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# $\mathcal{N} = 1$ superfield description of six-dimensional supergravity

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Abstract: We express the action of six-dimensional supergravity in terms of four-dimensional $\mathcal{N} = 1$ superfields, focusing on the moduli dependence of the action. The gauge invariance of the action in the tensor-vector sector is realized in a quite nontrivial manner, and it determines the moduli dependence of the action. The resultant moduli dependence is intricate, especially on the shape modulus. Our result is reduced to the known superfield actions of six-dimensional global SUSY theories and of five-dimensional supergravity by replacing the moduli superfields with their background values and by performing the dimensional reduction, respectively.

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## Contents





## 1 Introduction

Higher dimensional supergravity (SUGRA) theories provide interesting setups for supersymmetric (SUSY) models with extra dimensions, and are also regarded as effective theories of the superstring theory in some cases. For the purpose of analyzing SUSY extra-dimensional models, the $\mathcal{N} = 1$ superfield description of the action is quite useful [1]–[10].[1] It makes the derivation of four-dimensional (4D) effective theories transparent since the Kaluza-Klein mode expansion can be performed keeping the $\mathcal{N} = 1$ superspace structure. It also expresses the SUGRA action compactly, and allows us to work in general setups. In the global SUSY case, the $\mathcal{N} = 1$ superfield description of SUSY Yang-Mills theories from five to ten dimensions are provided in ref. [2]. However, we have to work in the context

[1] "$\mathcal{N} = 1$" denotes SUSY with four supercharges in this paper.

of SUGRA in order to treat the moduli, which are dynamical degrees of freedom corresponding to the "volume" or the "shape" of the compactified internal space. Such moduli often play important roles when we construct phenomenologically viable models. We also need to discuss the stabilization of the moduli to some finite values to obtain consistent extra-dimensional models.

Five-dimensional (5D) SUGRA provides the simplest setup for SUSY extra-dimensional models. The general action can be obtained by the superconformal formulation [11]–[18]. Based on this formulation, 5D SUGRA action with arbitrary numbers of hyper and vector multiplets has been expressed in terms of $\mathcal{N} = 1$ superfields [7, 8]. We have derived 4D effective theories of various 5D SUGRA models, and discussed their phenomenology [19]–[24].

The next simplest case is six-dimensional (6D) SUGRA [25, 26]. This has the smallest even extra-dimensions, and we can introduce magnetic flux that penetrates the compact space as a background. The shape modulus newly appears in addition to the volume modulus. These ingredients widen the possibility of model-building. Besides, we can also consider 6D SUGRA as a toy model of ten-dimensional superstring theories. With these reasons, 6D SUGRA is intriguing subject to investigate. As mentioned above, the $\mathcal{N} = 1$ superfield description is useful to discuss it, as was provided in ref. [2] in the global SUSY case. However, 6D action in ref. [2] cannot be promoted to SUGRA straightforwardly. As discussed in refs. [27, 28], the off-shell description of 6D SUGRA necessarily contains a tensor multiplet, which was not introduced in ref. [2]. It contains a self-dual antisymmetric tensor $B^+_{MN}$ $(M, N = 0, 1, \cdots, 5)$, and the 6D superconformal Weyl multiplet contains an anti-self-dual tensor $T^-_{MNL}$. In general, the (anti-)self-dual condition is an obstacle to the Lagrangian formulation, similar to that for type IIB SUGRA. Fortunately, we can evade this difficulty in 6D SUGRA. By combining $T^-_{MNL}$ with the field strength $F^+_{MNL} \equiv \partial_{[M}B^+_{NL]}$, we can define a new Weyl multiplet[2] that contains an unconstrained tensor $B_{MN}$. This new tensor field couples to the vector multiplets [27, 28]. Therefore we need to know how the tensor and the vector multiplets couple to each other in the $\mathcal{N} = 1$ superfield language.

In our previous work [30], we derived the $\mathcal{N} = 1$ superfield description of the tensor-vector couplings in 6D global SUSY theories, which is derived from the invariant action [29] in the projective superspace [31–33]. In this case, the tensor multiplet must be treated as external fields because we do not have the Weyl multiplet that contains $T^-_{MNL}$, and only have the constrained one $B^+_{MN}$. In this paper, we extend our result in ref. [30] to SUGRA. Since ref. [29] provides the projective superspace formulation of 6D SUGRA, we can in principle obtain its $\mathcal{N} = 1$ superfield description by integrating out half of the Grassmannian coordinates, as we did in the global SUSY case [30]. However, the procedure is not so straightforward as that in the global SUSY case because we need to separately treat the 4D part and the extra-dimensional part of the gravity sector that has a complicated structure in the projective superspace. Hence we adopt another strategy. We first identify the moduli superfields that originate from the extra-dimensional components of the 6D Weyl multiplet. Then, we insert them into the action in the global SUSY case under the

[2] This is called the "Weyl 2 multiplet" in ref. [28], and the "type-II Weyl multiplet" in ref. [29].



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following requirements.

1. The action is reduced to the global SUSY one if the moduli superfields are replaced with their background values.

2. It is consistent with the component field expression of the action.

3. It is invariant under the supergauge transformations.

The superfield action is uniquely determined by these requirements. As a nontrivial check, we show that our result reproduces the known superfield action of 5D SUGRA obtained in refs. [7, 8] after the dimensional reduction.

The paper is organized as follows. In the next section, we give a brief review of the superfield description of 6D global SUSY theories. In section 3, we promote it to the local SUSY case, and identify the desired superfield action of 6D SUGRA. In section 4, we explicitly show the gauge invariance of our result and the consistency with the known 5D SUGRA action through the dimensional reduction. Section 5 is devoted to the summary. We also collect some formulae and their derivation in the appendices.

## 2 6D global SUSY theory

Throughout the paper, we take the metric convention as $\eta_{MN} = \text{diag}(-1,1,1,1,1,1)$, and follow the notation of ref. [34] for the 2-component spinors.

### 2.1 Invariant action

We consider 6D (1,0) SUSY theories. The spacetime coordinates $x^M$ $(M = 0,1,\cdots,5)$ are decomposed into the 4D ones $x^\mu$ $(\mu = 0,1,2,3)$ and the extra dimensional ones $x^m$ $(m = 4,5)$. Before discussing 6D SUGRA, let us begin with its global SUSY limit. In this case, it is convenient to use the complex coordinates $z \equiv \frac{s}{2}(x^4 - ix^5)$ $(s \equiv e^{-\frac{\pi}{4}i})$ and its complex conjugate $\bar{z}$,[3] instead of $x^m$. Originally, the $\mathcal{N} = 1$ description of the action is provided in ref. [2]. For simplicity, we will consider Abelian gauge theories. The field content consists of hypermultiplets $\mathbb{H}^A$ $(A = 1,2,\cdots)$ and vector multiplets $\mathbb{V}^I$ $(I = 1,2,\cdots)$. They are decomposed into $\mathcal{N} = 1$ superfields as

$$\mathbb{H}^A = (H^{2A-1}, H^{2A}), \qquad \mathbb{V}^I = (V^I, \Sigma^I), \tag{2.1}$$

where $V^I$ is an $\mathcal{N} = 1$ real vector superfield, while the others are chiral superfields. By using these $\mathcal{N} = 1$ superfields, we can construct 6D global SUSY action as [2]

$$\begin{aligned} S_{\text{global}} &= \int d^6x \; (\mathcal{L}_{\text{V}} + \mathcal{L}_{\text{H}}) \,, \\ \mathcal{L}_{\text{V}} &\equiv \left\{ \int d^2\theta \, \frac{f_{IJ}}{2} \mathcal{W}^I \mathcal{W}^J + \text{h.c.} \right\} \\ &\quad + \int d^4\theta \; f_{IJ} \left\{ 4(\bar{\partial} V^I - \bar{\Sigma}^I)(\partial V^J - \Sigma^J) - 2\bar{\partial} V^I \partial V^J \right\} , \end{aligned}$$

[3] The definition of $z$ is different from that of ref. [30]. As we will see in the next section, this choice is convenient for the promotion to SUGRA.

$$\mathcal{L}_{\rm H} \equiv \int d^4\theta\, 2\left(H^\dagger_{\rm odd} e^V H_{\rm odd} + H^\dagger_{\rm even} e^{-V} H_{\rm even}\right)$$
$$- \left[\int d^2\theta\, \left\{H^t_{\rm odd}\left(\partial - \Sigma\right) H_{\rm even} - H^t_{\rm even}\left(\partial + \Sigma\right) H_{\rm odd}\right\} + {\rm h.c.}\right], \qquad (2.2)$$

where $\partial \equiv \partial_z = \bar{s}(\partial_4 + i\partial_5) = s^{-1}\partial_4 - s\partial_5$, and $H_{\rm odd}$ and $H_{\rm even}$ are column vectors that consist of $H^{2A-1}$ and $H^{2A}$, respectively. The contracted indices $I$ and $J$ are understood as being summed, and

$$\mathcal{W}^I_\alpha \equiv -\frac{1}{4}\bar{D}^2 D_\alpha V^I \qquad (2.3)$$

is the gauge-invariant field strength superfield. The coefficients $f_{IJ}$ are real constants and $f_{IJ} = f_{JI}$. The superfields without the indices $V$ and $\Sigma$ are defined as

$$V \equiv t_I V^I, \qquad \Sigma \equiv t_I \Sigma^I, \qquad (2.4)$$

where $t_I$ $(I = 1, 2, \cdots)$ are generators for the corresponding Abelian gauge groups. The Lagrangian (2.2) is invariant under the following (super)gauge transformation.

$$V^I \to V^I + \Lambda^I + \bar{\Lambda}^I, \qquad \Sigma^I \to \Sigma^I + \partial\Lambda^I,$$
$$H_{\rm odd} \to e^{-\Lambda} H_{\rm odd}, \qquad H_{\rm even} \to e^{\Lambda} H_{\rm even}, \qquad (2.5)$$

where the transformation parameter $\Lambda^I$ is a chiral superfield.

Unfortunately, (2.2) cannot be promoted to SUGRA straightforwardly. As mentioned in the introduction, a tensor multiplet $\mathbb{T} = \{B^+_{MN}, \cdots\}$ is necessary to describe 6D SUGRA. Thus we need to extend (2.2) including $\mathbb{T}$ in order to promote the action to the SUGRA one. This extension was provided in our previous work [30], which is directly derived from the invariant action in the 6D projective superspace [29]. We have to note that the tensor multiplet $\mathbb{T}$ cannot be off-shell in the global SUSY case [35]. We found that it is expressed by two $\mathcal{N} = 1$ superfields, i.e., a real linear superfield $\Phi_T$ and a chiral spinor superfield $\mathcal{W}_{T\alpha}$, which are subject to the constraints:

$$D^\alpha \mathcal{W}_{T\alpha} = -2\bar{\partial}\Phi_T,$$
$$\bar{D}^2 D_\alpha \Phi_T = -4\partial \mathcal{W}_{T\alpha}. \qquad (2.6)$$

From these relations, we obtain

$$\left(\Box_4 + \partial\bar{\partial}\right)\Phi_T = \left(\Box_4 + \partial\bar{\partial}\right)\mathcal{W}_{T\alpha} = 0, \qquad (2.7)$$

where $\Box_4 \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$. We have used that $\mathcal{P}_{\rm T}\Phi_T = \Phi_T$ and $\bar{D}^2 D^2 \mathcal{W}_{T\alpha} = 16\Box_4 \mathcal{W}_{T\alpha}$, where $\mathcal{P}_{\rm T} \equiv -\bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha}/(8\Box_4)$. Namely, $\Phi_T$ and $\mathcal{W}_{T\alpha}$ are on-shell, and thus should be treated as external superfields. Using these superfields, $\mathcal{L}_{\rm V}$ in (2.2) is extended to

$$\mathcal{L}_{\rm VT} = -\left[\int d^2\theta\, f_{IJ}\left\{2\Sigma^I \mathcal{W}^J \mathcal{W}_T + \frac{1}{4}\bar{D}^2\left(\Phi_T D^\alpha V^I \mathcal{W}^J_\alpha + \partial V^I D^\alpha V^J \mathcal{W}_{T\alpha}\right)\right\} + {\rm h.c.}\right]$$
$$+ \int d^4\theta\, 2 f_{IJ}\Phi_T\left\{V^I\left(\Box_4\mathcal{P}_{\rm T} + \partial\bar{\partial}\right)V^J + 2(\bar{\partial}V^I - \bar{\Sigma}^I)(\partial V^J - \Sigma^J)\right\}. \qquad (2.8)$$

For later convenience, we rewrite this Lagrangian as

$$\begin{aligned}\mathcal{L}_{\rm VT} = \int d^4\theta\, f_{IJ} \Bigg[ \bigg\{ -2\Sigma^I D^\alpha V^J \mathcal{W}_{T\alpha} + \frac{1}{2}\left(\partial V^I D^\alpha V^J - \partial D^\alpha V^I V^J\right) \mathcal{W}_{T\alpha} + \text{h.c.} \bigg\} \\ +\Phi_T \left\{ D^\alpha V^I \mathcal{W}^J_\alpha + \bar{D}_{\dot\alpha} V^I \bar{\mathcal{W}}^{J\dot\alpha} + V^I D^\alpha \mathcal{W}^J_\alpha \right. \\ \left. +4(\bar\partial V^I - \bar\Sigma^I)(\partial V^J - \Sigma^J) - 2\bar\partial V^I \partial V^J \right\} \Bigg], \end{aligned} \tag{2.9}$$

where we have dropped total derivatives and used the first constraint in (2.6). As we have shown in ref. [30], this Lagrangian is invariant under the gauge transformation (2.5)[4] up to total derivatives, and reduces to (2.2) in the limit of $\Phi_T = 1$ and $\mathcal{W}_{T\alpha} = 0$, which corresponds to the case where the tensor multiplet is absent.

The superfields $\Phi_T$ and $\mathcal{W}_{T\alpha}$ are expressed as

$$\begin{aligned}\Phi_T &= -2iD^\alpha \bar{D}^2 Y_\alpha + 2i\bar{D}_{\dot\alpha} D^2 \bar{Y}^{\dot\alpha}, \\ \mathcal{W}_{T\alpha} &= i\bar{D}^2 \left(D_\alpha \bar{X} + 4\bar\partial Y_\alpha\right),\end{aligned} \tag{2.10}$$

where $X$ and $Y_\alpha$ are complex superfields that are related through

$$\bar{D}^2 \left(D_\alpha X + 4\partial Y_\alpha\right) = 0. \tag{2.11}$$

This relation indicates that $Y_\alpha$ cannot be a general superfield. The first constraint in (2.6) is automatically satisfied if (2.11) is satisfied. Thus, independent constraints are (2.11) and the second constraint in (2.6). Note that $\Phi_T$ and $\mathcal{W}_{T\alpha}$ are the field strength superfields of the "gauge potentials" $X$ and $Y_\alpha$, and are invariant under

$$X \to X + \partial V_G - \Sigma_G, \qquad Y_\alpha \to Y_\alpha - \frac{1}{4} D_\alpha V_G, \tag{2.12}$$

where the transformation parameters $V_G$ and $\Sigma_G$ are $\mathcal{N} = 1$ real vector and chiral superfields, and form a 6D vector multiplet. The transformation (2.12) is the SUSY extension of the gauge transformation: $B^+_{MN} \to B^+_{MN} + \partial_M \lambda_N - \partial_N \lambda_M$ ($\lambda_M$: real transformation parameter).

Here we decompose $X$ as

$$X = s^{-1} X_4 - s X_5, \tag{2.13}$$

where $X_4$ and $X_5$ are real superfields. Then the second equation in (2.10) and (2.11) are rewritten as

$$\mathcal{W}_{T\alpha} = \bar{D}^2 \left\{ s^{-1} D_\alpha X_4 + s D_\alpha X_5 + 4\left(s^{-1}\partial_4 + s\partial_5\right) Y_\alpha \right\}, \tag{2.14}$$

and

$$\bar{D}^2 \left( s^{-1} D_\alpha X_4 - s D_\alpha X_5 + 4\partial Y_\alpha \right) = 0. \tag{2.15}$$

Using the constraint (2.15), $\mathcal{W}_{T\alpha}$ is also expressed as

$$\begin{aligned}\mathcal{W}_{T\alpha} &= 2s^{-1}\bar{D}^2 \left(D_\alpha X_4 + 4\partial_4 Y_\alpha\right) = s^{-1}\mathcal{W}_{4\alpha} + 8s^{-1}\partial_4 \bar{D}^2 Y_\alpha \\ &= 2s\bar{D}^2 \left(D_\alpha X_5 + 4\partial_5 Y_\alpha\right) = s\mathcal{W}_{5\alpha} + 8s\partial_5 \bar{D}^2 Y_\alpha,\end{aligned} \tag{2.16}$$

[4]The tensor multiplet $(\Phi_T, \mathcal{W}_{T\alpha})$ is invariant under the gauge transformation.



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where

$$\mathcal{W}_{4\alpha} \equiv 2\bar{D}^2 D_\alpha X_4, \qquad \mathcal{W}_{5\alpha} \equiv 2\bar{D}^2 D_\alpha X_5. \tag{2.17}$$

Thus, the tensor multiplet $\mathbb{T}$ is described by two constrained superfields $X_4$ (or $X_5$) and $Y_\alpha$.

### 2.2 Components of superfields

Each $\mathcal{N}=1$ superfield has the following components. Here we focus on the bosonic fields, for simplicity.

Hyperscalars $(\mathcal{A}_i^{2A-1}, \mathcal{A}_i^{2A})$ in $\mathbb{H}^A$, where $i=1,2$ is an $\mathrm{SU}(2)_{\mathbf{U}}$-doublet-index,[5] are embedded into $H^{2A-1}$ and $H^{2A}$ as

$$H^{2A-1} = \mathcal{A}_2^{2A-1} + \mathcal{O}(\theta), \qquad H^{2A} = \mathcal{A}_2^{2A} + \mathcal{O}(\theta). \tag{2.18}$$

A 6D vector field $A_M^I$ in $\mathbb{V}^I$ is embedded into $V^I$ and $\Sigma^I$ as

$$V^I = -(\theta\sigma^\mu\bar{\theta})A_\mu + \mathcal{O}(\theta^3), \qquad \Sigma^I = \left(s^{-1}A_4 - sA_5\right) + \mathcal{O}(\theta). \tag{2.19}$$

A 6D tensor field $B_{MN}^+$ and its scalar partner $\sigma$ in $\mathbb{T}$ are embedded into $\Phi_T$ and $\mathcal{W}_{T\alpha}$ as

$$\begin{aligned}\Phi_T &= \sigma + (\theta\sigma^\mu\bar{\theta})\epsilon_{\mu\nu\rho\lambda}\partial^\nu B^{+\rho\lambda} - \frac{1}{4}\theta^2\bar{\theta}^2\Box_4\sigma + \cdots, \\ \mathcal{W}_{T\alpha} &= \theta_\alpha\bar{\partial}\sigma + (\sigma^{\mu\nu}\theta)_\alpha\left(\bar{\partial}B_{\mu\nu}^+ + \partial_\mu C_\nu - \partial_\nu C_\mu\right) + \cdots,\end{aligned} \tag{2.20}$$

where $C_\mu \equiv -i\left(s^{-1}B_{\mu 4}^+ + sB_{\mu 5}^+\right)$, and $B_{MN}^+$ satisfies the self-dual condition:

$$\begin{aligned}\epsilon_{\mu\nu\rho\lambda}\partial^\nu B^{+\rho\lambda} &= -2\left\{\partial_\mu B_{45}^+ - \mathrm{Im}(\partial C_\mu)\right\}, \\ \bar{\partial}B_{\mu\nu}^+ + \partial_\mu C_\nu - \partial_\nu C_\mu &= \frac{i}{2}\epsilon_{\mu\nu\rho\lambda}\left(\bar{\partial}B^{+\rho\lambda} + \partial^\rho C^\lambda - \partial^\lambda C^\rho\right).\end{aligned} \tag{2.21}$$

The expressions in (2.20) are realized when $X$ and $Y_\alpha$ have the following components:

$$\begin{aligned}X &= \frac{1}{4}(\theta\sigma^\mu\bar{\theta})\bar{C}_\mu - \frac{1}{8}\theta^2\bar{\theta}^2\left(B_{45}^+ + \frac{i}{2}\sigma\right) + \cdots, \\ Y_\alpha &= \frac{1}{16}\theta_\alpha\bar{\theta}^2\left(B_{45}^+ + \frac{i}{2}\sigma\right) + \frac{i}{16}(\sigma^{\mu\nu}\theta)_\alpha\bar{\theta}^2 B_{\mu\nu}^+ + \cdots,\end{aligned} \tag{2.22}$$

where $\bar{C}_\mu = s^{-1}B_{\mu 4}^+ - sB_{\mu 5}^+$. The $B_{45}^+$-dependence is determined from the transformation property under (2.12).

## 3 Extension to 6D SUGRA

Now we extend the action in the previous section to the local SUSY case. Since we are interested in the moduli-dependence of the action, we focus on $e_m^{\ n}$ $(m,n=4,5)$ among the sechsbein $e_M^{\ N}$, and treat the other components as a background,[6] i.e., $e_\mu^{\ \underline{\nu}} = \delta_\mu^{\ \nu}$ and $e_\mu^{\ \underline{n}} = e_m^{\ \underline{\nu}} = 0$. Therefore, we do not discriminate the curved index $\mu$ from the flat index $\underline{\mu}$ for the 4D part in the following.

[5] $\mathrm{SU}(2)_{\mathbf{U}}$ is an automorphism of 6D superconformal algebra (see appendix A).

[6] The fluctuation modes of the 4D gravity multiplet can be easily taken into account by promoting the $d^4\theta$- and $d^2\theta$-integrals to the D-term and the F-term action formulae [36], respectively, in the superconformal formulation of 4D SUGRA [37–39].

### 3.1 Moduli superfields

First we identify the $\mathcal{N}=1$ superfields constructed from the extra-dimensional components of the 6D Weyl multiplet $\mathbb{E}=(e_M^{\ N}, \Psi^i_{M\underline{\alpha}}, V_M^{ij}, \cdots)$ (see appendix B). Notice that if a complex scalar $\mathcal{A}$ is the lowest component of a chiral superfield, it transforms under consecutive SUSY transformations as

$$\delta_\epsilon \delta_\eta \mathcal{A} = 2i(\eta\sigma^\mu\bar{\epsilon})\partial_\mu \mathcal{A} + \cdots , \tag{3.1}$$

and if a real scalar $\phi$ is the lowest component of a real general superfield, it transforms as

$$\delta_\epsilon \delta_\eta \phi = i(\eta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\eta})\partial_\mu \phi + \cdots , \tag{3.2}$$

where the 2-component spinors $\epsilon_\alpha$ and $\eta_\alpha$ are the transformation parameters, and the ellipses denote terms involving other fields. In order to identify combinations of $e_m^{\ n}$ that belong to $\mathcal{N}=1$ superfields, we focus on the $\mathcal{N}=1$ SUSY transformations at linearized level in the fluctuations $\tilde{e}_m^{\ n}$. Then, from (B.1), we obtain

$$\delta_\epsilon \delta_\eta u = \frac{1}{2\langle e^{(2)} \rangle}(\eta\sigma^\mu\bar{\epsilon})\langle \mathcal{M} \rangle \partial_\mu u + \text{c.c.} + \cdots , \tag{3.3}$$

where $e^{(2)} \equiv \det(e_m^{\ n}) = e_4^{\ 4} e_5^{\ 5} - e_4^{\ 5} e_5^{\ 4}$ and $u \equiv (\tilde{e}_4^{\ 4}, \tilde{e}_4^{\ 5}, \tilde{e}_5^{\ 4}, \tilde{e}_5^{\ 5})^t$. The matrix $\mathcal{M}$ is defined as

$$\mathcal{M} \equiv \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & -E_4 e_4^{\ 4} & -E_4 e_4^{\ 5} \\ -i\mathcal{M}_{11} & -i\mathcal{M}_{12} & iE_4 e_4^{\ 4} & iE_4 e_4^{\ 5} \\ E_5 e_5^{\ 4} & E_5 e_5^{\ 5} & \mathcal{M}_{33} & \mathcal{M}_{34} \\ -iE_5 e_5^{\ 4} & -iE_5 e_5^{\ 5} & -i\mathcal{M}_{33} & -i\mathcal{M}_{34} \end{pmatrix}, \tag{3.4}$$

where $E_m \equiv e_m^{\ 4} + ie_m^{\ 5}$, and

$$\begin{aligned} \mathcal{M}_{11} &\equiv 2E_5 e_4^{\ 4} - E_4 e_5^{\ 4}, \qquad \mathcal{M}_{12} \equiv 2E_5 e_4^{\ 5} - E_4 e_5^{\ 5}, \\ \mathcal{M}_{33} &\equiv E_5 e_4^{\ 4} - 2E_4 e_5^{\ 4}, \qquad \mathcal{M}_{34} \equiv E_5 e_4^{\ 5} - 2E_4 e_5^{\ 5}. \end{aligned} \tag{3.5}$$

There are three eigenvectors $v_a$ ($a=\pm,0$) that satisfy $v_a\langle \mathcal{M} \rangle = \lambda_a v_a$ and $v_a \langle \mathcal{M} \rangle^* = \lambda'_a v_a$ simultaneously ($\lambda_a$, $\lambda'_a$: eigenvalues).

$$\begin{aligned} (\lambda_-, \lambda'_-) = (0, -4i\langle e^{(2)} \rangle) \quad &: \quad v_- = \left(\langle \bar{E}_5 \rangle, -i\langle \bar{E}_5 \rangle, -\langle \bar{E}_4 \rangle, i\langle \bar{E}_4 \rangle\right), \\ (\lambda_0, \lambda'_0) = (2i\langle e^{(2)} \rangle, -2i\langle e^{(2)} \rangle) \quad &: \quad v_0 = \left(\langle e_5^{\ 5} \rangle, -\langle e_5^{\ 4} \rangle, -\langle e_4^{\ 5} \rangle, \langle e_4^{\ 4} \rangle\right), \\ (\lambda_+, \lambda'_+) = (4i\langle e^{(2)} \rangle, 0) \quad &: \quad v_+ = \left(\langle E_5 \rangle, i\langle E_5 \rangle, -\langle E_4 \rangle, -i\langle E_4 \rangle\right). \end{aligned} \tag{3.6}$$

Thus, we obtain

$$\begin{aligned} \delta_\epsilon \delta_\eta (v_- \cdot u) &= -2i(\epsilon\sigma^\mu\bar{\eta})\partial_\mu (v_- \cdot u) + \cdots , \\ \delta_\epsilon \delta_\eta (v_0 \cdot u) &= i(\eta\sigma^\mu\bar{\epsilon} - \epsilon\sigma^\mu\bar{\eta})\partial_\mu (v_0 \cdot u) + \cdots , \\ \delta_\epsilon \delta_\eta (v_+ \cdot u) &= 2i(\eta\sigma^\mu\bar{\epsilon})\partial_\mu (v_+ \cdot u) + \cdots . \end{aligned} \tag{3.7}$$

Therefore, we infer that $v_+ \cdot u = \langle E_5 \rangle \tilde{E}_4 - \langle E_4 \rangle \tilde{E}_5$ is the lowest component of a chiral superfield, and $v_0 \cdot u = \langle e_5^{\ 5} \rangle \tilde{e}_4^{\ 4} - \langle e_5^{\ 4} \rangle \tilde{e}_4^{\ 5} - \langle e_4^{\ 5} \rangle \tilde{e}_5^{\ 4} + \langle e_4^{\ 4} \rangle \tilde{e}_5^{\ 5}$ is the lowest component of a real general superfield.[7] Note that $v_+ \cdot u$ and $v_0 \cdot u$ are the linear parts of $E_4/E_5$ and $e^{(2)}$ in the fluctuations, respectively. In fact, we can show that

$$(\delta_\epsilon \delta_\eta - \delta_\eta \delta_\epsilon) \frac{E_4}{E_5} = 2i \, (\eta \sigma^\mu \bar{\epsilon} - \epsilon \sigma^\mu \bar{\eta}) \, \partial_\mu \left( \frac{E_4}{E_5} \right),$$
$$(\delta_\epsilon \delta_\eta - \delta_\eta \delta_\epsilon) e^{(2)} = 2i \, (\eta \sigma^\mu \bar{\epsilon} - \epsilon \sigma^\mu \bar{\eta}) \, \partial_\mu e^{(2)}, \tag{3.8}$$

at the full order in the fluctuation. Thus the correct SUSY algebra is realized on them, and they can be the components of the superfields. Namely, we find that the extra-dimensional components of the 6D Weyl multiplet $\mathbb{E}$ form a chiral superfield,[8]

$$S_E = \sqrt{\frac{E_4}{E_5}} + \mathcal{O}(\theta), \tag{3.9}$$

and a real general superfield,

$$V_E = e^{(2)} + \mathcal{O}(\theta). \tag{3.10}$$

In the superconformal formulation of 4D SUGRA [36]–[39], each superconformal multiplet is characterized by the Weyl weight $w$ and the chiral weight $n$, which are the charges of the dilatation and the automorphism $\mathrm{U}(1)_A$ of the superconformal algebra, respectively. From (A.6), we can see that $E_m$ $(m = 4, 5)$ have $(w, n) = (-1, -1)$. Thus, noting that $e^{(2)} = \mathrm{Im}(\bar{E}_4 E_5)$, we find that $S_E$ and $V_E$ have $(w, n) = (0, 0)$ and $(-2, 0)$, respectively. This is consistent with the fact that they are a chiral and a real general superfields [36]. From their forms of the lowest components, we can see that $V_E$ and $S_E$ correspond to the "volume" and the "shape" of the compact space.

In the following, we identify how these superfields appear in the 6D SUGRA action. We construct the action in such a way that it is reduced to the global SUSY one if the moduli superfields $V_E$ and $S_E$ are replaced with constant values 1 and $s = e^{-\frac{\pi}{4}i}$, respectively. These values correspond to the background values of the case that $\langle e_4{}^4 \rangle = \langle e_5{}^5 \rangle = 1$ and $\langle e_4{}^5 \rangle = \langle e_5{}^4 \rangle = 0$.

### 3.2 Hypermultiplet sector

Here we extend $\mathcal{L}_\mathrm{H}$ in (2.2) to the SUGRA version. In this case, we need to introduce the $n_C$ compensator hypermultiplets in addition to the $n_P$ physical ones. Thus, besides the dependence on $S_E$ and $V_E$, the Lagrangian in this sector is written as

$$\mathcal{L}_\mathrm{H} = -\int d^4\theta \, 2 \left( H^\dagger_\mathrm{odd} \tilde{d} e^V H_\mathrm{odd} + H^\dagger_\mathrm{even} \tilde{d} e^{-V} H_\mathrm{even} \right)$$
$$+ \left[ \int d^2\theta \, \left\{ H^t_\mathrm{odd} \tilde{d} \, (\partial - \Sigma) \, H_\mathrm{even} - H^t_\mathrm{even} \tilde{d} (\partial + \Sigma) H_\mathrm{odd} \right\} + \mathrm{h.c.} \right], \tag{3.11}$$

where $\tilde{d} = \mathrm{diag}(\mathbf{1}_{n_C}, -\mathbf{1}_{n_P})$ is the metric for the space spanned by the hyperscalars, and discriminates the compensators from the physical ones.

[7] $v_- \cdot u = (v_+ \cdot u)^*$ is the lowest component of an anti-chiral superfield.

[8] When $E_4/E_5$ is the lowest component of a chiral superfield, so is $(E_4/E_5)^p$ ($p$: real number). We choose $p = 1/2$ just for convenience.



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Now we consider the moduli dependence of the Lagrangian. Since $V_E$ cannot appear in the chiral superspace, $H_{\rm odd}$ and $H_{\rm even}$ must have $w = n = 3/2$. However, the 6D hyperscalars $\mathcal{A}_i^{2A-1}$ and $\mathcal{A}_i^{2A}$ have $w = n = 2$. Hence the component identification in (2.18) must be modified. Since we have to keep the condition $w = n$ for a chiral superfield, we need to adjust the weights by using $E_m = e_m^{\underline{4}} + ie_m^{\underline{5}}$ $(m = 4, 5)$ that has $w = n = -1$. We find that (2.18) should be modified as

$$
\begin{aligned}
H^{2A-1} &= E_4^p E_5^{1/2-p} \mathcal{A}_2^{2A-1} + \mathcal{O}(\theta), \\
H^{2A} &= E_4^q E_5^{1/2-q} \mathcal{A}_2^{2A} + \mathcal{O}(\theta),
\end{aligned}
\tag{3.12}
$$

where $p$ and $q$ are arbitrary real numbers. We can always set $p = q = 1/4$ by redefining the above chiral superfields as $S_E^{1/2-2p} H^{2A-1} \to H^{2A-1}$ and $S_E^{1/2-2q} H^{2A} \to H^{2A}$. Hence, in the following, we identify the lowest components of these chiral superfields as

$$
\begin{aligned}
H^{2A-1} &= (E_4 E_5)^{1/4} \mathcal{A}_2^{2A-1} + \mathcal{O}(\theta), \\
H^{2A} &= (E_4 E_5)^{1/4} \mathcal{A}_2^{2A} + \mathcal{O}(\theta).
\end{aligned}
\tag{3.13}
$$

Next we promote the derivative $\partial$ to the SUGRA version $\partial_E$ that depends on $S_E$. (This is independent of $V_E$ because it cannot appear in the chiral superspace.) In order to reproduce the correct 6D kinetic terms for the hyperscalars after eliminating the F-terms of $H_{\rm odd,even}$, the lowest component of $\partial_E$ should be proportional to $\partial_{\underline{4}} + i\partial_{\underline{5}}$ because $\left|\left(\partial_{\underline{4}} + i\partial_{\underline{5}}\right) \mathcal{A}\right|^2 = \partial^m \mathcal{A}^\dagger \partial_m \mathcal{A}$. Since

$$
\partial_{\underline{4}} + i\partial_{\underline{5}} = -\frac{i\sqrt{E_4 E_5}}{e^{(2)}} \left( \sqrt{\frac{E_5}{E_4}} \partial_4 - \sqrt{\frac{E_4}{E_5}} \partial_5 \right),
\tag{3.14}
$$

we define $\partial_E$ as

$$
\partial_E \equiv \frac{1}{S_E} \partial_4 - S_E \partial_5.
\tag{3.15}
$$

Then, its lowest component is

$$
\partial_E| = \frac{ie^{(2)}}{\sqrt{E_4 E_5}} \left(\partial_{\underline{4}} + i\partial_{\underline{5}}\right).
\tag{3.16}
$$

Here and hereafter, the symbol $|$ denotes the lowest component of a superfield. This promoted derivative $\partial_E$ is certainly reduced to the global SUSY one $\partial$ if we replace $S_E$ with its background value $s$.

From the counting of the Weyl and chiral weights, (3.11) should be modified as

$$
\begin{aligned}
\mathcal{L}_{\rm H} = &- \int d^4\theta \, 2V_E^{1/2} U_E(S_E, \bar{S}_E) \left( H_{\rm odd}^\dagger \tilde{d} e^V H_{\rm odd} + H_{\rm even}^\dagger \tilde{d} e^{-V} H_{\rm even} \right) \\
&+ \left[ \int d^2\theta \, \left\{ H_{\rm odd}^t \tilde{d} \left(\partial_E - \Sigma\right) H_{\rm even} - H_{\rm even}^t \tilde{d} \left(\partial_E + \Sigma\right) H_{\rm odd} \right\} + \text{h.c.} \right],
\end{aligned}
\tag{3.17}
$$



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where $U_E(S_E, \bar{S}_E)$ is a real function. From (3.9), (3.10) and (3.13), the lowest component of the integrand in the $d^4\theta$-integral is read off as

$$\begin{aligned} \boldsymbol{C} &\equiv V_E^{1/2} U_E(S_E, \bar{S}_E) \left( H_{\rm odd}^\dagger \tilde{d} e^V H_{\rm odd} + H_{\rm even}^\dagger \tilde{d} e^{-V} H_{\rm even} \right) \Big| \\ &= \sqrt{e^{(2)}} U_E \left( \sqrt{\frac{E_4}{E_5}}, \sqrt{\frac{\bar{E}_4}{\bar{E}_5}} \right) \cdot \left| (E_4 E_5)^{1/4} \right|^2 \left( \mathcal{A}_{\rm odd}^\dagger \tilde{d} \mathcal{A}_{\rm odd} + \mathcal{A}_{\rm even}^\dagger \tilde{d} \mathcal{A}_{\rm even} \right), \end{aligned} \quad (3.18)$$

where $\mathcal{A}_{\rm odd}$ and $\mathcal{A}_{\rm even}$ are column vectors that consist of $\mathcal{A}_2^{2A-1}$ and $\mathcal{A}_2^{2A}$, respectively. Note that $\boldsymbol{C}$ appears in front of the Ricci scalar when the $d^4\theta$-integral is promoted to the D-term action formula [36]. From the component expression of 6D SUGRA [27], on the other hand, the coefficient of the Ricci scalar should be $e^{(2)} \left( \mathcal{A}_{\rm odd}^\dagger \tilde{d} \mathcal{A}_{\rm odd} + \mathcal{A}_{\rm even}^\dagger \tilde{d} \mathcal{A}_{\rm even} \right)$.[9] Thus the function $U_E|$ is determined as

$$\begin{aligned} U_E^2| = \frac{e^{(2)}}{|E_4 E_5|} &= -\frac{i}{2\,|E_4 E_5|} \left( \bar{E}_4 E_5 - E_4 \bar{E}_5 \right) \\ &= -\frac{i}{2} \left( \sqrt{\frac{\bar{E}_4 E_5}{E_4 \bar{E}_5}} - \sqrt{\frac{\bar{E}_5 E_4}{E_5 \bar{E}_4}} \right) = {\rm Im}\, \frac{\bar{S}_E}{S_E} \Big| . \end{aligned} \quad (3.19)$$

Therefore, we obtain

$$U_E(S_E, \bar{S}_E) = \left( {\rm Im}\, \frac{\bar{S}_E}{S_E} \right)^{1/2}. \quad (3.20)$$

In fact, substituting (3.20) into (3.17) and eliminating the F-terms of $H_{\rm odd,even}$, we obtain the correct kinetic terms.

$$\mathcal{L}_{\rm H} = 2 e^{(2)} \left\{ \partial^M \mathcal{A}_{\rm odd}^\dagger \tilde{d} \partial_M \mathcal{A}_{\rm odd} + \partial^M \mathcal{A}_{\rm even}^\dagger \tilde{d} \partial_M \mathcal{A}_{\rm even} \right\} + \cdots . \quad (3.21)$$

Correspondingly to the promotion: $\partial \to \partial_E$, (2.19) is also modified as

$$\begin{aligned} V &= -(\theta \sigma^\mu \bar{\theta}) A_\mu + \mathcal{O}(\theta^3), \\ \Sigma &= \left( \sqrt{\frac{E_5}{E_4}} A_4 - \sqrt{\frac{E_4}{E_5}} A_5 \right) + \mathcal{O}(\theta) \\ &= \frac{i e^{(2)}}{\sqrt{E_4 E_5}} \left( A_{\underline{4}} + i A_{\underline{5}} \right) + \mathcal{O}(\theta). \end{aligned} \quad (3.22)$$

### 3.3 Vector-tensor sector

Next we consider the vector-tensor sector. The definition of the tensor (field-strength) superfield $\Phi_T$ is unchanged from (2.10),

$$\Phi_T \equiv -2i D^\alpha \bar{D}^2 Y_\alpha + 2i \bar{D}_{\dot{\alpha}} D^2 \bar{Y}^{\dot{\alpha}}, \quad (3.23)$$

[9] Note that $\det(e_M^{\ \underline{N}}) = e^{(2)}$ under our assumption.



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while that of $\mathcal{W}_{T\alpha}$ is now modified from (2.14) as

$$\mathcal{W}_{T\alpha} \equiv \bar{D}^2 \left( \frac{1}{S_E} D_\alpha X_4 + S_E D_\alpha X_5 + 4 S_E \mathcal{O}_E Y_\alpha \right), \tag{3.24}$$

where $X_4$ and $X_5$ are real superfields, and

$$\mathcal{O}_E \equiv \frac{1}{S_E^2} \partial_4 + \partial_5. \tag{3.25}$$

The constraint (2.15) is promoted to the SUGRA version:

$$\bar{D}^2 \left( \frac{1}{S_E} D_\alpha X_4 - S_E D_\alpha X_5 + 4 \partial_E Y_\alpha \right) = 0. \tag{3.26}$$

Under this constraint, $\mathcal{W}_{T\alpha}$ can be rewritten as

$$\begin{aligned} \mathcal{W}_{T\alpha} &= \frac{1}{S_E} \mathcal{W}_{4\alpha} + \frac{8}{S_E} \partial_4 \bar{D}^2 Y_\alpha \\ &= S_E \mathcal{W}_{5\alpha} + 8 S_E \partial_5 \bar{D}^2 Y_\alpha, \end{aligned} \tag{3.27}$$

which is the SUGRA version of (2.16). The field strength superfields $\mathcal{W}_{4\alpha}$ and $\mathcal{W}_{5\alpha}$ are defined as (2.17). The superfields $\Phi_T$, $\mathcal{W}_{T\alpha}$ and the constraint (3.26) are invariant under the gauge transformation:

$$\begin{aligned} \delta X_4 &= \partial_4 V_G - \text{Re}(S_E \Sigma_G), \qquad \delta X_5 = \partial_5 V_G + \text{Re}\left( \frac{\Sigma_G}{S_E} \right), \\ \delta Y_\alpha &= -\frac{1}{4} D_\alpha V_G, \end{aligned} \tag{3.28}$$

where the transformation parameters $V_G$ and $\Sigma_G$ are a real and a chiral superfields that form a 6D vector multiplet. From the expressions in (3.23) and (3.27), we can show that

$$D^\alpha \left( U_E^2 \mathcal{W}_{T\alpha} \right) = -2 \bar{\partial}_E \Phi_T + \frac{i \bar{D}_{\dot{\alpha}} \bar{S}_E}{\bar{S}_E} \bar{\mathcal{W}}_T^{\dot{\alpha}}, \tag{3.29}$$

which is the SUGRA extension of the first constraint in (2.6). From the gauge invariance of the action, the second constraint in (2.6) should be modified as

$$\bar{D}^2 D_\alpha (V_E \Phi_T) = -4 \left\{ \partial_E \mathcal{W}_{T\alpha} - (\mathcal{O}_E S_E) \mathcal{W}_{T\alpha} \right\}. \tag{3.30}$$

(See section 4.1.) The bosonic components of $X_4$, $X_5$ and $Y_\alpha$ are given by

$$\begin{aligned} X_4 &= \frac{1}{4} (\theta \sigma^\mu \bar{\theta}) B_{\mu 4} + \cdots, \qquad X_5 = \frac{1}{4} (\theta \sigma^\mu \bar{\theta}) B_{\mu 5} + \cdots, \\ Y_\alpha &= \frac{1}{16} \theta_\alpha \bar{\theta}^2 \left( B_{45} + \frac{i}{2} \sigma \right) + \frac{i}{16} (\sigma^{\mu\nu} \theta)_\alpha \bar{\theta}^2 B_{\mu\nu} + \cdots, \end{aligned} \tag{3.31}$$

where $B_{MN}$ is an unconstrained tensor field.

As explained in appendix C, the constraint (3.26) can be satisfied for arbitrary unconstrained superfields $Y_\alpha$ and $X_4$ by adjusting $S_E$ and $X_5$. This indicates that the latter two



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superfields are not independent. In fact, we can express the action without $X_5$ by adopting the first equation in (3.27) as the definition of $\mathcal{W}_{T\alpha}$. This reflects the fact that $X_5$ can be gauged away by (3.28). Of course, we can choose $Y_\alpha$ and $X_5$ as independent superfields.

Now we promote $\mathcal{L}_{\rm VT}$ in (2.9) to SUGRA by replacing $\partial$ with $\partial_E$ and inserting $V_E$ to match the Weyl weight of the integrand to 2, and obtain

$$\begin{aligned}\mathcal{L}_{\rm VT} = \int d^4\theta\; f_{IJ}\Bigg[&\left\{-2\Sigma^I D^\alpha V^J \mathcal{W}_{T\alpha} + \frac{1}{2}\left(\partial_E V^I D^\alpha V^J - \partial_E D^\alpha V^I V^J\right)\mathcal{W}_{T\alpha} + \text{h.c.}\right\}\\ &+\Phi_T V_E\left(D^\alpha V^I \mathcal{W}^I_\alpha + \bar{D}_{\dot\alpha} V^I \bar{\mathcal{W}}^{J\dot\alpha} + V^I D^\alpha \mathcal{W}^J_\alpha\right)\\ &+\frac{\Phi_T}{U_E^2}\left\{4(\bar\partial_E V^I - \bar\Sigma^I)(\partial_E V^J - \Sigma^J) - 2\bar\partial_E V^I \partial_E V^J\right\}\Bigg]. \end{aligned} \tag{3.32}$$

The factor $U_E^{-2}$ is necessary in order to obtain the correct component expression of the Lagrangian. Note that the third line in (3.32) provides the extra-dimensional components of the kinetic terms for the 6D vector fields. The lowest component of $U_E^{-2}$ cancels the unwanted factor in (3.16).

In order for the Lagrangian to be gauge-invariant, we need to add the following terms to (3.32). (See section 4.1.)

$$\mathcal{L}^{(\rm SG)}_{\Sigma^2} = \int d^4\theta\; 2f_{IJ}\frac{\Phi_T}{U_E^2}\left(\frac{S_E}{\bar S_E}\Sigma^I\Sigma^J + \frac{\bar S_E}{S_E}\bar\Sigma^I\bar\Sigma^J\right). \tag{3.33}$$

Note that this vanishes if $V_E$ and $S_E$ are replaced with their background values.

### 3.4 6D SUGRA action

In summary, the 6D SUGRA action is expressed as

$$\begin{aligned} S^{(\rm SG)} &= \int d^6x\,\left(\mathcal{L}^{(\rm SG)}_{\rm H} + \mathcal{L}^{(\rm SG)}_{\rm VT}\right),\\ \mathcal{L}^{(\rm SG)}_{\rm H} &= -\int d^4\theta\; 2V_E^{1/2} U_E(S_E,\bar S_E)\left(H^\dagger_{\rm odd}\tilde{d}e^V H_{\rm odd} + H^\dagger_{\rm even}\tilde{d}e^{-V}H_{\rm even}\right)\\ &\quad+\left[\int d^2\theta\,\left\{H^t_{\rm odd}\tilde{d}\left(\partial_E - \Sigma\right)H_{\rm even} - H^t_{\rm even}\tilde{d}\left(\partial_E+\Sigma\right)H_{\rm odd}\right\} + \text{h.c.}\right],\\ \mathcal{L}^{(\rm SG)}_{\rm VT} &= \int d^4\theta\; f_{IJ}\Bigg[\left\{-2\Sigma^I D^\alpha V^J\mathcal{W}_{T\alpha} + \frac{1}{2}\left(\partial_E V^I D^\alpha V^J - \partial_E D^\alpha V^I V^J\right)\mathcal{W}_{T\alpha} + \text{h.c.}\right\}\\ &\quad+\Phi_T V_E\left(D^\alpha V^I\mathcal{W}^J_\alpha + \bar D_{\dot\alpha}V^I\bar{\mathcal{W}}^{J\dot\alpha} + V^I D^\alpha\mathcal{W}^J_\alpha\right)\\ &\quad+\frac{\Phi_T}{U_E^2}\Bigg\{4(\bar\partial_E V^I - \bar\Sigma^I)(\partial_E V^J - \Sigma^J) - 2\bar\partial_E V^I\partial_E V^J\\ &\qquad+\frac{2S_E}{\bar S_E}\Sigma^I\Sigma^J + \frac{2\bar S_E}{S_E}\bar\Sigma^I\bar\Sigma^J\Bigg\}\Bigg]. \end{aligned} \tag{3.34}$$

This certainly reproduces the global SUSY action in the previous section when $V_E = 1$ and $S_E = s$.



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Here we comment on the constraints (3.26) and (3.30). They can be released by introducing the following terms.

$$\mathcal{L}_{\rm LM}^{\rm (SG)} = \int d^4\theta\, i\tilde{Z}^\alpha \bar{D}^2 \left( \frac{1}{S_E} D_\alpha X_4 - S_E D_\alpha X_5 + 4\partial_E Y_\alpha \right) + \int d^4\theta\, 2i\tilde{Y}^\alpha \left[ \bar{D}^2 D_\alpha (V_E \Phi_T) + 4\left\{ \partial_E \mathcal{W}_{T\alpha} - (\mathcal{O}_E S_E)\mathcal{W}_{T\alpha} \right\} \right] + {\rm h.c.}, \quad (3.35)$$

where the Lagrange multipliers $\tilde{Z}^\alpha$ and $\tilde{Y}^\alpha$ are unconstrained superfields.[10] These terms can be rewritten as

$$\begin{aligned} \mathcal{L}_{\rm LM}^{\rm (SG)} &= \int d^4\theta\, i\left\{ D^\alpha \bar{D}^2 (S_E \tilde{Z}_\alpha) - \bar{D}_{\dot\alpha} D^2 (\bar{S}_E \bar{\tilde{Z}}^{\dot\alpha}) \right\} X_5 \\ &\quad + \int d^4\theta\, \left\{ i(S_E \tilde{Z}^\alpha) \left( \frac{1}{2S_E^2} \mathcal{W}_{4\alpha} + \frac{4}{S_E} \partial_E \bar{D}^2 Y_\alpha \right) + {\rm h.c.} \right\} \\ &\quad + \int d^4\theta\, \left\{ V_E \Phi_T \tilde{\Phi}_T - 8i\partial_E \tilde{Y}^\alpha \mathcal{W}_{T\alpha} + 8i\bar{\partial}_E \bar{\tilde{Y}}_{\dot\alpha} \bar{\mathcal{W}}_T^{\dot\alpha} \right\}, \end{aligned} \quad (3.36)$$

where $\tilde{\Phi}_T \equiv -2iD^\alpha \bar{D}^2 \tilde{Y}_\alpha + 2i\bar{D}_{\dot\alpha} D^2 \bar{\tilde{Y}}^{\dot\alpha}$. We have dropped total derivatives. If we adopt the first equation in (3.27) as the definition of $\mathcal{W}_{T\alpha}$, a real superfield $X_5$ only appears in the first line of (3.36) and thus is regarded as a Lagrange multiplier. Then its equation of motion provides

$$D^\alpha \bar{D}^2 (S_E \tilde{Z}_\alpha) = \bar{D}_{\dot\alpha} D^2 (\bar{S}_E \bar{\tilde{Z}}^{\dot\alpha}), \quad (3.37)$$

which is understood as the Bianchi identity. Thus, this can be solved as

$$S_E \tilde{Z}_\alpha = \frac{1}{2} D_\alpha V_Z, \quad (3.38)$$

where $V_Z$ is a real superfield. Therefore, (3.36) is rewritten as

$$\mathcal{L}_{\rm LM}^{\rm (SG)} = \left[ \int d^2\theta\, \left\{ \frac{i}{4S_E^2} \mathcal{W}_Z \mathcal{W}_4 + \frac{2i}{S_E} \left( \mathcal{W}_Z^\alpha \partial_E \bar{D}^2 Y_\alpha + \mathcal{W}_4^\alpha \partial_E \bar{D}^2 \tilde{Y}_\alpha \right) + \frac{16i}{S_E} \partial_E \bar{D}^2 \tilde{Y}^\alpha \partial_4 \bar{D}^2 Y_\alpha \right\} + {\rm h.c.} \right] + \int d^4\theta\, V_E \Phi_T \tilde{\Phi}_T, \quad (3.39)$$

where $\mathcal{W}_{Z\alpha} \equiv -\frac{1}{4}\bar{D}^2 D_\alpha V_Z$. Note that all the superfields are now unconstrained in this expression. Needless to say, we can choose $X_4$ instead of $X_5$ as the Lagrange multiplier, and adopt the second equation in (3.27) as the definition of $\mathcal{W}_{T\alpha}$.

## 4 Consistency checks

In this section, we show that our result (3.34) is gauge-invariant, and is reduced to the known superfield expression of 5D SUGRA after the dimensional reduction.

[10]If we identify $\tilde{Y}_\alpha$ as a superfield coming from another 6D tensor multiplet, we can understand the second line of (3.35) as the $\mathcal{N} = 1$ superfield description of (3.53) of ref. [29], which is described in the projective superspace.

### 4.1 Gauge invariance

The (super)gauge transformation is given by

$$\begin{aligned} &V_E \to V_E, && S_E \to S_E, \\ &H_{\rm odd} \to e^{-\Lambda} H_{\rm odd}, && H_{\rm even} \to e^{\Lambda} H_{\rm even}, \\ &V^I \to \Lambda^I + \bar{\Lambda}^I, && \Sigma^I \to \Sigma^I + \partial_E \Lambda^I, \\ &Y_\alpha \to Y_\alpha, && X_4 \to X_4, \qquad X_5 \to X_5. \end{aligned} \tag{4.1}$$

Under this transformation, $\mathcal{L}_{\rm H}^{(\rm SG)}$ is manifestly invariant, while the invariance of the remaining part $\mathcal{L}_{\rm VT}^{(\rm SG)}$ is quite nontrivial because it is invariant only up to total derivatives. In the following, we neglect total derivative terms. Note that the following formulae hold.

$$\begin{aligned} (\partial_E A) B &= -A \partial_E B + (\mathcal{O}_E S_E) AB, \\ D^\alpha \partial_E A &= \partial_E D^\alpha A - (D^\alpha S_E) \mathcal{O}_E A. \end{aligned} \tag{4.2}$$

The variation of $\mathcal{L}_{\rm VT}^{(\rm SG)}$ is

$$\begin{aligned} \delta\mathcal{L}_{\rm VT}^{(\rm SG)} &= \int d^4\theta\; f_{IJ} \Bigg[ \Bigg\{ -2\partial_E \Lambda^I D^\alpha V^J \mathcal{W}_{T\alpha} - 2\Sigma^I D^\alpha \Lambda^J \mathcal{W}_{T\alpha} \\ &\qquad + \frac{1}{2} \left\{ \partial_E \left(\Lambda^I + \bar{\Lambda}^I\right) D^\alpha V^J + \partial_E V^I D^\alpha \Lambda^J - \partial_E D^\alpha \Lambda^I V^J \right. \\ &\qquad \left. - \partial_E D^\alpha V^I \left(\Lambda^J + \bar{\Lambda}^J\right) \right\} \mathcal{W}_{T\alpha} + \text{h.c.} \Bigg\} \\ &\qquad + \Phi_T V_E \left\{ D^\alpha \Lambda^I \mathcal{W}_\alpha^J + \bar{D}_{\dot{\alpha}} \bar{\Lambda}^I \bar{\mathcal{W}}^{J\dot{\alpha}} + \left(\Lambda^I + \bar{\Lambda}^I\right) D^\alpha \mathcal{W}_\alpha^J \right\} \\ &\qquad + \frac{\Phi_T}{U_E^2} \Bigg\{ 4\bar{\partial}_E \Lambda^I \left(\partial_E V^J - \Sigma^J\right) + 4\left(\bar{\partial}_E V^I - \bar{\Sigma}^I\right) \partial_E \bar{\Lambda}^J \\ &\qquad - 2\bar{\partial}_E \left(\Lambda^I + \bar{\Lambda}^I\right) \partial_E V^J - 2\bar{\partial}_E V^I \partial_E \left(\Lambda^J + \bar{\Lambda}^J\right) \\ &\qquad + \frac{4S_E}{\bar{S}_E} \partial_E \Lambda^I \Sigma^J + \frac{4\bar{S}_E}{S_E} \bar{\partial}_E \bar{\Lambda}^I \bar{\Sigma}^J \Bigg\} \Bigg] \\ &= \int d^4\theta\; f_{IJ} \Bigg[ \frac{1}{2} \left\{ \partial_E \left(-3\Lambda^I + \bar{\Lambda}^I\right) D^\alpha V^J + D^\alpha \Lambda^I \partial_E V^J \right. \\ &\qquad \left. - \partial_E D^\alpha \Lambda^I V^J - \left(\Lambda^I + \bar{\Lambda}^I\right) \partial_E D^\alpha V^J \right\} \mathcal{W}_{T\alpha} \\ &\qquad + \Phi_T V_E \left( D^\alpha \Lambda^I \mathcal{W}_\alpha^J + \Lambda^I D^\alpha \mathcal{W}_\alpha^J \right) \\ &\qquad + \frac{2\Phi_T}{U_E^2} \bar{\partial}_E \left(\Lambda^I - \bar{\Lambda}^I\right) \partial_E V^J + \text{h.c.} \Bigg]. \end{aligned} \tag{4.3}$$

At the second equality, we have used the following equation:

$$\begin{aligned} \int d^4\theta\; \frac{\Phi_T}{U_E^2} \frac{\bar{S}_E}{S_E} \partial_E \Lambda^I \Sigma^J &= \int d^4\theta\; \frac{\Phi_T}{U_E^2} \frac{\bar{S}_E}{S_E} \left( \frac{1}{S_E} \partial_4 - S_E \partial_5 \right) \Lambda^I \Sigma^J \\ &= \int d^4\theta\; \frac{\Phi_T}{U_E^2} \left\{ \left( 2iU_E^2 + \frac{S_E}{\bar{S}_E} \right) \frac{1}{S_E} \partial_4 \Lambda^I - \bar{S}_E \partial_5 \Lambda^I \right\} \Sigma^J \\ &= \int d^4\theta\; \Phi_T \bar{\partial}_E \Lambda^I \Sigma^J. \end{aligned} \tag{4.4}$$



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The last equality holds because of the property of $\Phi_T$ as a linear superfield. By means of (4.2), we can show that

$$\begin{aligned}
&\frac{1}{2}\left\{\partial_E\left(-3\Lambda^I+\bar{\Lambda}^I\right)D^\alpha V^J+D^\alpha\Lambda^I\partial_E V^J-\partial_E D^\alpha\Lambda^I V^J-\left(\Lambda^I+\bar{\Lambda}^I\right)\partial_E D^\alpha V^J\right\}\\
&\quad=\frac{1}{2}\left\{\partial_E D^\alpha\left(\bar{\Lambda}^I-\Lambda^I\right)V^J+\partial_E\left(\bar{\Lambda}^I-\Lambda^I\right)D^\alpha V^J\right.\\
&\qquad\quad\left.-D^\alpha\left(\bar{\Lambda}^I-\Lambda^I\right)\partial_E V^J-\left(\bar{\Lambda}^I-\Lambda^I\right)\partial_E D^\alpha V^J\right\}-\partial_E\Lambda^I D^\alpha V^J-\Lambda^I\partial_E D^\alpha V^J\\
&\quad=\frac{1}{2}D^\alpha\left\{\partial_E\left(\bar{\Lambda}^I-\Lambda^I\right)V^J-\left(\bar{\Lambda}^I-\Lambda^I\right)\partial_E V^J\right\}\\
&\qquad+\frac{D^\alpha S_E}{2}\left\{\mathcal{O}_E\left(\bar{\Lambda}^I-\Lambda^I\right)V^J-\left(\bar{\Lambda}^I-\Lambda^I\right)\mathcal{O}_E V^J\right\}-\partial_E\left(\Lambda^I D^\alpha V^J\right).
\end{aligned}\tag{4.5}$$

Thus (4.3) is rewritten as

$$\begin{aligned}
\delta\mathcal{L}_{\mathrm{VT}}^{(\mathrm{SG})}&=\int d^4\theta\, f_{IJ}\left[\frac{1}{2}D^\alpha\left\{\partial_E\left(\bar{\Lambda}^I-\Lambda^I\right)V^J-\left(\bar{\Lambda}^I-\Lambda^I\right)\partial_E V^J\right\}\mathcal{W}_{T\alpha}\right.\\
&\qquad+\frac{D^\alpha S_E}{2}\left\{\mathcal{O}_E\left(\bar{\Lambda}^I-\Lambda^I\right)V^J-\left(\bar{\Lambda}^I-\Lambda^I\right)\mathcal{O}_E V^J\right\}\mathcal{W}_{T\alpha}\\
&\qquad-\partial_E\left(\Lambda^I D^\alpha V^J\right)\mathcal{W}_{T\alpha}+V_E\Phi_T D^\alpha\left(\Lambda^I\mathcal{W}^J_\alpha\right)\\
&\qquad\left.+\frac{2\Phi_T}{U_E^2}\bar{\partial}_E\left(\Lambda^I-\bar{\Lambda}^I\right)\partial_E V^J+\text{h.c.}\right]\\
&=\int d^4\theta\, f_{IJ}\left[\frac{1}{2}\left\{\partial_E\left(\Lambda^I-\bar{\Lambda}^I\right)V^J-\left(\Lambda^I-\bar{\Lambda}^I\right)\partial_E V^J\right\}D^\alpha\mathcal{W}_{T\alpha}\right.\\
&\qquad-\frac{D^\alpha S_E}{2}\left\{\mathcal{O}_E\left(\Lambda^I-\bar{\Lambda}^I\right)V^J-\left(\Lambda^I-\bar{\Lambda}^I\right)\mathcal{O}_E^\alpha V^J\right\}\mathcal{W}_{T\alpha}\\
&\qquad\left.+\frac{2\Phi_T}{U_E^2}\bar{\partial}_E\left(\Lambda^I-\bar{\Lambda}^I\right)\partial_E V^J+\text{h.c.}\right].
\end{aligned}\tag{4.6}$$

At the second equality, we have used that

$$\begin{aligned}
&-\partial_E\left(\Lambda^I D^\alpha V^J\right)\mathcal{W}_{T\alpha}+V_E\Phi_T D^\alpha\left(\Lambda^I\mathcal{W}^J_\alpha\right)\\
&\quad=\Lambda^I D^\alpha V^J\left\{\partial_E\mathcal{W}_{T\alpha}-(\mathcal{O}_E S_E)\mathcal{W}_{T\alpha}\right\}-D^\alpha\left(V_E\Phi_T\right)\Lambda^I\mathcal{W}^J_\alpha\\
&\quad=\frac{1}{4}\Lambda^I D^\alpha V^J\left\{\bar{D}^2 D_\alpha(V_E\Phi_T)+4\left(\partial_E\mathcal{W}_{T\alpha}-(\mathcal{O}_E S_E)\mathcal{W}_{T\alpha}\right)\right\}=0,
\end{aligned}\tag{4.7}$$

where (3.30) is used at the last step.

Using (3.27), we find that

$$\begin{aligned}
&\frac{1}{2}\partial_E\left(\Lambda^I-\bar{\Lambda}^I\right)V^J D^\alpha\mathcal{W}_{T\alpha}+\text{h.c.}\\
&\quad=\frac{1}{2S_E}\partial_4\left(\Lambda^I-\bar{\Lambda}^I\right)V^J D^\alpha\left\{S_E\left(\mathcal{W}_{5\alpha}+8\partial_5\bar{D}^2Y_\alpha\right)\right\}\\
&\qquad-\frac{S_E}{2}\partial_5\left(\Lambda^I-\bar{\Lambda}^I\right)V^J D^\alpha\left\{\frac{1}{S_E}\left(\mathcal{W}_{4\alpha}+8\partial_4\bar{D}^2Y_\alpha\right)\right\}+\text{h.c.}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{D^\alpha S_E}{2}\mathcal{O}_E\left(\Lambda^I-\bar{\Lambda}^I\right)V^J\mathcal{W}_{T\alpha} \\
&\quad +\frac{1}{2}\partial_4\left(\Lambda^I-\bar{\Lambda}^I\right)V^J D^\alpha\left(\mathcal{W}_{5\alpha}+8\partial_5\bar{D}^2Y_\alpha\right) \\
&\quad -\frac{1}{2}\partial_5\left(\Lambda^I-\bar{\Lambda}^I\right)V^J D^\alpha\left(\mathcal{W}_{4\alpha}+8\partial_4\bar{D}^2Y_\alpha\right)+\text{h.c.} \\
&= \left\{\frac{D^\alpha S_E}{2}\mathcal{O}_E\left(\Lambda^I-\bar{\Lambda}^I\right)V^J\mathcal{W}_{T\alpha}+\text{h.c.}\right\} \\
&\quad +2i\partial_4\left(\Lambda^I-\bar{\Lambda}^I\right)V^J\partial_5\Phi_T+2i\partial_5\left(\Lambda^I-\bar{\Lambda}^I\right)V^J\partial_4\Phi_T. \qquad (4.8)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
-\frac{1}{2}\left(\Lambda^I-\bar{\Lambda}^I\right)\partial_E V^J D^\alpha\mathcal{W}_{T\alpha}+\text{h.c.} &= \left\{-\frac{D^\alpha S_E}{2}\left(\Lambda^I-\bar{\Lambda}^I\right)\mathcal{O}_E V^J\mathcal{W}_{T\alpha}+\text{h.c.}\right\} \\
&\quad -2i\left(\Lambda^I-\bar{\Lambda}^I\right)\left\{\partial_4 V\partial_5\Phi_T-\partial_5 V\partial_4\Phi_T\right\}. \qquad (4.9)
\end{aligned}
$$

Furthermore, we can see that

$$
\begin{aligned}
&\frac{2\Phi_T}{U_E^2}\Phi_T\bar{\partial}_E\left(\Lambda^I-\bar{\Lambda}^I\right)\partial_E V^J+\text{h.c.} \\
&\quad = 4i\Phi_T\left\{\left(\Lambda^I-\bar{\Lambda}^I\right)\partial_5 V^J-\partial_5\left(\Lambda^I-\bar{\Lambda}^I\right)\partial_4 V^J\right\}. \qquad (4.10)
\end{aligned}
$$

By means of these equations, we find that

$$
\begin{aligned}
\delta\mathcal{L}_{\text{VT}}^{(\text{SG})} &= \int d^4\theta\; f_{IJ}\left[2iV^I\left\{\partial_4\left(\Lambda^J-\bar{\Lambda}^J\right)\partial_5\Phi_T-\partial_5\left(\Lambda^J-\bar{\Lambda}^J\right)\partial_4\Phi_T\right\}\right. \\
&\qquad -2i\left(\Lambda^I-\bar{\Lambda}^I\right)\left(\partial_4 V^J\partial_5\Phi_T-\partial_5 V^J\partial_4\Phi_T\right) \\
&\qquad \left.+4i\Phi_T\left\{\partial_4\left(\Lambda^I-\bar{\Lambda}^I\right)\partial_5 V^J-\partial_5\left(\Lambda^I-\bar{\Lambda}^I\right)\partial_4 V^J\right\}\right] \\
&= \int d^4\theta\; f_{IJ}\left[-2i\Phi_T\left\{\partial_5 V^I\partial_4\left(\Lambda^J-\bar{\Lambda}^J\right)-\partial_4 V^I\partial_5\left(\Lambda^J-\bar{\Lambda}^J\right)\right\}\right. \\
&\qquad +2i\Phi_T\left\{\partial_5\left(\Lambda^I-\bar{\Lambda}^I\right)\partial_4 V^J-\partial_4\left(\Lambda^I-\bar{\Lambda}^I\right)\partial_5 V^J\right\} \\
&\qquad \left.+4i\Phi_T\left\{\partial_4\left(\Lambda^I-\bar{\Lambda}^I\right)\partial_5 V^J-\partial_5\left(\Lambda^I-\bar{\Lambda}^I\right)\partial_4 V^J\right\}\right] \\
&= 0. \qquad (4.11)
\end{aligned}
$$

Namely, the 6D SUGRA action (3.34) is gauge-invariant.

### 4.2 Dimensional reduction to 5D

Here we show that the our result (3.34) reproduces the known 5D SUGRA action after the dimensional reduction. We drop the $x^5$-dependence of the superfields in (3.34).[11] Then the differential operators become

$$
\partial_E \to \frac{1}{S_E}\partial_4, \qquad \mathcal{O}_E \to \frac{1}{S_E^2}\partial_4. \qquad (4.12)
$$

[11] The case that the $x^4$-dependence is dropped is essentially the same.

Hence the hyper-sector Lagrangian $\mathcal{L}_{\rm H}^{\rm (SG)}$ in (3.34) becomes

$$\begin{aligned}\mathcal{L}_{\rm H}^{\rm (5D)} &= -\int d^4\theta\; 2V_E^{1/2}U_E\left(H_{\rm odd}^\dagger \tilde{d}e^V H_{\rm odd}+H_{\rm even}^\dagger \tilde{d}e^{-V}H_{\rm even}\right)\\ &\quad+\left[\int d^2\theta\;\left\{H_{\rm odd}^t\tilde{d}\left(\frac{1}{S_E}\partial_4-\Sigma\right)H_{\rm even}-H_{\rm even}^t\tilde{d}\left(\frac{1}{S_E}\partial_4+\Sigma\right)H_{\rm odd}\right\}+{\rm h.c.}\right]\\ &= -\int d^4\theta\; 2\hat{V}_E\left(\hat{H}_{\rm odd}^\dagger\tilde{d}e^V\hat{H}_{\rm odd}+\hat{H}_{\rm even}^\dagger\tilde{d}e^{-V}\hat{H}_{\rm even}\right)\\ &\quad+\left[\int d^2\theta\;\left\{\hat{H}_{\rm odd}^t\tilde{d}\left(\partial_4-\hat{\Sigma}\right)\hat{H}_{\rm even}-\hat{H}_{\rm even}^t\tilde{d}\left(\partial_4+\hat{\Sigma}\right)\hat{H}_{\rm odd}\right\}+{\rm h.c.}\right], \qquad (4.13)\end{aligned}$$

where

$$\begin{aligned}\hat{V}_E &\equiv V_E^{1/2}U_E\left|S_E\right|, & \hat{\Sigma}^I &\equiv S_E\Sigma^I,\\ \hat{H}_{\rm odd} &\equiv S_E^{-1/2}H_{\rm odd}, & \hat{H}_{\rm even} &\equiv S_E^{-1/2}H_{\rm even}. \qquad (4.14)\end{aligned}$$

Next we consider the vector-tensor sector Lagrangian $\mathcal{L}_{\rm VT}^{\rm (SG)}$. From (3.27), $\mathcal{W}_{T\alpha}$ becomes

$$\mathcal{W}_{T\alpha}\to S_E\mathcal{W}_{5\alpha}. \qquad (4.15)$$

Then, $\mathcal{L}_{\rm VT}^{\rm (SG)}$ becomes

$$\begin{aligned}\mathcal{L}_{\rm VT}^{\rm (5D)} &= \int d^4\theta\; f_{IJ}\left[\left\{-2\Sigma^I D^\alpha V^J S_E\mathcal{W}_{5\alpha}+\frac{1}{2}\left(\partial_4V^ID^\alpha V^J-\partial_4D^\alpha V^IV^J\right)\mathcal{W}_{5\alpha}+{\rm h.c.}\right\}\right.\\ &\qquad+\Phi_TV_E\left(D^\alpha V^I\mathcal{W}_\alpha^J+\bar{D}_{\dot\alpha}V^I\bar{\mathcal{W}}^{J\dot\alpha}+V^ID^\alpha\mathcal{W}_\alpha^J\right)\\ &\qquad+\frac{\Phi_T}{U_E^2\left|S_E\right|^2}\left\{4\left(\partial_4V^I-\bar{S}_E\bar{\Sigma}^I\right)\left(\partial_4-S_E\Sigma^J\right)-2\partial_4V^I\partial_4V^J\right.\\ &\qquad\qquad\left.\left.+2S_E^2\Sigma^I\Sigma^J+2\bar{S}_E^2\bar{\Sigma}^I\bar{\Sigma}^J\right\}\right]\\ &= \int d^4\theta\; f_{IJ}\left[\left\{-2\hat{\Sigma}^ID^\alpha V^J\mathcal{W}_{5\alpha}+\frac{1}{2}\left(\partial_4V^ID^\alpha V^J-\partial_4D^\alpha V^IV^J\right)\mathcal{W}_{5\alpha}+{\rm h.c.}\right\}\right.\\ &\qquad+V_E\Phi_T\left(D^\alpha V^I\mathcal{W}_\alpha^J+\bar{D}_{\dot\alpha}V^I\bar{\mathcal{W}}^{J\dot\alpha}+V^ID^\alpha\mathcal{W}_\alpha^J\right)\\ &\qquad+\frac{2V_E\Phi_T}{\hat{V}_E^2}\left\{\partial_4V^I\partial_4V^J-2\partial_4V^I\left(\hat{\Sigma}^J+\bar{\hat{\Sigma}}^J\right)+2\bar{\hat{\Sigma}}^I\hat{\Sigma}^J\right.\\ &\qquad\qquad\left.\left.+\hat{\Sigma}^I\hat{\Sigma}^J+\bar{\hat{\Sigma}}^I\bar{\hat{\Sigma}}^J\right\}\right], \qquad (4.16)\end{aligned}$$

where we have used (4.14). Here, note that the constraint (3.30) is now

$$\begin{aligned}\bar{D}^2D_\alpha(V_E\Phi_T) &= -\frac{4}{S_E}\left\{\partial_4(S_E\mathcal{W}_{5\alpha})-\partial_4S_E\mathcal{W}_{5\alpha}\right\}\\ &= -4\partial_4\mathcal{W}_{5\alpha}=-8\partial_4\bar{D}^2D_\alpha X_5. \qquad (4.17)\end{aligned}$$

This can be solved as

$$V_E\Phi_T=\partial_4V_5-\Sigma_5-\bar{\Sigma}_5, \qquad (4.18)$$



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where $V_5 \equiv -8X_5$,[12] and $\Sigma_5$ is a chiral superfield. Substituting this into (4.16), we obtain

$$\begin{aligned}\mathcal{L}_{\rm VT}^{(5D)} = \int d^4\theta\, f_{IJ} \Bigg[ & \left\{ -2\hat{\Sigma}^I D^\alpha V^J \mathcal{W}_{5\alpha} + \frac{1}{2}\left(\partial_4 V^I D^\alpha V^J - \partial_4 D^\alpha V^I V^J\right)\mathcal{W}_{5\alpha} + \text{h.c.} \right\} \\ & + \left(\partial_4 V_5 - \Sigma_5 - \bar{\Sigma}_5\right)\left(D^\alpha V^I \mathcal{W}^J_\alpha + \bar{D}_{\dot\alpha} V^I \bar{\mathcal{W}}^{J\dot\alpha} + V^I D^\alpha \mathcal{W}^J_\alpha\right) \\ & + \frac{2\left(\partial_4 V_5 - \Sigma_5 - \bar{\Sigma}_5\right)}{\hat{V}_E^2}\left(\partial_4 V^I - \hat{\Sigma}^I - \bar{\hat{\Sigma}}^I\right)\left(\partial_4 V^J - \hat{\Sigma}^J - \bar{\hat{\Sigma}}^J\right)\Bigg] .\end{aligned} \tag{4.19}$$

Notice that the "shape-modulus" superfield $S_E$ completely disappears from the Lagrangian by the field redefinition (4.14).

Since it follows that

$$\begin{aligned} & \left(\partial_4 V_5 - \Sigma_5 - \bar{\Sigma}_5\right)\left(D^\alpha V^I \mathcal{W}^J_\alpha + \bar{D}_{\dot\alpha} V^I \bar{\mathcal{W}}^{J\dot\alpha} + V^I D^\alpha \mathcal{W}^J_\alpha\right) \\ & = \partial_4 V_5 D^\alpha V^I \mathcal{W}^J_\alpha + \frac{1}{2}\partial_4 V_5 V^I D^\alpha \mathcal{W}^J_\alpha - \Sigma_5\left(D^\alpha V^I \mathcal{W}^J_\alpha + \bar{D}_{\dot\alpha} V^I \bar{\mathcal{W}}^{J\dot\alpha} + V^I D^\alpha \mathcal{W}^J_\alpha\right) + \text{h.c.} \\ & = \partial_4 V_5 D^\alpha V^I \mathcal{W}^J_\alpha - \frac{1}{2} D^\alpha\left(\partial_4 V_5 V^I\right)\mathcal{W}^J_\alpha - \Sigma_5 D^\alpha V^I \mathcal{W}^J_\alpha \\ & \quad + \Sigma_5 V^I \bar{D}_{\dot\alpha} \bar{\mathcal{W}}^{J\dot\alpha} - \Sigma_5 V^I D^\alpha \mathcal{W}^J_\alpha + \text{h.c.} \\ & = \frac{1}{2}\left(\partial_4 V_5 D^\alpha V^I - \partial_4 D^\alpha V_5 V^I\right)\mathcal{W}^J_\alpha - \Sigma_5 D^\alpha V^I \mathcal{W}^J_\alpha + \text{h.c.},\end{aligned} \tag{4.20}$$

the above Lagrangian is rewritten as

$$\begin{aligned}\mathcal{L}_{\rm VT}^{(5D)} = \int d^4\theta\, f_{IJ} \Bigg[ & \Bigg\{ -2\hat{\Sigma}^I D^\alpha V^J \mathcal{W}_{5\alpha} + \frac{1}{2}\left(\partial_4 V^I D^\alpha V^J - \partial_4 D^\alpha V^I V^J\right)\mathcal{W}_{5\alpha} \\ & - \Sigma_5 D^\alpha V^I \mathcal{W}^J_\alpha + \frac{1}{2}\left(\partial_4 V_5 D^\alpha V^I - \partial_4 D^\alpha V_5 V^I\right)\mathcal{W}^J_\alpha + \text{h.c.} \Bigg\} \\ & + \frac{2\left(\partial_4 V_5 - \Sigma_5 - \bar{\Sigma}_5\right)}{\hat{V}_E^2}\left(\partial_4 V^I - \hat{\Sigma}^I - \bar{\hat{\Sigma}}^I\right)\left(\partial_4 V^J - \hat{\Sigma}^J - \bar{\hat{\Sigma}}^J\right)\Bigg] .\end{aligned} \tag{4.21}$$

As shown in appendix D, we find that

$$\begin{aligned} & f_{IJ}\left\{\left(\partial_4 V^I D^\alpha V^J - \partial_4 D^\alpha V^I V^J\right)\mathcal{W}_{5\alpha} + \left(\partial_4 V_5 D^\alpha V^I - \partial_4 D^\alpha V_5 V^I\right)\mathcal{W}^J_\alpha\right\} + \text{h.c.} \\ & \quad = 2 f_{IJ}\left(\partial_4 V^I D^\alpha V_5 - \partial_4 D^\alpha V^I V_5\right)\mathcal{W}^J_\alpha + \text{h.c.}. \end{aligned} \tag{4.22}$$

[12] Thus, $\mathcal{W}_{5\alpha}$ is expressed as $\mathcal{W}_{5\alpha} = -\frac{1}{4}\bar{D}^2 D_\alpha V_5$.



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By means of this relation, (4.21) is further rewritten as

$$\mathcal{L}^{(5\mathrm{D})}_{\mathrm{VT}} = \int d^4\theta\, f_{IJ}\Bigg[\bigg\{-2\hat{\Sigma}^I D^\alpha V^J \mathcal{W}_{5\alpha} - \Sigma_5 D^\alpha V^I \mathcal{W}^J_\alpha + \frac{1}{3}\left(\partial_4 V^I D^\alpha V^J - \partial_4 D^\alpha V^I V^J\right)\mathcal{W}_{5\alpha}$$
$$+\frac{1}{3}\left(\partial_4 V_5 D^\alpha V^I - \partial_4 D^\alpha V_5 V^I\right)\mathcal{W}^J_\alpha + \frac{1}{3}\left(\partial_4 V^I D^\alpha V_5 - \partial_4 D^\alpha V^I V_5\right)\mathcal{W}^J_\alpha + \mathrm{h.c.}\bigg\}$$
$$+\frac{2\left(\partial_4 V_5 - \Sigma_5 - \bar{\Sigma}_5\right)}{\hat{V}_E^2}\left(\partial_4 V^I - \hat{\Sigma}^I - \bar{\hat{\Sigma}}^I\right)\left(\partial_4 V^J - \hat{\Sigma}^J - \bar{\hat{\Sigma}}^J\right)\Bigg]. \tag{4.23}$$

Here we relabel $(V_5, \Sigma_5)$ as $(V^0, \Sigma^0)$. Then this Lagrangian is expressed as

$$\begin{aligned}\mathcal{L}^{(5\mathrm{D})}_{\mathrm{VT}} &= \left[-\int d^2\theta\, C_{IJK}\Sigma^I \mathcal{W}^J \mathcal{W}^K + \mathrm{h.c.}\right] \\ &\quad + \int d^4\theta\, \frac{C_{IJK}}{3}\left\{\left(\partial_4 V^I D^\alpha V^J - \partial_4 D^\alpha V^I V^J\right)\mathcal{W}^K_\alpha + \mathrm{h.c.}\right\} \\ &\quad + \int d^4\theta\, \frac{2C_{IJK}}{3\hat{V}_E^2}\mathcal{V}^I\mathcal{V}^J\mathcal{V}^K,\end{aligned} \tag{4.24}$$

where the indices $I, J, K$ now run from 0, the completely symmetric constant tensor $C_{IJK}$ is defined as $C_{IJ0} = f_{IJ}$ $(I, J \neq 0)$ and the other components are zero, and

$$\mathcal{V}^I \equiv \partial_4 V^I - \Sigma^I - \bar{\Sigma}^I, \tag{4.25}$$

which is the extra-dimensional component of the field strength superfield.

The 5D Lagrangians (4.13) and (4.24) perfectly agree with the $\mathcal{N} = 1$ superfield description of 5D SUGRA derived in refs. [7, 8].

## 5 Summary

We have found the $\mathcal{N} = 1$ superfield description of 6D SUGRA, and clarified how the moduli superfields appear in the action. We identified the combinations of the bosonic component fields that form $\mathcal{N} = 1$ superfields. By acting the SUSY transformations on them, we can identify the fermionic components of the superfields, which are expected to have complicated forms. Our result (3.34) reproduces the action in the global SUSY case by replacing the moduli superfields $V_E$ and $S_E$ with their constant background values. We have also shown that it is gauge-invariant both under (3.28) and (4.1), and is consistent with the known superfield action of 5D SUGRA through the dimensional reduction.

Compared to 5D SUGRA, the existence of the tensor multiplet and the "shape" modulus $S_E$ make the construction of the action complicated. In the global SUSY limit, the tensor multiplet is described by on-shell superfields that are subject to the constraints in (2.6). When the theory is promoted to SUGRA, this multiplet becomes off-shell and the superfields $X_4$ (or $X_5$) and $Y_\alpha$ can be treated as unconstrained independent superfields. As shown in section 4.1, the gauge invariance of the action in the vector-tensor sector is realized in a quite nontrivial manner because the Lagrangian is invariant only up to total



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derivatives. The gauge invariance strictly restricts the $S_E$-dependence of the action. It appears in the action through $\partial_E$ and $U_E(S_E, \bar{S}_E)$ defined in (3.15) and (3.20), respectively. We should also note that the $S_E$-dependence is absorbed by the field redefinition and completely disappears when one of the extra dimensions is reduced. This is another nontrivial check for our result.

In this work, we have neglected the fluctuation modes of $e_\mu{}^{\underline{\nu}}$, $e_\mu{}^{\underline{n}}$ and $e_m{}^{\underline{\nu}}$ ($\mu, \nu = 0,1,2,3$; $m,n = 4,5$). As mentioned in the footnote 6, the fluctuations of $e_\mu{}^{\underline{\nu}}$ can be taken into account by using the invariant action formulae in the superconformal formulation of 4D SUGRA. As for the "off-diagonal" components $e_\mu{}^{\underline{n}}$ and $e_m{}^{\underline{\nu}}$, further effort is necessary. However, we expect that it is not very difficult to incorporate them at linear order by means of the linearized SUGRA formulation [40–42], just like the 5D SUGRA case discussed in refs. [6, 10].

Our superfield description is useful to derive 4D effective theories of various 6D SUGRA models, as we did in the 5D SUGRA case [19–21]. Especially, we can treat a case that there exists the background magnetic flux penetrating the compact space or that the compact space has nonvanishing curvature. An explicit derivation of 4D effective theory will be discussed in a subsequent paper.


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## A 6D and 4D superconformal algebras

The 6D superconformal algebra consists of the translation $P_A$ ($A = 0, 1, \cdots, 5$), the local Lorentz transformation $M_{AB}$, the dilatation $D$, the special conformal transformation $K_A$, the $\mathrm{SU}(2)_{\mathbf{U}}$ automorphism $U^{ij}$, SUSY $Q^i_{\underline{\alpha}}$ and the conformal SUSY $S^i_{\underline{\alpha}}$.[13] Here, $\underline{\alpha} = 1,2,3,4$ is the 6D Weyl spinor index, and $i = 1,2$ is the $\mathrm{SU}(2)_{\mathbf{U}}$-doublet index. They satisfy the following algebra.

$$\begin{aligned}
[M_{AB}, M_{CD}] &= i\left(\eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC} + \eta_{AD}M_{BC}\right), \\
[M_{AB}, P_C] &= i\left(\eta_{BC}P_A - \eta_{AC}P_B\right), \\
[M_{AB}, K_C] &= i\left(\eta_{BC}K_A - \eta_{AC}K_B\right), \\
[M_{AB}, D] &= 0, \qquad [D, P_A] = iP_A, \qquad [D, K_A] = -iK_A, \\
[P_A, K_B] &= 2i\left(\eta_{AB}D + M_{AB}\right),
\end{aligned} \tag{A.1}$$

[13] Note that $Q^i_{\underline{\alpha}}$ and $S^{i\underline{\alpha}}$ are $\mathrm{SU}(2)_{\mathbf{U}}$-Majorana-Weyl spinors. We follow the notation of ref. [30] for 6D spinors.

and

$$\begin{aligned}
[M_{AB}, Q^i_{\underline{\alpha}}] &= \frac{i}{2}\left(\gamma_{AB}Q^i\right)_{\underline{\alpha}}, \qquad [D, Q^i_{\underline{\alpha}}] = \frac{i}{2}Q^i_{\underline{\alpha}},\\
[P_A, Q^i_{\underline{\alpha}}] &= 0, \qquad [K_A, Q^i_{\underline{\alpha}}] = (\gamma_A S^i)_{\underline{\alpha}},\\
[M_{AB}, S^{i\underline{\alpha}}] &= \frac{i}{2}\left(\tilde{\gamma}_{AB}S^i\right)^{\underline{\alpha}}, \qquad [D, S^{i\underline{\alpha}}] = -\frac{i}{2}S^{i\underline{\alpha}},\\
[P_A, S^{i\underline{\alpha}}] &= \left(\tilde{\gamma}_A Q^i\right)^{\underline{\alpha}}, \qquad [K_A, S^{i\underline{\alpha}}] = 0,\\
\left\{Q^1_{\underline{\alpha}}, Q^2_{\underline{\beta}}\right\} &= 2\left(\gamma^A C^{-1}\right)_{\underline{\alpha}\underline{\beta}} P_A,\\
\left\{Q^i_{\underline{\alpha}}, S^{j\underline{\beta}}\right\} &= -i\epsilon^{ij}\left\{\left(\gamma^{AB}\tilde{C}^{-1}\right)_{\underline{\alpha}}{}^{\underline{\beta}} M_{AB} - 2\left(\tilde{C}^{-1}\right)_{\underline{\alpha}}{}^{\underline{\beta}} D\right\} + 8\left(\tilde{C}^{-1}\right)_{\underline{\alpha}}{}^{\underline{\beta}} U^{ij},\\
\left\{S^{1\underline{\alpha}}, S^{2\underline{\beta}}\right\} &= 2\left(\tilde{\gamma}^A\tilde{C}^{-1}\right)^{\underline{\alpha}\underline{\beta}} K_A,\\
\left[U^{ij}, U^{kl}\right] &= \epsilon^{li}U^{kj} - \epsilon^{jk}U^{il},\\
\left[U^{ij}, Q^k_{\underline{\alpha}}\right] &= -\epsilon^{jk}Q^i_{\underline{\alpha}} - \frac{1}{2}\epsilon^{ij}Q^k_{\underline{\alpha}}, \qquad \left[U^{ij}, S^{k\underline{\alpha}}\right] = -\epsilon^{jk}S^{i\underline{\alpha}} - \frac{1}{2}\epsilon^{ij}S^{k\underline{\alpha}}.
\end{aligned} \tag{A.2}$$

Here we decompose the 4-component spinors into 2-component ones as

$$\begin{aligned}
Q^1_{\underline{\alpha}} &= \begin{pmatrix} Q^1_\alpha \\ -\bar{Q}^{2\dot{\alpha}} \end{pmatrix}, \qquad Q^2_{\underline{\alpha}} = \begin{pmatrix} Q^2_\alpha \\ \bar{Q}^{1\dot{\alpha}} \end{pmatrix},\\
S^{1\underline{\alpha}} &= \begin{pmatrix} S^{1\alpha} \\ -\bar{S}^2_{\dot{\alpha}} \end{pmatrix}, \qquad S^{2\underline{\alpha}} = \begin{pmatrix} S^{2\alpha} \\ \bar{S}^1_{\dot{\alpha}} \end{pmatrix}.
\end{aligned} \tag{A.3}$$

The $SU(2)_{\mathbf{U}}$ generators $U^{ij}$ are also expressed as

$$U^i{}_j = \epsilon_{jk}U^{ik} = \sum_{a=1}^{3} u^a(\sigma^a)^i{}_j. \tag{A.4}$$

From (A.2), we obtain

$$\begin{aligned}
[M_{\mu\nu}, Q^1_\alpha] &= i\left(\sigma^{\mu\nu}Q^1\right)_\alpha, \qquad [M_{\mu\nu}, S^2_\alpha] = i\left(\sigma^{\mu\nu}S^2\right)_\alpha,\\
[M_{45}, Q^1_\alpha] &= -\frac{1}{2}Q^1_\alpha, \qquad [M_{45}, S^2_\alpha] = \frac{1}{2}S^2_\alpha,\\
[D, Q^1_\alpha] &= \frac{i}{2}Q^1_\alpha, \qquad [D, S^2_\alpha] = -\frac{i}{2}S^2_\alpha,\\
[K_\mu, Q^1_\alpha] &= \left(\sigma_\mu\bar{S}^2\right)_\alpha, \qquad [P_\mu, S^2_\alpha] = \left(\sigma_\mu\bar{Q}^1\right)_\alpha,\\
\left\{Q^1_\alpha, \bar{Q}^1_{\dot{\beta}}\right\} &= -2\sigma^\mu_{\alpha\dot{\beta}}P_\mu, \qquad \left\{S^2_\alpha, \bar{S}^2_{\dot{\beta}}\right\} = -2\sigma^\mu_{\alpha\dot{\beta}}K_\mu,\\
\left\{Q^1_\alpha, S^{2\beta}\right\} &= 2i(\sigma^{\mu\nu})_\alpha{}^\beta M_{\mu\nu} - 2\delta_\alpha{}^\beta\left(M_{45} - 4u^3 + iD\right),\\
[u^3, Q^1_\alpha] &= -\frac{1}{2}Q^1_\alpha, \qquad [u^3, S^2_\alpha] = \frac{1}{2}S^1_\alpha,
\end{aligned} \tag{A.5}$$

in the 2-component-spinor notation. This is the 4D $\mathcal{N}=1$ superconformal algebra, and we can identify the generator of the $\mathrm{U}(1)_A$ automorphism as

$$\mathcal{Q}_A = M_{45} - 4u^3. \tag{A.6}$$

We have normalized $\mathcal{Q}_A$ so that $Q^1_\alpha$ and $S^2_\alpha$ have the charges 3/2 and $-3/2$, respectively.

## B SUSY transformation of 6D Weyl multiplet

The 6D Weyl multiplet consists of the sechsbein $e_M{}^{\underline{N}}$, the gravitino $\Psi^i_{M\underline{\alpha}}$, the gauge fields for the dilatation $b_M$ and for the $\mathrm{SU(2)_U}$ automorphism $V^a_M$ ($a=1,2,3$), the anti-self-dual tensor $T^-_{\underline{MNL}}$, and some auxiliary fields. The SUSY transformations of the (extra-dimensional-components of) 6D Weyl multiplet [14, 27] are expressed in the 2-component spinor notation as follows.[14]

$$\begin{aligned}
\delta_\epsilon e_m{}^{\underline{4}} &= 2\left(\epsilon^1\psi^2_m - \epsilon^2\psi^1_m\right) + \text{h.c.}, \\
\delta_\epsilon e_m{}^{\underline{5}} &= -2i\left(\epsilon^1\psi^2_m - \epsilon^2\psi^1_m\right) + \text{h.c.}, \\
\delta_\epsilon \psi^1_m &= \left\{\partial_m + \frac{1}{2}b_m - \frac{1}{2}\left(\omega_m{}^{\mu\nu}\sigma_{\mu\nu} + i\omega_m{}^{\underline{45}}\right) - iV^3_m + \frac{e_m{}^{\underline{4}} - ie_m{}^{\underline{5}}}{4}\left(T_{\mu\nu\underline{4}} + iT_{\mu\nu\underline{5}}\right)\sigma^{\mu\nu}\right\}\epsilon^1 \\
&\quad - i\left(V^1_m - iV^2_m\right)\epsilon^2 \\
&\quad + \left\{\frac{i}{2}\left(\omega_m{}^{\mu\underline{4}} + i\omega_m{}^{\mu\underline{5}}\right)\sigma_\mu - \frac{e_m{}^{\underline{4}} + ie_m{}^{\underline{5}}}{24}\epsilon^{\mu\nu\rho\lambda}T^-_{\mu\nu\rho}\sigma_\lambda + 6T^-_{\mu\underline{45}}\sigma^\mu\right\}\bar{\epsilon}^2, \\
\delta_\epsilon \psi^2_m &= \left\{\partial_m + \frac{1}{2}b_m - \frac{1}{2}\left(\omega_m{}^{\mu\nu}\sigma_{\mu\nu} + i\omega_m{}^{\underline{45}}\right) + iV^3_m + \frac{e_m{}^{\underline{4}} - ie_m{}^{\underline{5}}}{4}\left(T_{\mu\nu\underline{4}} + iT_{\mu\nu\underline{5}}\right)\sigma^{\mu\nu}\right\}\epsilon^2 \\
&\quad - \left\{\frac{i}{2}\left(\omega_m{}^{\mu\underline{4}} + i\omega_m{}^{\mu\underline{5}}\right)\sigma_\mu - \frac{e_m{}^{\underline{4}} + ie_m{}^{\underline{5}}}{24}\left(\epsilon^{\mu\nu\rho\lambda}T^-_{\mu\nu\rho}\sigma_\lambda + 6T^-_{\mu\underline{45}}\sigma^\mu\right)\right\}\bar{\epsilon}^1 \\
&\quad - i\left(V^1_m + iV^2_m\right)\epsilon^1, \\
&\ \ \vdots
\end{aligned} \tag{B.1}$$

where the 2-component spinors $\psi^i_m$ ($i=1,2$) are embedded into the 4-component ones as

$$\Psi^1_{m\underline{\alpha}} = \begin{pmatrix}\psi^1_{m\alpha} \\ -\bar{\psi}^{2\dot{\alpha}}_m\end{pmatrix}, \qquad \Psi^2_{m\underline{\alpha}} = \begin{pmatrix}\psi^2_{m\alpha} \\ \bar{\psi}^{1\dot{\alpha}}_m\end{pmatrix}, \tag{B.2}$$

which have positive 6D chiralities. In section 3.1, we focus on a half of the whole SUSY parameterized by $\epsilon^1_\alpha$ and $\bar{\epsilon}^1_{\dot{\alpha}}$.

[14] Since we neglect the fluctuations of $e_\mu{}^{\underline{\nu}}$, $e_\mu{}^{\underline{n}}$ and $e_m{}^{\underline{\nu}}$, we do not discriminate the curved indices from the flat ones for the 4D part.

## C Component expression of constraint (3.26)

Here we express the constraint (3.26) in terms of the component fields, and clarify the independent degrees of freedom. Note that (3.26) is rewritten as

$$\bar{D}^2\left(D_\alpha X_5 + 4\partial_5 Y_\alpha\right) = \frac{1}{S_E^2}\bar{D}^2\left(D_\alpha X_4 + 4\partial_4 Y_\alpha\right). \tag{C.1}$$

Since $\bar{D}^2 D_\alpha X_m$ $(m = 4, 5)$ are field strength superfields, $4\partial_m \bar{D}^2 Y_\alpha$ are chiral spinor superfields and $1/S_E^2$ is a chiral scalar superfield, they are expanded as

$$\begin{aligned}
\bar{D}^2 D_\alpha X_m &= \lambda_{m\alpha} + \theta_\alpha D_m + i(\sigma^{\mu\nu}\theta)_\alpha v_{m\mu\nu} - i\theta^2(\sigma^\mu\partial_\mu\bar{\lambda}_m)_\alpha, \\
4\partial_m \bar{D}^2 Y_\alpha &= \omega_{m\alpha} + \theta_\alpha K_m + i(\sigma^{\mu\nu}\theta)_\alpha K_{m\mu\nu} + \theta^2\tau_{m\alpha}, \\
\frac{1}{S_E^2} &= a + \theta\psi + \theta^2 F,
\end{aligned} \tag{C.2}$$

where $D_m$ is a real scalar, $v_{m\mu\nu} \equiv \partial_\mu v_{m\nu} - \partial_\nu v_{m\mu}$ is a field strength, $K_m$ is a complex scalar and $K_{m\mu\nu}$ is a real antisymmetric tensor. Then, we calculate

$$\begin{aligned}
4\partial_5\bar{D}^2 Y_\alpha &= \frac{1}{S_E^2}\bar{D}^2\left(D_\alpha X_4 + 4\partial_4 Y_\alpha\right) - \bar{D}^2 D_\alpha X_5 \\
&= a\left(\lambda_4 + \omega_4\right)_\alpha - \lambda_{5\alpha} \\
&\quad + \theta_\alpha\left\{a\left(D_4 + K_4 + \frac{1}{2}\psi(\lambda_4+\omega_4)\right) - D_5\right\} \\
&\quad + i(\sigma^{\mu\nu}\theta)_\alpha\left(\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}C_{4\mathrm{R}}^{\rho\lambda} + C_{4\mathrm{I}\mu\nu} - v_{5\mu\nu}\right) \\
&\quad + \theta^2\left\{F(\lambda_4+\omega_4)_\alpha - \frac{1}{2}\psi_\alpha(D_4+K_4) - \frac{i}{2}(\sigma^{\mu\nu}\psi)_\alpha(v_{4\mu\nu}+K_{4\mu\nu})\right. \\
&\qquad\qquad \left. + a\left(\tau_4 - i\sigma^\mu\partial_\mu\bar{\lambda}_4\right)_\alpha + i(\sigma^\mu\partial_\mu\bar{\lambda}_5)_\alpha\right\},
\end{aligned} \tag{C.3}$$

where

$$\begin{aligned}
C_{4\mathrm{R}\mu\nu} &\equiv (\mathrm{Re}\, a)\left(v_{4\mu\nu} + K_{4\mu\nu}\right) - \mathrm{Re}\left\{\frac{a}{2}\psi\sigma_{\mu\nu}\left(\lambda_4+\omega_4\right)\right\}, \\
C_{4\mathrm{I}\mu\nu} &\equiv (\mathrm{Im}\, a)\left(v_{4\mu\nu} + K_{4\mu\nu}\right) - \mathrm{Im}\left\{\frac{a}{2}\psi\sigma_{\mu\nu}\left(\lambda_4+\omega_4\right)\right\}.
\end{aligned} \tag{C.4}$$

We have used that

$$\begin{aligned}
(\theta\psi)\tilde{\lambda}_\alpha &= \frac{1}{2}\left\{(\psi\tilde{\lambda})\theta_\alpha - (\psi\sigma^{\mu\nu}\tilde{\lambda})(\sigma_{\mu\nu}\theta)_\alpha\right\}, \\
\left(C_{4\mathrm{R}\mu\nu} + iC_{4\mathrm{I}\mu\nu}\right)(\sigma^{\mu\nu}\theta)_\alpha &= i\left(\frac{1}{2}\epsilon_{\mu\nu\rho\lambda}C_{4\mathrm{R}}^{\rho\lambda} + C_{4\mathrm{I}\mu\nu}\right)(\sigma^{\mu\nu}\theta)_\alpha,
\end{aligned} \tag{C.5}$$

where $\tilde{\lambda}_\alpha \equiv \lambda_{4\alpha} + \omega_{4\alpha}$.

From (C.3), we can see that the constraint (3.26) can be satisfied for a given $X_4$ and $Y_\alpha$ by adjusting $X_5$ and $S_E$. Specifically, for given values of $\bar{D}^2 D_\alpha X_4$ and $4\partial_4\bar{D}^2 Y_\alpha$, we can realize any values for $\omega_{5\alpha}$, $K_5$, $K_{5\mu\nu}$ and $\tau_{5\alpha}$ in $4\partial_5\bar{D}^2 Y_\alpha$ by tuning $\lambda_{5\alpha}$, $D_5$ and $a$, $v_{5\mu}$ and two real degrees of freedom in $\psi_\alpha$, and $F$ and the remaining degrees of freedom in $\psi_\alpha$, respectively.

## D Derivation of eq. (4.22)

Here we derive the relation (4.22). We neglect total derivatives. Then we obtain

$$\begin{aligned} A &\equiv f_{IJ}\left(\partial_4 V_5 D^\alpha V^I - \partial_4 D^\alpha V_5 V^I\right)\mathcal{W}^J_\alpha + \text{h.c.} \\ &= -f_{IJ}\left(V_5\partial_4 D^\alpha V^I - D^\alpha V_5 \partial_4 V^I\right)\mathcal{W}^J_\alpha + B + \text{h.c.}, \end{aligned} \tag{D.1}$$

where

$$B \equiv -f_{IJ}\left(V_5 D^\alpha V^I - D^\alpha V_5 V^I\right)\partial_4 \mathcal{W}^J_\alpha. \tag{D.2}$$

We can show that

$$\begin{aligned} B + \text{h.c.} &= \frac{f_{IJ}}{4}\bar{D}^2\left(V_5 D^\alpha V^I - D^\alpha V_5 V^I\right)\partial_4 D_\alpha V^J + \text{h.c.} \\ &= f_{IJ}\mathcal{W}^\alpha_5 V^I \partial_4 D_\alpha V^J + C + \text{h.c.}, \end{aligned} \tag{D.3}$$

where

$$\begin{aligned} C \equiv \frac{f_{IJ}}{4}\big(&\bar{D}^2 V_5 D^\alpha V^I + 2\bar{D}_{\dot\alpha} V_5 \bar{D}^{\dot\alpha} D^\alpha V^I + V_5 \bar{D}^2 D^\alpha V^I \\ &+ 2\bar{D}_{\dot\alpha} D^\alpha V_5 \bar{D}^{\dot\alpha} V^I - D^\alpha V_5 \bar{D}^2 V^I\big)\partial_4 D_\alpha V^J. \end{aligned} \tag{D.4}$$

Here, it follows that

$$\begin{aligned} C + \text{h.c.} &= -\frac{f_{IJ}}{4}D^\alpha\big(\bar{D}^2 V_5 D_\alpha V^I + 2\bar{D}_{\dot\alpha} V_5 \bar{D}^{\dot\alpha} D_\alpha V^I + V_5 \bar{D}^2 D_\alpha V^I \\ &\qquad + 2\bar{D}_{\dot\alpha} D_\alpha V_5 \bar{D}^{\dot\alpha} V^I - D_\alpha V_5 \bar{D}^2 V^I\big)\partial_4 V^J + \text{h.c.} \\ &= -\frac{f_{IJ}}{4}\big(D^\alpha \bar{D}^2 V_5 D_\alpha V^I - 2\bar{D}_{\dot\alpha} V_5 D^\alpha \bar{D}^{\dot\alpha} D_\alpha V^I + D^\alpha V_5 \bar{D}^2 D_\alpha V^I \\ &\qquad + V_5 D^\alpha \bar{D}^2 D_\alpha V^I + 2D^\alpha \bar{D}_{\dot\alpha} D_\alpha V_5 D_\alpha V^I + D_\alpha V_5 D^\alpha \bar{D}^2 V^I\big)\partial_4 V^J + \text{h.c.} \\ &= -\frac{f_{IJ}}{4}\big(\bar{D}^2 D^\alpha V_5 D_\alpha V^I + 4i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \bar{D}^{\dot\alpha} V_5 D^\alpha V^I + 2\bar{D}_{\dot\alpha} V_5 D^2 \bar{D}^{\dot\alpha} V^I \\ &\qquad - 4i\sigma^\mu_{\alpha\dot\alpha}\bar{D}^{\dot\alpha} V_5 \partial_\mu D^\alpha V^I + V_5 D^\alpha \bar{D}^2 D_\alpha V^I - 2D^2 \bar{D}_{\dot\alpha} V_5 \bar{D}^{\dot\alpha} V^I \\ &\qquad - 4i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu D^\alpha V_5 \bar{D}^{\dot\alpha} V^I + 4i\sigma^\mu_{\alpha\dot\alpha} D^\alpha V_5 \partial_\mu \bar{D}^{\dot\alpha} V^I\big)\partial_4 V^J + \text{h.c.} \\ &= f_{IJ}\left(-\mathcal{W}^\alpha_5 D_\alpha V^I + 2\bar{D}_{\dot\alpha} V_5 \bar{\mathcal{W}}^{I\dot\alpha} + V_5 D^\alpha \mathcal{W}^I_\alpha\right)\partial_4 V^J + \text{h.c.} \\ &= f_{IJ}\left[-D^\alpha V^I \partial_4 V^J \mathcal{W}_{5\alpha} + \left\{2D^\alpha V_5 \partial_4 V^I - D^\alpha\left(V_5\partial_4 V^I\right)\right\}\mathcal{W}^J_\alpha\right] + \text{h.c.} \\ &= f_{IJ}\left\{-D^\alpha V^I \partial_4 V^J \mathcal{W}_{5\alpha} + \left(D^\alpha V_5 \partial_4 V^I - V_5 \partial_4 D^\alpha V^I\right)\mathcal{W}^J_\alpha\right\} + \text{h.c.}. \end{aligned} \tag{D.5}$$

We have used the commutation relations:

$$\left\{D_\alpha, \bar{D}_{\dot\alpha}\right\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu, \qquad \left[D_\alpha, \bar{D}^2\right] = -4i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \bar{D}^{\dot\alpha}. \tag{D.6}$$



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Therefore, (D.1) is calculated as

$$\begin{aligned} A =& -f_{IJ}\left(V_5\partial_4 D^\alpha V^I - D^\alpha V_5 \partial_4 V^I\right)\mathcal{W}^J_\alpha + f_{IJ}\mathcal{W}^\alpha_5 V^I \partial_4 D_\alpha V^J \\ &-f_{IJ} D^\alpha V^I \partial_4 V^J \mathcal{W}_{5\alpha} + f_{IJ}\left(D^\alpha V_5 \partial_4 V^I - V_5 \partial_4 D^\alpha V^I\right)\mathcal{W}^J_\alpha + \text{h.c.} \\ =& \, 2 f_{IJ}\left(\partial_4 V^I D^\alpha V_5 - \partial_4 D^\alpha V^I V_5\right)\mathcal{W}^J_\alpha \\ &-f_{IJ}\left(\partial_4 V^I D^\alpha V^J - \partial_4 D^\alpha V^I V^J\right)\mathcal{W}_{5\alpha} + \text{h.c..} \end{aligned} \tag{D.7}$$

Namely, we obtain

$$\begin{aligned} & f_{IJ}\left\{\left(\partial_4 V^I D^\alpha V^J - \partial_4 D^\alpha V^I V^J\right)\mathcal{W}_{5\alpha} + \left(\partial_4 V_5 D^\alpha V^I - \partial_4 D^\alpha V_5 V^I\right)\mathcal{W}^J_\alpha\right\} + \text{h.c.} \\ &\quad = 2 f_{IJ}\left(\partial_4 V^I D^\alpha V_5 - \partial_4 D^\alpha V^I V_5\right)\mathcal{W}^J_\alpha + \text{h.c..} \end{aligned} \tag{D.8}$$



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