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Character relations and replication identities in 2d Conformal Field Theory

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Abstract: We study replication identities satisfied by conformal characters of a 2D CFT, providing a natural framework for a physics interpretation of the famous Hauptmodul property of Monstrous Moonshine, and illustrate the underlying ideas in simple cases.

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Contents

1 Introduction

The remarkable interaction between mathematics and physics around the turn of the century has been to a large part spurred by the new mathematical structures underlying String Theory [25, 29], leading to such interesting new mathematical concepts as Vertex Operator Algebras [10, 21] and Modular Tensor Categories [1, 34]. These developments in turn were strongly influenced by Monstrous Moonshine [14, 30, 31], the amazing connection between the representation theory of the Monster $\mathbb{M}$, the largest sporadic finite simple group, with the classical theory of modular forms. Actually, VOA theory grew out from the need to provide a conceptual explanation of Moonshine.

It has been recognized pretty early [17] that, to a large extent, the Moonshine conjectures find a natural physics explanation by interpreting the relevant quantities as describing string propagation in a suitable (rather exotic) background, the Moonshine orbifold, obtained as the result of orbifolding [18] the Moonshine module by the Monster. From this point of view, many strange-looking properties [28] of the Thompson-McKay series involved in Moonshine follow from general physical principles, with one notable exception: the so-called Hauptmodul property, which states (roughly speaking) that Thompson-McKay series generate the field of meromorphic functions of suitable genus zero Riemann surfaces, does not find any obvious interpretation from a physics perspective [22, 23].

There has been several attempts to remedy this situation and find a physics explanation of the Hauptmodul property, see e.g. [19, 32, 33], but none proposed to this date seems completely satisfactory. The aim of the present paper is to present a new approach to the problem, based on the notion of character relations and replication identities, which generalizes to arbitrary 2D Conformal Field Theories [9, 20], and which provides an equivalent formulation of the Hauptmodul property in the special case of the Moonshine orbifold. Roughly speaking, this approach relates the Hauptmodul property to symmetries of second quantized string propagation [16] on the Moonshine orbifold. While the precise nature of these symmetries is still unclear (because identifying them would require a thorough analysis of the higher symmetric products of the Moonshine orbifold, a pretty challenging task

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in view of the intricate computations involved), the above identification could prove to be a first step in a better understanding of the problem. That the above approach can be made to work is demonstrated in the comparatively much simpler case of the Ising model, where the analysis can be explicitly performed (at least for low degrees), and the resulting replication identities related precisely to actual symmetries of symmetric products.

2 Conformal characters, the modular representation and character relations

Among the important characteristics of a 2D CFT [9, 20], a prominent role is played by the conformal characters of the 'primaries', the trace functions of irreducible modules in the language of (C_2-cofinite rational) Vertex Operator Algebras. As a consequence of conformal symmetry, the chiral symmetry algebra contains the Virasoro algebra, whose zero mode L_0 plays the role of (chiral) Hamiltonian. The commutation rules of the Virasoro generators imply that, in each irreducible module separately, the eigenvalues of L_0 are integrally spaced, hence the spectrum of L_0 can be characterized by specifying the lowest eigenvalue, called the conformal weight of the primary, and the generating function of the eigenvalue multiplicities. For a primary p of conformal weight $\mathtt{h}_p$, the conformal character reads

$$\chi_p(q) = q^{-c/24} \sum_{n=0}^{\infty} d_n q^{n+\mathtt{h}_p} \tag{2.1}$$

where d_n denotes the multiplicity of $n+\mathtt{h}_p$ as an eigenvalue of L_0 and c the central charge of the model. One can show that the above (fractional) power series is absolutely convergent in the disk $|q| < 1$, hence defines an analytic function there.

Besides characterizing the spectrum of L_0 in the irreducible modules, the conformal characters also provide the basic building blocks of the torus partition function. In the simplest case of diagonal theories, the torus partition function reads

$$Z(\tau, \overline{\tau}) = \sum_p \left|\chi_p\left(\mathtt{e}^{2\mathtt{i}\pi\tau}\right)\right|^2 \tag{2.2}$$

where τ denotes the modular parameter of the torus, and the sum runs over all primaries; more generally, the torus partition function is a sesquilinear combination of the conformal characters. Combining this observation with the invariance [13] of the torus partition function under modular transformations i.e. transformations of the modular parameter τ that do not change the conformal equivalence class, one arrives at the conclusion that the modular group $\mathrm{SL}_2(\mathbb{Z})$ is represented on the linear span of the characters, i.e. for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ there exists a unitary representation matrix $M = \rho\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that

$$\chi_p\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_s M_{ps} \chi_s(\tau) \tag{2.3}$$

Two remarks are in order here: first, the modular representation is actually a *matrix* representation, meaning that each individual modular matrix element has an invariant meaning. This is particularly clear when considering Verlinde's celebrated formula [35] expressing the fusion rules of the theory in terms of modular matrix elements, or its various generalizations [6, 27]. From a technical point of view, this means that the linear space $\mathscr{V}$ affording the modular representation comes equipped with a distinguished basis $\mathfrak{B} = \{\mathfrak{b}_p\}$ labeled by the primaries, and a different choice of basis would correspond to a different theory.

The second observation is that the transformation rule eq. (2.3) does not always determine the modular representation matrices. The reason for this is that the conformal characters, as functions of the modular parameter τ, are not necessarily linearly independent, i.e. there may exist nontrivial relations of the form

$$\sum_p R_p \chi_p(\tau) = 0 \tag{2.4}$$

with coefficients R_p independent of τ. The existence of such nontrivial character relations is actually pretty common, e.g. the characters of charge conjugate primaries are automatically equal

$$\chi_{\overline{p}}(\tau) = \chi_p(\tau) \tag{2.5}$$

As a consequence of the character relations, the linear span $\mathscr{W}$ of the characters is usually only a subspace of $\mathscr{V}$, and the individual modular matrix elements cannot be determined from eq. (2.3), only suitable linear combinations of them. Actually, the example of charge conjugation is a good indication for the origin of such character relations: they are the reflections of (possibly hidden) global symmetries of the theory.[1]

To illustrate this last point, let us consider the orbifold line of $c{=}1$ theories [24]. It is well known that, at compactification radii for which $N{=}2r_{\mathtt{orb}}^2$ is an integer, these theories have exactly $N{+}7$ primary fields with conformal characters

$$\begin{aligned}
u_{\pm}(\tau) &= \frac{1}{2\eta(\tau)}\theta_3(2N\tau) \pm \sqrt{\frac{\eta}{2\theta_2}}(\tau) = \frac{1}{2\eta(\tau)}\left\{\Theta\begin{bmatrix}0\\0\end{bmatrix}(2N\tau) \pm \theta_4(2\tau)\right\}\\
\chi_k(\tau) &= \frac{1}{\eta(\tau)}\Theta\begin{bmatrix}\frac{k}{2N}\\0\end{bmatrix}(2N\tau) \quad \text{for } k{=}1,\ldots,N-1\\
\phi_{\pm}(\tau) &= \frac{1}{2\eta(\tau)}\theta_2(2N\tau) = \frac{1}{2\eta(\tau)}\Theta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(2N\tau)\\
\sigma_{\pm}(\tau) &= \frac{1}{2}\left\{\sqrt{\frac{\eta}{\theta_4}}(\tau) + \sqrt{\frac{\eta}{\theta_3}}(\tau)\right\} = \frac{1}{2\eta(\tau)}\left\{\theta_2\left(\frac{\tau}{2}\right) + \mathrm{e}^{-\frac{\pi\mathrm{i}}{24}}\theta_2\left(\frac{\tau+1}{2}\right)\right\}\\
\tau_{\pm}(\tau) &= \frac{1}{2}\left\{\sqrt{\frac{\eta}{\theta_4}}(\tau) - \sqrt{\frac{\eta}{\theta_3}}(\tau)\right\} = \frac{1}{2\eta(\tau)}\left\{\theta_2\left(\frac{\tau}{2}\right) - \mathrm{e}^{-\frac{\pi\mathrm{i}}{24}}\theta_2\left(\frac{\tau+1}{2}\right)\right\}
\end{aligned} \tag{2.6}$$

where

$$\Theta\begin{bmatrix}a\\b\end{bmatrix}(\tau) = \sum_{n\in\mathbb{Z}} \mathtt{e}^{\mathtt{i}\pi\tau(n-a)^2}\mathtt{e}^{-2\pi\mathtt{i}bn} \tag{2.7}$$

[1] Indeed, character relations, as linear relations between suitable (chiral) correlators, may be considered as Ward identities related to some global symmetry. Of course, the precise nature of the relevant symmetry might be pretty hard to pin down.

and

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \tag{2.8}$$

denotes Dedekind's eta function (with $q = \mathrm{e}^{2\mathrm{i}\pi\tau}$), while

$$\begin{aligned}
\theta_2 = \vartheta\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}(\tau) &= 2q^{1/8} \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n)^2 \\
\theta_3 = \vartheta\begin{bmatrix}0\\0\end{bmatrix}(\tau) &= \prod_{n=1}^{\infty} (1 - q^n)\left(1 + q^{n-1/2}\right)^2 \\
\theta_4 = \vartheta\begin{bmatrix}0\\\frac{1}{2}\end{bmatrix}(\tau) &= \prod_{n=1}^{\infty} (1 - q^n)\left(1 - q^{n-1/2}\right)^2
\end{aligned}$$

are the classical theta functions of Jacobi.

Let's restrict our attention to the models with even N. Since charge conjugation is trivial in this case, the obvious character relations

$$\begin{aligned}
\phi_- &= \phi_+ \\
\sigma_- &= \sigma_+ \\
\tau_- &= \tau_+
\end{aligned} \tag{2.9}$$

must have a different origin: they are a manifestation of the dihedral $\mathbb{D}_4$ symmetry underlying these models [24], which follows from the fact that the orbifold line may be obtained as the conformal limit of Ashkin-Teller models, i.e. two Ising spins coupled locally via their energy density. Clearly, the transformations that flip each Ising spin separately, together with the one that exchanges the two, form a $\mathbb{D}_4$ symmetry group, explaining the above character relations. In case $N = 4$ (corresponding to the 4-state Potts model) this symmetry is extended to a full $\mathbb{S}_4$, resulting in the extra character relations

$$\begin{aligned}
\phi_\pm &= u_- \\
\chi_1 &= \sigma_\pm \\
\chi_3 &= \tau_\pm
\end{aligned} \tag{2.10}$$

Another interesting case is that of $N = 16$, when the generic character relations eq. (2.9) get supplemented by

$$\begin{aligned}
\chi_8 - \phi_\pm &= u_- \\
\chi_2 + \chi_{14} &= \sigma_\pm \\
\chi_6 + \chi_{10} &= \tau_\pm
\end{aligned} \tag{2.11}$$

More generally, such extra character relations occur whenever N is the square of an even

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integer, $N=(2n)^2$, when one has

$$\begin{aligned}\sum_{k=1}^{n-1}(-1)^{k-1}\chi_{4nk}-(-1)^n\phi_\pm &= u_-\\ \sum_{k=0}^{\left[\frac{n-1}{2}\right]}\left\{\chi_{n(8k+1)}+\chi_{n(8k+7)}\right\} &= \sigma_\pm\\ \sum_{k=0}^{\left[\frac{n-1}{2}\right]}\left\{\chi_{n(8k+3)}+\chi_{n(8k+5)}\right\} &= \tau_\pm\end{aligned} \tag{2.12}$$

as a consequence of the general identity

$$\sum_{k=0}^{N-1}\mathsf{e}^{2\mathrm{i}\pi\frac{kb}{N}}\,\theta\!\begin{bmatrix}a+\frac{k}{N}\\0\end{bmatrix}\!(\tau)=\theta\!\begin{bmatrix}-Na\\\frac{b}{N}\end{bmatrix}\!\left(\frac{\tau}{N^2}\right) \tag{2.13}$$

valid for integer b and N, as well as the theta relations[2]

$$\begin{aligned}\theta_4(2\tau) &= \sqrt{\theta_3(\tau)\,\theta_4(\tau)}\\ \theta_2\!\left(\frac{\tau}{2}\right) &= \sqrt{2\theta_2(\tau)\,\theta_3(\tau)}\\ \theta_2\!\left(\frac{\tau+1}{2}\right) &= \mathsf{e}^{\frac{\mathrm{i}\pi}{16}}\sqrt{2\theta_2(\tau)\,\theta_4(\tau)}\end{aligned} \tag{2.14}$$

The origin of these extra relations eq. (2.12) may be traced back to the fact that the corresponding models may be constructed as dihedral orbifolds of the compactified boson at radius $r={}^1\!/\!\sqrt{2}$ [24].

From a technical point of view, nontrivial character relations indicate that the modular representation ρ is reducible. Indeed, as a consequence of the τ independence of the coefficients R_p in eq. (2.4), the linear span $\mathscr{W}$ of the characters (considered as a subspace of $\mathscr{V}$) is invariant under ρ. In particular, this means that in order to fully characterize the modular properties of the characters, it is not enough to specify the matrix representation ρ, but one should amend this by a description of the invariant subspace $\mathscr{W}$ (e.g. by specifying a basis of it). Formally, one could think that this last step can be avoided by directly reducing the modular representation to the invariant subspace $\mathscr{W}$: after all, this subspace is the linear span of the conformal characters, thus it contains all the physically relevant information. But this is far from being true. For example, application of Verlinde's formula [27, 35], one of the cornerstones of the whole theory, necessitates the consideration of the full modular representation, with all individual matrix elements. Similarly, computation of Frobenius-Schur indicators [2], or the application of the trace identities of [6] require the knowledge of each matrix element separately.

[2] An interesting consequence of eq. (2.12) is that in this case the characters of the orbifold can be expressed as linear combinations of the characters of the original theory, i.e. the compactified boson at radius $r=\sqrt{2}n$.

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3 Symmetric products and replication identities

Consider a system made up of n identical subsystems, each described by the same CFT $\mathcal{C}$. The whole system will be still conformally invariant, described by the n-fold tensor power of $\mathcal{C}$, and any permutation of the identical subsystems will leave the whole system invariant. Consequently, for any permutation group $\Omega < \mathbb{S}_n$ of degree n, one could consider the permutation orbifold[3] $\mathcal{C} \wr \Omega$ obtained by orbifolding the tensor power by the twist group Ω [3, 11, 26]. Because of the universal nature of the action of Ω, all relevant quantities (like correlation and partition functions, fusion rules, modular matrix elements, etc.) of $\mathcal{C}\wr\Omega$ may be expressed in terms of the relevant quantities of $\mathcal{C}$, namely as polynomial expressions of these quantities evaluated on suitable n-sheeted covering surfaces of the world sheet, see [4, 5] for details. In particular, the conformal characters of the permutation orbifold are completely determined by those of $\mathcal{C}$ and the twist group Ω [3]. We note that all relevant relations can be subsumed under a general group theoretic construct, the orbifold transform, described in detail in [8].

A particularly interesting case is when the twist group Ω is maximal, i.e. when Ω is the full symmetric group $\mathbb{S}_n$ of degree n: the resulting permutation orbifold $\mathcal{C}\wr\mathbb{S}_n$ is called the n-th symmetric product of $\mathcal{C}$, and plays an important role in the description of second quantized strings [7, 15, 16]. The analysis of symmetric products is greatly simplified by the exponential identity [8], a general combinatorial identity satisfied by the orbifold transform, which provides closed expressions for the characteristic quantities of symmetric products.

According to the general theory [3], the conformal characters (evaluated at some specific modulus τ) of the n-fold symmetric product $\mathcal{C} \wr \mathbb{S}_n$ may be expressed as polynomial expressions of the conformal characters of $\mathcal{C}$ evaluated on the different n-sheeted (unbranched) coverings of a torus with modulus τ. But all theses coverings have genus 1, hence each connected component is itself a torus of modulus

$$\frac{a\tau + b}{d} \tag{3.1}$$

for suitable non-negative integers a, b, d characterizing the relevant covering. The precise form of the polynomial expressions is irrelevant at this point, the only thing to note is that all possible coverings occur in the process. This means that a character relation of the symmetric product $\mathcal{C} \wr \mathbb{S}_n$ is nothing but a polynomial relation between quantities of the form

$$\chi_p\left(\frac{a\tau + b}{d}\right)$$

We shall call such relations replication identities, because in the specific case of the Moonshine orbifold they yield precisely the replication formulas satisfied by the generalized Thompson-McKay series. It should be emphasized that the above notion of replication identities is pretty general, far from being confined to derivatives of the Moonshine module or to rational conformal models.

As explained in the previous section, character relations for a given CFT are usually reflections of outer symmetries relating different irreducible modules of the chiral algebra.

[3]The origin of the wreath product notation for permutation orbifolds is explained in [3].

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Consequently, one may view replication identities as an indication to the existence of suitable symmetries of the higher symmetric products. To illustrate the above ideas, let us consider the Ising model, i.e. the Virasoro minimal model of central charge $c{=}1/2$. In this case there are three primary fields, $\mathbf{0}$, $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$, of respective conformal weights $0, 1/2$ and $1/16$, with conformal characters

$$
\begin{aligned}
\chi_{\mathbf{0}} &= \frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}}+\sqrt{\frac{\theta_4}{\eta}}\right)\\
\chi_{\boldsymbol{\epsilon}} &= \frac{1}{2}\left(\sqrt{\frac{\theta_3}{\eta}}-\sqrt{\frac{\theta_4}{\eta}}\right)\\
\chi_{\boldsymbol{\sigma}} &= \sqrt{\frac{\theta_2}{2\eta}}
\end{aligned}
\tag{3.2}
$$

Note that

$$
\begin{aligned}
\sqrt{\frac{\theta_3}{\eta}} &= q^{-1/48}\prod_{n=0}^{\infty}\left(1+q^{n+\frac{1}{2}}\right) &&= \frac{\eta(\tau)^2}{\eta\left(\frac{\tau}{2}\right)\eta(2\tau)}\\
\sqrt{\frac{\theta_4}{\eta}} &= q^{-1/48}\prod_{n=0}^{\infty}\left(1-q^{n+\frac{1}{2}}\right) &&= \frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)}\\
\sqrt{\frac{\theta_2}{2\eta}} &= q^{1/24}\prod_{n=1}^{\infty}\left(1+q^{n}\right) &&= \frac{\eta(2\tau)}{\eta(\tau)}
\end{aligned}
\tag{3.3}
$$

The modular representation, characterized by the matrix

$$
S=\frac{1}{2}\begin{pmatrix} 1 & 1 & \sqrt{2}\\ 1 & 1 & -\sqrt{2}\\ \sqrt{2} & -\sqrt{2} & 0\end{pmatrix}
\tag{3.4}
$$

is irreducible, hence has no non-trivial invariant subspace; consequently, the conformal characters of Ising are linearly independent.

As a consequence of the identities

$$
\begin{aligned}
\sqrt{\frac{\theta_4(2\tau)}{\eta(2\tau)}} &= \frac{\sqrt{\theta_3(\tau)\,\theta_4(\tau)}}{\eta(\tau)}\\
\sqrt{\frac{\theta_2\left(\frac{\tau}{2}\right)}{\eta\left(\frac{\tau}{2}\right)}} &= \frac{\sqrt{\theta_2(\tau)\,\theta_3(\tau)}}{\eta(\tau)}\\
\mathrm{e}^{-\frac{\pi\mathrm{i}}{24}}\sqrt{\frac{\theta_2\left(\frac{\tau+1}{2}\right)}{\eta\left(\frac{\tau+1}{2}\right)}} &= \frac{\sqrt{\theta_2(\tau)\,\theta_4(\tau)}}{\eta(\tau)}
\end{aligned}
\tag{3.5}
$$

that follow easily from eqs. (2.14), one gets that

$$
\begin{aligned}
\chi_{\mathbf{0}}(2\tau)-\chi_{\boldsymbol{\epsilon}}(2\tau) &= \chi_{\mathbf{0}}(\tau)^2-\chi_{\boldsymbol{\epsilon}}(\tau)^2\\
\chi_{\boldsymbol{\sigma}}\left(\frac{\tau}{2}\right) &= \frac{\chi_{\mathbf{0}}(\tau)+\chi_{\boldsymbol{\epsilon}}(\tau)}{2}\chi_{\boldsymbol{\sigma}}(\tau)\\
\chi_{\boldsymbol{\sigma}}\left(\frac{\tau+1}{2}\right) &= \mathrm{e}^{\frac{\pi\mathrm{i}}{24}}\,\frac{\chi_{\mathbf{0}}(\tau)-\chi_{\boldsymbol{\epsilon}}(\tau)}{2}\chi_{\boldsymbol{\sigma}}(\tau)
\end{aligned}
\tag{3.6}
$$

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These are prime examples of replication identities, involving values of characters on different covering surfaces. Consequently, they should be related to character relations, and ultimately to symmetries of symmetric products of the Ising model. Let's see how this comes about!

According to the general theory [3], the 2-fold symmetric product of Ising has central charge $c=1$ (twice the central charge of the Ising model) and a total of $\frac{3(3+7)}{2}=15$ primary fields, whose conformal characters read

$$\begin{aligned}
\boldsymbol{\Phi}_{\langle p,q\rangle}(\tau) &= \chi_p(\tau)\,\chi_q(\tau) \qquad\qquad \text{for}\quad p\neq q\\
\mathfrak{u}_p^{(\pm)}(\tau) &= \frac{1}{2}\left\{\chi_p(\tau)^2 \pm \chi_p(2\tau)\right\}\\
\mathfrak{t}_p^{(\pm)}(\tau) &= \frac{1}{2}\left\{\chi_p\left(\frac{\tau}{2}\right) \pm \mathsf{e}^{-\mathtt{i}\pi(\mathtt{h}_p-1/48)}\chi_p\left(\frac{\tau+1}{2}\right)\right\}
\end{aligned} \tag{3.7}$$

for $p,q\in\{\mathbf{0},\boldsymbol{\epsilon},\boldsymbol{\sigma}\}$, with $\mathtt{h}_p$ denoting the conformal weight of the primary p. By inspecting the q-expansions of these characters, one arrives at the character relations

$$\begin{aligned}
\mathfrak{u}_{\mathbf{0}}^{(-)}(\tau) &= \mathfrak{u}_{\boldsymbol{\epsilon}}^{(-)}(\tau)\\
\boldsymbol{\Phi}_{\langle\mathbf{0},\boldsymbol{\sigma}\rangle}(\tau) &= \mathfrak{t}_{\boldsymbol{\sigma}}^{(+)}(\tau)\\
\boldsymbol{\Phi}_{\langle\boldsymbol{\epsilon},\boldsymbol{\sigma}\rangle}(\tau) &= \mathfrak{t}_{\boldsymbol{\sigma}}^{(-)}(\tau)
\end{aligned} \tag{3.8}$$

which reduce to

$$\begin{aligned}
\chi_{\mathbf{0}}(\tau)^2-\chi_{\mathbf{0}}(2\tau) &= \chi_{\boldsymbol{\epsilon}}(\tau)^2-\chi_{\boldsymbol{\epsilon}}(2\tau)\\
\chi_{\boldsymbol{\sigma}}\left(\frac{\tau}{2}\right)+\mathrm{e}^{-\frac{\pi\mathrm{i}}{24}}\chi_{\boldsymbol{\sigma}}\left(\frac{\tau+1}{2}\right) &= \chi_{\mathbf{0}}(\tau)\,\chi_{\boldsymbol{\sigma}}(\tau)\\
\chi_{\boldsymbol{\sigma}}\left(\frac{\tau}{2}\right)-\mathrm{e}^{-\frac{\pi\mathrm{i}}{24}}\chi_{\boldsymbol{\sigma}}\left(\frac{\tau+1}{2}\right) &= \chi_{\boldsymbol{\epsilon}}(\tau)\,\chi_{\boldsymbol{\sigma}}(\tau)
\end{aligned} \tag{3.9}$$

upon taking into account the expressions eqs. (3.7). Clearly, these are equivalent to the replication identities eqs. (3.6), and we see that, indeed, the latter are nothing but character relations for the second symmetric product. What remains to do is to find out which symmetries are responsible for this.

Since the full moduli space of $c=1$ conformal models is known, it is a simple matter to identify the second symmetric product of the Ising model: it lies on the orbifold line at radius $r_{\mathtt{orb}}=2$, i.e. at $N=8$. Furthermore, it is an easy exercise to identify the respective primary fields, in particular one gets

$$\begin{array}{lllll}
u_+\leftrightarrow\mathfrak{u}_{\mathbf{0}}^{(+)} & \chi_2\leftrightarrow\mathfrak{u}_{\boldsymbol{\sigma}}^{(+)} & \chi_5\leftrightarrow\mathfrak{t}_{\boldsymbol{\epsilon}}^{(-)} & \phi_+\leftrightarrow\mathfrak{u}_{\mathbf{0}}^{(-)} & \phi_-\leftrightarrow\mathfrak{u}_{\boldsymbol{\epsilon}}^{(-)}\\
u_-\leftrightarrow\mathfrak{u}_{\boldsymbol{\epsilon}}^{(+)} & \chi_3\leftrightarrow\mathfrak{t}_{\boldsymbol{\epsilon}}^{(+)} & \chi_6\leftrightarrow\mathfrak{u}_{\boldsymbol{\sigma}}^{(-)} & \sigma_+\leftrightarrow\boldsymbol{\Phi}_{\langle\mathbf{0},\boldsymbol{\sigma}\rangle} & \sigma_-\leftrightarrow\mathfrak{t}_{\boldsymbol{\sigma}}^{(+)}\\
\chi_1\leftrightarrow\mathfrak{t}_{\mathbf{0}}^{(+)} & \chi_4\leftrightarrow\boldsymbol{\Phi}_{\langle\mathbf{0},\boldsymbol{\epsilon}\rangle} & \chi_7\leftrightarrow\mathfrak{t}_{\mathbf{0}}^{(-)} & \tau_+\leftrightarrow\boldsymbol{\Phi}_{\langle\boldsymbol{\epsilon},\boldsymbol{\sigma}\rangle} & \tau_-\leftrightarrow\mathfrak{t}_{\boldsymbol{\sigma}}^{(-)}
\end{array} \tag{3.10}$$

But, as explained in the previous section, this model exhibits a dihedral $\mathbb{D}_4$ symmetry as a consequence of its Ashkin-Teller origin, leading to the character relations eqs. (2.9), which, taking into account the field identifications eq. (3.10), yield precisely eqs. (3.8). In this case the analysis of the relevant symmetries is relatively easy thanks to the identification of the symmetric product as an Ashkin-Teller model, but the underlying idea should be clear.

The third symmetric product of Ising has central charge $c = 3/2$, and can be identified with an isolated $N = 1$ superconformal model [12], which has a total of 49 primaries. This superconformal model has 9 independent character relations, but it turns out that all of these follow from the replication identities (3.6). New replication identities could come from the character relations of the 4-fold symmetric product: unfortunately, this latter model of central charge $c = 2$ has 171 different primary fields, with 59 independent relations between their characters, whose connection to the symmetries of the model is far from being easy to determine.

4 Outlook and conclusion

Trying to find a physics interpretation of the Hauptmodul property of Monstrous Moonshine, we considered the question of character relations and replication identities in Conformal Field Theory. Character relations play an important role in understanding the structure of specific models, and should be viewed as one of the basic ingredients (besides the modular representation) to fully specify their modular properties, while replication identities are nothing but character relations for symmetric products. Since character relations can be traced back ultimately to suitable symmetries of the model under study, replication identities should correspond to symmetries of its symmetric products.

The Hauptmodul property of Monstrous Moonshine is a consequence of the replication identities obeyed by the (generalized) Thompson-McKay series. Based on this, we suggest that it is actually a manifestation of the inherent symmetries of second quantized string propagation on the Moonshine orbifold, the string background obtained by orbifolding the Moonshine module by the Monster, and whose primary characters are linear combinations of the Thompson-McKay series. Let us stress that this approach does not give us an alternate proof of the Hauptmodul property, just a possible physics interpretation for it. However, if correct, it could have interesting consequences even from a purely mathematical perspective, e.g. providing suitably generalized versions of the replication identities for higher genus analogues of the Thompson-McKay series.

While the arguments leading to the above could seem straightforward, the actual implementation, i.e. the identification of the relevant symmetries might be far from simple. The proliferation of character relations in higher symmetric products makes the analysis pretty difficult even for the Ising model, and one should expect worse in more complicated cases. But there are various arguments suggesting that, notwithstanding all computational difficulties involved, the identification of the relevant symmetries might be nevertheless carried out.

The first observation is that, for any two permutation groups Ω_1 and Ω_2 such that Ω_1 is a subgroup of Ω_2, the character relations of the Ω_1 permutation orbifold are inherited by the Ω_2 permutation orbifold. This is actually the reason why it is sufficient to look only at symmetric products when considering replication identities. Combining this with the obvious embeddings of wreath products into symmetric groups and the transitivity property of permutation orbifolds [3, 5], one can see that many of the replication identities of a given degree are trivial consequences of lower degree ones, and in particular of character relations, which are nothing but replication identities of degree one. As a result, it is enough to understand the 'primitive' identities that do not follow from identities coming from lower degrees, and these are clearly much less abundant, hopefully forming a set that can be dealt with.

The second point is that one does not even need the precise identification of all of the symmetries responsible for the primitive identities, it is enough to identify only a generating set, which can turn out to be pretty small. Since all replication identities for Moonshine are known, this should simplify the job to a large extent. Of course, even in case of a few generators the actual identification of the relevant symmetries could require some ingenuity, but one could expect that special properties of the Monster and the Moonshine module should allow the use of ad hoc techniques to solve this problem: after all, such considerations allow the determination of the character table of the Monster (with cca. 10^{54} elements), while a brute force computation for a group with only a few million of elements is already a time and resource consuming task.

Even if the above program can be completed and all relevant symmetries responsible for the replication identities of the Moonshine orbifold identified, there would still remain the question of what is so special about this particular model. After all, while non-trivial replication identities are not uncommon for rational models, they are usually not restrictive enough to force the chiral characters to be actually Hauptmoduls; this seems to be connected with a particularly high degree of symmetry inherent to symmetric products of the Moonshine orbifold. It would be interesting to find out other models that show similar features, and whether this could be linked with other approaches [19, 32] to the Hauptmodul property of Moonshine. We believe that further elaboration of these issues could lead to a better understanding of the whole subject.

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