega_2} \gamma^{(2)}(y; \omega_1, \omega_2), \quad (3.1)$$

where $\gamma^{(2)}(y; \omega_1, \omega_2)$ is the hyperbolic gamma function. The parameter v approaches to 0 along the positive real axis and parameters ω_1 and ω_2 have positive real parts: $\text{Re}(\omega_1) > 0$ and $\text{Re}(\omega_2) > 0$.

The function $\gamma^{(2)}(y; \omega_1, \omega_2)$ has the integral representation

$$\gamma^{(2)}(y; \omega_1, \omega_2) = \exp \left(- \int_0^\infty \left(\frac{\sinh(2y - \omega_1 - \omega_2)x}{2 \sinh(\omega_1 x) \sinh(\omega_2 x)} - \frac{2y - \omega_1 - \omega_2}{2\omega_1\omega_2 x} \right) \frac{dx}{x} \right), \quad (3.2)$$

and obeys the equations:

$$\frac{\gamma^{(2)}(y + \omega_1; \omega_1, \omega_2)}{\gamma^{(2)}(y; \omega_1, \omega_2)} = 2 \sin \frac{\pi y}{\omega_2}, \quad \frac{\gamma^{(2)}(y + \omega_2; \omega_1, \omega_2)}{\gamma^{(2)}(y; \omega_1, \omega_2)} = 2 \sin \frac{\pi y}{\omega_1}. \quad (3.3)$$

Setting

$$z = e^{-2\pi v y/r}, \quad p = e^{-2\pi v \omega_1/r}, \quad q = e^{-2\pi v \omega_2/r}, \quad (3.4)$$

one can write

$$\begin{aligned} q^{r-m} z (pq)^k &= e^{-2\pi v [\frac{y}{r} + \omega_2(1 - \frac{m}{r}) + (\omega_1 + \omega_2)\frac{k}{r}]}, \\ p^m z (pq)^k &= e^{-2\pi v [\frac{y}{r} + \frac{m}{r}\omega_1 + (\omega_1 + \omega_2)\frac{k}{r}]}. \end{aligned}$$

Now one can show that:

$$\gamma_e \left(e^{-\frac{2\pi v y}{r}}, m; e^{-\frac{2\pi v \omega_1}{r}}, e^{-\frac{2\pi v \omega_2}{r}} \right) \underset{v \rightarrow 0}{=} e^{-\pi(2y - \omega_1 - \omega_2)/12v\omega_1\omega_2} \Lambda(y, m; \omega_1, \omega_2), \quad (3.5)$$

where the function $\Lambda(y, m; \omega_1, \omega_2)$ is defined as follows. For $0 \leq m \leq r$ one has

$$\begin{aligned} \Lambda(y, m; \omega_1, \omega_2) &= \prod_{k=0}^{m-1} \gamma^{(2)} \left(\frac{y}{r} + \omega_2 \left(1 - \frac{m}{r} \right) + (\omega_1 + \omega_2) \frac{k}{r}; \omega_1, \omega_2 \right) \\ &\quad \times \prod_{k=0}^{r-m-1} \gamma^{(2)} \left(\frac{y}{r} + \frac{m}{r} \omega_1 + (\omega_1 + \omega_2) \frac{k}{r}; \omega_1, \omega_2 \right), \end{aligned} \quad (3.6)$$

for $m < 0$

$$\Lambda(y, m; \omega_1, \omega_2) = \frac{\prod_{k=0}^{r-m-1} \gamma^{(2)} \left(\frac{y}{r} + \frac{m}{r} \omega_1 + (\omega_1 + \omega_2) \frac{k}{r}; \omega_1, \omega_2 \right)}{\prod_{k=1}^{-m} \gamma^{(2)} \left(\frac{y}{r} + \omega_2 \left(1 - \frac{m}{r} \right) - (\omega_1 + \omega_2) \frac{k}{r}; \omega_1, \omega_2 \right)}, \quad (3.7)$$

and for $m > r$

$$\Lambda(y, m; \omega_1, \omega_2) = \frac{\prod_{k=0}^{m-1} \gamma^{(2)} \left(\frac{y}{r} + \omega_2 \left(1 - \frac{m}{r} \right) + (\omega_1 + \omega_2) \frac{k}{r}; \omega_1, \omega_2 \right)}{\prod_{k=1}^{m-r} \gamma^{(2)} \left(\frac{y}{r} + \frac{m}{r} \omega_1 - (\omega_1 + \omega_2) \frac{k}{r}; \omega_1, \omega_2 \right)}. \quad (3.8)$$

In fact it is enough to consider functions $\Lambda(y, m; \omega_1, \omega_2)$ only for $0 \leq m \leq r$. Recall the quasiperiodicity property [33]:

$$\frac{\gamma_e(z, m + kr; p, q)}{\gamma_e(z, m; p, q)} = \left(-\frac{\sqrt{pq}}{z} \right)^{mk+r\frac{k(k-1)}{2}} \left(\frac{q}{p} \right)^{k\left(\frac{m^2}{2} + mr\frac{k-1}{2} + r^2\frac{(k-1)(2k-1)}{12}\right)}, \quad k \in \mathbb{Z}. \quad (3.9)$$

In the limit (3.5) it implies

$$\Lambda(y, m + kr; \omega_1, \omega_2) = (-1)^{mk+r\frac{k(k-1)}{2}} \Lambda(y, m; \omega_1, \omega_2) \quad (3.10)$$

Let us study the function $\Lambda(y, m; \omega_1, \omega_2)$ for the particular choice $r = 2$. Eq. (3.10) implies that in this case we have only two functions corresponding to $m = 0, 1$. For $m = 0$ we have:

$$\Lambda(y, 0; \omega_1, \omega_2) = \gamma^{(2)} \left(\frac{y}{2}; \omega_1, \omega_2 \right) \gamma^{(2)} \left(\frac{y}{2} + \frac{\omega_1 + \omega_2}{2}; \omega_1, \omega_2 \right), \quad (3.11)$$

and for $m = 1$

$$\Lambda(y, 1; \omega_1, \omega_2) = \gamma^{(2)} \left(\frac{y}{2} + \frac{\omega_2}{2}; \omega_1, \omega_2 \right) \gamma^{(2)} \left(\frac{y}{2} + \frac{\omega_1}{2}; \omega_1, \omega_2 \right). \quad (3.12)$$

Setting $\omega_2 = b$ and $\omega_1 = \frac{1}{b}$ and $Q = b + \frac{1}{b}$ and using the notation accepted in conformal field theory literature

$$\gamma^{(2)}(z; b, 1/b) = S_b(z), \quad (3.13)$$

we obtain that

$$\Lambda(y, 0; b^{-1}, b) = S_b \left(\frac{y}{2} \right) S_b \left(\frac{y}{2} + \frac{Q}{2} \right) \equiv S_{\text{NS}}(y) \equiv S_1(y), \quad (3.14)$$

$$\Lambda(y, 1; b^{-1}, b) = S_b \left(\frac{y}{2} + \frac{b}{2} \right) S_b \left(\frac{y}{2} + \frac{b^{-1}}{2} \right) \equiv S_{\text{R}}(y) \equiv S_0(y). \quad (3.15)$$

The functions $S_{\text{NS}}(y)$ and $S_{\text{R}}(y)$ appear in numerous aspects of $N = 1$ supersymmetric Liouville conformal field theory. Subscripts NS and R refer to the Neveu-Schwarz and Ramond sectors respectively. First defined in [12] for calculation of the boundary two-point functions, they played important role in writing down fusion and braiding matrices of conformal blocks [14]. It was suggested in [15] to denote them as $S_1(y)$ and $S_0(y)$, respectively, to write in compact way the corresponding star-triangle relation.

Consider now the functions $\Lambda(y; m; \omega_1, \omega_2)$ for arbitrary r :

$$\Lambda(y, m; b^{-1}, b) = \prod_{k=0}^{m-1} S_b \left(\frac{y}{r} + b \left(1 - \frac{m}{r} \right) + Q \frac{k}{r} \right) \prod_{k=0}^{r-m-1} S_b \left(\frac{y}{r} + \frac{m}{r} b^{-1} + Q \frac{k}{r} \right). \quad (3.16)$$

Compare them with the $\Upsilon_m^{(r)}(y)$ functions defined in [4] for the purpose of calculation of three-point functions in the parafermionic Liouville field theory:

$$\Upsilon_m^{(r)}(y) = \prod_{j=1}^{r-m} \Upsilon_b \left(\frac{y + mb^{-1} + (j-1)Q}{r} \right) \prod_{j=r-m+1}^r \Upsilon_b \left(\frac{y + (m-r)b^{-1} + (j-1)Q}{r} \right). \quad (3.17)$$

Let us replace Υ_b by S_b in expression (3.17). Then the substitution $j = k + 1$ in its first product yields precisely the second product in (3.16). Similarly, the substitution $j = k + r - m + 1$ converts its second product to the first one in (3.16) because $b^{-1} - Q = -b$. So, we have intriguing exact structural correspondence between the functions (3.17) and (3.16).

For this reason we call $\Lambda(y; m; \omega_1, \omega_2)$ the parafermionic hyperbolic gamma function. It should play the same role in the construction of parafermionic fusion matrices as $\Upsilon_m^{(r)}(y)$ serves the correlation functions. Applying the limit (3.5) to expression (2.2) one can derive another expression for it

$$\Lambda(y, m; \omega_1, \omega_2) = \gamma^{(2)} \left(\frac{y + m\omega_1}{r}; \omega_1, \frac{\omega_1 + \omega_2}{r} \right) \gamma^{(2)} \left(\frac{y + (r-m)\omega_2}{r}; \omega_2, \frac{\omega_1 + \omega_2}{r} \right), \quad (3.18)$$

which was obtained in [13, 16, 20]. Using equations (3.3) one can easily show that (3.18) satisfies (3.10).

4 Integral identities for parafermionic hyperbolic gamma functions

Now we apply the limit (3.5) to the rarefied elliptic beta integral evaluation (2.12). For that we set additionally to (3.4) the parameterization:

$$t_a = e^{-\frac{2\pi v s_a}{r}}, \quad \sum_{a=1}^6 s_a = \omega_1 + \omega_2 \quad (4.1)$$

and take the limit $v \rightarrow 0^+$. As a result, we obtain the following identity representing a rarefied hyperbolic beta integral evaluation

$$\begin{aligned} & \int_{-i\infty}^{i\infty} \sum_{m=0}^{r-1} \frac{\prod_{a=1}^6 \Lambda(y + s_a, n_a + m + \epsilon; \omega_1, \omega_2) \Lambda(-y + s_a, n_a - m; \omega_1, \omega_2)}{\Lambda(2y, 2m + \epsilon; \omega_1, \omega_2) \Lambda(-2y, -(2m + \epsilon); \omega_1, \omega_2)} \frac{dy}{i\sqrt{\omega_1 \omega_2}} \\ &= 2r(-1)^\epsilon \prod_{1 \leq a < b \leq 6} \Lambda(s_a + s_b, n_a + n_b + \epsilon; \omega_1, \omega_2). \end{aligned} \quad (4.2)$$

For $\epsilon = 0$ this evaluation was derived in [13] and for $\epsilon = 1$ it is a new result. Reductions of the ordinary $r = 1$ elliptic hypergeometric integrals to the hyperbolic level are systematically considered in [6]. They are based on a rigorous justification for such transitions established in [26].

To derive from (4.2) the parafermionic star-triangle relation we should elaborate asymptotic properties of the $\Lambda(y, m; \omega_1, \omega_2)$ function.

The function $\gamma^{(2)}(y; \omega_1, \omega_2)$ has the following asymptotics [19]:

$$\lim_{y \rightarrow \infty} e^{\frac{i\pi}{2} B_{2,2}(y; \omega_1, \omega_2)} \gamma^{(2)}(y; \omega_1, \omega_2) = 1, \quad \text{for } \arg \omega_1 < \arg y < \arg \omega_2 + \pi, \quad (4.3)$$

$$\lim_{y \rightarrow \infty} e^{-\frac{i\pi}{2} B_{2,2}(y; \omega_1, \omega_2)} \gamma^{(2)}(y; \omega_1, \omega_2) = 1, \quad \text{for } \arg \omega_1 - \pi < \arg y < \arg \omega_2, \quad (4.4)$$

where $B_{2,2}(y; \omega_1, \omega_2)$ is the second order Bernoulli polynomial:

$$B_{2,2}(y; \omega_1, \omega_2) = \frac{y^2}{\omega_1 \omega_2} - \frac{y}{\omega_1} - \frac{y}{\omega_2} + \frac{1}{6} \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} \right) + \frac{1}{2}. \quad (4.5)$$

Because of (3.18) this implies that $\Lambda(y, m, \omega_1, \omega_2)$ function has similar asymptotics with $B_{2,2}(y; \omega_1, \omega_2)$ replaced by:

$$\begin{aligned} & B_{2,2} \left(\frac{y + m\omega_1}{r}; \omega_1, \frac{\omega_1 + \omega_2}{r} \right) + B_{2,2} \left(\frac{y + (r-m)\omega_2}{r}; \omega_2, \frac{\omega_1 + \omega_2}{r} \right) \\ &= \frac{y^2}{\omega_1 \omega_2 r} - \frac{y}{r\omega_1} - \frac{y}{r\omega_2} + \frac{1}{6r} \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} \right) + \frac{m^2}{r} - m + \frac{r}{6} + \frac{1}{3r} \\ &= \frac{1}{r} B_{2,2}(y; \omega_1, \omega_2) + \frac{m^2}{r} - m + \frac{r}{6} - \frac{1}{6r}. \end{aligned} \quad (4.6)$$

Let us reparameterize s_a in (4.2) in the following asymmetric way

$$s_a = f_a + i\mu, \quad a = 1, 2, 3, \quad s_{a+3} = g_a - i\mu, \quad a = 1, 2, 3, \quad (4.7)$$

which preserves the balancing condition. We denote also

$$n_{a+3} \equiv l_a, \quad a = 1, 2, 3. \quad (4.8)$$

So, we have

$$\sum_{a=1}^3 (f_a + g_a) = \omega_1 + \omega_2 \quad (4.9)$$

and

$$\sum_{a=1}^3 (n_a + l_a) = -3\epsilon. \quad (4.10)$$

Now we shift the integration variable $y \rightarrow y - i\mu$ and take the limit $\mu \rightarrow +\infty$ using the asymptotics of $\Lambda(y, m, \omega_1, \omega_2)$. Since the integrand is an even function (in fact the parity

transformation reshuffles the terms keeping the sum intact), one can write:

$$\begin{aligned}
 & 2 \int_0^{i\infty} \sum_{m=0}^{r-1} \left[\frac{\prod_{a=1}^3 \Lambda(y + f_a + i\mu, n_a + m + \epsilon; \omega_1, \omega_2) \Lambda(y + g_a - i\mu, l_a + m + \epsilon; \omega_1, \omega_2)}{\Lambda(2y, 2m + \epsilon; \omega_1, \omega_2) \Lambda(-2y, -(2m + \epsilon); \omega_1, \omega_2)} \right. \\
 & \quad \left. \times \prod_{a=1}^3 \Lambda(-y + f_a + i\mu, n_a - m; \omega_1, \omega_2) \Lambda(-y + g_a - i\mu, l_a - m; \omega_1, \omega_2) \right] \frac{dy}{i\sqrt{\omega_1 \omega_2}} \\
 & = 2 \int_{-i\mu}^{i\infty} \sum_{m=0}^{r-1} \prod_{a=1}^3 \Lambda(y + f_a, n_a + m + \epsilon; \omega_1, \omega_2) \Lambda(-y + g_a, l_a - m; \omega_1, \omega_2) e^{\frac{i\pi}{2} \sigma_1} \frac{dy}{i\sqrt{\omega_1 \omega_2}},
 \end{aligned} \tag{4.11}$$

where in the limit $\mu \rightarrow \infty$

$$\begin{aligned}
 \sigma_1 = & \frac{1}{r} \sum_{a=1}^3 [B_{2,2}(y + g_a - 2i\mu) - B_{2,2}(-y + f_a + 2i\mu)] \\
 & - \frac{1}{r} B_{2,2}(2y - 2i\mu) + \frac{1}{r} B_{2,2}(-2y + 2i\mu) \\
 & + \sum_{a=1}^3 \left[\frac{1}{r} (l_a + m + \epsilon)^2 - (l_a + m + \epsilon) - \frac{1}{r} (n_a - m)^2 + (n_a - m) \right] + 4m + 2\epsilon.
 \end{aligned} \tag{4.12}$$

On the right-hand side we have

$$2(-1)^\epsilon r \prod_{a,b=1}^3 \Lambda(f_a + g_b, n_a + l_b + \epsilon; \omega_1, \omega_2) e^{\frac{i\pi}{2} \sigma_2}, \tag{4.13}$$

where

$$\begin{aligned}
 \sigma_2 = & \frac{1}{r} \sum_{1 \leq a < b \leq 3} [B_{2,2}(g_a + g_b - 2i\mu) - B_{2,2}(f_a + f_b + 2i\mu)] \\
 & + \sum_{1 \leq a < b \leq 3} \left[\frac{1}{r} (l_a + l_b + \epsilon)^2 - (l_a + l_b + \epsilon) - \frac{1}{r} (n_a + n_b + \epsilon)^2 + (n_a + n_b + \epsilon) \right].
 \end{aligned} \tag{4.14}$$

Similar to the considerations of [6, 32] for $r = 1$ case, it can be checked that all $B_{2,2}$ -terms appearing in (4.12) and (4.14) cancel each other. Taking care about the rest yields:

$$\begin{aligned}
 & \int_{-i\infty}^{i\infty} \sum_{m=0}^{r-1} (-1)^m \prod_{a=1}^3 \Lambda(y + f_a, n_a + m + \epsilon; \omega_1, \omega_2) \Lambda(-y + g_a, l_a - m; \omega_1, \omega_2) \frac{dy}{i\sqrt{\omega_1 \omega_2}} \\
 & = (-1)^{\epsilon + n_1 + n_2 + n_3} r \prod_{a,b=1}^3 \Lambda(f_a + g_b, n_a + l_b + \epsilon; \omega_1, \omega_2).
 \end{aligned} \tag{4.15}$$

This is the desired star-triangle relation for the parafermionic hyperbolic gamma functions. Note that using (3.10) we can always bring all the functions to the basic domain $0 \leq m \leq r$. For $r = 1$ one gets the star-triangle relation for the Faddeev-Volkov model [2, 35].

5 Supersymmetric Liouville model case

In this section we study in detail relation (4.15) for supersymmetric hyperbolic gamma functions, which correspond to the choice $r = 2$. So, we set in (4.15) $r = 2$ and $\omega_1 = 1/b$, $\omega_2 = b$. Also we represent integer variables n_a and l_a in the form

$$\begin{aligned} n_a &= 2k_a + \nu_a, & \nu_a &= 0, 1, & a &= 1, 2, 3, \\ l_a &= 2h_a + \mu_a, & \mu_a &= 0, 1, & a &= 1, 2, 3, \end{aligned} \quad (5.1)$$

for some integers k_a and h_a .

Consider first the case $\epsilon = 1$. Using (3.14), (3.15) and (3.10), we can write:

$$\Lambda(y + f_a, n_a + m + 1; b^{-1}, b) = (-1)^{k_a(\nu_a + m + 1)} (-1)^{\nu_a m} S_{\nu_a + m}(y + f_a), \quad (5.2)$$

$$\Lambda(-y + g_a, l_a - m; b^{-1}, b) = (-1)^{h_a(\mu_a - m)} (-1)^{(\mu_a + 1)m} S_{\mu_a + m + 1}(-y + g_a), \quad (5.3)$$

$$\Lambda(f_a + g_b, n_a + l_b + 1; b^{-1}, b) = (-1)^{(k_a + h_b)(\nu_a + \mu_b + 1)} (-1)^{\nu_a \mu_b} S_{\nu_a + \mu_b}(f_a + g_b). \quad (5.4)$$

The subscript a in $S_a(y)$ is defined mod 2: $S_{a+2k}(y) \equiv S_a(y)$.

Inserting (5.2)–(5.4) in (4.15) we obtain

$$\begin{aligned} \sum_{m=0,1} (-1)^{m(1+\sum_a(\nu_a+\mu_a))/2} \int \frac{dx}{i} \prod_{a=1}^3 S_{m+\nu_a}(x + f_a) S_{1+m+\mu_a}(-x + g_a) \\ = 2(-1)^{(\sum \mu_a)(1+\sum_a(\nu_a+\mu_a))/2} \prod_{a,b=1}^3 S_{\nu_a+\mu_b}(f_a + g_b), \end{aligned} \quad (5.5)$$

$$\sum_a (\nu_a + \mu_a) = 1 \pmod{2}, \quad (5.6)$$

and

$$\sum_a (f_a + g_a) = Q. \quad (5.7)$$

Consider now the case $\epsilon = 0$. Using (3.14), (3.15) and (3.10) we can write:

$$\Lambda(y + f_a, n_a + m; b^{-1}, b) = (-1)^{k_a(\nu_a + m)} S_{\nu_a + m + 1}(y + f_a), \quad (5.8)$$

$$\Lambda(-y + g_a, l_a - m; b^{-1}, b) = (-1)^{h_a(\mu_a - m)} (-1)^{(\mu_a + 1)m} S_{\mu_a + m + 1}(-y + g_a), \quad (5.9)$$

$$\Lambda(f_a + g_b, n_a + l_b; b^{-1}, b) = (-1)^{(k_a + h_b)(\nu_a + \mu_b)} S_{\nu_a + \mu_b + 1}(f_a + g_b). \quad (5.10)$$

Inserting (5.8)–(5.10) in (4.15) we obtain

$$\begin{aligned} \sum_{m=0,1} (-1)^{m(\sum_a(\mu_a - \nu_a))/2} \int \frac{dx}{i} \prod_{a=1}^3 S_{m+\nu_a+1}(x + f_a) S_{1+m+\mu_a}(-x + g_a) \\ = 2(-1)^{(\sum \mu_a)(\sum_a(\mu_a - \nu_a))/2} \prod_{a,b=1}^3 S_{\nu_a+\mu_b+1}(f_a + g_b), \end{aligned} \quad (5.11)$$

$$\sum_a (\nu_a + \mu_a) = 0 \pmod{2}, \quad (5.12)$$

and

$$\sum_a (f_a + g_a) = Q. \quad (5.13)$$

It is obvious that (5.5), (5.6) and (5.11), (5.12) are related by the transformation $\nu_a \rightarrow 1 - \nu_a$, $a = 1, 2, 3$, i.e. we have only one independent relation.

Comparing (5.5) with the star-triangle relation found in [15], we see that they coincide in all aspects besides of the overall sign in the right-hand side $(-1)^{(\sum \mu_a)(1 + \sum_a (\nu_a + \mu_a))/2}$ present in our formula. We suggest the following independent check of the presence of this multiplier in a particular case, when it is equal to -1 . Such a situation takes place only when both $\sum \mu_a$ and $\frac{(1 + \sum_a (\nu_a + \mu_a))}{2}$ are odd. The following choice of the parameters obviously satisfies both conditions:

$$\nu_1 = 0, \quad \nu_2 = 0, \quad \nu_3 = 0 \quad (5.14)$$

and

$$\mu_1 = 1, \quad \mu_2 = 0, \quad \mu_3 = 0. \quad (5.15)$$

Substituting these values in (5.5) we obtain:

$$\begin{aligned} & \int \frac{dx}{i} [S_0(x + f_1)S_0(x + f_2)S_0(x + f_3)S_0(-x + g_1)S_1(-x + g_2)S_1(-x + g_3) \\ & \quad - S_1(x + f_1)S_1(x + f_2)S_1(x + f_3)S_1(-x + g_1)S_0(-x + g_2)S_0(-x + g_3)] \\ & = -2S_1(f_1 + g_1)S_0(f_1 + g_2)S_0(f_1 + g_3) \\ & \quad \times S_1(f_2 + g_1)S_0(f_2 + g_2)S_0(f_2 + g_3)S_1(f_3 + g_1)S_0(f_3 + g_2)S_0(f_3 + g_3). \end{aligned} \quad (5.16)$$

Let us study this integral directly in the limit $f_1 + g_1 \rightarrow 0$ and compare it with the suggested right-hand side expression.

As a warm-up exercise consider at the beginning this question for the “bosonic” star-triangle identity

$$\int \frac{dx}{i} \prod_{j=1}^3 S_b(x + f_j)S_b(-x + g_j) = \prod_{j,k=1}^3 S_b(f_j + g_k). \quad (5.17)$$

Recall that the function $S_b(x)$ is meromorphic with poles at $x = -nb - mb^{-1}$, and zeros at $x = Q + nb + mb^{-1}$, where n and m are non-negative integers. Around zero $x = 0$ the $S_b(x)$ function has the behavior:

$$\lim_{x \rightarrow 0} x S_b(x) = \frac{1}{2\pi}. \quad (5.18)$$

Take the limit $f_1 + g_1 \rightarrow 0$ in a way that $-f_1$ and g_1 approach to a point A of imaginary axis ($A \in i\mathbb{R}$) from different sides. Without loss of generality we can assume that $-f_1$ moves to this point from the left side and g_1 comes from the right side. This results in the pinching of the integration contour (the imaginary axis) by two poles. Consider the left-hand side integral as a function of parameters f_i and g_i . Let us show that pinching of the contour results in the pole singularity of this function $1/(f_1 + g_1)$ and compute its leading

asymptotics. For that we deform the integration contour and force it to cross over the point $x = -f_1$ and pick up the corresponding pole residue determined by the integral over small circle around $-f_1$. The integral over deformed contour is finite and the singularity can arise only from the taken residue. According to (5.18) the integrand around the point $x = -f_1 \approx g_1$ takes the asymptotic form:

$$\frac{1}{4i\pi^2(x+f_1)(-x+g_1)} S_b(x+f_2)S_b(x+f_3)S_b(-x+g_2)S_b(-x+g_3). \quad (5.19)$$

Then, by the Cauchy theorem the integral over small circle around this point is equal to

$$\frac{1}{2\pi(f_1+g_1)} S_b(-f_1+f_2)S_b(-f_1+f_3)S_b(f_1+g_2)S_b(f_1+g_3). \quad (5.20)$$

On the other hand, we see that the right-hand side expression in (5.17) indeed has the pole singularity at $f_1+g_1 \rightarrow 0$ coming from the $j=k=1$ multiplier. The rest can be seen to yield the same result due to the balancing condition, which in this limit takes the form $f_2+f_3+g_2+g_3=Q$, and relation $S_b(x)S_b(Q-x)=1$. The same situation will take place if we take the limit $f_1+g_1 \rightarrow 0$ in an asymmetric way, i.e. for an arbitrary eventual value of f_1 . For instance, we may deform the integration contour close to a fixed point $-f_1$ and in the limit $g_1 \rightarrow -f_1$ we come inevitably to pinching of the contour which leads to the same singular asymptotics for the integral.

Now let us get back to the integral (5.16). First let us indicate necessary properties of the functions $S_0(x)$ and $S_1(x)$. The function $S_0(x)$ has zeros at $x=Q+nb+mb^{-1}$ and poles at $x=-mb-nb^{-1}$, where m and n are both non-negative integers and $m+n$ is odd. The function $S_1(x)$ has zeros at $x=Q+nb+mb^{-1}$ and poles at $x=-mb-nb^{-1}$, where m and n are both non-negative integers and $m+n$ is even. The function $S_1(x)$ near zero has the behavior:

$$\lim_{x \rightarrow 0} xS_1(x) = \frac{1}{\pi}. \quad (5.21)$$

Also we have

$$S_0(x)S_0(Q-x)=1, \quad S_1(x)S_1(Q-x)=1. \quad (5.22)$$

In the same limit $f_1+g_1 \rightarrow 0$ the poles of $S_0(x)$ functions in (5.16) do not pinch the contour ($S_0(0)$ is regular) and the contribution from the first term in the integrand remains finite. The pole singularity is produced only by the second term in the integrand. Using (5.21) one can see that around the point $x = -f_1$ the integrand asymptotically takes the form

$$-\frac{1}{i\pi^2(x+f_1)(-x+g_1)} S_1(x+f_2)S_1(x+f_3)S_0(-x+g_2)S_0(-x+g_3). \quad (5.23)$$

Again, by the Cauchy theorem the integral over the small circle around $x = -f_1$ is equal to

$$-\frac{2}{\pi(f_1+g_1)} S_1(-f_1+f_2)S_1(-f_1+f_3)S_0(f_1+g_2)S_0(f_1+g_3). \quad (5.24)$$

It is easy to see that, due to the balancing condition and properties (5.21), (5.22), the asymptotics of the right-hand side expression in (5.16) indeed coincides with (5.24) with the correct sign.

6 Conclusion

To conclude, in this work we established a link between the superconformal indices of 4d SCFTs on the lens space, the corresponding rarefied elliptic hypergeometric functions and the parafermionic Liouville model. The parafermionic star-triangle relation (4.15) should play a proper role in the consideration of corresponding LFT fusion matrices. Following the logic of the present work it would be also interesting to investigate the hyperbolic degeneration of the rarefied elliptic hypergeometric function $V^{(r)}$ constructed in [33] and search for its proper parafermionic, or supersymmetric for $r = 2$ interpretation.

One of the relevant topics which we skipped in the present note concerns partition functions of supersymmetric 3d field theories described by hyperbolic integrals. Our relations (4.2) and (4.15) should describe dualities of certain models on the manifold $S^3/\mathbb{Z}_r$ similar to the $r = 1$ cases [28]. Indeed, in [16] a number of such dualities has been investigated, but coincidence of dual partition functions was established only numerically. It would be interesting to analyze whether the corresponding conjectural identities are consequences of (4.2) or hyperbolic limits of other identities from [33], or they describe somewhat different systems.

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