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Holographic duals of large- c torus conformal blocks

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ABSTRACT: We study CFT₂ conformal blocks on a torus and their holographic realization. The classical conformal blocks arising in the regime where conformal dimensions grow linearly with the large central charge are shown to be holographically dual to the geodesic networks stretched in the thermal AdS bulk space. We discuss the n -point conformal blocks and their duals, the 2-point case is elaborated in full detail. We develop various techniques to calculate both quantum and classical conformal block functions. In particular, we show that exponentiated global torus blocks reproduce classical torus blocks in the specific perturbative regimes of the conformal parameter space.

KEYWORDS: Conformal Field Theory, AdS-CFT Correspondence, Conformal and W Symmetry

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1 Introduction

The study of the $\text{AdS}_3/\text{CFT}_2$ correspondence in the large central charge approximation has lead to the simple formula relating the classical conformal block f_{class} and the length of geodesic network L_{dual} in the dual spacetime

$$f_{\text{class}}(\epsilon, \tilde{\epsilon}|z) \cong L_{\text{dual}}(\epsilon, \tilde{\epsilon}|w), \quad (1.1)$$

where coordinates $z = z(w)$ are given by the conformal map from the AdS boundary to the plane CFT, $\epsilon, \tilde{\epsilon}$ are conformal dimensions, the equality symbol $\cong$ means that both sides

are equal modulo logarithmic terms defined by the conformal map [1–8] (for the further development see [9–20]). The large- c conformal blocks can be defined by a number of heavy or light operators with dimensions growing linearly with c . The corresponding bulk geometry can be either conical singularity/BTZ [2] or thermal AdS [15] in the case of the boundary spherical or toric CFT, respectively.

In this paper we continue to study large- c CFT₂ on a torus from the holographic perspective. Basically, we consider the 2-point torus blocks generalizing to the n -point case wherever possible. We calculate the corresponding classical blocks within various approximations and show that they are holographically realized as geodesic networks on the thermal AdS in keeping with the block/length correspondence formula (1.1).

The outline of the paper is as follows. In section 2 we first shortly discuss n -point correlation functions on a torus and then focus on the $n = 2$ case. In order to calculate the block functions we develop two methods, (i) a straightforward evaluation of the constituent matrix elements, (ii) a combinatorial (AGT) representation, see appendices A, B, and C, respectively.¹

In section 3–5 we discuss various limiting torus blocks paying particular attention to the so called classical and global blocks.² In section 6 we discuss the *classical global* torus blocks that arise as exponentiation of the global blocks in the regime of large conformal dimensions. We also show that classical global blocks describe the linearized part of the standard classical blocks within particular perturbation theory when some conformal dimensions are larger than the others. This yields the method to calculate approximate classical blocks that bypass the full quantum Virasoro block analysis.

The holographic interpretation of classical torus blocks is discussed in section 7. Here we propose the general scheme and formulate the system of differential and algebraic equations that describes the dual network. However, similar to the n -point sphere case, exact solutions to the equation system are not known yet. Instead, in section 8 we propose a particular perturbation theory and find approximate solutions in the 2-point case.

We close with a brief conclusion in section 9. All technicalities and subsidiary discussions are collected in appendices A–E.

2 Torus conformal blocks

An n -point torus correlation function of arbitrary primary operators $\phi_i(z_i, \bar{z}_i)$ with conformal dimensions $(\Delta_i, \bar{\Delta}_i)$ is given by

$$\langle \phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n) \rangle = (q\bar{q})^{-\frac{c}{24}} \text{Tr} \left(q^{L_0} \bar{q}^{\bar{L}_0} \phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n) \right), \quad (2.1)$$

where $q = e^{2\pi i \tau_{\text{cft}}}$, $\tau_{\text{cft}} \in \mathbb{C}$ is the torus modular parameter, L_0 and $\bar{L}_0$ are the Virasoro generators, c is the central charge. Since a torus can be realized as a cylinder with the edges rotated and glued together, the torus correlation functions can be defined using the plane CFT notation (we always assume that (2.1) is supplemented by the conformal map

¹These results are complementary to the recursive representations of 2-point torus blocks [21].

²Global blocks and their holographic duals were recently discussed in [19].

Figure 1. Two-point conformal blocks in the s -channel (right) and t -channel (left). The s -channel loop consists of two segments with different conformal dimensions.

from the plane to the cylinder) [22]. Assuming that the space of states is generated by primary operators with dimensions denoted as $(\tilde{\Delta}_1, \bar{\Delta}_1)$ and evaluating the trace we find that the torus correlation functions is given by a power series in the modular parameter,

$$\begin{aligned} \langle \phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n) \rangle &= (q\bar{q})^{-\frac{c}{24}} \sum_{(\tilde{\Delta}_1, \bar{\Delta}_1)} \sum_{m=0}^{\infty} q^{\tilde{\Delta}_1+m} \bar{q}^{\bar{\Delta}_1+m} \times \\ &\times \sum_{m=|M|=|N|} \sum_{m=|\bar{M}|=|\bar{N}|} B^{M|N} B^{\bar{M}|\bar{N}} F(\Delta_i, \bar{\Delta}_i, \tilde{\Delta}_1, \bar{\Delta}_1, M, N, \bar{M}, \bar{N} | z_i, \bar{z}_i), \end{aligned} \quad (2.2)$$

where the matrix element of n primary operators

$$F(\Delta_i, \bar{\Delta}_i, \tilde{\Delta}_1, \bar{\Delta}_1, M, N, \bar{M}, \bar{N} | z, \bar{z}) = \langle \tilde{\Delta}_1, \bar{\Delta}_1, M, \bar{M} | \phi_1(z_1, \bar{z}_1) \cdots \phi_n(z_n, \bar{z}_n) | N, \bar{N}, \tilde{\Delta}_1, \bar{\Delta}_1 \rangle, \quad (2.3)$$

is given in the standard basis $|M, \bar{M}, \tilde{\Delta}_1, \bar{\Delta}_1\rangle = \bar{L}_{-n_1}^{j_1} \cdots \bar{L}_{-n_l}^{j_l} L_{-m_1}^{i_1} \cdots L_{-m_k}^{i_k} |\tilde{\Delta}_1, \bar{\Delta}_1\rangle$. Here, descendant vectors in the Verma module are generated from the primary state $|\tilde{\Delta}_1, \bar{\Delta}_1\rangle$, indices $M, \bar{M}$ label basis monomials, $|M| = i_1 m_1 + \cdots + i_k m_k$ and $|\bar{M}| = j_1 n_1 + \cdots + j_l n_l$. The matrix $B^{M|N}$ is the inverse of the Gram matrix.

In what follows we focus on the $n = 2$ point case and expand the matrix elements (2.3) into conformal blocks. Different exchanged channels can be obtained by plugging resolutions of identity into the matrix element (2.3) and/or using the OPE. Note that conformal dimensions $(\tilde{\Delta}_1, \bar{\Delta}_1)$ associated to the q -expansion (2.2) partially describe possible exchanged channels and, in fact, define an (un)closed loop part of the corresponding diagrams.

In the 2-point case, the torus correlation functions can be expanded in two channels that we call s -channel and t -channel, see figure 1. They directly follow from (2.1): the trace over the space of states can be understood as a sum of 4-point correlation functions on a sphere over two outermost descendant operators in points ∞ (left) and 0 (right). Then, using the OPE and recalling that on a sphere there are three different 4-point exchanged channels we can see that identifying two external legs we get just two topologically non-equivalent configurations, figure 1.

s -channel. Two-point torus conformal block in the s -channel (left diagram on figure 1) is defined by inserting the resolution of identity between two primary operators in the matrix element (2.3),

$$\mathbb{K} = \sum_{\Delta, \bar{\Delta}} \sum_{m=0}^{\infty} \sum_{|S|=|T|=m} \sum_{|\bar{S}|=|\bar{T}|=m} |S, \bar{S}, \Delta, \bar{\Delta}\rangle B^{S|T} B^{\bar{S}|\bar{T}} \langle \Delta, \bar{\Delta}, T, \bar{T} |, \quad (2.4)$$

where the inverse Gram matrix $B^{S|T}$ enters the formula because the standard basis is non-diagonal. Then (2.3) splits into products of two 3-point functions of primary operator with two descendant operators. Recalling that $\langle \tilde{\Delta}_m | \phi_k(z_k) | \tilde{\Delta}_l \rangle = C_{\tilde{\Delta}_m \Delta_k \tilde{\Delta}_k} z_k^{\tilde{\Delta}_m - \Delta_k - \tilde{\Delta}_l}$, where $C_{\tilde{\Delta}_m \Delta_k \tilde{\Delta}_k}$ are the structure constants, we decompose the 2-point correlation function as follows

$$\begin{aligned} \langle \phi_1(z_1, \bar{z}_1) \phi_2(z_2, \bar{z}_2) \rangle &= \sum_{(\tilde{\Delta}_1, \tilde{\Delta}_1)} \sum_{(\tilde{\Delta}_2, \tilde{\Delta}_2)} C_{\tilde{\Delta}_1 \Delta_1 \tilde{\Delta}_2} C_{\tilde{\Delta}_2 \Delta_2 \tilde{\Delta}_1} \times \\ &\times z_1^{\tilde{\Delta}_1 - \Delta_1 - \tilde{\Delta}_2} \bar{z}_1^{\tilde{\Delta}_1 - \bar{\Delta}_1 - \tilde{\Delta}_2} z_2^{\tilde{\Delta}_2 - \Delta_2 - \tilde{\Delta}_1} \bar{z}_2^{\tilde{\Delta}_2 - \bar{\Delta}_2 - \tilde{\Delta}_1} \left[\mathcal{V}_c^{\Delta_1, 2, \tilde{\Delta}_1, 2}(q, z_{1,2}) \mathcal{V}_c^{\tilde{\Delta}_1, 2, \tilde{\Delta}_1, 2}(\bar{q}, \bar{z}_{1,2}) \right], \end{aligned} \quad (2.5)$$

where the s -channel (holomorphic) conformal block is³

$$\begin{aligned} \mathcal{V}_c^{\Delta_1, 2, \tilde{\Delta}_1, 2}(q, z_{1,2}) &= \\ &= q^{c/24 - \tilde{\Delta}_1} \sum_{n,m=0}^{\infty} q^n \sum_{\substack{|M|=|N|=n \\ |S|=|T|=m}} B_1^{M|N} \frac{\langle \tilde{\Delta}_1, M | \phi_1(z_1) | S, \tilde{\Delta}_2 \rangle}{\langle \tilde{\Delta}_1 | \phi_1(z_1) | \tilde{\Delta}_2 \rangle} B_2^{S|T} \frac{\langle \tilde{\Delta}_2, T | \phi_2(z_2) | N, \tilde{\Delta}_1 \rangle}{\langle \tilde{\Delta}_2 | \phi_2(z_2) | \tilde{\Delta}_1 \rangle}, \end{aligned} \quad (2.6)$$

where $\tilde{\Delta}_{1,2}$ are exchanged conformal dimensions. Given that all inner products in (2.6) have been explicitly calculated we arrive at the double power series in the modular parameter q and the ratio $x = z_2/z_1$ with the expansion coefficients being rational functions of conformal parameters,

$$\mathcal{V}_c^{\Delta_1, 2, \tilde{\Delta}_1, 2}(q, x) = q^{c/24 - \tilde{\Delta}_1} \sum_{n \in \mathbb{N}_0} \sum_{m \in \mathbb{Z}} \mathcal{V}_{c(n,m)}^{\Delta_1, 2, \tilde{\Delta}_1, 2} q^n x^m. \quad (2.7)$$

Using explicit formulas from appendix A we write down the block function in the lowest orders,⁴

$$\begin{aligned} \mathcal{V}_c^{\Delta_1, 2, \tilde{\Delta}_1, 2}(q, x) &= \\ &= \left[1 + \frac{(\Delta_1 + \tilde{\Delta}_2 - \tilde{\Delta}_1)(\Delta_2 + \tilde{\Delta}_2 - \tilde{\Delta}_1)}{2\tilde{\Delta}_2} x + \mathcal{O}(x^2) \right] + q \left[\frac{(\tilde{\Delta}_1 + \Delta_1 - \tilde{\Delta}_2)(\tilde{\Delta}_1 + \Delta_2 - \tilde{\Delta}_2)}{2\tilde{\Delta}_1} x^{-1} \right. \\ &\quad + \frac{((\tilde{\Delta}_2 + \Delta_1 - \tilde{\Delta}_1)(\tilde{\Delta}_1 + \Delta_1 - \tilde{\Delta}_2 - 1) + 2\tilde{\Delta}_2)((\tilde{\Delta}_1 + \Delta_2 - \tilde{\Delta}_2)(\tilde{\Delta}_2 + \Delta_2 - \tilde{\Delta}_1 - 1) + 2\tilde{\Delta}_1)}{4\tilde{\Delta}_1 \tilde{\Delta}_2} \\ &\quad \left. + \mathcal{O}(x^1) \right] + \mathcal{O}(x^m, q^k). \end{aligned} \quad (2.8)$$

Setting $\Delta_2 = 0$, $\Delta_1 \equiv \Delta$ and equating $\tilde{\Delta}_1 = \tilde{\Delta}_2 \equiv \tilde{\Delta}$ we reproduce the 1-point torus block [25] with external dimension Δ and exchanged dimension $\tilde{\Delta}$. From the form of coefficients in (2.8) it follows that *vacuum* ($\tilde{\Delta}_1 = 0$ or $\tilde{\Delta}_2 = 0$) blocks in this channel are absent.

³Differently parameterized Virasoro torus blocks were considered in the 2-point case in ref. [23], in the n -point case in ref. [21], and for minimal models in ref. [24]. For a related discussion of n -point global blocks see [19].

⁴In principle, the matrix approach can be used to find coefficients in any order. Instead, in appendix C we develop the *combinatorial representation* of the n -point conformal torus blocks in the s -channel. Using Mathematica we apply this representation to compute the block coefficients up to high enough order in q and x .

t -channel. Alternatively, we can use the OPE of two primary operators in the matrix element (2.3). In this case (2.3) reduces to summing over 3-point functions of three descendant operators with the OPE coefficients. Namely, we fuse two primaries

$$\phi_1(z_1)\phi_2(z_2) = \sum_{\tilde{\Delta}_2} C_{\Delta_1\Delta_2\tilde{\Delta}_2} (z_1 - z_2)^{\tilde{\Delta}_2 - \Delta_1 - \Delta_2} \psi_{\tilde{\Delta}_2}(z_1, z_2), \quad (2.9)$$

where the resulting operator is given by

$$\psi_{\tilde{\Delta}_2}(z_1, z_2) = \phi_{\tilde{\Delta}_2}(z_2) + \beta_1(z_1 - z_2)L_{-1}\phi_{\tilde{\Delta}_2}(z_2) + \dots, \quad \beta_1 = \frac{\tilde{\Delta}_2 + \Delta_1 - \Delta_2}{2\tilde{\Delta}_2}. \quad (2.10)$$

Plugging the OPE (2.9) into the correlation function (2.2) we find

$$\begin{aligned} \langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2) \rangle &= (q\bar{q})^{-\frac{c}{24}} \sum_{(\tilde{\Delta}_1, \tilde{\bar{\Delta}}_1)} \sum_{(\tilde{\Delta}_2, \tilde{\bar{\Delta}}_2)} C_{\Delta_1\Delta_2\tilde{\Delta}_2} C_{\tilde{\Delta}_1\tilde{\Delta}_2\tilde{\bar{\Delta}}_2} \times \\ &\times (z_1 - z_2)^{\tilde{\Delta}_2 - \Delta_1 - \Delta_2} (\bar{z}_1 - \bar{z}_2)^{\tilde{\bar{\Delta}}_2 - \tilde{\Delta}_1 - \tilde{\bar{\Delta}}_2} z_2^{-\tilde{\Delta}_2} \bar{z}_2^{-\tilde{\bar{\Delta}}_2} \left[\mathcal{V}_c^{\Delta_1, 2, \tilde{\Delta}_1, 2}(q, z_{1,2}) \mathcal{V}_c^{\tilde{\Delta}_1, 2, \tilde{\bar{\Delta}}_1, 2}(\bar{q}, \bar{z}_{1,2}) \right], \end{aligned} \quad (2.11)$$

where the t -channel (holomorphic) conformal block is

$$\mathcal{V}_c^{\Delta_1, 2, \tilde{\Delta}_1, 2}(q, z_{1,2}) = \sum_{n=0}^{\infty} q^n \sum_{|M|=|N|=n} B_1^{M|N} \frac{\langle \tilde{\Delta}_1, M | \psi_{\tilde{\Delta}_2}(z_1, z_2) | N, \tilde{\Delta}_1 \rangle}{\langle \tilde{\Delta}_1 | \phi_{\tilde{\Delta}_2}(z_2) | \tilde{\Delta}_1 \rangle}, \quad (2.12)$$

where $\tilde{\Delta}_{1,2}$ are exchanged conformal dimensions. Given that 3-point functions of three descendants in (2.12) have been explicitly calculated we arrive at the double power series in the modular parameter q and the ratio $w = (z_1 - z_2)/z_2$ with the expansion coefficients being rational functions of conformal parameters,

$$\mathcal{V}_c^{\Delta_1, 2, \tilde{\Delta}_1, 2}(q, w) = q^{c/24 - \tilde{\Delta}_1} \sum_{n \in \mathbb{N}_0} \sum_{m \in \mathbb{N}_0} \mathcal{V}_c^{\Delta_1, 2, \tilde{\Delta}_1, 2}(n, m) q^n w^m. \quad (2.13)$$

Note that the t -channel block is the power series in w -variable, while the s -channel block is the Laurent series in x -variable, cf. (2.7). Using explicit formulas from appendix B we write down the block function in the lowest orders,

$$\begin{aligned} \mathcal{V}_c^{\Delta_1, 2, \tilde{\Delta}_1, 2}(q, w) &= \left[1 - \frac{\tilde{\Delta}_2 + \Delta_1 - \Delta_2}{2} w + \mathcal{O}(w^2) \right] + q \left[\frac{2\tilde{\Delta}_1 + \tilde{\Delta}_2(\tilde{\Delta}_2 - 1)}{2\tilde{\Delta}_1} \right. \\ &\quad \left. - \frac{(2\tilde{\Delta}_1 + (\tilde{\Delta}_2 - 1)\tilde{\Delta}_2)(\Delta_1 + \tilde{\Delta}_2 - \Delta_2)}{4\tilde{\Delta}_1} w + \mathcal{O}(w^2) \right]. \end{aligned} \quad (2.14)$$

The 1-point block with dimensions $\Delta, \tilde{\Delta}$ is reproduced by setting $\Delta_1 = 0, \Delta_2 = \tilde{\Delta}_2 \equiv \Delta$, and $\tilde{\Delta}_1 \equiv \tilde{\Delta}$. Let us note that there exist *vacuum* t -channel conformal blocks which arise when the second exchanged operator (an intermediate straight line on the right diagram on figure 1) is the unity operator. Thus, setting $\tilde{\Delta}_2 = 0$ supplemented by the fusion condition $\Delta_1 = \Delta_2 \equiv \Delta$ we find the vacuum block function which depends on the external dimension Δ , and the loop exchanged dimension $\tilde{\Delta}$.

3 Classical two-point torus blocks

The parameter space of the conformal block functions $\mathcal{V}_c^{\Delta, \tilde{\Delta}}(q|z)$ includes external and intermediate dimensions, modular parameters, and the central charge. In this section we shall discuss *semiclassical* blocks that correspond to different asymptotics in the parameter space when the conformal dimensions scale differently with the central charge. Near the point $c = \infty$ we distinguish between heavy and light dimensions depending on how a given dimension scales with the central charge: $\Delta_{\text{light}} \approx \epsilon$ or $\Delta_{\text{heavy}} \approx c\epsilon$, where ϵ are classical dimensions. Assuming that a conformal block depends on a number of light and heavy operators and expanding around $c = \infty$ we find that

$$\mathcal{V}_c^{\Delta, \tilde{\Delta}}(q, z) \cong \sum_{n \in \mathbb{N}} \frac{v_n^{\epsilon, \tilde{\epsilon}}(q, z)}{c^n}, \quad (3.1)$$

where the expansion coefficients are power series in the modular parameter q with coefficients being rational functions of classical dimensions $\epsilon, \tilde{\epsilon}$.

The principal part of (3.1) vanishes in the large- c limit, while the form of the regular part defines a particular semiclassical block. It is known that there are different types of semiclassical blocks, including the light blocks, various heavy-light blocks, and the classical block (see e.g. discussion in [7, 21, 26, 27]).⁵ Using the expansion (3.1) we see that e.g. the light block is given by $v_0^{\epsilon, \tilde{\epsilon}}(q, z)$, while other coefficients vanish, $v_n^{\epsilon, \tilde{\epsilon}}(q, z) = 0$, $n > 0$. The classical block has non-vanishing coefficients $v_n^{\epsilon, \tilde{\epsilon}}(q, z) \neq 0$, $\forall n \in \mathbb{N}_0$. In particular, the regular part of the classical block is claimed to be an exponential functional linear in c .

In what follows we focus on the classical torus block which therefore is given by

$$\mathcal{V}_c^{\Delta_{1,2}, \tilde{\Delta}_{1,2}}(q, z_{1,2}) \cong \exp \left[\frac{c}{6} f^{\epsilon_{1,2}, \tilde{\epsilon}_{1,2}}(q, z_{1,2}) \right] \quad \text{as} \quad c \rightarrow \infty, \quad (3.2)$$

where function $f^{\epsilon_{1,2}, \tilde{\epsilon}_{1,2}}$ is the corresponding classical conformal block conveniently parameterized by the classical conformal dimensions

$$\epsilon_i = \frac{\Delta_i}{k}, \quad \tilde{\epsilon}_i = \frac{\tilde{\Delta}_i}{k}, \quad \text{where} \quad k = \frac{c}{6}. \quad (3.3)$$

s. Using (2.8) we find the s -channel block function

$$f^{\epsilon_{1,2}, \tilde{\epsilon}_{1,2}}(q, x) = (\tilde{\epsilon}_1 - 1/4) \log q + \sum_{n=0}^{\infty} q^n f_n^{(1)}(\epsilon, \tilde{\epsilon}|x), \quad (3.4)$$

where a few lowest level coefficients are given by

$$\begin{aligned} f_0^{(1)}(\epsilon, \tilde{\epsilon}|x) &= \frac{(\epsilon_1 + \tilde{\epsilon}_2 - \tilde{\epsilon}_1)(\epsilon_2 + \tilde{\epsilon}_2 - \tilde{\epsilon}_1)}{2\tilde{\epsilon}_2} x + \mathcal{O}(x^2), \\ f_1^{(1)}(\epsilon, \tilde{\epsilon}|x) &= \frac{(\epsilon_1 + \tilde{\epsilon}_1 - \tilde{\epsilon}_2)(\epsilon_2 + \tilde{\epsilon}_1 - \tilde{\epsilon}_2)}{2\tilde{\epsilon}_1} x^{-1} + \mathcal{O}(x^0). \end{aligned} \quad (3.5)$$

⁵Generally speaking, global torus blocks cannot be obtained in this way, see section 5.

t. Using (2.14) we find the t -channel block function

$$f^{\epsilon_{1,2},\tilde{\epsilon}_{1,2}}(q,w) = (\tilde{\epsilon}_1 - 1/4) \log q + \sum_{n=0}^{\infty} q^n f_n^{(2)}(\epsilon, \tilde{\epsilon}|w), \quad (3.6)$$

where a few lowest level coefficients are given by

$$f_0^{(2)}(\epsilon, \tilde{\epsilon}|w) = -\frac{(\tilde{\epsilon}_2 + \epsilon_1 - \epsilon_2)}{2} w + \mathcal{O}(w^2), \quad f_1^{(2)}(\epsilon, \tilde{\epsilon}|w) = \frac{\tilde{\epsilon}_2^2}{2\tilde{\epsilon}_1} + \mathcal{O}(w). \quad (3.7)$$

4 Perturbative classical s -channel torus blocks

The analysis of classical blocks simplifies within various approximations where some of dimensions are much larger than the others, see, e.g. [2, 6, 8, 13, 15]. In the torus case, already from the first coefficients (3.5) (for simplicity, we choose pairwise equal dimensions) $f_0 = \frac{\epsilon_1^2}{2\tilde{\epsilon}_1}$ we immediately conclude that it blows up when $\tilde{\epsilon}_1 \ll \epsilon_1$ and is smooth in the opposite regime $\epsilon_1 \ll \tilde{\epsilon}_1$. In what follows we study the torus s -channel blocks assuming that the exchanged channels with equal dimensions $\tilde{\epsilon} \equiv \tilde{\epsilon}_1 = \tilde{\epsilon}_2$ are much heavier than the external operators ϵ_1 and ϵ_2 , i.e. $\epsilon_i/\tilde{\epsilon} \ll 1$. This step provides the first level perturbation expansion because the external dimensions remain unrelated to each other. On the second level, we find two possible perturbation expansions. In the first version called the superlight expansion [8, 10, 26] we assume that the second external dimension is much less than the first one, $\epsilon_1/\epsilon_2 \ll 1$. In the second version that we call the double leg expansion the external dimensions are equated.

Superlight expansion. In this case the dimensions are restricted as

$$\tilde{\epsilon} \equiv \tilde{\epsilon}_1 = \tilde{\epsilon}_2, \quad \delta \equiv \frac{\epsilon_1}{\tilde{\epsilon}} \ll 1, \quad \nu \equiv \frac{\epsilon_2}{\epsilon_1} \ll 1. \quad (4.1)$$

The first constraint is the fusion rule guaranteeing that $\epsilon_1 = 0$ and/or $\epsilon_2 = 0$ are consistent. The corresponding conformal block is parameterized by three parameters $\tilde{\epsilon}, \delta, \nu$ so that we arrive at the triple deformation theory: we expand around $\tilde{\epsilon} = \infty$ and then around $\delta = 0$ and $\nu = 0$. Keeping terms linear in $\tilde{\epsilon}$ we arrive at the quartic series expansion

$$f_{\text{lin}}^{\tilde{\epsilon},\delta,\nu}(q,x) \equiv (\tilde{\epsilon} - 1/4) \log q + \tilde{\epsilon} \sum_{n=1}^{\infty} f_{2n}^{(1)} \delta^{2n} + \tilde{\epsilon} \nu \sum_{n=2}^{\infty} f_n^{(2)} \delta^n + \mathcal{O}(\tilde{\epsilon}^2, \nu^2), \quad (4.2)$$

where the second term is the perturbative 1-point block with coefficients [15, 27]

$$f_2^{(1)} = \frac{q}{2} \frac{1}{1-q}, \quad f_4^{(1)} = -\frac{q^2}{48} \frac{q-3}{(1-q)^3}, \quad \dots, \quad (4.3)$$

because setting $\nu = 0$ we automatically reproduce the 1-point case, and the third term gives the ν -correction, where a few first coefficients read

$$f_2^{(2)} = \frac{x}{2} + \left(\frac{x}{2} + \frac{1}{2x} \right) q + \dots, \quad f_3^{(2)} = \frac{x^2}{8} + \left(\frac{x^2}{4} - \frac{1}{4} \right) q + \dots. \quad (4.4)$$

More higher order terms can be found in (D.1).

Double leg expansion. The conformal dimensions are pairwise equal and satisfy the constraints

$$\tilde{\epsilon}_1 = \tilde{\epsilon}_2 \equiv \tilde{\epsilon}, \quad \epsilon_1 = \epsilon_2 \equiv \epsilon, \quad \delta \equiv \frac{\epsilon}{\tilde{\epsilon}} \ll 1. \quad (4.5)$$

The corresponding block depends on two parameters $\tilde{\epsilon}$ and δ and, therefore, we consider the double perturbation theory: we expand around $\tilde{\epsilon} = \infty$ and then around $\delta = 0$. Keeping terms linear in $\tilde{\epsilon}$ we arrive at the triple series expansion

$$f_{\text{lin}}^{\tilde{\epsilon}, \delta}(q, x) \equiv (\tilde{\epsilon} - 1/4) \log q + \tilde{\epsilon} \sum_{n=2}^{\infty} g_n(q, x) \delta^n + \mathcal{O}(\tilde{\epsilon}^2, \nu^2), \quad (4.6)$$

where a few first coefficients read

$$g_2(x, q) = \frac{x}{2} + q \left(\frac{x}{2} + \frac{1}{2x} + 1 \right) + \dots, \quad g_3(x, q) = \frac{x^2}{4} + q \left(\frac{x^2}{2} - \frac{1}{2} \right) + \dots \quad (4.7)$$

The higher order expression can be found in (D.2).

5 Global torus blocks

Global blocks of CFT_2 are associated to $sl(2, \mathbb{R}) \subset \text{Vir}$. Equivalently, the global blocks can be obtained by considering particularly contracted Virasoro algebra at $c \rightarrow \infty$ while keeping both external and exchanged conformal dimensions independent of the central charge, $\Delta_i = \mathcal{O}(c^0)$ and $\tilde{\Delta}_j = \mathcal{O}(c^0)$ [27]. Note that just sending the central charge to infinity while keeping the dimensions fixed yields the so called *light* blocks which are generically different from the global blocks by the truncated Virasoro character prefactor (see [27] and [21] for details, and our discussion below (3.1)). In what follows we calculate the global blocks from scratch using the definitions (2.6) and (2.12) and restricting the Virasoro generators to $sl(2)$ ones.⁶

s-channel. Associating the 2-point s -channel block (2.6) to the $sl(2)$ algebra we find

$$\mathcal{F}^{\Delta_{1,2}, \tilde{\Delta}_{1,2}}(q, z_{1,2}) = \sum_{n,m=0}^{\infty} \frac{\tau_{n,m}(\tilde{\Delta}_1, \Delta_1, \tilde{\Delta}_2) \tau_{m,n}(\tilde{\Delta}_2, \Delta_2, \tilde{\Delta}_1)}{n! m! (2\tilde{\Delta}_1)_n (2\tilde{\Delta}_2)_m} q^n x^{n-m}, \quad (5.1)$$

where $x = z_1/z_2$ and the coefficients $\tau_{m,n} = \tau_{m,n}(\Delta_a, \Delta_b, \Delta_c)$ defining the $sl(2)$ 3-point function of a primary operator Δ_b and descendant operators $\Delta_{a,c}$ on the levels n, m are given by [26]

$$\begin{aligned} \tau_{n,m}(\Delta_{a,b,c}) &= \sum_{p=0}^{\min[n,m]} \frac{n!}{p!(n-p)!} (2\Delta_c + m - 1)^{(p)} m^{(p)} \\ &\quad \times (\Delta_c + \Delta_b - \Delta_a)_{m-p} (\Delta_a + \Delta_b - \Delta_c + p - m)_{n-p}, \end{aligned} \quad (5.2)$$

⁶On the other hand, global torus blocks can be viewed as solutions to the second order Casimir equations arising from the $sl(2)$ Ward identities on a torus [19]. See also related considerations of global blocks within the recursive representations [21].

where $(a)_k = a(a+1)\dots(a+k-1)$ and $(a)^{(k)} = a(a-1)\dots(a-k+1)$. Setting $\Delta_2 = 0$, $\Delta_1 \equiv \Delta$ and equating $\tilde{\Delta}_1 = \tilde{\Delta}_2 \equiv \tilde{\Delta}$ we will reproduce the 1-point torus block. Indeed, in this case $\tau_{n,m}(\tilde{\Delta}, 0, \tilde{\Delta}) = \delta_{n,m} m! (2\tilde{\Delta})_m$ and therefore the n -th global block coefficient (5.1) is given by $\frac{\tau_{n,n}(\tilde{\Delta}, \Delta, \tilde{\Delta})}{n! (2\tilde{\Delta})_n}$. It defines the 1-point global block with external/exchanged dimensions Δ and $\tilde{\Delta}$ and can be expressed as the hypergeometric function coefficients [25].

In the n -point case we can consider the generalized s -channel block defined by a diagram consisting of a loop with n external legs. Let $x_i = z_i/z_{i-1}$, where $x_i = 2, \dots, n$ and $q = x_1 x_2 \dots x_n$. With the identification $\Delta_1 = \Delta_{n+1}$, $\tilde{\Delta}_1 = \tilde{\Delta}_{n+1}$ and $s_1 = s_{n+1}$ the n -point global block in the generalized s -channel is given by

$$\mathcal{F}^{\Delta_i, \tilde{\Delta}_j}(x) = \sum_{s_1, \dots, s_{n-1}=0}^{\infty} \prod_{m=1}^n \frac{\tau_{s_m, s_{m+1}}(\tilde{\Delta}_m, \Delta_m, \tilde{\Delta}_{m+1})}{s_m! (2\tilde{\Delta}_m)_{s_m}} x_m^{s_m}, \quad (5.3)$$

where τ -coefficients are given by (5.2).

t -channel. To find the 2-point global block in the t -channel (2.12) we use the OPE for quasi-primary fields, see e.g. [28–30],

$$\phi_1(z_1) \phi_2(z_2) = \sum_{\tilde{\Delta}_2} C_{\Delta_1 \Delta_2 \tilde{\Delta}_2} (z_1 - z_2)^{\tilde{\Delta}_2 - \Delta_1 - \Delta_2} {}_1F_1\left(\tilde{\Delta}_2 + \Delta_1 - \Delta_2, 2\tilde{\Delta}_2 \middle| (z_1 - z_2)L_{-1}\right) \phi_{\tilde{\Delta}_2}(z_2), \quad (5.4)$$

where the OPE coefficients are now packed into the confluent hypergeometric function, cf. (2.9)–(2.10). Substituting the above OPE into the block function (2.12) restricted to the $sl(2)$ subalgebra we find the global t -channel block

$$\mathcal{F}^{\Delta_{1,2}, \tilde{\Delta}_{1,2}}(q, z_{1,2}) = \sum_{n,m=0}^{\infty} \frac{\sigma_m(\Delta_1, \Delta_2, \tilde{\Delta}_2) \tau_{n,n}(\tilde{\Delta}_1, \tilde{\Delta}_2, \tilde{\Delta}_1)}{n! m! (2\tilde{\Delta}_1)_n (2\tilde{\Delta}_2)_m} q^n w^m, \quad (5.5)$$

where $w = (z_1 - z_2)/z_2$, the τ -function is given by (5.2), the σ -function reads

$$\sigma_m(\Delta_1, \Delta_2, \tilde{\Delta}_2) = (-)^m (\tilde{\Delta}_2 + \Delta_1 - \Delta_2)_m (\tilde{\Delta}_2)_m. \quad (5.6)$$

Setting $\Delta_1 = 0$, $\Delta_2 = \tilde{\Delta}_2 \equiv \Delta$, and $\tilde{\Delta}_1 \equiv \tilde{\Delta}$ we will reproduce the 1-point torus block because in this case $\sigma_m(0, \Delta, \Delta) = \delta_{m,0}$ and, therefore, the n -th global block coefficient (5.5) is given by $\frac{\tau_{n,n}(\tilde{\Delta}, \Delta, \tilde{\Delta})}{n! (2\tilde{\Delta})_n}$ which is exactly the 1-point block coefficient. Let us note that the 1-point torus block factorizes from the 2-point expression. Moreover, (5.5) is a product of two hypergeometric functions in accordance with the analysis in [19]. The higher-point generalizations can be obtained in the standard fashion by successively applying the OPE (5.4) in the torus correlation function (2.2) associated to the $sl(2)$ algebra.

6 Exponentiating global blocks

Let us consider the regime of large dimensions, when all conformal dimensions are rescaled using a large parameter κ in a coherent manner,

$$\Delta_i = \kappa \sigma_i, \quad \tilde{\Delta}_i = \kappa \tilde{\sigma}_i, \quad \kappa \gg 1, \quad (6.1)$$

where σ_i and $\tilde{\sigma}_j$ can be referred to as *classical global* dimensions. We expect that in this regime the global blocks are exponentiated

$$\mathcal{F}^{\Delta_i, \tilde{\Delta}_j}(x) \cong \exp [\varkappa g^{\sigma_i, \tilde{\sigma}_j}(x)] \quad \text{as} \quad \varkappa \rightarrow \infty, \quad (6.2)$$

where $g^{\sigma_i, \tilde{\sigma}_j}(x)$ is a *classical global* block. Indeed, n -point global blocks satisfy the Casimir channel equations which are second order partial differential equations with coefficients being rational functions of the conformal dimensions Δ_i, Δ_j [19, 26, 27, 31]. At the same time from the general theory of differential equations it follows that once the equation coefficients depend on some large parameter in a specific way then the leading asymptotics in the solution space are given by exponentials.⁷

Matching the central charge c of the Virasoro block and the scale κ of the global blocks we see that to some extent the classical global blocks (6.2) are similar to the standard classical blocks, cf. (3.2).⁸ Most probably, the standard classical block and classical global blocks are not related for general values of conformal dimensions. However, there is a lot of evidence that they can be related to each other in particular perturbative regimes when some conformal dimensions Δ_{bgr} are much larger than the others Δ_{prt} ,

$$\Delta_{\text{prt}}/\Delta_{\text{bgr}} \ll 1. \quad (6.3)$$

(Note that this assumption is equally translated both to the standard classical and global classical dimensions.) For example, in the sphere case with two background external operators the perturbative classical block coincides with respective perturbative classical global block [7, 26]. In the 1-point torus case, the relevant perturbation theory assumes that the exchanged channel (loop) dimension is much large than the external dimension. In this case we also observe that the perturbative classical block can be reproduced from the global block [27].

Superlight expansion. Using the superlight expansion assumption (4.1) rewritten in terms of classical global dimensions σ_i and $\tilde{\sigma}_j$ we find out that modulo the logarithmic term the respective perturbative classical block coincides with the perturbative classical global block. Indeed, the corresponding classical global block is given by

$$g^{\tilde{\sigma}, \delta, \nu}(q, x) = \tilde{\sigma} \sum_{n=1}^{\infty} f_{2n}^{(1)}(x, q) \delta^{2n} + \tilde{\sigma} \nu \sum_{n=2}^{\infty} f_n^{(2)}(x, q) \delta^n + \mathcal{O}(\tilde{\sigma}^2, \nu^2), \quad (6.4)$$

where the first term defines the 1-point block with coefficients (4.3), and the second term is the first order ν -correction, where a few first coefficients read

$$f_2^{(2)}(x, q) = \frac{x}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left[x + \frac{1}{x} \right] q^n, \quad f_3^{(2)}(x, q) = \frac{x^2}{8} + \frac{1}{8} \sum_{n=1}^{\infty} \left[(n+1)x^2 - 2n + \frac{n-1}{x^2} \right] q^n, \quad (6.5)$$

$$f_4^{(2)}(x, q) = \frac{x^3}{24} + \frac{1}{48} \sum_{n=1}^{\infty} \left[(n+1)(n+2)x^3 - 3n(n+1)x - \frac{3(n-1)n}{x} + \frac{(n-1)(n-2)}{x^3} \right] q^n. \quad (6.6)$$

⁷See e.g. the monograph [32]. The 1-point case was considered in [27], the n -point case was discussed in [19].

⁸Contrary to the case of global blocks, the Virasoro blocks do not satisfy any differential equations (except for special values of conformal dimensions that results in the BPZ equation). In particular, this is why the exponentiation of the Virasoro blocks has not been rigorously proven yet (for related considerations see [33, 34]).

We observe that in a given order this expression coincides with the perturbative conformal block calculated using the combinatorial representation (D.1). Also, as a consistency check, setting $x = 1$ we reproduce the 1-point perturbative torus block with the external conformal dimension $(\epsilon_1 + \epsilon_2)$. Technically, at this point the second term in (6.4) coincides with the first term up to a prefactor fixed in the linear order in ν by that $(\frac{\epsilon_1 + \epsilon_2}{\epsilon})^n \approx (1 + n\nu)\delta^n$. Indeed, we notice that the polynomial in the square bracket of $f_3^{(2)}$ has a root $x = 1$, and, therefore, we find that $f_{2k+1}^{(2)}(x = 1) = \text{const}$ and $f_{2k}^{(2)}(x = 1) = 2kf_{2k}^{(1)} + \text{const}$.

Double leg expansion. Similarly, using the double leg expansion assumption (4.5) rewritten in terms of the classical global dimensions σ_i and $\tilde{\sigma}_j$ we find out that the corresponding block is given by

$$g^{\tilde{\sigma}, \delta}(q, x) = \tilde{\sigma} \sum_{n=1}^{\infty} g_n(x, q) \delta^n + \mathcal{O}(\tilde{\sigma}^2), \quad (6.7)$$

where a few first coefficients are found to be

$$\begin{aligned} g_2(x, q) &= \frac{x}{2} + \frac{1}{2} \sum_{n=1}^{\infty} \left[x + \frac{1}{x} + 2 \right] q^n, \quad g_3(x, q) = \frac{x^2}{4} + \frac{1}{4} \sum_{n=1}^{\infty} \left[(n+1)x^2 - 2n + \frac{n-1}{x^2} \right] q^n, \\ g_4(x, q) &= \frac{10x^3 - 3x^2}{48} + \frac{1}{48} \sum_{n=1}^{\infty} \left[5(n+1)(n+2)x^3 - 3(n+1)^2x^2 - 9n(n+1)x \right. \\ &\quad \left. - 2(n-1)(n+1) - \frac{9n(n-1)}{x} - \frac{3(n-1)^2}{x^2} + \frac{5(n-1)(n-2)}{x^3} \right] q^n. \end{aligned} \quad (6.9)$$

In a given order, the above expression reproduces the perturbative conformal block calculated using the combinatorial representation (D.2). Noticing that $g_3 = 2f_3^{(2)}$ and setting $x = 1$ we are left with the 1-point perturbative torus block with the external conformal dimension 2ϵ because $g_{2k+1}^{(1)}(x = 1) = \text{const}$ and $g_{2k}^{(1)}(x = 1) = 2^{2k}f_{2k}^{(1)} + \text{const}$. Indeed, going to the 1-point case produces factors $(\frac{2\epsilon}{\epsilon})^n \approx 2^n \delta^n$.

t-channel case. The relevant perturbation theory in the t -channel leads to imposing the following constraints,

$$\epsilon_1 = \epsilon_2, \quad \delta = \frac{\tilde{\epsilon}_2}{\epsilon_1} \ll 1, \quad (6.10)$$

where the first constraint is the fusion rule guaranteeing that $\tilde{\epsilon}_2 = 0$ approximation is consistent. The corresponding perturbative classical global block is found to be

$$g^{\tilde{\sigma}_1, \delta}(q, w) = \tilde{\sigma} \sum_{n=1}^{\infty} h_n(q, w) \delta^n + \mathcal{O}(\tilde{\sigma}^2), \quad (6.11)$$

where a first few coefficients can be represented as

$$h_1(q, w) = \text{Arccosh} \frac{i}{\sqrt{w}} - \log \frac{2i}{\sqrt{w}}, \quad (6.12)$$

$$h_2(q, w) = \frac{q}{2} \frac{1}{1-q}, \quad h_3(q, w) = -\frac{q^2}{48} \frac{q-3}{(1-q)^3}, \quad \dots \quad (6.13)$$

We notice that the first coefficient h_1 does not depend on q , while higher coefficients h_n at $n \geq 2$ do not depend on w and coincide with the 1-point coefficients (4.3).

Figure 2. n -point s -channel conformal blocks with the modular parameter τ in the s -channel holographically realized as the necklace graph on two-dimensional slice of the thermal AdS. Time runs along the non-contractible cycle $t \sim t + \beta$. Primary operators are inserted at x_i at the boundary. These points are mapped to boundary attachment points y_i in the bulk.

7 Holographic duals of the classical s -channel torus blocks

In this section we advocate that in the s -channel the perturbative classical block function f_{class} considered in section 4 can be represented as⁹

$$-f_{\text{class}}(x, q) \cong S_{\text{therm}} + L_{\text{dual}}(y, \beta), \quad (7.1)$$

where S_{therm} is the holomorphic part of the 3d gravity action evaluated on the thermal AdS space, $L_{\text{dual}}(y, \beta)$ is the length of the dual necklace graph attached to the boundary points y_i , $i = 1, \dots, n$, see figure 2. The gravitational action expressed in terms of the rescaled central charge $c/6$ is $S_{\text{thermal}} = i\pi\tau/2$ [35–37]. In terms of the modular parameter the action is $S_{\text{thermal}} = 1/4 \log q$ that reproduces the classical conformal block (3.4) at zeroth dimensions. The block/length correspondence (7.1) is supplemented with the identification of the modular parameter and coordinates of the primary operators as

$$\beta = -\log q, \quad y_i = -\log x_i. \quad (7.2)$$

Let us consider the thermal AdS space that is a solid torus with time running along the non-contractible cycle,

$$ds^2 = \left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\varphi^2, \quad (7.3)$$

where $t \sim t + \beta$, $\varphi \sim \varphi + 2\pi$, $r \geq 0$, the AdS radius is set $l = 1$. The time period β defines the modular parameter $\tau_{\text{ads}} = i\beta/2\pi$ and the temperature $\beta \sim T^{-1}$. The equivalent form of the metric can be obtained by rescaling the time coordinate $t \rightarrow -i\tau_{\text{ads}}t$ so that the metric coefficients explicitly depend on the modular parameter while $t \sim t + 2\pi$. However, the metric (7.3) is more convenient in practice because the modular parameter shows up

⁹The case of 1-point classical torus blocks was considered in [15]. The holographic correspondence for higher point global blocks was discussed recently in [19].

only when integrating along the non-contractible cycle. Otherwise, the local dynamics is the same as in the AdS space with non-periodic time.

Let us notice that in order to describe the correspondence between holomorphic conformal blocks and geodesic networks on the two-dimensional slice (7.1) we assume that the modular parameter q and coordinates of the primary operators are real (7.2). In this way we obtain that the holomorphic conformal block, being by definition a complex function, is equated to the real geodesic length on the thermal AdS with pure imaginary modulus τ_{ads} .¹⁰ Since $q = e^{2\pi i \tau_{\text{cft}}}$ we find that up to the modular transformations the parameter τ_{cft} can take one of two values $\tau_{\text{cft}} = 0 + ia$ or $\tau_{\text{cft}} = \frac{1}{2} + ia$, where $\forall a \in \mathbb{R}$. Therefore,

$$\tau_{\text{cft}} = \tau_{\text{ads}} \quad \text{or} \quad \tau_{\text{cft}} = \tau_{\text{ads}} + \frac{1}{2}. \quad (7.4)$$

In the first case, the torus where our CFT lives is indeed the conformal boundary of the bulk space. This is realized by the map (7.2). In the second case, the modular parameters are different but in the low-temperature approximation $\tau_{\text{ads}} \rightarrow \infty$, we again arrive at the standard duality with $\tau_{\text{cft}} \approx \tau_{\text{ads}}$ [15]. The two corresponding bulk solutions are discussed below in section 8.1.1.

7.1 Worldline formulation

In what follows we shortly review the worldline formulation on the thermal AdS background with the metric (7.3), see [6, 8, 15] for more details. A geodesic segment can be described by the action

$$S = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{g_{mn}(x) \dot{x}^m \dot{x}^n}, \quad (7.5)$$

where local coordinates are $x^m = (t, \phi, r)$ and the metric coefficients $g_{mn}(x)$ are read off from (7.3). The reparametrization invariance allows us to impose the normalization condition $|g_{mn}(x) \dot{x}^m \dot{x}^n| = 1$ so that the on-shell action is now given by $S = \lambda_2 - \lambda_1$. Using the Killing vectors of (7.3) we can restrict the dynamics to the constant angle $\varphi = 0$ surface (the annulus on figure 2). The corresponding conserved momentum is $p_\phi = 0$, while the other conserved momentum p_t is the motion constant that defines the shape of geodesics. In this case, the normalization condition is given by

$$\dot{r} = \sqrt{r^2 - s^2 + 1}, \quad s \equiv |p_t|. \quad (7.6)$$

- The length of a loop segment is given by

$$L_{\text{loop}} = \frac{1}{\tilde{s}} \int_{t_1}^{t_2} dt (1 + r^2(t)), \quad (7.7)$$

where $r(t)$ being the radial deviation, and $t_{1,2}$ are initial/final time positions.

¹⁰The geodesics network stretched between boundary points on different two-dimensional slices computes the full classical conformal block which is the sum of holomorphic and antiholomorphic conformal blocks. See [6] for an extended discussion in the sphere case.

- The length of an external leg with one endpoint attached to the conformal boundary is

$$L_{\text{leg}} = \int_{\rho}^{\Lambda} \frac{dr}{\sqrt{1+r^2-s^2}} = -\log\left(\rho + \sqrt{\rho^2 - s^2 + 1}\right), \quad (7.8)$$

where ρ is the vertex radial coordinate, and the cutoff parameter $\Lambda \rightarrow \infty$ is introduced to regularize the conformal boundary position.

The radial deviation is governed by the following evolution equation

$$\frac{dr}{dt} = \pm \frac{1}{s}(1+r^2)\sqrt{r^2 - s^2 + 1}, \quad (7.9)$$

that can be explicitly integrated. Using (7.9) we can express the momentum s in terms of the radius and its time derivative and substitute then into (7.7). The resulting action is (7.5) on the $\phi = \text{const}$ slice, where the metric is given by (7.3). The corresponding equations of motion are second-order ODE having a general solution that depends on two integration constants. The formulation described above is partially integrated form of these equations. Indeed, one of integration constants is given by the time momentum that drastically simplifies the analysis of geodesic lines, while the residual dynamical equation (7.9) is first-order ODE with a general solution depending on the other integration constant.

Integrating the evolution equation (7.9) we get

$$e^{2(t-t_0)} = \frac{(i+r_0)(r(t)-i)\left(r_0 - s\sqrt{r_0^2 - s^2 + 1} + is^2 - i\right)\left(s\sqrt{r(t)^2 - s^2 + 1} + r(t) - is^2 + i\right)}{(i-r_0)(r(t)+i)\left(r_0 + s\sqrt{r_0^2 - s^2 + 1} - is^2 + i\right)\left(s\sqrt{r(t)^2 - s^2 + 1} - r(t) - is^2 + i\right)}, \quad (7.10)$$

where $r(t_0) = r_0$ are initial conditions. Solving this equation we can express the radial coordinate as a function of parameters r_0 and s , i.e. $r = r(t|r_0, s)$.

7.2 Dual geodesic networks

Let us first describe the kinematics of the dual networks in the n -point case. The graph on figure 2 is drawn on two-dimensional annulus with coordinates (r, t) and consists of n external legs and n loop segments stretched between n vertex points. Any trivalent vertex connects two loop segments and one external leg. The radial and time vertex coordinates are denoted as (ρ_m, γ_m) , $m = 1, \dots, n$. The external legs are attached to the conformal boundary at points y_m , where $m = 1, \dots, n$. The motion constants (time momenta) of loop segments and external legs are respectively $\tilde{s}_m$ and s_k , where $m, k = 1, \dots, n$.

The geodesic length can be associated to the mechanical system of massive test particles propagating in the bulk space. Hence, the total length of the geodesic network is given by

$$L_{\text{dual}} = \sum_{m=1}^n \left(\epsilon_m L_{\text{leg}}^m + \tilde{\epsilon}_m L_{\text{loop}}^m \right), \quad (7.11)$$

where classical conformal dimensions $\sim$ masses, and the length of each geodesic segment (7.7), (7.8) reads

$$L_{\text{loop}}^m = \frac{1}{\tilde{s}_m} \int_{\gamma_1^m}^{\gamma_2^m} dt (1 + R_m^2(t)), \quad L_{\text{leg}}^m = -\log(\rho_m + \sqrt{\rho_m^2 - s_m^2 + 1}), \quad (7.12)$$

where $R_m(t)$ is a radius of m -th loop segment, s_m and $\tilde{s}_m$ are the leg and loop momenta, ρ_m is the radial vertex coordinate, the integration limits $\gamma_{1,2}^m$ parameterize endpoints of each loop segment:

$$\gamma_1^m \equiv \gamma_m, \quad \gamma_2^m \equiv \gamma_{m+1}, \quad m = 1, \dots, n, \quad (7.13)$$

$$\gamma_{n+1} \equiv \gamma_1 + \beta. \quad (7.14)$$

To find the total length of the geodesic network we have to know the radial functions $R_m(t)$ explicitly as well as the vertex position ρ_m , the momenta $s_m, \tilde{s}_m$, and the times γ_m .

Time evolution. The radial functions $R_m(t)$ in (7.12) are subjected to the evolution equations (7.9),

$$\frac{dR_m}{dt} = \frac{1}{\tilde{s}_m} (1 + R_m^2) \sqrt{R^2 - \tilde{s}_m^2 + 1}, \quad m = 1, \dots, n, \quad (7.15)$$

whose general solution $R_m = R_m(t | c_m, \tilde{s}_m)$ is parameterized by two constants, an integration constant c_m and the loop momentum $\tilde{s}_m$. In order to fix them we impose the vertex boundary conditions

$$R_m(\gamma_m) = \rho_m, \quad R_m(\gamma_{m+1}) = \rho_{m+1}, \quad m = 1, \dots, n, \quad (7.16)$$

supplemented with the condition (7.14). Whence, we have $2m$ equations for $2m$ variables that allows us to solve the evolution equations in terms of the vertex radial positions.

Time intervals. In order to find initial/final time positions (7.13)–(7.14) explicitly we use the general formula for time intervals stretched between given two radial positions (7.10). Let us consider the m -th leg stretched from the vertex point (ρ_m, γ_m) to the boundary attachment point (∞, y_m) . In this case, using formula (7.10) we find the time interval relation

$$e^{2(\gamma_m - y_m)} = \frac{\rho_m^2 + (\rho_m^2 - 1) s_m^2 - 2\rho_m s_m \sqrt{\rho_m^2 - s_m^2 + 1} + 1}{(\rho_m^2 + 1)(s_m - 1)^2}, \quad m = 1, \dots, n, \quad (7.17)$$

which can be explicitly solved as $\gamma_m = \gamma_m(\rho_m, s_m, y_m)$.

Momentum conservation conditions. Both the vertex radial positions and momenta are constrained by the conservation conditions following from the least action principle for the total action (8.1). There are three possible configurations of external legs according to values of their momenta: radial, convex, or concave legs. In what follows we consider the case of one concave and $n - 1$ convex legs. Other possible options can be shown to be equivalent to this configuration or be inconsistent.

The momentum conservation condition in each vertex reads $\tilde{\epsilon}_m \tilde{p}_m + \tilde{\epsilon}_{m+1} \tilde{p}_{m+1} + \epsilon_m p_m = 0$, where $m = 1, \dots, n$ and $p_m, \tilde{p}_m$ are two-component conserved momenta on the annulus (for the general discussion of trivalent vertices on the hyperbolic spaces see [38]). The corresponding time and radial components are given by

$$\epsilon_m s_m + \tilde{\epsilon}_m \tilde{s}_m - \tilde{\epsilon}_{m+1} \tilde{s}_{m+1} = 0, \quad (7.18)$$

$$\tilde{\epsilon}_m \sqrt{\rho_m^2 - \tilde{s}_m^2 + 1} + \tilde{\epsilon}_{m+1} \sqrt{\rho_m^2 - \tilde{s}_{m+1}^2 + 1} - \epsilon_m \sqrt{\rho_m^2 - s_m^2 + 1} = 0, \quad (7.19)$$

where indices run $m = 1, \dots, n$ with the identification $(n+1) \rightarrow 1$. Remarkably, these constraints can be explicitly solved as

$$s_m = \frac{\tilde{\epsilon}_m \tilde{s}_m - \tilde{\epsilon}_{m+1} \tilde{s}_{m+1}}{\epsilon_m}, \quad \rho_m^2 = -\frac{(\tilde{s}_m^2 + \tilde{s}_{m+1}^2 - 2\sigma_m \tilde{s}_m \tilde{s}_{m+1})}{1 - \sigma_m^2} - 1, \quad (7.20)$$

where we introduced

$$\sigma_m = \frac{\tilde{\epsilon}_m^2 + \tilde{\epsilon}_{m+1}^2 - \epsilon_m^2}{2\tilde{\epsilon}_m \tilde{\epsilon}_{m+1}}. \quad (7.21)$$

Following the general analysis of trivalent vertices on hyperbolic spaces [38] we can show that there are real vertex solutions $\rho_m^2 \geq 0$ provided that the conformal dimensions satisfy the triangle inequalities

$$\tilde{\epsilon}_m + \tilde{\epsilon}_{m+1} \geq \epsilon_m, \quad \tilde{\epsilon}_m + \epsilon_m \geq \tilde{\epsilon}_{m+1}, \quad \tilde{\epsilon}_{m+1} + \epsilon_m \geq \tilde{\epsilon}_m. \quad (7.22)$$

Final assembling. Now, we gather all ingredients discussed above. In general, there are $2n$ equations (7.16), $2n$ equations (7.18) and (7.19), and n equations (7.17) for $5n$ variables $\rho_m, s_m, \tilde{s}_m, \gamma_m$, and integration constants $c_m, m = 1, \dots, n$. Thus, we unambiguously fix all these variables in terms of the modular parameter β , the boundary attachment positions y_m , and classical dimensions $\epsilon_m, \tilde{\epsilon}_m, m = 1, \dots, n$. For convenience, using the rotational invariance of the boundary circle we can set $y_1 = 0$. Finally, we will obtain the total length as $L_{\text{dual}} = L_{\text{dual}}(\beta, y_m, \epsilon_m, \tilde{\epsilon}_m)$.

8 Dual networks for the 2-point block

We shall apply the general scheme described above to the 2-point case. The respective geodesic network is shown on figure 3. It consists of four different segments: two loop segments and two legs stretched from the boundary attachment points to trivalent vertex points.

The total length is given by

$$L_{\text{dual}} = \tilde{\epsilon}_1 L_{\text{loop}}^1 + \tilde{\epsilon}_2 L_{\text{loop}}^2 + \epsilon_1 L_{\text{leg}}^1 + \epsilon_2 L_{\text{leg}}^2, \quad (8.1)$$

where the segments are described by (7.12)–(7.14). The radial coordinates of each loop segment $R_m(t)$ in (7.12) are described by the evolution equation (7.15) subjected to the vertex boundary conditions (7.16), which in this case take the form

$$R_1(\gamma_1) = \rho_1, \quad R_1(\gamma_2) = \rho_2, \quad R_2(\gamma_2) = \rho_2, \quad R_2(\gamma_1 + \beta) = \rho_1. \quad (8.2)$$

The general solutions $R_1 = R_1(t|c_1, \tilde{s}_1)$ and $R_2 = R_2(t|c_2, \tilde{s}_2)$ are parameterized by two integrations constants c_1 and c_2 .

On top of that there are four momentum conservation conditions (7.18) and (7.19). Thus, we have to solve 8 equations to fix 8 variables $\rho_{1,2}, s_{1,2}, \tilde{s}_{1,2}$, and $c_{1,2}$. Exact solutions to this algebraic systems are not known that is a common problem arising in the bulk analysis, for a review see, e.g. [38].

Figure 3. The necklace graph on the annulus. The inner and outer black solid circles represent the conformal boundary. The blue dashed circle goes along the $r = 0$ radius and corresponds to the 0-point block. The blue solid loop is a deformation of the dashed circle by external operators. On the conformal boundary, we choose $y_1 = 0$ and denote $y_2 \equiv y$.

In what follows we use different approximations analogous to those developed for the classical conformal blocks in section 4. In general, we assume that the loop segments are more massive than the external legs,

$$\tilde{\epsilon}_m/\epsilon_k \ll 1. \quad (8.3)$$

To simplify our consideration we assume that $\tilde{\epsilon}_k = \tilde{\epsilon}_m \equiv \tilde{\epsilon}$ that gives rise to the natural heavy parameter. Expanding around $\tilde{\epsilon} = \infty$ we arrive at the perturbative geodesic network length function which is linear in $\tilde{\epsilon}$ and depends on $\delta_i = \epsilon_i/\tilde{\epsilon} \ll 1$. In the 2-point case, there are two small parameters that define further possible perturbation expansions: (i) $\delta_2 \ll \delta_1$, (ii) $\delta_1 = \delta_2$. In this section we explicitly consider these two opposite regimes referring to them respectively as superlight and double leg expansions.

8.1 Superlight expansion

In this case, the 2-point block can be considered as a small perturbation of the 1-point block. Recalling the constraints (4.1),

$$\tilde{\epsilon}_1 = \tilde{\epsilon}_2 \equiv \tilde{\epsilon}, \quad \delta = \frac{\epsilon_1}{\tilde{\epsilon}} \ll 1, \quad \nu = \frac{\epsilon_2}{\epsilon_1} \ll 1, \quad (8.4)$$

we expand any quantity F as follows $F(\delta, \nu) = F_0(\delta) + \nu F_1(\delta) + \nu^2 F_2(\delta) + \dots$, where $F_0(\delta)$ is a seed function corresponding to the 1-point case, while expansion coefficients $F_i(\delta)$ are expanded in δ .

8.1.1 The seed solution: 1-point block

The seed solution arising at $\nu = 0$ is given by the tadpole geodesic graph corresponding to the 1-point perturbative torus block [15]. However, it is not exact in δ because the remaining geodesic equations are still too complicated to be solved exactly.

In the 1-point case the conservation conditions are reduced to

$$\tilde{s}_1 = \tilde{s}_2, \quad s_1 = 0, \quad \rho_1 = \sqrt{\frac{4\tilde{s}_1^2}{4 - \delta^2} - 1}. \quad (8.5)$$

Following [15] we expand $\tilde{s}_1 \equiv s$, $R_1(t) \equiv R(t)$, and $\rho_1 \equiv \rho$ as

$$\begin{aligned} s &= 1 + s_2\delta^2 + s_4\delta^4 + \mathcal{O}(\delta^6), \\ R(t) &= R_1(t)\delta^1 + R_3(t)\delta^3 + \mathcal{O}(\delta^5), \\ \rho &= \rho_1\delta^1 + \rho_3\delta^3 + \mathcal{O}(\delta^5). \end{aligned} \quad (8.6)$$

We note that the time coordinate of the vertex defined by (7.17) does not depend on δ . Indeed, $s_1 = 0$, i.e. the leg is stretched along the radial direction, and, therefore, from (7.17) it follows that $\gamma_2 = y_2 = 0$.

Now, we see that the boundary conditions (7.16) in the 1-point case are given by $R(0) = \rho$ and $R(0) = R(\beta)$. Using the first condition we find that the time interval equation (7.10) can be reduced to

$$e^{2t} = \frac{(-1+i\rho)(R(t)-i) \left(-i\rho + is\sqrt{\rho^2 - s^2 + 1} + s^2 - 1 \right) \left(s\sqrt{R(t)^2 - s^2 + 1} + R(t) - is^2 + i \right)}{(\rho-i)(-1+iR(t)) \left(\rho + s\sqrt{\rho^2 - s^2 + 1} - is^2 + i \right) \left(is\sqrt{R(t)^2 - s^2 + 1} - iR(t) + s^2 - 1 \right)}. \quad (8.7)$$

Expanding (8.5), (8.7) in δ using (8.6) we find in the first non-trivial order that there are *two* possible sets of solutions

$$\rho_1 = \frac{1}{2} \coth \frac{\beta}{2}, \quad s_2 = \frac{1}{8} \operatorname{csch}^2 \frac{\beta}{2}, \quad (8.8)$$

$$\rho_1 = \frac{1}{2} \tanh \frac{\beta}{2}, \quad s_2 = -\frac{1}{8} \operatorname{sech}^2 \frac{\beta}{2}, \quad (8.9)$$

which can be used to calculate the corresponding first order correction $R_1(t)$.

The solution set (8.9) and the length of the corresponding tadpole graph has been analyzed in [15]. One can show that in this case the loop goes through the point $t = \beta/2$ and $r = 0$ in all orders in δ . It implies that the deformation caused by the leg pulling the loop is rapidly damped near the vertex point. In what follows we use the solution set (8.8). In this case the radial deviations of the loop segments are not vanishing near $t = \beta/2$ meaning that the leg produces perturbations along the whole loop.¹¹ Most importantly, these two types of solutions correspond to boundary CFTs with two different modular parameters, see our discussion below (7.4).

To summarize, the seed solution is given by

$$R(t) = \frac{1}{2} \left(\coth \frac{\beta}{2} \cosh t - \sinh t \right) \delta + \mathcal{O}(\delta^3) \quad \rho_1 = \frac{\delta}{2} \coth \frac{\beta}{2} + \mathcal{O}(\delta^3), \quad (8.10)$$

$$s_1 = 0, \quad \tilde{s}_1 = \tilde{s}_2, \quad \tilde{s}_1 = 1 + \frac{\delta^2}{8} \operatorname{csch}^2 \frac{\beta}{2} + \mathcal{O}(\delta^4). \quad (8.11)$$

We note that the zeroth order s_2 , ρ_2 , and γ_2 associated to the second vertex turn out to be non-vanishing power series in δ , see below.¹² They can also be treated as the seed quantities not seen at ν^0 order.

¹¹We are grateful to Per Kraus for useful discussion of this point.

¹²For consistency check, we can step aside from the first order formulation used to find (8.10)–(8.11) and reproduce the same answer solving the geodesic equations of motion that follow from the standard second-order formulation, see section 7.1 for more details.

8.1.2 First order corrections

Following appendix E we find the first non-trivial corrections

$$R_\alpha(t) = \frac{\delta}{2} \left(\coth \frac{\beta}{2} \cosh t - \sinh t \right) - \frac{\nu\delta}{2} \operatorname{csch} \frac{\beta}{2} \cosh \left((-)^{\alpha+1} \frac{\beta}{2} + t - y \right) + \mathcal{O}(\nu^2, \delta^3), \quad (8.12)$$

and

$$\tilde{s}_\alpha = 1 + \frac{\delta^2}{8} \operatorname{csch}^2 \frac{\beta}{2} + \frac{\nu\delta^2}{2} \frac{\cosh(y - \delta_{\alpha,1}\beta)}{1 - \cosh \beta} + \mathcal{O}(\nu^2, \delta^4). \quad (8.13)$$

Momenta s_1 and s_2 can be calculated using the conservation condition (E.2). Substituting the above expansions into the total action (7.11) and evaluating the integrals (7.12) we find that

$$L_{\text{dual}} = \tilde{\epsilon}\beta - \frac{\tilde{\epsilon}\delta^2}{4} \coth \frac{\beta}{2} + \frac{\tilde{\epsilon}\nu\delta^2}{2} \left(\sinh y - \coth \frac{\beta}{2} \cosh y \right) + \mathcal{O}(\nu^2, \delta^3), \quad (8.14)$$

where the leading contribution is the known length of the 1-point graph, terms $\mathcal{O}(\nu\delta)$ can be shown to be absent, while the first non-trivial correction is $\mathcal{O}(\nu\delta^2)$. This decomposition is consistent with what we would expect from the boundary side, cf. (D.1).

Using the map (7.2) which in our case takes the form $\beta = -\log q$ and $y = -\log x$ we find that the total length (8.14) goes to

$$L_{\text{dual}} := -\tilde{\epsilon} \log q - \frac{\epsilon\delta^2}{4} \frac{1+q}{1-q} + \frac{\tilde{\epsilon}\nu\delta^2}{2} \frac{q+x^2}{x-qx} + \mathcal{O}(\nu^2, \delta^3). \quad (8.15)$$

Adding the thermal AdS term $S_{\text{thermal}} = \frac{1}{4} \log q$ and expanding in the modular parameter q we find that up to unimportant additive constant $S_{\text{thermal}} + L_{\text{dual}}(\beta, y) = -f^{\tilde{\epsilon}, \delta, \nu}(q, x)$, where the perturbative block is given by (D.1), or, in a more refined form by (6.4). In this way we reproduce the identification formula (7.1).

8.2 Double leg expansion

We will now move on to consider the second perturbation theory. In this case we assume that conformal dimensions are constrained by (4.5). Most of the steps are similar to that of the previous section so here we only write down the resulting expressions. A more detailed analysis is relegated to appendix E.

The radial functions of two loop segments and the corresponding momenta are

$$R_\alpha(t) = \delta \operatorname{csch} \frac{\beta}{2} \cosh \left(\frac{y}{2} - \delta_{\alpha,1} \frac{\beta}{2} \right) \cosh \left(t - \frac{y}{2} - \delta_{\alpha,0} \frac{\beta}{2} \right) + \mathcal{O}(\delta^2), \quad (8.16)$$

and

$$s_\alpha = \frac{\delta}{2} \left(\coth \frac{\beta}{2} \sinh y - \cosh y \right) + \mathcal{O}(\nu^2, \delta^2), \quad (8.17)$$

$$\tilde{s}_\alpha = 1 + \frac{\delta^2}{8} \operatorname{csch}^2 \frac{\beta}{2} + \frac{\nu\delta^2}{2} \frac{\cosh(y - \delta_{\alpha,1}\beta)}{1 - \cosh \beta} + \mathcal{O}(\nu^2, \delta^4). \quad (8.18)$$

We note that in the first order the external momenta coincide. Moreover, both $s_\alpha \neq 0$, i.e. contrary to the 1-point case the legs are curved, cf. figure 3. It follows that the loop segments have different momenta and, therefore, the whole loop is inhomogeneously stretched despite that the loop dimensions are equal.

Substituting the above expansions into (7.11) and (7.12) we obtain that in the first non-trivial order the total action is given by,

$$L_{\text{dual}} = \tilde{\epsilon}\beta - \tilde{\epsilon}\delta^2 \operatorname{csch} \frac{\beta}{2} \cosh \frac{y}{2} \cosh \frac{\beta - y}{2} + \mathcal{O}(\delta^3), \quad (8.19)$$

and, finally, using the map (7.2) we arrive at

$$L_{\text{dual}} = -\tilde{\epsilon} \log q - \frac{\tilde{\epsilon}\delta^2}{2} \frac{(1+x)(q+x)}{x(1-q)} + \mathcal{O}(\delta^3). \quad (8.20)$$

Expanding in the modular parameter q we find that modulo an additive constant $S_{\text{thermal}} + L_{\text{dual}}(\beta, y) = -f^{\delta, \tilde{\epsilon}}(q, x)$, where the perturbative block is given by (D.2), or, in a more refined form by (6.7). This proves the identification formulas (7.1) and (7.2).

9 Concluding remarks

We considered the semiclassical holographic duality in the case of Virasoro CFTs on the torus. We studied classical torus blocks from various perspectives, in particular, we showed that they are dual to geodesic networks stretched in the thermal AdS bulk space, figure 2. In the n -point case we formulated the system of differential and algebraic equations that govern the dual network. Using various approximation schemes we explicitly solved the system in the 2-point case.

Our analysis in this paper was mainly focused on the perturbative classical torus blocks in the s -channel. We explicitly showed that the perturbative classical blocks within one or another perturbation theory are equal to the classical global blocks in the limit of large dimensions. Nevertheless, we also analyzed different forms of the t -channel torus blocks: quantum, global and classical. In particular, we calculated the perturbative classical global block in the case of equal external dimensions and heavy loop channel. In this respect our results prove one of the main conjectures of [19] that global blocks are related to the perturbative classical blocks.

In this paper the geodesic networks are described using the first order formulation which results from partially integrating the standard second-order geodesic equations. In this case the integration constants arising when going to the first order formulation are conserved momenta. In the sphere CFT case, it turns out that momenta are holographically related to the accessory parameters of the monodromy method and this observation is instrumental in proving the holographic duality between classical blocks and geodesic lengths in the n -point case [6, 10, 38]. In this respect, the monodromy method on the torus [39–42] which is essentially based on Virasoro symmetry provides an interesting possibility to go beyond the $sl(2)$ Casimir equation analysis of [19].

The study of the large- c torus CFT and semiclassical duality can be extended in many ways. For example, conformal blocks considered in this paper were defined in two channels

only. The next natural step would be to generalize to other channels and identify the corresponding bulk backgrounds that is intimately related to perturbation schemes we use. Also, it would be interesting to analyze the torus correlation functions with heavy insertions in the Liouville theory along the lines of refs. [43, 44].

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A The s -channel torus block

For illustrative purposes, using the general formula (2.6) we explicitly find the block coefficients in the linear order in the modular parameter,

$$\mathcal{V}_c^{\Delta_{1,2}, \tilde{\Delta}_{1,2}}(q, z_{1,2}) = \mathcal{A}_c^{\Delta_{1,2}, \tilde{\Delta}_{1,2}}(z_{1,2}) + q \mathcal{B}_c^{\Delta_{1,2}, \tilde{\Delta}_{1,2}}(z_{1,2}) + \mathcal{O}(q^2), \quad (\text{A.1})$$

with coefficients

$$\begin{aligned} \mathcal{A}_c^{\Delta_{1,2}, \tilde{\Delta}_{1,2}}(z_{1,2}) &= \sum_{m=0}^{\infty} \sum_{|S|=|T|=m} \frac{\langle \tilde{\Delta}_1 | \phi_1(z_1) | S, \tilde{\Delta}_2 \rangle}{\langle \tilde{\Delta}_1 | \phi_1(z_1) | \tilde{\Delta}_2 \rangle} B_2^{S|T} \frac{\langle \tilde{\Delta}_2, T | \phi_2(z_2) | \tilde{\Delta}_1 \rangle}{\langle \tilde{\Delta}_2 | \phi_2(z_2) | \tilde{\Delta}_1 \rangle}, \\ \mathcal{B}_c^{\Delta_{1,2}, \tilde{\Delta}_{1,2}}(z_{1,2}) &= \sum_{m=0}^{\infty} \sum_{|S|=|T|=m} \frac{1}{2\tilde{\Delta}_1} \frac{\langle \tilde{\Delta}_1 | L_1 \phi_1(z_1) | S, \tilde{\Delta}_2 \rangle}{\langle \tilde{\Delta}_1 | \phi_1(z_1) | \tilde{\Delta}_2 \rangle} B_2^{S|T} \frac{\langle \tilde{\Delta}_2, T | \phi_2(z_2) L_{-1} | \tilde{\Delta}_1 \rangle}{\langle \tilde{\Delta}_2 | \phi_2(z_2) | \tilde{\Delta}_1 \rangle}, \end{aligned} \quad (\text{A.2})$$

$$(\text{A.3})$$

where the Gram matrix on the first $m = 0, 1, 2$ levels in the basis $\{\mathbb{1}, L_{-1}, L_{-2}, L_{-1}^2, \dots\}$ is given by

$$B_{M|N} = 1, \quad B_{M|N} = 2\tilde{\Delta}, \quad B_{M|N} = \begin{pmatrix} \frac{c}{2} + 4\tilde{\Delta} & 6\tilde{\Delta} \\ 6\tilde{\Delta} & 4\tilde{\Delta}(2\tilde{\Delta} + 1) \end{pmatrix}. \quad (\text{A.4})$$

In a given order, the block coefficients can be represented as the matrix product,

$$\mathcal{V} = \varkappa \text{Tr} B_1^{-1} L B_2^{-1} R, \quad (\text{A.5})$$

where $B_{1,2}$ are the Gram matrices, L and R are matrix elements of the primary operators $\phi_m(z_m)$, and $\varkappa = \varkappa_1 \varkappa_2 = z_1^{\Delta_1} z_2^{\Delta_2} (z_2/z_1)^{\tilde{\Delta}_1 - \tilde{\Delta}_2}$, where $\varkappa_1 = 1/\langle \tilde{\Delta}_1 | \phi_1(z_1) | \tilde{\Delta}_2 \rangle$ and $\varkappa_2 = 1/\langle \tilde{\Delta}_2 | \phi_2(z_2) | \tilde{\Delta}_1 \rangle$.

Let $x = z_2/z_1$. Then, the expansion coefficients up to the second order in x are given by

$$\mathcal{A} = 1 + \mathcal{A}_1 x + \mathcal{A}_2 x^2 + \dots, \quad \mathcal{B} = \mathcal{B}_0 x^{-1} + \mathcal{B}_1 + \mathcal{B}_2 x + \dots, \quad (\text{A.6})$$

where $\mathcal{A}_i$ and $\mathcal{B}_j$ can be read off from the original expressions (A.2)–(A.3). In what follows we calculate them explicitly.

$\mathcal{A}$ -coefficients. We easily find the first coefficient

$$\mathcal{A}_1 = \frac{(\Delta_1 + \tilde{\Delta}_2 - \tilde{\Delta}_1)(\Delta_2 + \tilde{\Delta}_2 - \tilde{\Delta}_1)}{2\tilde{\Delta}_2}. \quad (\text{A.7})$$

Now, we calculate the second coefficient defined as $\mathcal{A}_2 x^2 = \varkappa \text{Tr} L B_2^{-1} R$, where the Gram matrix on the second level is given by (A.4), and matrices L and R are given by

$$L = \left(\langle \tilde{\Delta}_1 | \phi_1(z_1) L_{-2} | \tilde{\Delta}_2 \rangle \quad \langle \tilde{\Delta}_1 | \phi_1(z_1) L_{-1}^2 | \tilde{\Delta}_2 \rangle \right), \quad R = \begin{pmatrix} \langle \tilde{\Delta}_2 | L_2 \phi_2(z_2) | \tilde{\Delta}_1 \rangle \\ \langle \tilde{\Delta}_2 | L_1^2 \phi_2(z_2) | \tilde{\Delta}_1 \rangle \end{pmatrix}. \quad (\text{A.8})$$

A direct calculation yields

$$\begin{aligned} \langle \tilde{\Delta}_1 | \phi_1(z_1) L_{-2} | \tilde{\Delta}_2 \rangle &= \varkappa_1^{-1} (\tilde{\Delta}_2 + 2\Delta_1 - \tilde{\Delta}_1) z_1^{-2} \\ \langle \tilde{\Delta}_1 | \phi_1(z_1) L_{-1}^2 | \tilde{\Delta}_2 \rangle &= \varkappa_1^{-1} (\tilde{\Delta}_1 - \Delta_1 - \tilde{\Delta}_2) (\tilde{\Delta}_1 - \Delta_1 - \tilde{\Delta}_2 - 1) z_1^{-2} \\ \langle \tilde{\Delta}_2 | L_2 \phi_2(z_2) | \tilde{\Delta}_1 \rangle &= \varkappa_2^{-1} (\tilde{\Delta}_2 + 2\Delta_2 - \tilde{\Delta}_1) z_2^2 \\ \langle \tilde{\Delta}_2 | L_1^2 \phi_2(z_2) | \tilde{\Delta}_1 \rangle &= \varkappa_2^{-1} (\tilde{\Delta}_2 + \Delta_2 - \tilde{\Delta}_1) (\tilde{\Delta}_2 + \Delta_2 - \tilde{\Delta}_1 + 1) z_2^2 \end{aligned} \quad (\text{A.9})$$

where $\varkappa_{1,2}$ are defined below (A.5). Using (A.9) we can finally assemble the coefficient $\mathcal{A}_2$ as follows

$$\begin{aligned} \mathcal{A}_2 &= \frac{1}{8c\tilde{\Delta}_2 + 4c + 8\tilde{\Delta}_2(8\tilde{\Delta}_2 - 5)} \left[(-\tilde{\Delta}_1 + \tilde{\Delta}_2 + \Delta_2)(-\tilde{\Delta}_1 + \tilde{\Delta}_2 + \Delta_2 + 1) \times \right. \\ &\quad \times \left(\frac{(c + 8\tilde{\Delta}_2)(\tilde{\Delta}_1 - \Delta_1 - \tilde{\Delta}_2 - 1)(\tilde{\Delta}_1 - \Delta_1 - \tilde{\Delta}_2)}{\tilde{\Delta}_2} + 12(\tilde{\Delta}_1 - 2\Delta_1 - \tilde{\Delta}_2) \right) + \\ &\quad \left. + 4(3\tilde{\Delta}_1^2 - \tilde{\Delta}_1(6\Delta_1 + 2\tilde{\Delta}_2 + 1) + (\Delta_1 - \tilde{\Delta}_2)(3\Delta_1 + \tilde{\Delta}_2 - 1))(\tilde{\Delta}_1 - \tilde{\Delta}_2 - 2\Delta_2) \right]. \end{aligned} \quad (\text{A.10})$$

$\mathcal{B}$ -coefficients. Analogously, we find

$$\begin{aligned} \mathcal{B}_0 &= \frac{(\tilde{\Delta}_1 + \Delta_1 - \tilde{\Delta}_2)(\tilde{\Delta}_1 + \Delta_2 - \tilde{\Delta}_2)}{2\tilde{\Delta}_1}, \\ \mathcal{B}_1 &= \frac{\left((\tilde{\Delta}_2 + \Delta_1 - \tilde{\Delta}_1)(\tilde{\Delta}_1 + \Delta_1 - \tilde{\Delta}_2 - 1) + 2\tilde{\Delta}_2 \right) \left((\tilde{\Delta}_1 + \Delta_2 - \tilde{\Delta}_2)(\tilde{\Delta}_2 + \Delta_2 - \tilde{\Delta}_1 - 1) + 2\tilde{\Delta}_1 \right)}{4\tilde{\Delta}_1\tilde{\Delta}_2}. \end{aligned} \quad (\text{A.11})$$

Now, we calculate $\mathcal{B}_2 = \frac{\varkappa}{2\tilde{\Delta}_1} \text{Tr} L B_2^{-1} R$, where new matrices L and R are given by

$$L = \left(\langle \tilde{\Delta}_1 | L_1 \phi_1(z_1) L_{-2} | \tilde{\Delta}_2 \rangle \quad \langle \tilde{\Delta}_1 | L_1 \phi_1(z_1) L_{-1}^2 | \tilde{\Delta}_2 \rangle \right), \quad R = \begin{pmatrix} \langle \tilde{\Delta}_2 | L_2 \phi_2(z_2) L_{-1} | \tilde{\Delta}_1 \rangle \\ \langle \tilde{\Delta}_2 | L_1^2 \phi_2(z_2) L_{-1} | \tilde{\Delta}_1 \rangle \end{pmatrix}. \quad (\text{A.12})$$

The matrix elements are given by

$$\begin{aligned}
 \langle \tilde{\Delta}_1 | L_1 \phi_1(z_1) L_{-2} | \tilde{\Delta}_2 \rangle &= \kappa_1^{-1} (\tilde{\Delta}_2 - \Delta_1 - \tilde{\Delta}_1) (\tilde{\Delta}_1 - 2\Delta_1 - \tilde{\Delta}_2 + 1) z_1^{-1} \\
 \langle \tilde{\Delta}_1 | L_1 \phi_1(z_1) L_{-1}^2 | \tilde{\Delta}_2 \rangle &= \kappa_1^{-1} (\tilde{\Delta}_2 + \Delta_1 - \tilde{\Delta}_1) \\
 &\quad \times \left[(\tilde{\Delta}_2 + \Delta_1 - \tilde{\Delta}_1 + 1) (\tilde{\Delta}_1 + \Delta_1 - \tilde{\Delta}_2 - 2) + 2(2\tilde{\Delta}_2 + 1) \right] z_1^{-1} \\
 \langle \tilde{\Delta}_2 | L_2 \phi_2(z_2) L_{-1} | \tilde{\Delta}_1 \rangle &= \kappa_2^{-1} (\tilde{\Delta}_1 + \Delta_2 - \tilde{\Delta}_2) (\tilde{\Delta}_2 + 2\Delta_2 - \tilde{\Delta}_1 - 1) z_2 \\
 \langle \tilde{\Delta}_2 | L_1^2 \phi_2(z_2) L_{-1} | \tilde{\Delta}_1 \rangle &= \kappa_2^{-1} (\tilde{\Delta}_2 + \Delta_2 - \tilde{\Delta}_1) \\
 &\quad \times \left[(\tilde{\Delta}_2 + \Delta_2 - \tilde{\Delta}_1 + 1) (\tilde{\Delta}_1 + \Delta_2 - \tilde{\Delta}_2 - 2) + 2(2\tilde{\Delta}_2 + 1) \right] z_2 \quad (\text{A.13})
 \end{aligned}$$

Finally, we assemble the coefficient $\mathcal{B}_2$ as follows

$$\begin{aligned}
 \mathcal{B}_2 &= \frac{1}{8\tilde{\Delta}_1(2c\tilde{\Delta}_2 + c + 2\tilde{\Delta}_2(8\tilde{\Delta}_2 - 5))} \left[(-\tilde{\Delta}_1 + \tilde{\Delta}_2 + \Delta_2) ((\tilde{\Delta}_1 - \tilde{\Delta}_2 + \Delta_2 - 2) (-\tilde{\Delta}_1 + \tilde{\Delta}_2 + \Delta_2 + 1) \right. \\
 &\quad \left. + 4\tilde{\Delta}_2 + 2) \left(\frac{(c + 8\tilde{\Delta}_2) (-\tilde{\Delta}_1 + \Delta_1 + \tilde{\Delta}_2) ((\tilde{\Delta}_1 + \Delta_1 - \tilde{\Delta}_2 - 2) (-\tilde{\Delta}_1 + \Delta_1 + \tilde{\Delta}_2 + 1) + 4\tilde{\Delta}_2 + 2)}{\tilde{\Delta}_2} \right) \right. \\
 &\quad \left. + 12(\tilde{\Delta}_1 + \Delta_1 - \tilde{\Delta}_2) (\tilde{\Delta}_1 - 2\Delta_1 - \tilde{\Delta}_2 + 1) + 4(3\tilde{\Delta}_1^3 - \tilde{\Delta}_1^2(3\Delta_1 + 5\tilde{\Delta}_2 + 7) + \right. \\
 &\quad \left. + \tilde{\Delta}_1(-3\Delta_1^2 + 2\Delta_1(\tilde{\Delta}_2 + 5) + \tilde{\Delta}_2^2 + 6\tilde{\Delta}_2 + 2) + 3\Delta_1^3 - \Delta_1^2(5\tilde{\Delta}_2 + 7) + \right. \\
 &\quad \left. + \Delta_1(\tilde{\Delta}_2^2 + 6\tilde{\Delta}_2 + 2) + \tilde{\Delta}_2(\tilde{\Delta}_2^2 + \tilde{\Delta}_2 - 2) \right) (\tilde{\Delta}_1 - \tilde{\Delta}_2 + \Delta_2) (\tilde{\Delta}_1 - \tilde{\Delta}_2 - 2\Delta_2 + 1) \Big]. \quad (\text{A.14})
 \end{aligned}$$

We note that the above calculations can be effectively extended to any order using a recursive technique elaborated in [45].

B The t -channel torus block

As an example, in this appendix we calculate a first few t -channel block coefficients by directly calculating the matrix elements in (2.12). The OPE of two primary fields (2.9) is defined by the descendent operator

$$\begin{aligned}
 \psi_{\tilde{\Delta}_2}(z_{1,2}) &= \phi_{\tilde{\Delta}_2}(z_2) + \beta_1(z_1 - z_2) \phi_{\tilde{\Delta}_2}^{(-1,0)}(z_2) \\
 &\quad + \beta_{21}(z_1 - z_2)^2 \phi_{\tilde{\Delta}_2}^{(-1,-1)}(z_2) + \beta_{22}(z_1 - z_2)^2 \phi_{\tilde{\Delta}_2}^{(-2,0)}(z_2) + \dots, \quad (\text{B.1})
 \end{aligned}$$

where $\phi_{\tilde{\Delta}_2}^{(-1,0)}(z_2) = (L_{-1}\phi_{\tilde{\Delta}_2})(z_2)$, $\phi_{\tilde{\Delta}_2}^{(-1,-1)}(z_2) = (L_{-1}^2\phi_{\tilde{\Delta}_2})(z_2)$, and $\phi_{\tilde{\Delta}_2}^{(-2,0)}(z_2) = (L_{-2}\phi_{\tilde{\Delta}_2})(z_2)$ are descendants of the primary field $\phi_{\tilde{\Delta}_2}(z_2)$. The lowest level β -coefficients are explicitly known (see, e.g., [46])

$$\begin{aligned}
 \beta_1 &= \frac{\tilde{\Delta}_2 + \Delta_1 - \Delta_2}{2\tilde{\Delta}_2}, \quad \beta_{21} = \frac{(\tilde{\Delta}_2 + \Delta_1 - \Delta_2)(\tilde{\Delta}_2 + \Delta_1 - \Delta_2 + 1)}{4\tilde{\Delta}_2(2\tilde{\Delta}_2 + 1)} - \frac{3}{2(2\tilde{\Delta}_2 + 1)}\beta_{22}, \\
 \beta_{22} &= \left(\frac{\Delta_1 + \Delta_2}{2} + \frac{\tilde{\Delta}_2(\tilde{\Delta}_2 - 1)}{2(2\Delta_2 + 1)} - \frac{3(\Delta_1 - \Delta_2)^2}{2(2\Delta_2 + 1)} \right) / \left(4\tilde{\Delta}_2 + \frac{c}{2} - \frac{9\tilde{\Delta}_2}{2\tilde{\Delta}_2 + 1} \right). \quad (\text{B.2})
 \end{aligned}$$

Using the Gram matrix expressions (A.4) we find in the first order in the modular parameter

$$\mathcal{V}_c^{\Delta_{1,2}, \tilde{\Delta}_{1,2}}(q, z_{1,2}) = \mathcal{A}_c^{\Delta_{1,2}, \tilde{\Delta}_{1,2}}(q, z_{1,2}) + q \mathcal{B}_c^{\Delta_{1,2}, \tilde{\Delta}_{1,2}}(q, z_{1,2}) + \mathcal{O}(q^2), \quad (\text{B.3})$$

where

$$\begin{aligned}\mathcal{A}_c^{\Delta_{1,2},\tilde{\Delta}_{1,2}}(q, z_{1,2}) &= \varkappa \langle \tilde{\Delta}_1 | \psi_{\tilde{\Delta}_2}(z_{1,2}) | \tilde{\Delta}_1 \rangle, \\ \mathcal{B}_c^{\Delta_{1,2},\tilde{\Delta}_{1,2}}(q, z_{1,2}) &= \frac{\varkappa}{2\tilde{\Delta}_1} \langle \tilde{\Delta}_1 | L_1 \psi_{\tilde{\Delta}_2}(z_{1,2}) L_{-1} | \tilde{\Delta}_1 \rangle,\end{aligned}\quad (\text{B.4})$$

where $\varkappa^{-1} = \langle \tilde{\Delta}_1 | \phi_{\tilde{\Delta}_2}(z_2) | \tilde{\Delta}_1 \rangle$. Let $w = (z_1 - z_2)/z_2$. Then, the expansion coefficients (B.4) up to the second order in w can be represented as

$$\mathcal{A} = 1 + \mathcal{A}_1 w + \mathcal{A}_2 w^2 + \dots, \quad \mathcal{B} = \mathcal{B}_0 + \mathcal{B}_1 w + \mathcal{B}_2 w^2 + \dots. \quad (\text{B.5})$$

The matrix elements are found to be

$$\begin{aligned}\langle \tilde{\Delta}_1 | \phi_{\tilde{\Delta}_2}^{(-1,0)} | \tilde{\Delta}_1 \rangle &= -\varkappa^{-1} \tilde{\Delta}_2 w^{-1}, & \langle \tilde{\Delta}_1 | \phi_{\tilde{\Delta}_2}^{(-1,-1)} | \tilde{\Delta}_1 \rangle &= \varkappa^{-1} \tilde{\Delta}_2 (\tilde{\Delta}_2 + 1) w^{-2}, \\ \langle \tilde{\Delta}_1 | \phi_{\tilde{\Delta}_2}^{(-2,0)} | \tilde{\Delta}_1 \rangle &= \varkappa^{-1} (\tilde{\Delta}_1 + \tilde{\Delta}_2) w^{-2}, & \langle \tilde{\Delta}_1 | L_1 \phi_{\tilde{\Delta}_2} L_{-1} | \tilde{\Delta}_1 \rangle &= \varkappa^{-1} \left[2\tilde{\Delta}_1 + \tilde{\Delta}_2 (\tilde{\Delta}_2 - 1) \right], \\ \langle \tilde{\Delta}_1 | L_1 \phi_{\tilde{\Delta}_2}^{(-1,0)} L_{-1} | \tilde{\Delta}_1 \rangle &= -\varkappa^{-1} \tilde{\Delta}_2 \left[2\tilde{\Delta}_1 + \tilde{\Delta}_2 (\tilde{\Delta}_2 - 1) \right] w^{-1}, \\ \langle \tilde{\Delta}_1 | L_1 \phi_{\tilde{\Delta}_2}^{(-1,-1)} L_{-1} | \tilde{\Delta}_1 \rangle &= \varkappa^{-1} \tilde{\Delta}_2 (\tilde{\Delta}_2 + 1) \left[2\tilde{\Delta}_1 + \tilde{\Delta}_2 (\tilde{\Delta}_2 - 1) \right] w^{-2}, \\ \langle \tilde{\Delta}_1 | L_1 \phi_{\tilde{\Delta}_2}^{(-2,0)} L_{-1} | \tilde{\Delta}_1 \rangle &= \varkappa^{-1} \left(2\tilde{\Delta}_1 (\tilde{\Delta}_1 + \tilde{\Delta}_2) + (\tilde{\Delta}_2 + 1) \left[(\tilde{\Delta}_2 + 2)(\tilde{\Delta}_1 + \tilde{\Delta}_2) - 3\tilde{\Delta}_2 \right] \right) w^{-2}.\end{aligned}\quad (\text{B.6})$$

For the sake of simplicity, we represent the coefficients in terms of the β -coefficients as follows

$$\mathcal{A}_1 = -\tilde{\Delta}_2 \beta_1, \quad \mathcal{A}_2 = \tilde{\Delta}_2 (\tilde{\Delta}_2 + 1) \beta_{21} + (\tilde{\Delta}_1 + \tilde{\Delta}_2) \beta_{22}, \quad (\text{B.7})$$

and

$$\begin{aligned}\mathcal{B}_0 &= \frac{2\tilde{\Delta}_1 + \tilde{\Delta}_2 (\tilde{\Delta}_2 - 1)}{2\tilde{\Delta}_1}, & \mathcal{B}_1 &= \frac{-\tilde{\Delta}_2 \left(2\tilde{\Delta}_1 + \tilde{\Delta}_2 (\tilde{\Delta}_2 - 1) \right) \beta_1}{2\tilde{\Delta}_1}, \\ \mathcal{B}_2 &= \frac{1}{2\tilde{\Delta}_1} \left(\tilde{\Delta}_2 (\tilde{\Delta}_2 + 1) \left[2\tilde{\Delta}_1 + \tilde{\Delta}_2 (\tilde{\Delta}_2 - 1) \right] \beta_{21} + \right. \\ &\quad \left. + \left(2\tilde{\Delta}_1 (\tilde{\Delta}_1 + \tilde{\Delta}_2) + (\tilde{\Delta}_2 + 1) \left[(\tilde{\Delta}_2 + 2)(\tilde{\Delta}_1 + \tilde{\Delta}_2) - 3\tilde{\Delta}_2 \right] \right) \beta_{22} \right).\end{aligned}\quad (\text{B.8})$$

C Combinatorial representation of the s -channel block

Here we explicitly elaborate the combinatorial AGT [47] representation of the torus conformal multi-point blocks. First, we set convenient Liouville-like parametrization. Instead of central charge c and conformal dimensions Δ_k we will use parameters b and p_k according to

$$c = 1 + 6Q^2, \quad Q = b + \frac{1}{b}, \quad \Delta_k = \frac{Q^2}{4} - p_k. \quad (\text{C.1})$$

Let $F(\mathbf{q})$, where $\mathbf{q} = (q_1, \dots, q_N)$, be the conformal block of N primary fields $\phi_{\Delta_k}(z_k)$, $k = 1, \dots, N$, on the torus with the modular parameter q , associated with the (left) diagram on figure 2 and with the intermediate channel parameters between two external lines

p_k and p_{k+1} being $\tilde{p}_k$. The modular parameter q and the holomorphic coordinates z_k are expressed in terms of $\{q_i\}$ via

$$q = \prod_{i=1}^N q_i, \quad q_k = \frac{z_{k+1}}{z_k}, \quad k = 1, \dots, N-1. \quad (\text{C.2})$$

According to AGT the Virasoro conformal blocks obey the following factorization property

$$F(\mathbf{q}) = F^H(\mathbf{q}) \mathcal{Z}(\mathbf{q}), \quad (\text{C.3})$$

where $F^H(\mathbf{q})$ is the N -point conformal block of Heisenberg primaries $\tilde{\phi}_k(z_k)$ associated with the same diagram. The Heisenberg algebra is defined by

$$[a_n, a_m] = \frac{n}{2} \delta_{n+m,0}, \quad n, m \in \mathbb{Z}, \quad (\text{C.4})$$

and $\tilde{\phi}_k(z_k)$ are related to $\phi_{\Delta_k}(z_k)$ by the requirement that

$$[a_{\pm n}, \tilde{\phi}_k(z)] = \left(p_k \pm \frac{Q}{2}\right) z^{\pm n} \tilde{\phi}_k(z), \quad n \in \mathbb{N}, \quad (\text{C.5})$$

and p_k is the conformal parameter of the corresponding Virasoro primary field $\phi_{\Delta_k}(z_k)$.

Using commutation relations (C.4), (C.5) and diagonal form of the Heisenberg Gram matrix the Heisenberg conformal block $F^H(\mathbf{q})$ coefficients can be computed order by order.¹³ For example, on the two lower levels we find

$$\sim q_i : \quad M^{11} \langle \tilde{\phi}_{\tilde{p}_i} | \tilde{\phi}_{p_i} a_{-1} | \tilde{\phi}_{\tilde{p}_{i+1}} \rangle \langle \tilde{\phi}_{\tilde{p}_{i+1}} | a_1 \tilde{\phi}_{p_{i+1}} | \tilde{\phi}_{\tilde{p}_{i+2}} \rangle = 2 \left(p_i - \frac{Q}{2}\right) \left(p_{i+1} + \frac{Q}{2}\right), \quad (\text{C.6})$$

and

$$\begin{aligned} \sim q_i q_{i+1} : \quad & (M^{11})^2 \langle \tilde{\phi}_{\tilde{p}_i} | a_1 \tilde{\phi}_{p_i} a_{-1} | \tilde{\phi}_{\tilde{p}_{i+1}} \rangle \langle \tilde{\phi}_{\tilde{p}_{i+1}} | a_1 \tilde{\phi}_{p_{i+1}} a_{-1} | \tilde{\phi}_{\tilde{p}_{i+2}} \rangle = \\ & = 4 \left[\left(p_i + \frac{Q}{2}\right) \langle \tilde{\phi}_{\tilde{p}_i} | \tilde{\phi}_{p_i}(1) a_{-1} | \tilde{\phi}_{\tilde{p}_{i+1}} \rangle + \frac{1}{2} \langle \tilde{\phi}_{\tilde{p}_i} | \tilde{\phi}_{p_i}(1) | \tilde{\phi}_{\tilde{p}_{i+1}} \rangle \right] \times \left[i \rightarrow i+1 \right] \\ & = \left[2 \left(p_i - \frac{Q}{2}\right) \left(p_i + \frac{Q}{2}\right) + 1 \right] \left[2 \left(p_{i+1} - \frac{Q}{2}\right) \left(p_{i+1} + \frac{Q}{2}\right) + 1 \right], \end{aligned} \quad (\text{C.7})$$

where M^{11} stands for the inverse (first-level block) Gram matrix and the normalization $\langle \tilde{\phi}_{\tilde{p}_i} | \tilde{\phi}_{p_j} | \tilde{\phi}_{\tilde{p}_k} \rangle = 1$ is taken into account.

In the case of 2-point s -channel torus Heisenberg block $F_{2pt}^H(\mathbf{q})$ which contributes to (C.3), we find up to forth degree in q_i the following expression

$$\prod_{n=0}^{\infty} (1 - q^n q_1)^{\alpha_{1,2}} (1 - q^n q_2)^{\alpha_{2,1}} (1 - q^{n+1})^{\alpha_{1,1} + \alpha_{2,2}}, \quad (\text{C.8})$$

where $\frac{w}{z} = q_1$ and $q = q_1 q_2$, and $\alpha_{i,j} = 2(p_i - \frac{Q}{2})(p_j + \frac{Q}{2})$.

¹³For the analogous consideration on the sphere see [48]. Alternative considerations are discussed in [49, 50].

The second factor in (C.3) has the following combinatorial representation

$$\mathcal{Z}(\mathbf{q}) = \sum_{k_i \in \mathbb{N}_0} q_1^{k_1} q_2^{k_2} \cdots q_N^{k_N} \mathcal{Z}_k(\Delta, \tilde{\Delta}, c), \quad (\text{C.9})$$

where $\mathbb{N}_0$ stands for non-negative integers, and

$$\mathcal{Z}_k(\Delta, \tilde{\Delta}, c) = \sum_{\vec{\lambda}_1, \dots, \vec{\lambda}_N} \frac{Z(p_1|\tilde{p}_0, \lambda_0; \tilde{p}_1, \vec{\lambda}_1) Z(p_2|\tilde{p}_1, \vec{\lambda}_1; \tilde{p}_2, \vec{\lambda}_2) \cdots Z(p_N|\tilde{p}_{N-1}, \vec{\lambda}_{N-1}; \tilde{p}_N, \lambda_N)}{Z(\frac{Q}{2}|\tilde{p}_1, \vec{\lambda}_1; \tilde{p}_1, \vec{\lambda}_1) \cdots Z(\frac{Q}{2}|\tilde{p}_N, \vec{\lambda}_N; \tilde{p}_N, \vec{\lambda}_N)}. \quad (\text{C.10})$$

Here, the sum goes over N pairs of Young tableaux $\vec{\lambda}_j = (\lambda_j^{(1)}, \lambda_j^{(2)})$,¹⁴ with the total number of cells $|\vec{\lambda}_j| \equiv |\lambda_j^{(1)}| + |\lambda_j^{(2)}| = k_j$ and we identify $\tilde{p}_0 = \tilde{p}_N$, $\lambda_0 = \lambda_N$. The explicit form of Z reads

$$Z(P''|P', \vec{\mu}; P, \vec{\lambda}) = \prod_{i,j=1}^2 \left[\prod_{s \in \lambda_i} \left(P'' - E_{\lambda_i, \mu_j}((-1)^j P' - (-1)^i P|s) + \frac{Q}{2} \right) \times \prod_{t \in \mu_j} \left(P'' + E_{\mu_j, \lambda_i}((-1)^i P - (-1)^j P'|t) - \frac{Q}{2} \right) \right], \quad (\text{C.11})$$

where

$$E_{\lambda, \mu}(x|s) = x - b l_\mu(s) + b^{-1}(a_\lambda(s) + 1). \quad (\text{C.12})$$

For a cell $s = (m, n)$ such that m and n label a respective row and a column, the arm-length function $a_\lambda(s) = (\lambda)_m - n$ and the leg-length function $l_\lambda(s) = (\lambda)_n^T - m$, where $(\lambda)_m$ is the length of m -th row of the Young tableau λ , and $(\lambda)_n^T$ the height of the n -th column, where $(\lambda)^T$ stands for the transposed Young tableau.

Using the final combinatorial expression for the 2-point s -channel torus block we find the block coefficients reproducing those in appendix A.

D Perturbative classical s -channel torus blocks

Using the combinatorial representation of the 2-point torus block elaborated in appendix C we find the perturbative classical s -channel block:

Superlight expansion.

$$f^{\tilde{\epsilon}, \delta, \nu}(q, x) = (\tilde{\epsilon} - 1/4) \log q + \tilde{\epsilon} \left[\delta^2 \left(\frac{q}{2} + \frac{q^2}{2} + \dots \right) + \delta^4 \left(\frac{q^2}{16} + \dots \right) \right] + \\ + \tilde{\epsilon} \nu \left[\delta^2 \left(\frac{q^2}{2x} + \frac{qx}{2} + \frac{q}{2x} + \frac{x}{2} + \dots \right) + \delta^3 \left(\frac{q^3}{4x^2} + \frac{q^2}{8x^2} - \frac{q^2}{2} + \frac{qx^2}{4} - \frac{q}{4} + \frac{x^2}{8} + \dots \right) + \right. \\ \left. \delta^4 \left(\frac{q^3}{24x^3} - \frac{q^2}{8x} - \frac{qx}{8} + \frac{x^3}{24} + \dots \right) + \delta^5 \left(\frac{q^4}{64x^4} - \frac{q^3}{16x^2} + \frac{3q^2}{32} - \frac{qx^2}{16} + \frac{x^4}{64} + \dots \right) + \dots \right], \quad (\text{D.1})$$

where the ellipses denote higher orders in q, x and $\tilde{\epsilon}, \delta, \nu$.

¹⁴The pairs of diagrams label the elements of the orthogonal basis in the representation space of the composite algebra $H \otimes Vir$ [51].

Double leg expansion.

$$\begin{aligned}
 f^{\delta, \tilde{\epsilon}}(q, x) = & (\tilde{\epsilon} - 1/4) \log q + \\
 & + \tilde{\epsilon} \delta^2 \left(q^2 \left(\frac{1}{2x} + 1 \right) + q \left(\frac{x}{2} + \frac{1}{2x} + 1 \right) + \frac{x}{2} \right) \\
 & + \tilde{\epsilon} \delta^3 \left(\frac{q^3}{2x^2} + q^2 \left(\frac{1}{4x^2} - 1 \right) + q \left(\frac{x^2}{2} - \frac{1}{2} \right) + \frac{x^2}{4} \right) \\
 & + \tilde{\epsilon} \delta^4 \left(q^3 \left(\frac{5}{24x^3} - \frac{1}{4x^2} \right) + q^2 \left(-\frac{1}{16x^2} - \frac{3}{8x} - \frac{1}{8} \right) + q \left(-\frac{x^2}{4} - \frac{3x}{8} \right) + \frac{5x^3}{24} - \frac{x^2}{16} \right) \\
 & + \tilde{\epsilon} \delta^5 \left(\frac{7q^4}{32x^4} + q^3 \left(-\frac{1}{8x^3} - \frac{3}{8x^2} \right) + q^2 \left(\frac{1}{8x} + \frac{5}{16} \right) + q \left(-\frac{3x^2}{8} + \frac{x}{8} \right) + \frac{7x^4}{32} - \frac{x^3}{8} \right) \\
 & + \tilde{\epsilon} \delta^6 \left(-\frac{7q^4}{32x^4} + q^3 \left(\frac{1}{48x^3} + \frac{5}{16x^2} \right) + \frac{5qx^2}{16} - \frac{7x^4}{32} + \frac{x^3}{48} \right) \\
 & + \tilde{\epsilon} \delta^7 \left(\frac{5q^4}{64x^4} - \frac{q^3}{16x^2} - \frac{q^2}{32} - \frac{qx^2}{16} + \frac{5x^4}{64} \right) \\
 & + \tilde{\epsilon} \delta^8 \left(-\frac{5q^4}{512x^4} - \frac{5x^4}{512} \right) + \dots, \tag{D.2}
 \end{aligned}$$

where the ellipses denote higher orders in q, x and $\tilde{\epsilon}, \delta$.

E Perturbation theory in the bulk

Superlight expansion. Let $F = \{s_\alpha, \tilde{s}_\alpha, \gamma_\alpha, \rho_\alpha, R_\alpha, L_{\text{loop}}^\alpha, L_{\text{leg}}^\alpha \mid \alpha = 1, 2\}$ be a double series in the deformation parameters ν and δ ,

$$F = \sum_{m,n=0}^{\infty} F_{[mn]} \nu^m \delta^n. \tag{E.1}$$

Using the general formulas (7.20) we solve the conservation conditions as

$$s_1 = \frac{\tilde{s}_2 - \tilde{s}_1}{\delta}, \quad s_2 = \frac{s_1}{\nu}, \tag{E.2}$$

and

$$\rho_1 = \left[\tilde{s}_1 \tilde{s}_2 + \frac{(\tilde{s}_1 - \tilde{s}_2)^2}{\delta^2} + \frac{\delta^2}{4} - 1 \right]^{1/2} \left(1 - \frac{\delta^2}{4} \right)^{-1/2}, \quad \rho_2 = \rho_1|_{\delta \rightarrow \delta\nu}. \tag{E.3}$$

In what follows we discuss several peculiarities of geodesic equations not seen previously in the 1-point torus case and in the n -point sphere case. The main observation is that the angle (time) positions must be known explicitly because they are the integration limits in the loop segment integrals.

(I). First of all, we note that defining the radial vertex positions as $\rho_\alpha = R_\alpha(\gamma_\alpha)$ and then decomposing in ν we find in the lowest orders relations like

$$\rho_{\alpha[0]} = R_{\alpha[0]}(\gamma_{\alpha[0]}), \quad \rho_{\alpha[1]} = R_{\alpha[1]}(\gamma_{\alpha[0]}) + \gamma_{\alpha[1]} \dot{R}_{\alpha[0]}(\gamma_{\alpha[0]}), \quad \dots \tag{E.4}$$

where the dot denotes a time derivative. Such decompositions do not arise in the 1-point case because the external leg is radial, and, therefore, the angle position is fixed and coincides with the boundary attachment position.

When defining the superlight expansions (E.1) we have to assume that ν -corrections start with δ^1 . It follows that setting $\delta = 0$ we are guaranteed that all ν -dependent terms vanish, i.e. once the first leg is switched off than the second leg is vanished identically. Hence,

$$R_{\alpha[n0]}(t) = 0, \quad \tilde{s}_{\alpha[n0]} = 0, \quad s_{\alpha[n0]} = 0, \quad \rho_{\alpha[n0]} = 0, \quad \gamma_{\alpha[n0]} = 0, \quad n \geq 1. \quad (\text{E.5})$$

Using these formulas we find that in the first non-trivial orders

$$\rho_{\alpha[00]} = 0, \quad \rho_{\alpha[01]} = R_{\alpha[01]}(\gamma_{\alpha[00]}), \quad \rho_{\alpha[10]} = 0, \quad \rho_{\alpha[11]} = R_{\alpha[11]}(\gamma_{\alpha[00]}). \quad (\text{E.6})$$

In other words, the radial vertex positions are defined by radial functions at the seed angle values, $\gamma_{\alpha[00]} \equiv y_{\alpha} = \{0, y\}$. This property holds only in the first non-trivial order, while in higher orders we will have to use the general expansion formulas like (E.4). Note that relations (E.6) considerably simplify the analysis of the evolution equation and the vertex boundary conditions.

To find how the angles flow with the deformation parameters we use the interval equation (7.17). Assuming that the seed solution is given by (8.5) we find that

$$\gamma_1 = \frac{\tilde{s}_{1[12]} - \tilde{s}_{2[12]}}{2} \nu \delta + \dots, \quad \gamma_2 = y + \frac{\tilde{s}_{1[12]} - \tilde{s}_{2[12]}}{2} \delta + \frac{\tilde{s}_{1[22]} - \tilde{s}_{2[22]}}{2} \nu \delta + \dots, \quad (\text{E.7})$$

where the ellipses denote higher order terms. The above relations say that the angle (time) positions of the boundary attachment points and vertex points are generally different: in the zeroth order they coincide, but switching the interaction on they are starting to fall apart.

(II). The parameter flow of the angle positions (E.7) makes integrating along the loop segments (7.12) technically involved because the integration limits are now double power series in δ and ν . Indeed, for a given integral (7.12) there is the standard expansion formula

$$\begin{aligned} \int_{\gamma_1(\nu, \delta)}^{\gamma_2(\nu, \delta)} F(x, \nu, \delta) dx &= \int_{\gamma_1(0, \delta)}^{\gamma_2(0, \delta)} F(x, 0, \delta) dx + \\ &+ \nu \left[\int_{\gamma_1(0, \delta)}^{\gamma_2(0, \delta)} \partial_{\nu} F(x, 0, \delta) dt + F(\gamma_{\alpha}(0, \delta), 0, \delta) \partial_{\nu} \gamma_{\alpha}(0, \delta) \Big|_{\alpha=1}^{\alpha=2} \right] + \mathcal{O}(\nu^2), \end{aligned} \quad (\text{E.8})$$

where each term is to be further expanded in powers of δ . We can show that in the first non-trivial orders the right-hand said of (E.8) reduces to the first and second terms, i.e. terms $\partial \gamma_{\alpha}$ do not contribute. Analogously, expanding the remainder in δ we find that corrections in γ_{α} do not contribute as well. In other words, in the first non-trivial order in ν and δ we can assume that the integration limits are of the zeroth order, $\gamma_{\alpha} \approx \gamma_{\alpha[0,0]}$, while the integrands are expanded in the standard fashion. In higher orders this is not generally true.

(III). Following the general scheme described in section 7 we shall find the general solution of the evolution equations. To this end we will solve the system of algebraic equations involving the vertex positions, integration constants, and momenta. Finally, the resulting radial functions $R_m(t)$ are used to integrate along the loop.

Let us decompose (E.3) in ν and then in δ . To avoid poles in ν we set $\tilde{s}_{\alpha[11]} = 0$ (additionally to (E.5)). In the lowest orders we find that radial vertex and momentum corrections are related as

$$\rho_{1[01]} = \frac{1}{2} \sqrt{8\tilde{s}_{1[02]} + 1}, \quad \rho_{2[01]} = \sqrt{(\tilde{s}_{1[12]} - \tilde{s}_{2[12]})^2 + 2\tilde{s}_{1[02]}}, \quad (\text{E.9})$$

$$\rho_{1[11]} = \frac{\tilde{s}_{1[12]} + \tilde{s}_{2[12]}}{\sqrt{8\tilde{s}_{1[02]} + 1}}, \quad \rho_{2[11]} = \frac{\tilde{s}_{1[12]} + \tilde{s}_{2[12]} + 2(\tilde{s}_{1[12]} - \tilde{s}_{2[12]})(\tilde{s}_{1[22]} - \tilde{s}_{2[22]})}{2\sqrt{(\tilde{s}_{1[12]} - \tilde{s}_{2[12]})^2 + 2\tilde{s}_{1[02]}}}. \quad (\text{E.10})$$

To solve the above relations we note that they depend on three combinations of the first order loop momenta $\tilde{s}_{1[12]} \pm \tilde{s}_{2[12]}$ and $\tilde{s}_{1[22]} - \tilde{s}_{2[22]}$.

The first relation in (E.9) is just the first order expansion coefficient of (8.5) of the 1-point case. It follows that $\rho_{1[01]}$ and $\tilde{s}_{1[02]}$ read off from (8.10)–(8.11) satisfy this relation. The second equation allows us to fix $\tilde{s}_{1[22]} - \tilde{s}_{2[22]}$. Indeed, from (E.6) we know that $\rho_{2[01]} = R_{2[01]}(y)$. Recalling (8.10)–(8.11) we find that

$$\tilde{s}_{1[22]} - \tilde{s}_{2[22]} = \frac{1}{2} \left(\coth \frac{\beta}{2} \sinh(y) - \cosh y \right). \quad (\text{E.11})$$

Now, we expand the evolution equation (7.15) in powers of ν, δ . The first non-trivial correction reads

$$\dot{R}_{\alpha[11]}(t) + \frac{\tilde{s}_{\alpha[12]} - R_{\alpha[01]}(t)R_{\alpha[11]}(t)}{\dot{R}_{\alpha[01]}(t)} = 0, \quad (\text{E.12})$$

where the dot denotes a time derivative, while $R_{\alpha[01]}(t)$ is a known function read off from (8.10). The general solution is given by

$$R_{\alpha[11]}(t) = c_{\alpha} \sinh(t - \beta/2) + \tilde{s}_{\alpha[12]}(\sinh t - \sinh(t - \beta)), \quad (\text{E.13})$$

where c_{α} are integration constants. We now use this function to solve the boundary conditions (8.2) that take the form

$$\begin{aligned} R_{1[11]}(0) &= \rho_{1[11]}, & R_{1[11]}(y) &= \rho_{2[11]}, \\ R_{2[11]}(y) &= \rho_{2[11]}, & R_{2[11]}(\beta) &= \rho_{1[11]}. \end{aligned} \quad (\text{E.14})$$

This is four linear equations for four variables $c_{\alpha}, \tilde{s}_{\alpha[12]}$. Also, we add two equations (E.10) which relate radial corrections $\rho_{\alpha[11]}$ and loop momenta $\tilde{s}_{\alpha[12]}$. The full system of six equations on six variables can be solved unambiguously in terms of parameters β and y . Then, the roots can be used to find the time dependence $R_{\alpha[11]}(t)$ (E.13). All in all, we get

$$R_{\alpha[11]}(t) = -\frac{1}{2} \operatorname{csch} \frac{\beta}{2} \cosh \left((-)^{\alpha+1} \frac{\beta}{2} + t - y \right), \quad (\text{E.15})$$

and

$$\tilde{s}_{1[12]} = \frac{\cosh(y - \beta)}{2 - 2 \cosh \beta}, \quad \tilde{s}_{2[12]} = \frac{\cosh y}{2 - 2 \cosh \beta}. \quad (\text{E.16})$$

These functions completely fix the form of the 2-point geodesic graph in the first non-trivial order, cf. (8.12)–(8.13). The leg momenta can be found using the relation (E.2).

(IV). Within the superlight expansion the total action (7.11) can be represented as

$$L_{\text{dual}} = \tilde{\epsilon} [L_{\text{loop}} + \delta L_{\text{leg}}^1 + \nu \delta L_{\text{leg}}^2] = \tilde{\epsilon} [L_{\text{tot}[0]} + \nu \delta^2 L_{\text{tot}[12]} + \mathcal{O}(\nu^2, \delta^3)], \quad (\text{E.17})$$

where the leading contribution is the known length of the 1-point graph, terms $\mathcal{O}(\nu\delta)$ can be shown to be absent, while the first non-trivial correction is given by $L_{\text{tot}[1,2]}$. This decomposition agrees with what we would expect from the boundary side, cf. (D.1).

The zeroth order correction $L_{\text{tot}[0]}$ is given by [15]

$$L_{\text{tot}[0]} = \beta + \delta^2 \int_0^\beta dt (R_{1[01]}(t) R_{1[01]}(t) - \tilde{s}_{1[02]}) - \delta^2 R_{1[01]}(0), \quad (\text{E.18})$$

where the first and second terms are loop contributions, while the third term is the leg contribution. Substituting the explicit solution (8.10)–(8.11) into (E.18) we get

$$L_{\text{tot}[0]} = \beta - \frac{1}{4} \coth \frac{\beta}{2}. \quad (\text{E.19})$$

To find the first order correction $L_{\text{tot}[12]}$ we use (7.11) and (7.12) and show that

$$L_{\text{tot}[12]} = L_{\text{loop}[12]}^1 + L_{\text{loop}[12]}^2 + L_{\text{leg}[11]}^1 + L_{\text{leg}[11]}^2, \quad (\text{E.20})$$

where

$$\begin{aligned} L_{\text{loop}[12]}^m &= \int_0^y dt (2R_{m[11]}(t) R_{m[0]}(t) - \tilde{s}_{m[12]}) , & m = 1, 2 \\ L_{\text{leg}[11]}^1 &= -R_{1[11]}(0), & L_{\text{leg}[11]}^2 &= -R_{2[01]}(y). \end{aligned} \quad (\text{E.21})$$

Here, the integrands are explicitly given by (8.10)–(8.11) and (E.15)–(E.16). We find

$$L_{\text{tot}[12]} = \frac{1}{2} \left(\sinh y - \coth \frac{\beta}{2} \cosh y \right). \quad (\text{E.22})$$

Combining (E.19) and (E.22) we arrive at the final formula (8.14).

Double leg expansion. Let $F = \{\tilde{s}_\alpha, \gamma_\alpha, s, \rho, R_\alpha, L_{\text{loop}}^\alpha, L_{\text{leg}}^\alpha \mid \alpha = 1, 2\}$ be a power series in the lightness parameter δ ,

$$F = \sum_{n=0}^{\infty} F_{[n]} \delta^n. \quad (\text{E.23})$$

As the seed solution $\mathcal{O}(\delta^0)$ we choose the circle going along the constant zeroth radius and radial legs, cf. (7.6),

$$\tilde{s}_{\alpha[0]} = 1, \quad s_{[0]} = 0, \quad R_{\alpha[0]}(t) = 0, \quad \gamma_{\alpha[0]} = y_\alpha, \quad (\text{E.24})$$

where $y_1 = 0$ and $y_2 = y$. Moreover, similar to the analysis of [15] we assume that $\tilde{s}_{\alpha[2n+1]} = 0$, $s_{\alpha[2n]} = 0$, $R_{\alpha[2n]}(t) = 0$ for $n \in \mathbb{N}_0$.

Using constraints (4.5) along with the identity $\tilde{\epsilon}\delta^n = \epsilon\delta^{n-1}$ we find from (7.11) that the total length is expanded as

$$L_{\text{dual}} = \tilde{\epsilon}\beta + \tilde{\epsilon} \sum_{n=2}^{\infty} \left[L_{\text{loop}[n]}^1 + L_{\text{loop}[n]}^2 + L_{\text{leg}[n-1]}^1 + L_{\text{leg}[n-1]}^2 \right] \delta^n, \quad (\text{E.25})$$

where we used that in the zeroth approximation the loop is the constant radius circle and the legs are vanishing, i.e. $L_{\text{loop}[0]}^1 + L_{\text{loop}[0]}^2 = \tilde{\epsilon}\beta$ and $L_{\text{leg}[0]}^{\alpha} = 0$. In particular, the sum in (E.25) starts with $n = 2$ term. Also, $L_{\text{loop}[2n+1]}^{\alpha} = 0$ for $n \in \mathbb{N}_0$.

Using explicit expressions (7.12) and taking into comments (I)–(III) above we find that in the first non-trivial order

$$\begin{aligned} L_{\text{loop}}^1 &= y + \delta^2 \int_0^y dt (R_{1[1]}(t)R_{1[1]}(t) - \tilde{s}_{1[2]}) + \mathcal{O}(\delta^4), \\ L_{\text{loop}}^2 &= (\beta - y) + \delta^2 \int_y^{\beta-y} dt (R_{2[1]}(t)R_{2[1]}(t) - \tilde{s}_{2[2]}) + \mathcal{O}(\delta^4), \end{aligned} \quad (\text{E.26})$$

and

$$L_{\text{leg}}^1 = L_{\text{leg}}^2 = -\rho_{[1]}\delta + \mathcal{O}(\delta^2). \quad (\text{E.27})$$

Now, the conservation conditions are solved by the general formulas (7.20) as follows

$$\rho = \rho_1 = \rho_2 = \left[\tilde{s}_1 \tilde{s}_2 + \frac{(\tilde{s}_1 - \tilde{s}_2)^2}{\delta^2} + \frac{\delta^2}{4} - 1 \right]^{1/2} \left(1 - \frac{\delta^2}{4} \right)^{-1/2}, \quad s \equiv s_1 = s_2 = \frac{\tilde{s}_2 - \tilde{s}_1}{\delta}, \quad (\text{E.28})$$

that mean the two legs are symmetrically attached to the loop. To find the radial functions $R_{\alpha[1]}$ we expand the evolution equations in δ . In the first non-trivial order we find the following equations

$$\frac{R^{\alpha[1]}(t)}{dt} - \sqrt{R^{\alpha[1]}(t)^2 - 2\tilde{s}_{\alpha[2]}} = 0 \quad \text{solved as} \quad R^{\alpha[1]}(t) = \frac{1}{2} (2\tilde{s}_{\alpha[2]}e^{-c_{\alpha}-t} + e^{c_{\alpha}+t}). \quad (\text{E.29})$$

Integration constants c_{α} can be fixed by the boundary conditions (8.2) given in the first non-trivial order. Finally, we find that the radial functions read

$$R_{1[1]}(t) = \rho_{[1]} \left(\cosh t - \tanh \frac{\gamma}{2} \sinh t \right), \quad R_{2[1]}(t) = \rho_{[1]} \frac{e^{\beta+y-t} + e^t}{e^{\beta} + e^{\gamma}}, \quad (\text{E.30})$$

where $\rho_{[1]}$ is the vertex correction, and

$$\tilde{s}_{1[2]} = \frac{\rho_{[1]}^2}{\cosh y + 1}, \quad \tilde{s}_{2[2]} = \frac{2e^{\beta+y}\rho_{[1]}^2}{(e^{\beta} + e^y)^2}. \quad (\text{E.31})$$

On the other hand, expanding (E.28) we find that $\rho_{[1]}$ and $\tilde{s}_{\alpha[2]}$ are related as

$$\rho_{[1]} = \sqrt{(\tilde{s}_{1[2]} - \tilde{s}_{2[2]})^2 + (\tilde{s}_{1[2]} + \tilde{s}_{2[2]})} + \frac{1}{4} \rightarrow \rho_{[1]} = \text{csch} \frac{\beta}{2} \cosh \frac{y}{2} \cosh \frac{\beta - y}{2}. \quad (\text{E.32})$$

All in all, we find that the radial functions and momenta are given by (8.16)–(8.18). Substituting these expressions into (E.26)–(E.27) we obtain (8.19).

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