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ϵ -expansion in the Gross-Neveu

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ABSTRACT: We use the recently developed CFT techniques of Rychkov and Tan to compute anomalous dimensions in the $O(N)$ Gross-Neveu model in $d = 2 + \epsilon$ dimensions. To do this, we extend the “cowpie contraction” algorithm of [arXiv:1506.06616](https://arxiv.org/abs/1506.06616) to theories with fermions. Our results match perfectly with Feynman diagram computations.

KEYWORDS: Conformal and W Symmetry, Field Theories in Lower Dimensions

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1 Introduction

In recent work, Rychkov and Tan [1] have shown that the power of conformal invariance can be used to compute ϵ -expansions at the Wilson-Fisher fixed point (see also [2]). This approach is not reliant on Feynman diagrams (and in that sense is non-perturbative¹), and uses only conformal symmetry and analyticity in ϵ as inputs.

The results of Rychkov-Tan were generalized to other dimensions and other fixed point theories in [2]. The computations require a systematic approach to handling contractions of fields in these theories, and a systematic approach for doing this was developed for scalar $O(N)$ theories [2]. One of the goals of this paper is to generalize this to CFTs with fermions.

Concretely, we will work with $O(N)$ Gross-Neveu model in $d = 2 + \epsilon$ dimensions [3].² This theory is interesting for various reasons: there is a huge literature on this theory, and its large- N expansion and asymptotic freedom (among various other features) have been thoroughly investigated in the last decades. We generalize the approach of [1, 2] to this

¹This should be taken with a pinch of salt — the epsilon expansion is afterall perturbative. The idea here is that the perturbative parameter in the present approach is not (at least manifestly) the coupling constant.

²The multiplicative renormalizability of Gross-Neveu model in $2 + \epsilon$ dimensions is discussed in [4, 5], our results are unchanged for the $U(N)$ model as well.

theory, and verify that the results agree with existing perturbative results in the literature, where they overlap.

The paper is organized as follows. In section 2 we introduce the Gross-Neveu as a Wilson-Fisher CFT along with the axioms that help us along in the computation. In section 3 we give a recursive algorithm, based on [2], to compute OPE coefficients in the free theory and in section 4 we show how these results are matched with that of interacting theory in the $\epsilon \rightarrow 0$ limit which help us determine anomalous dimensions of various composite operators.

Note added. The paper [12] also discusses the same problem, and even though the details of the algorithm are different, our results agree.

2 $O(N)$ Gross-Neveu model in $2 + \epsilon$ dimensions

The Gross-Neveu model action in $d = 2 + \epsilon$ dimensions is given by

$$S = \frac{1}{2\pi} \int d^d \sigma \left[\bar{\psi}^A \not{\partial} \psi^A + \frac{1}{2} g \mu^{-\epsilon} (\bar{\psi}^A \psi^A)^2 \right] \quad (2.1)$$

In 2 dimensions, this theory is renormalizable with a dimensionless coupling constant. The coupling constant is proportional to ϵ and hence this theory describes a weakly coupled fixed point for small values of ϵ . We have introduced a scale μ to make the coupling constant dimensionless.

The engineering dimension of the fields is fixed by the action

$$[\psi] \equiv \delta = \frac{1 + \epsilon}{2} \quad (2.2)$$

The equations of motion for this theory are given by

$$\gamma^\mu \partial_\mu \psi^A + g \mu^{-\epsilon} (\bar{\psi}^B \psi^B) \psi^A = 0 \quad (2.3)$$

$$\partial_\mu \bar{\psi}^A \gamma^\mu - g \mu^{-\epsilon} (\bar{\psi}^B \psi^B) \bar{\psi}^A = 0 \quad (2.4)$$

According to [1], this equation has to be seen as a conformal multiplet shortening condition, where in the free theory, the operators $(\bar{\psi}^B \psi^B) \psi^A$ and $(\bar{\psi}^B \psi^B) \bar{\psi}^A$ is a primary, but in the interacting theory it is made secondary by above equations. Following [1], we formalize the relationship between operators in the free and interacting case by means of following axioms:

- The interacting theory enjoys conformal symmetry.
- For any operator in the interacting theory, there is a corresponding operator in the free theory, which the interacting theory operator approaches to in the $\epsilon \rightarrow 0$ limit.

For definiteness, we call the interacting theory operators as V_{2n} , $V_{2n+1\ a}^A$ and $\bar{V}_{2n+1\ a}^A$,³ which in the free limit goes to

$$\begin{aligned} V_{2n} &\rightarrow (\bar{\psi}^A \psi^A)^n \\ V_{2n+1\ a}^A &\rightarrow (\bar{\psi}^B \psi^B)^n \psi_a^A \\ \bar{V}_{2n+1\ a}^A &\rightarrow (\bar{\psi}^B \psi^B)^n \bar{\psi}_a^A \end{aligned} \quad (2.5)$$

- Operators $V_{3\ a}^A$ and $\bar{V}_{3\ a}^A$ are not primaries, instead they are related to the primaries by the multiplet shortening conditions

$$\begin{aligned} \gamma^\mu \partial_\mu V_1^A &= -\alpha(\epsilon) V_3^A \\ \partial_\mu \bar{V}_1^A \gamma^\mu &= \alpha(\epsilon) \bar{V}_3^A \end{aligned} \quad (2.6)$$

This puts restrictions on the dimensions of these operators

$$\Delta_3 = \Delta_1 + 1 \quad (2.7)$$

The proportionality constant $\alpha(\epsilon)$ can be fixed later using the axioms above. All other operators V_m , $m \neq 3$, are primaries.

The two-point function of two primaries of same dimension Δ_1 is

$$\langle V_{1\ a}^A(x_1) \bar{V}_{1\ b}^B(x_2) \rangle = \frac{(\not{x}_{12})_{ab}}{(x_{12}^2)^{\Delta_1 + \frac{1}{2}}} \delta^{AB} \quad (2.8)$$

In the free limit this becomes

$$\langle \psi_a^A(x_1) \bar{\psi}_b^B(x_2) \rangle = \frac{(\not{x}_{12})_{ab}}{x_{12}^2} \delta^{AB} \quad (2.9)$$

The anomalous dimension is defined as the difference between the actual scaling dimension of the operator and the engineering dimension, i.e, $\Delta_n = n\delta + \gamma_n$. We also make the crucial assumption that the anomalous dimensions are analytic functions of ϵ and therefore admits a power series expansion

$$\gamma_n = y_{n,1}\epsilon + y_{n,2}\epsilon^2 + \dots \quad (2.10)$$

Our first task is to fix α in (2.6). Differentiating (2.8) and substituting appropriate factors of γ matrices, we obtain

$$\begin{aligned} (\gamma^\mu)_{ca} \langle \partial_{1\ \mu} V_{1\ a}^A(x_1) \partial_{2\ \nu} \bar{V}_{1\ b}^B(x_2) \rangle (\gamma^\nu)_{bd} &= (\gamma^\mu)_{ca} \partial_{1\ \mu} \partial_{2\ \nu} \left(\frac{(\not{x}_{12})_{ab}}{(x_{12}^2)^{\Delta_1 + \frac{1}{2}}} \right) (\gamma^\nu)_{bd} \delta^{AB} \\ &= -(2\Delta_1 + 1)(2\Delta_1 + 1 - d) \frac{(\not{x}_{12})_{cd}}{(x_{12}^2)^{\Delta_1 + \frac{3}{2}}} \delta^{AB} \end{aligned} \quad (2.11)$$

³A word on notations: small latin indices a, b, \dots are the spinor indices whereas A, B , etc stand for $O(N)$ indices.

Left hand side of (2.11) takes the form

$$-\alpha^2 \langle V_{3c}^A(x) \bar{V}_{3d}^B(y) \rangle \quad (2.12)$$

which in the free limit evaluates to

$$-\alpha^2(\epsilon)(N-1) \frac{(\not{x}_{12})_{cd}}{(x_{12}^2)^2} \delta^{AB} \quad (2.13)$$

Comparing both sides, we obtain

$$\alpha = \sigma \sqrt{\frac{4\gamma_1}{N-1}} \quad (2.14)$$

where $\sigma = \pm 1$. The exact sign will be determined later. Following [1, 2], we consider correlators of the form

$$\langle V_{2n}(x_1) V_{2n+1a}^A(x_2) \bar{V}_{1b}^B(x_3) \rangle, \quad \langle V_{2n}(x_1) V_{2n+1a}^A(x_2) \bar{V}_{3b}^B(x_3) \rangle \quad (2.15)$$

which in the free limit goes to

$$\langle \phi_{2n}(x_1) \phi_{2n+1a}^A(x_2) \bar{\phi}_{1b}^B(x_3) \rangle, \quad \langle \phi_{2n}(x_1) \phi_{2n+1a}^A(x_2) \bar{\phi}_{3b}^B(x_3) \rangle \quad (2.16)$$

where we have introduced operators ϕ_{2n} and ϕ_{2n+1a}^A as a shorthand for $(\bar{\psi}_b^B \psi_b^B)^n$ and $(\bar{\psi}_b^B \psi_b^B)^n \psi_a^A$. The reason we are interested in these correlators is because of its sensitivity to multiplet recombination. To see this, we notice that in the free theory, $\phi_{2n} \times \phi_{2n+1a}^A$ OPE contains operators ψ_a^A and $(\bar{\psi}_b^B \psi_b^B) \psi_a^A$ whereas in the interacting theory $V_{2n} \times V_{2n+1a}^A$ OPE only contains V_1 as the primary. The coefficients in both cases are independently computable and by Axiom:2, we expect them to match in the limit $\epsilon \rightarrow 0$.

In the free case, we have following OPE

$$\phi_{2n}(x_1) \times \phi_{2n+1a}^A(x_2) \supset f_{2n}(x_{12}^2)^{-n} (\psi_a^A + \rho_{2n}(\not{x}_{12})_{ab} (\bar{\psi}\psi) \psi_b^A) \quad (2.17)$$

The coefficients f_{2n} and ρ_{2n} can be determined by counting the number of Wick contractions. In next section, we provide an algorithm, based on [2], to determine these coefficients for arbitrary n . This is matched with the interacting theory OPE

$$V_{2n}(x_1) \times V_{2n+1a}^A(x_2) \supset \tilde{f}_{2n}(x_{12}^2)^{-\frac{1}{2}[\Delta_{2n} + \Delta_{2n+1} - \Delta_1]} \left[\delta_{ac} + q_1 \delta_{ac} x_{12}^\mu \partial_{2\mu} + q_2 (\not{x}_{12} \not{\partial}_2)_{ac} \right] V_{1c}^A(x_2) \quad (2.18)$$

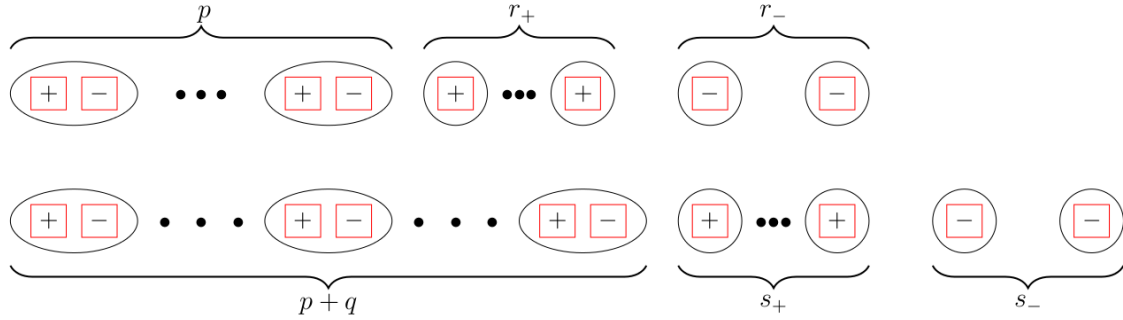
3 Counting contractions

We now turn our attention to computing f and ρ coefficients in (2.17). Apart from (2.17), we also need OPE's of the form

$$\phi_{2n+1a}^A(x_1) \times \phi_{2n+2}(x_2) \supset f_{2n+1}(x_{12}^2)^{-(n+1)} \left[(\not{x}_{12})_{ab} \psi_b^A + \rho_{2n+1} x_{12}^2 (\bar{\psi}\psi) \psi_a^A \right] \quad (3.1)$$

which are used to fix the anomalous dimensions of odd operators. In [2] a recursive algorithm was used to count Wick contractions, which can be adapted for the fermions. The

Wick contractions can then be viewed as various ways of connecting upper and lower rows, resulting in recursive equations. In the case of fermions, the principle is essentially the same, but the contractions have a bit more structure. We use '+' and '-' to denote $\bar{\psi}$ and ψ respectively. To capture the contractions, we introduce the quantity $F_{p+q,s_+,s_-;m_+,m_-}^{p,r_+,r_-}$, where p is the number of upper double cow-pies which stand for $\bar{\psi}\psi$, r_+ is the number of upper single cow-pies of '+' type, r_- is the number of upper single cow-pies of '-' type, $p+q$ is the number of lower double cow-pies, s_{\pm} is the number of lower single cow-pies of type \pm , m_{\pm} is the number of uncontracted $\bar{\psi}$ s and ψ s respectively. A contraction is always between an upper + and a lower - or vice-versa.



The various coefficients f s and ρ s in our notation becomes

$$\begin{aligned} f_{2p} &= F_{p,0,1;0,1}^{p,0,0} & f_{2p}\rho_{2p} &= F_{p,0,1;1,2}^{p,0,0} \\ f_{2p+1} &= F_{p+1,0,0;0,1}^{p,0,1} & f_{2p+1}\rho_{2p+1} &= F_{p+1,0,0;1,2}^{p,0,0} \end{aligned} \quad (3.2)$$

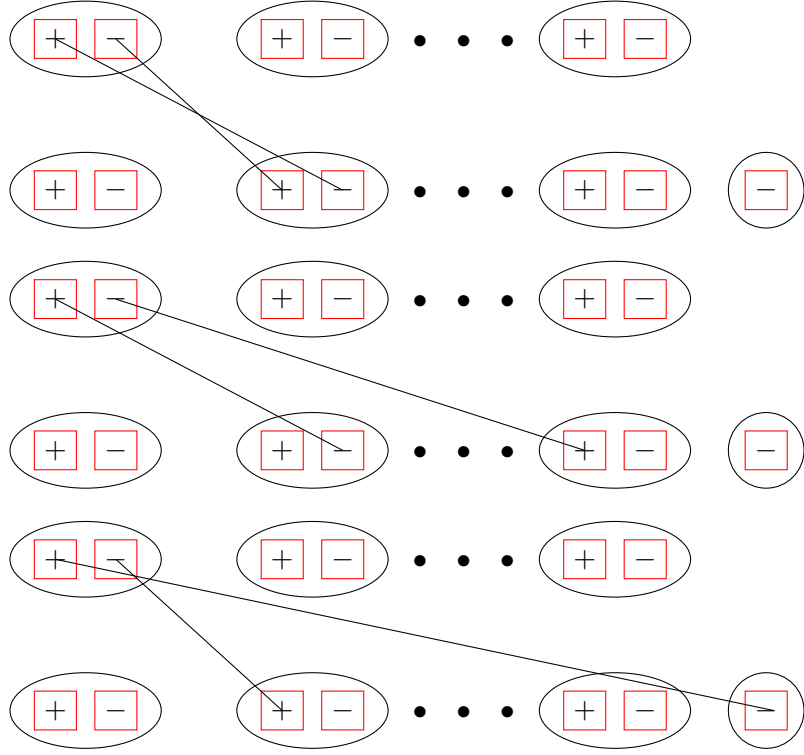
3.1 f_{2p}

There are 3 different kind of contractions that are possible. Of the first type, the two kernels of the p^{th} double cow-pie are contracted with two kernels of same lower cow-pie. This gives a factor of Np . The second possibility is to contract the two kernels of upper cow-pie to two different kernels of lower double cow-pie resulting in a factor of $-p(p-1)$ and the last possibility is to contract one of the kernels of upper double cow-pie with a kernel in lower double cow-pie and the second kernel of upper cow-pie with the single kernel of lower row. This gives a factor of $-p$. So, the resulting contraction can now be expressed as following recursion equation

$$F_{p,0,1;0,1}^{p,0,0} = (Np - p(p-1) - p)F_{p-1,1;0,1}^{p-1,0} \quad (3.3)$$

This recursion equation can be solved along with the launching condition $F_{0,1;0,1}^{0,0} = 1$ and we obtain

$$f_{2p} = p!(N-1)(N-2)\cdots(N-p) \quad (3.4)$$



3.2 f_{2p+1}

There are only two types of contractions possible, analogous to the first two types above. The recursion equation can therefore be written by inspection

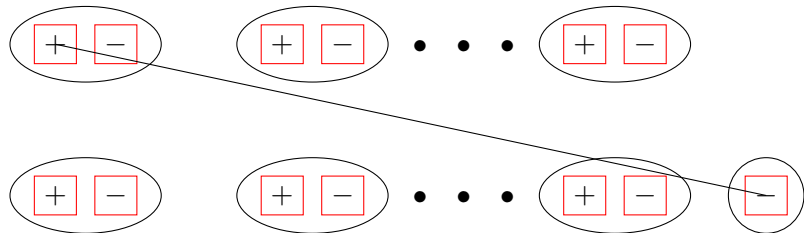
$$F_{p+1,0;0,1}^{p;1} = (N(p+1) - p(p+1))F_{p,0;0,1}^{p-1,0,1} \quad (3.5)$$

with the launching condition $F_{1,0;0,1}^{0;1} = 1$ which gives

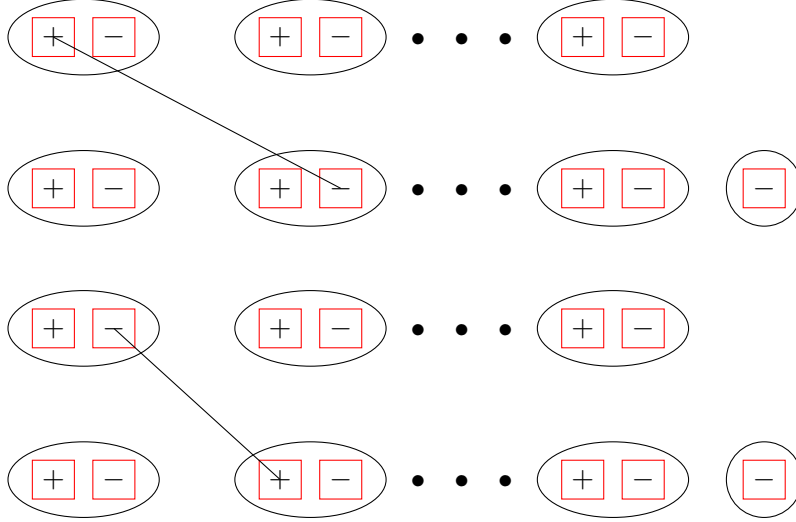
$$f_{2p+1} = (p+1)!(N-1)(N-2)\cdots(N-p) \quad (3.6)$$

3.3 $f_{2p}\rho_{2p}$

Here the first possibility involves contracting both the kernels of upper double cow-pie with lower cow-pies analogous to the computation of f_{2p} . This gives a factor of $Np - p(p-1) - p$. Another possibility involves contracting '+' of an upper double cow-pie with the single '-' in the lower row. This gives a factor of $-F_{p-1,0;1,1}^{p-1;0}$.



Other possibilities involve single contractions of upper kernels, which can be done in following ways: (a) '+' of the upper double cow-pie contracted with a '-' of the lower double cow-pie, (b) '-' from the upper double cow-pie with a + from lower double cow-pie. One can see by explicit computation for the lower orders (i.e. $p = 2, 3, \dots$) that their contribution is given by $-pF_{p-1,0,1,2}^{p-1,0,0}$. Notice that the coefficient is different from the naive expectation because not all single contractions are independent and we must be careful to avoid over-counting and to keep track of the index structure.



Thus we get the recursion equation

$$F_{p,0,1,2}^{p,0,0} = p[N - p - 1] F_{p-1,0,1,2}^{p-1,0,0} - F_{p,0,0,1,1}^{p-1,0,1} \quad (3.7)$$

$F_{p,0,0,1,1}^{p-1,0,1}$ can be again evaluated using the cow-pie formalism and its recursion equation is given by

$$F_{p,0,1,1}^{p,0} = (p+1)(N-p)F_{p,0,1,1}^{p-1,0} \quad (3.8)$$

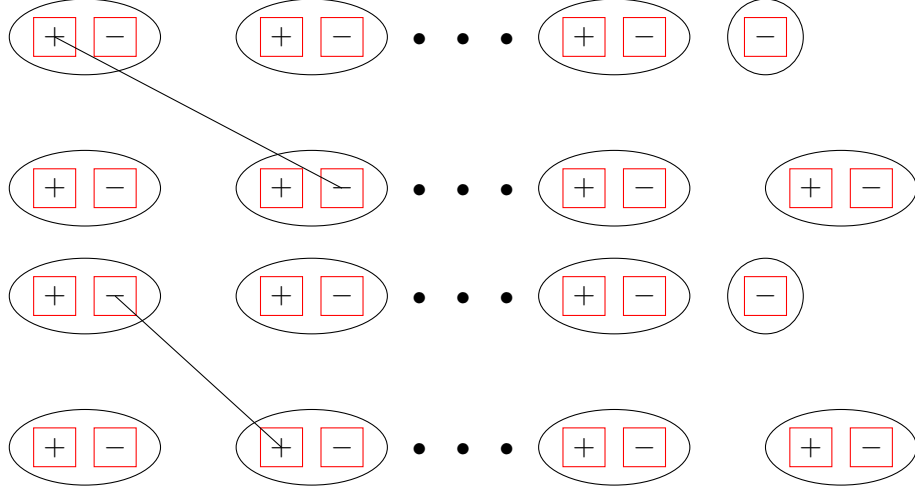
This system of recursion equations can be solved using the launching condition $F_{0,0,1,2}^{0,0,0} = 0$ and $F_{1,0,0,1,1}^{0,0,0} = 1$. Using the expression for f_{2p} in (3.4) we get,

$$\rho_{2p} = -\frac{p}{N-1} \quad (3.9)$$

3.4 $f_{2p+1}\rho_{2p+1}$

We again have three cases to consider: (a) Both kernels of the upper cow-pie contracted with lower cow-pies, (b) Both kernel remain uncontracted, and (c) Only one of the kernels is contracted.

Case (a) is similar to the computation of f_{2p+1} and gives a factor of $(p+1)(N-p)F_{p-1,0,1,2}^{p,0,0}$. Case (b) does not contribute as we do not obtain the desired operator. Case (c) is similar to the case of $f_{2p}\rho_{2p}$. Once again, by explicit computation for the lowest order, we can see that its contribution is $-(p+1)F_{p-1,0,1,2}^{p,0,0}$.



So we have following recursion equation

$$F_{p+1,0,0;1,2}^{p,0,1} = (p+1)(N-p-1)F_{p,0,0;1,2}^{p-1,0,1} \quad (3.10)$$

which can be solved with the launching condition $F_{0,0,1;1,2}^{1,0,0} = 1$. Using (3.6) along with the recursion equations above, we get

$$\rho_{2p+1} = 1 - \frac{p}{N-1} \quad (3.11)$$

In the appendix we provide an alternate derivation of (3.9) and (3.11) using cow-pies.

4 Matching with the free theory

Having fixed the OPE coefficients of the free theory, we are now in a position to compute the anomalous dimensions of the interacting theory operators. This involves analyzing 3-point functions with $V_{2n} \times V_{2n+1\ a}^A$ OPEs in (2.18) and demanding that in the $\epsilon \rightarrow 0$, they go to corresponding quantities in free theory. In particular, we analyze 3-point correlators of the form

$$\begin{aligned} \langle V_{2n}(x_1) V_{2n+1\ a}^A(x_2) \bar{V}_{1\ b}^B(x_3) \rangle &\rightarrow \langle \phi_{2n}(x_1) \phi_{2n+1\ a}^A(x_2) \bar{\phi}_{1\ b}^B(x_3) \rangle \\ &\sim f_{2n}(x_{12}^2)^{-n} \langle \psi_a^A(x_2) \bar{\psi}_b^B(x_3) \rangle \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} \langle V_{2n}(x_1) V_{2n+1\ a}^A(x_2) \bar{V}_{3\ b}^B(x_3) \rangle &\rightarrow \langle \phi_{2n}(x_1) \phi_{2n+1\ a}^A(x_2) \bar{\phi}_{3\ b}^B(x_3) \rangle \\ &\sim f_{2n} \rho_{2n} (x_{12}^2)^{-n} (\not{x}_{12})_{ab} \langle \phi_{3\ a}^A(x_2) \bar{\phi}_{3\ b}^B(x_3) \rangle \end{aligned} \quad (4.2)$$

The l.h.s. of (4.2) can be evaluated, to the leading order, using $V_{2n} \times V_{2n+1\ a}^A$ OPE of (2.18) and the fact that V_3 is a descendent of V_1 , i.e,

$$\langle V_{1\ a}^A(x_1) \bar{V}_{3\ b}^B \rangle = \alpha^{-1}(\epsilon) \partial_{2\ \mu} \langle V_{1\ a}^A(x_1) \bar{V}_{1\ c}^B \rangle (\gamma^\mu)_{cb} = \sigma \sqrt{(N-1)\gamma_1} \frac{\delta_{ab} \delta^{AB}}{(x_{12}^2)^{\Delta_1 + \frac{1}{2}}} \quad (4.3)$$

Since this is proportional to $\sqrt{\gamma_1}$, it vanishes in the $\epsilon \rightarrow 0$ limit. Therefore, to reproduce (2.17) and (3.1) we need q_1 to remain finite in this limit. We also need q_2 to blow up as $\epsilon \rightarrow 0$ limit such that

$$q_2^i \alpha(\epsilon) \rightarrow \rho_i, \quad i = 2p, 2p+1 \quad (4.4)$$

The coefficients q_i are determined by conformal symmetry whose details and explicit form can be found in appendix. As alluded before, we find that q_1 is indeed finite in the free limit. The asymptotic behavior of q_2 is given by

$$q_2^{2n} \approx \frac{(\gamma_1 + \gamma_{2n} - \gamma_{2n+1})}{4\gamma_1}, \quad q_2^{2n+1} \approx \frac{(\gamma_1 + \gamma_{2n+1} - \gamma_{2n+2})}{4\gamma_1} \quad (4.5)$$

It is evident that for q_2 to blow up $y_{1,1}$ has to vanish. This gives us following telescoping series

$$y_{2n,1} - y_{2n+1,1} = 2\sigma \sqrt{(N-1)y_{1,2} \rho_{2n}}, \quad n = 1, 2, \dots \quad (4.6)$$

$$y_{2n+1,1} - y_{2n+2,1} = 2\sigma \sqrt{(N-1)y_{1,2} \rho_{2n+1}} \quad n = 0, 1, \dots \quad (4.7)$$

Together this can be written as

$$y_{i,1} - y_{i+1,1} = 2\sigma \sqrt{(N-1)y_{1,2} \rho_i}, \quad i = 1, 2, \dots \quad (4.8)$$

Summing the telescoping series gives

$$y_{n,1} = K \sum_{m=1}^{n-1} \rho_m \quad (4.9)$$

This gives the anomalous dimensions of all the odd and even primaries in the theory once we fix the numerical value of K . To fix this we make use of (2.7) which can be written as

$$2\delta + \gamma_3 = \gamma_1 + 1 \quad (4.10)$$

This gives $y_{3,1} = -1$ which can now be used to fix K by setting $n = 3$ in (4.9).

$$y_{3,1} = K(\rho_1 + \rho_2) \quad (4.11)$$

This gives $K = -\frac{(N-1)}{(N-2)}$ which fixes $\sigma = -1$ and furthermore fixes $y_{1,2}$ also. Thus we obtain

$$\gamma_1 = \frac{(N-1)}{4(N-2)^2} \epsilon^2 \quad (4.12)$$

One can also compute the anomalous dimensions of V_2 which we obtain to be

$$\gamma_2 = -\frac{(N-1)}{(N-2)} \epsilon \quad (4.13)$$

which are in perfect agreement with the results of [6–8].

Acknowledgments

I thank Chethan Krishnan for suggesting the problem, collaboration at various stages and several useful discussions, without which this paper would not have seen the daylight.

A OPE coefficients from 3-point function

As mentioned in section 2, the OPE coefficients, q_i , are completely determined by the conformal symmetry [9]. Here we outline a procedure for obtaining these coefficients from an expansion of 3-point functions. For the case in hand, the coefficients are computed from a scalar-fermion-antifermion 3-pt correlator which takes following form [10, 11]

$$\begin{aligned} \langle V_{2n}(x_1) V_{2n+1\ a}^A(x_2) \bar{V}_{1\ b}^B(x_3) \rangle &= C_{123} \frac{(\not{x}_{23})_{ab} \delta^{AB}}{(x_{12}^2)^{l_3} (x_{23}^2)^{l_1} (x_{31}^2)^{l_2}} \\ &+ C'_{123} \frac{(\not{x}_{12} \not{x}_{31})_{ab} \delta^{AB}}{(x_{12}^2)^{l'_3} (x_{23}^2)^{l'_1} (x_{31}^2)^{l'_2}} \end{aligned} \quad (\text{A.1})$$

where l_1 , l_2 and l_3 and their primed counterparts are determined in terms of the scaling dimensions of the operators

$$\begin{aligned} l_1 &= \frac{1}{2} [1 - \Delta_{2n} + \Delta_{2n+1} + \Delta_1] \\ l_2 &= \frac{1}{2} [\Delta_{2n} - \Delta_{2n+1} + \Delta_1] \\ l_3 &= \frac{1}{2} [\Delta_{2n} + \Delta_{2n+1} - \Delta_1] \end{aligned} \quad (\text{A.2})$$

and $l'_1 = l_1 - 1/2$, $l'_2 = l_2 + 1/2$ and $l'_3 = l_3 + 1/2$. The functional form of the 3-pt function (A.1) is essentially fixed by imposing conformal symmetry, i.e.

$$\begin{aligned} 0 &= \langle G_I V_{2n}(x_1) V_{2n+1\ a}^A(x_2) \bar{V}_{1\ b}^B(x_3) \rangle \\ &= \langle [G_I, V_{2n}(x_1)] V_{2n+1\ a}^A(x_2) \bar{V}_{1\ b}^B(x_3) \rangle + \langle V_{2n}(x_1) [G_I, V_{2n+1\ a}^A(x_2)] \bar{V}_{1\ b}^B(x_3) \rangle \\ &\quad + \langle V_{2n}(x_1) V_{2n+1\ a}^A(x_2) [G_I, \bar{V}_{1\ b}^B(x_3)] \rangle \end{aligned} \quad (\text{A.3})$$

where G_I collectively stand for the generators of the conformal group and $V_{2n}(x_1)$, $V_{2n+1\ a}^A(x_2)$ and $\bar{V}_{1\ b}^B(x_3)$ are all assumed to transform as primary under the action of G_I s.

The case of $n = 1$ has to be treated separately since V_3 is not primary. For the time being, we consider the simpler case of all three operators being primary and return to $n = 1$ case in the end. Now we imagine a scenario where the first two operators, $V_{2n}(x_1)$ and $V_{2n+1\ a}^A(x_2)$, are coming together such that $|x_{12}| \ll |x_{31}|$ and $|x_{12}| \ll |x_{23}|$. This allows us to expand the 3-pt function (A.1) by eliminating x_{31} using the relation

$$x_{31}^2 = x_{23}^2 \left(1 + \frac{2x_{12} \cdot x_{23}}{x_{23}^2} + \frac{x_{12}^2}{x_{23}^2} \right) \quad (\text{A.4})$$

Substituting this in (A.1) and keeping leading terms in x_{12} we obtain following series

$$\begin{aligned}
 \langle V_{2n}(x_1) V_{2n+1\ a}^A(x_2) \bar{V}_{1\ b}^B(x_3) \rangle &\equiv C_{123} \frac{(\not{x}_{23})_{ab} \delta^{AB}}{(x_{12}^2)^{l_3} (x_{23}^2)^{l_1} (x_{31}^2)^{l_2}} \\
 &\approx C_{123} (x_{12}^2)^{-\frac{1}{2}[\Delta_{2n} + \Delta_{2n+1} - \Delta_1]} \left[\frac{(\not{x}_{23})_{ab}}{(x_{23}^2)^{\Delta_1 + \frac{1}{2}}} - 2l_2 \frac{(\not{x}_{23})_{ab} (x_{12} \cdot x_{23})}{(x_{23}^2)^{\Delta_1 + \frac{3}{2}}} \right] \\
 &\quad - C'_{123} (x_{12}^2)^{-\frac{1}{2}[\Delta_{2n} + \Delta_{2n+1} - \Delta_1]} \left[\frac{(x_{12}^2)^{3/2} \delta_{ab}}{(x_{23}^2)^{\Delta_1 + \frac{1}{2}}} + \frac{(x_{12}^2)^{1/2} (\not{x}_{12} \not{x}_{23})_{ab}}{(x_{23}^2)^{\Delta_1 + \frac{1}{2}}} \right]
 \end{aligned} \tag{A.5}$$

Since the operators V_{2n} and $V_{2n+1\ a}^A(x_2)$ are close, we may use OPE (2.18). Substituting this into the l.h.s. of (A.1), we obtain

$$\begin{aligned}
 \langle V_{2n}(x_1) V_{2n+1\ a}^A(x_2) \bar{V}_{1\ b}^B(x_3) \rangle &\approx \tilde{f}(x_{12}^2)^{-\frac{1}{2}[\Delta_{2n} + \Delta_{2n+1} - \Delta_1]} \left[\langle V_{1\ a}^A(x_2) \bar{V}_{1\ b}^B(x_3) \rangle \right. \\
 &\quad \left. + q_1 x_{12}^\mu \partial_{2\ \mu} \langle V_{1\ a}^A(x_2) \bar{V}_{1\ b}^B(x_3) \rangle + q_2 (\not{x}_{12} \not{\partial}_2)_{ac} \langle V_{1\ c}^A(x_2) \bar{V}_{1\ b}^B(x_3) \rangle \right]
 \end{aligned} \tag{A.6}$$

This evaluates to

$$\begin{aligned}
 \langle V_{2n}(x_1) V_{2n+1\ a}^A(x_2) \bar{V}_{1\ b}^B(x_3) \rangle &\approx \tilde{f}(x_{12}^2)^{-\frac{1}{2}[\Delta_{2n} + \Delta_{2n+1} - \Delta_1]} \left[\frac{(\not{x}_{23})_{ab}}{(x_{23}^2)^{\Delta_1 + \frac{1}{2}}} \right. \\
 &\quad \left. + q_1 \left(\frac{(\not{x}_{23})_{ab}}{(x_{12}^2)^{\Delta_1 + \frac{1}{2}}} - \frac{(2\Delta_1 + 1)(x_{12} \cdot x_{23})(\not{x}_{23})_{ab}}{(x_{23}^2)^{\Delta_1 + \frac{3}{2}}} \right) + q_2 \frac{(d - 2\Delta_1 - 1)(\not{x}_{12})_{ab}}{(x_{23}^2)^{\Delta_1 + \frac{1}{2}}} \right] \delta^{AB}
 \end{aligned} \tag{A.7}$$

Comparing this with the 3-pt expansion (A.1), we see that the terms proportional to C'_{123} does not match with the terms in (A.7). Therefore we have $C'_{123} = 0$ and

$$\begin{aligned}
 q_1 &= \frac{\Delta_{2n} - \Delta_{2n+1} + \Delta_1}{2\Delta_1 + 1} \\
 q_2 &= \frac{\Delta_{2n} - \Delta_{2n+1} + \Delta_1}{(2\Delta_1 + 1)(2\Delta_1 + 1 - d)}
 \end{aligned} \tag{A.8}$$

The fermion-scalar-anti-fermion 3-point function is given by

$$\begin{aligned}
 \langle V_{2n+1\ a}^A(x_1) V_{2n+2}(x_2) \bar{V}_{1\ b}^B(x_3) \rangle &= C_{123} \frac{(\not{x}_{12} \not{x}_{23})_{ab} \delta^{AB}}{(x_{12}^2)^{m_3} (x_{23}^2)^{m_1} (x_{31}^2)^{m_2}} \\
 &\quad + C'_{123} \frac{(\not{x}_{31})_{ab} \delta^{AB}}{(x_{12}^2)^{m'_3} (x_{23}^2)^{m'_1} (x_{31}^2)^{m'_2}}
 \end{aligned} \tag{A.9}$$

with

$$\begin{aligned}
 m_1 &= \frac{1}{2} [1 - \Delta_{2n+1} + \Delta_{2n+2} + \Delta_1] \\
 m_2 &= \frac{1}{2} [\Delta_{2n+1} - \Delta_{2n+2} + \Delta_1] \\
 m_3 &= \frac{1}{2} [1 + \Delta_{2n+1} + \Delta_{2n+2} - \Delta_1]
 \end{aligned} \tag{A.10}$$

and $m'_1 = m_1 + 1/2$, $m'_2 = m_2 - 1/2$ and $m'_3 = m_3 - 1/2$. Here we may set $C'_{123} = 0$ which follows directly from the arguments of the previous case. Proceeding in a similar manner, the 3-point function expansion takes the form

$$\begin{aligned} \langle V_{2n+1\ a}^A(x_1) V_{2n+2}(x_2) \bar{V}_{1\ b}^B(x_3) \rangle &\equiv C_{123} \frac{(\not{x}_{12} \not{x}_{23})_{ab} \delta^{AB}}{(x_{12}^2)^{m_3} (x_{23}^2)^{m_1} (x_{31}^2)^{m_2}} \\ &\approx C_{123} (x_{12}^2)^{-\frac{1}{2}[\Delta_{2n+1} + \Delta_{2n+2} - \Delta_1 + 1]} (\not{x}_{12})_{ac} \left[\frac{(\not{x}_{23})_{cb}}{(x_{23}^2)^{\Delta_1 + \frac{1}{2}}} - 2m_2 \frac{(x_{12} \cdot x_{23})(\not{x}_{23})_{cb}}{(x_{23}^2)^{\Delta_1 + \frac{3}{2}}} \right] \end{aligned} \quad (\text{A.11})$$

On the other hand, OPE of the first two operators is given by

$$\begin{aligned} V_{2n+1\ a}^A(x_1) \times V_{2n+2}(x_2) &\approx \tilde{f}(x_{12}^2)^{-\frac{1}{2}[\Delta_{2n+1} + \Delta_{2n+2} - \Delta_1 + 1]} (\not{x}_{12})_{ac} \\ &\times \left[\delta_{cd} + q_1 \delta_{cd} x_{12}^\mu \partial_{2\ \mu} + q_2 (\not{x}_{12} \not{\partial}_2)_{cd} \right] V_{1\ d}^A(x_2) \end{aligned} \quad (\text{A.12})$$

Substituting this in the l.h.s. of (A.9) and comparing with (A.11), we get

$$\begin{aligned} q_1 &= \frac{\Delta_{2n+1} - \Delta_{2n+2} + \Delta_1}{2\Delta_1 + 1} \\ q_2 &= \frac{\Delta_{2n+1} - \Delta_{2n+2} + \Delta_1}{(2\Delta_1 + 1)(2\Delta_1 + 1 - d)} \end{aligned} \quad (\text{A.13})$$

We now return to the case of $n = 1$ in (A.1). Here we have

$$\begin{aligned} \langle V_2(x_1) V_{3\ a}^A(x_2) \bar{V}_{1\ b}^B(x_3) \rangle &= -\frac{1}{\alpha(\epsilon)} (\not{\partial}_2)_{ac} \langle V_2(x_1) V_{1\ c}^A(x_2) \bar{V}_{1\ b}^B(x_3) \rangle \\ &- \frac{1}{\alpha(\epsilon)} \left[(d + 2l_1) \delta_{ab} - 2l_3 \frac{(\not{x}_{12} \not{x}_{23})_{ab}}{x_{12}^2} \right] \frac{1}{(x_{12}^2)^{l_3} (x_{23}^2)^{l_1} (x_{31}^2)^{l_2}} \end{aligned} \quad (\text{A.14})$$

Expanding the above expression for $|x_{12}| \ll |x_{31}|$ and $|x_{12}| \ll |x_{23}|$ as previously, and comparing against the OPE gives the required coefficient

$$q_2^2 \approx \frac{(\gamma_{2,1} + 1)}{4\gamma_{1,2}\epsilon} \quad (\text{A.15})$$

where we have used the fact that $\gamma_1 = \gamma_{1,2}\epsilon^2$ and $\gamma_2 = \gamma_{2,1}\epsilon$. Similarly, from the $\langle V_{1\ a}^A(x_1) V_4(x_2) \bar{V}_{1\ b}^B(x_3) \rangle$ 3-pt function one obtains

$$q_2^3 \approx -\frac{(1 + \gamma_{4,1})}{4\gamma_{1,2}\epsilon} \quad (\text{A.16})$$

B Computing $f_{2p}\rho_{2p}$ and $f_{2p+1}\rho_{2p+1}$ from cow-pies

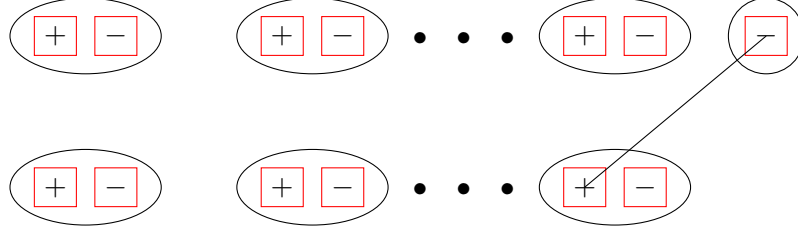
In this appendix we give an alternate way to obtain $f_{2p}\rho_{2p}$ and $f_{2p+1}\rho_{2p+1}$ coefficients using cow-pie contractions. This works as a double check of our results, because there are not many results other than (4.12) and (4.13) that we can check in the literature.

B.1 $f_{2p}\rho_{2p}$

For the ease of counting, we invert the cow-pie and start the contractions from the single kernel. There are two cases to consider,

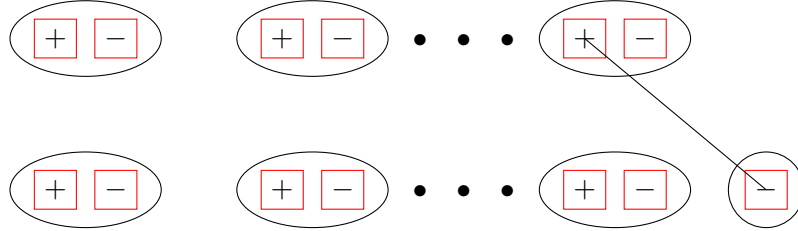
Case I: The ‘ $-$ ’ kernel remains uncontracted. It is easy to see that its contribution is zero because rest of the contractions cannot give the desired operator.

Case II: The ‘ $-$ ’ kernel is contracted with ‘ $+$ ’ from the double cow-pie.



which contributes $-pF_{p,0,0;1,1}^{p-1,0,1}$ which can again be evaluated using cow-pies. Once again, we invert the cow-pie diagram and start the contractions from the uncontracted ‘ $-$ ’ in the double cow-pie. As can be readily seen, there are two cases to consider:

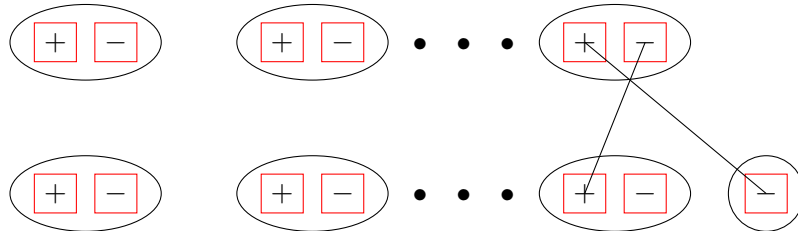
(a): ‘ $-$ ’ remains uncontracted. This gives a contribution of $F_{p+1,0,0;1,1}^{p,0,0}$.



$F_{p+1,0,0;1,1}^{p,0,0}$ can in turn be evaluated using cow-pies. This is similar to the case of f_{2p} and f_{2p+1} and its recursion equation is given by

$$F_{p+1,0,0;1,1}^{p,0,0} = (p+1)(N-p)F_{p,0,0;1,1}^{p-1,0,0} \quad (\text{B.1})$$

(b): ‘ $-$ ’ can be contracted with one of the double cow-pies. This gives a factor of $(p+1)F_{p,0,1;1,2}^{p,0,0}$



Putting all the pieces together, we have the following system of recursion equations

$$\begin{aligned} F_{p,0,1;1,2}^{p,0,0} &= -pF_{p,0,0;1,1}^{p-1,0,1} \\ F_{p+1,0,0;1,1}^{p,0,1} &= F_{p+1,0,0;1,1}^{p,0,0} + (p+1)F_{p,0,1;1,2}^{p,0,0} \\ F_{p+1,0,0;1,1}^{p,0,0} &= (p+1)(N-p)F_{p,0,0;1,1}^{p-1,0,0} \end{aligned} \quad (\text{B.2})$$

which can be solved along with the launching conditions $F_{0,0,1;1,2}^{0,0,0} = 0$, $F_{1,0,0;1,1}^{0,0,1} = 1$ and $F_{1,0,0;1,1}^{0,0,0} = 1$. Using the expression for f_{2p} in (3.4), we obtain

$$\rho_{2p} = -\frac{p}{N-1} \quad (\text{B.3})$$

B.2 $f_{2p+1}\rho_{2p+1}$

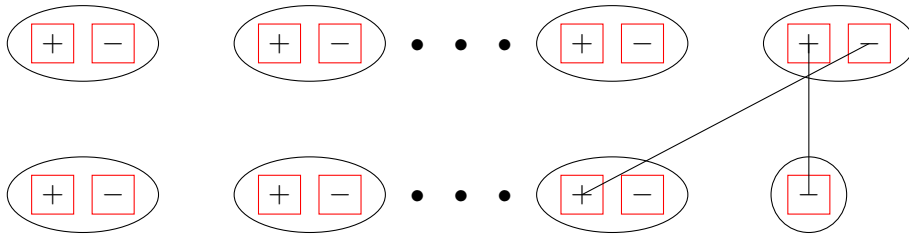
We proceed analogous to the even case, i.e. $f_{2p}\rho_{2p}$. As in the previous case, we start the contractions with the single kernel ‘−’. Again, we have two cases:

Case I: ‘−’ remains uncontracted. This indeed contributes, with a factor of $F_{p+1,0,0;1,1}^{p,0,0}$ which can be further evaluated using cow-pies. It can be seen that the recursion equation for $F_{p+1,0,0;1,1}^{p,0,0}$ is given by

$$F_{p+1,0,0;1,1}^{p,0,0} = (p+1)(N-p)F_{p,0,0;1,1}^{p-1,0,0} \quad (\text{B.4})$$

Case II: ‘−’ is contracted with one of the double cow-pies. This contributes a factor of $(p+1)F_{p,0,1;1,2}^{p,0,0}$, where $F_{p,0,1;1,2}^{p,0,0}$ can be furthermore evaluated using cow-pies. To this end, we invert the cow-pie diagram and once again consider two separate cases

- a: ‘−’ of the double cow-pie remains uncontracted. It can be seen that this does not contribute as we do not get the desired operator.
- b: ‘−’ of the double cow-pie is contracted with one of the double cow-pies. This contributes $-pF_{p,0,0;1,2}^{p-1,0,1}$.



Thus we have following system of recursion equations, which can be solved along with the launching conditions $F_{0,0,1,0,0;1,2} = 1$, $F_{1,0,0;1,1}^{0,0,0} = 1$, $F_{0,0,1;1,2}^{0,0,0} = 0$.

$$\begin{aligned} F_{p+1,0,0;1,2}^{p,0,1} &= F_{p+1,0,0;1,1}^{p,0,0} + (p+1)F_{p,0,1;1,2}^{p,0,0} \\ F_{p+1,0,0;1,1}^{p,0,0} &= (p+1)(N-p)F_{p,0,0;1,1}^{p-1,0,0} \\ F_{p,0,1;1,2}^{p,0,0} &= -pF_{p,0,1;1,2}^{p-1,0,1} \end{aligned} \quad (\text{B.5})$$

Using (3.6), along with above set of recursion equations gives

$$\rho_{2p+1} = 1 - \frac{p}{N-1} \quad (\text{B.6})$$

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