

Topological string on elliptic CY 3-folds and the ring of Jacobi forms

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ABSTRACT: We give evidence that the all genus amplitudes of topological string theory on compact elliptically fibered Calabi-Yau manifolds can be written in terms of meromorphic Jacobi forms whose weight grows linearly and whose index grows quadratically with the base degree. The denominators of these forms have a simple universal form with the property that the poles of the meromorphic form lie only at torsion points. The modular parameter corresponds to the fibre class while the rôle of the string coupling is played by the elliptic parameter. As a consequence the topological string amplitudes are modular and quasi periodic in the string coupling. This leads to very strong all genus results on these geometries, which are checked against results from curve counting. The structure can be viewed as an indication that an $N = 2$ analog of the reciprocal of the Igusa cusp form exists that might govern the topological string theory on these Calabi-Yau manifolds completely.

KEYWORDS: Topological Strings, String Duality

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1 Introduction and summary

Topological strings on non-compact Calabi-Yau geometries are solvable and have a very interesting structure with a wealth of connections to gauge theories, integrable models, large N-dualities, Chern-Simons theories, supersymmetric localisation and matrix models. In fact topological string techniques had considerable influence on these areas and solved important problems, see [18] for a recent example. As a benchmark problem in topological string theory with a similar wealth of connections to quantum gravity problems, remains the solution of the topological string on compact Calabi-Yau manifolds. In this paper we report on some progress in this matter.

It follows from the Feynman rules¹ of [7] that the genus g topological string amplitudes $F_g(S^{ij}, S^j, S; \underline{z})$ are inhomogeneous polynomials of weighted degree $3g - 3$ in the anholomorphic propagators, $S^{ij}, S^j := S^{j\varphi}, S := S^{\varphi\varphi}$, which have respective weights $(1, 2, 3)$, with rational functions of the moduli \underline{z} as coefficients. The holomorphic anomaly determines all of F_g except the weight zero piece f_g , the so called holomorphic ambiguity [7] whose determination is the key conceptual obstacle to solving the topological string on compact Calabi-Yau spaces. A careful analysis of the conifold gap condition [44], regularity at the orbifold and Castelnuovo's criterium for the vanishing of higher genus curves reveals that the f_g can be determined up genus 51,² for the quintic [44]. While for non-compact models the same conditions lead to a complete solution of the closed topological string on these geometries [29],³ on compact Calabi-Yau manifolds one needs additional boundary condition.

One purpose of this article is to investigate the additional symmetries and boundary conditions that are specific for elliptic fibered Calabi-Yau threefolds.⁴ Beside the motivation to solve the topological string in general the elliptically fibered cases are very interesting in their own right as the topological string amplitudes predict terms in the low energy

¹See the vertex rule (6.10) [7].

²That condition has been derived under the assumption that the ring of modular generators on the quintic has no relations at high weight, which has been checked up to the weight needed for genus 30.

³In fact our key example $X_{18}(1, 1, 1, 6, 9)$ is an elliptic CY manifold over \mathbb{P}^2 and decompactifying the elliptic fibre, leads to a local model solved in [29].

⁴Note that the transitions between Calabi-Yau manifolds [68] and mirror symmetry make it possible to predict from the solution of the topological string on one Calabi-Yau 3 fold with fibration structure the ones of such without fibration structure, if the complex moduli space of the latter is embedded into the one of the former, see [32] for many examples.

effective actions of F-theory and calculate indices of 6d conformal theories in particular the E -string and its generalizations with gauge theories [28].

One possible constraint could come from the $SL(2, \mathbb{Z})$ modularity of the amplitudes in the Kähler parameter of the elliptic fibre that has been argued in [3, 54] based on a monodromy analysis in one global case [12] and calculations for more general elliptic threefolds [52]. It also extends observations for local elliptic surfaces inside Calabi-Yau spaces [55] and [31, 34, 35].

We find that the constraint from the fibre modularity is equivalent to the constraint that comes from a \mathbb{Z}_2 *involution* symmetry I , that acts on the moduli space of elliptic fibrations as an R symmetry, i.e. it is not realized as a symplectic transformation like the monodromies of the model. This symmetry I acts also on the F_g by $(-1)^{(g-1)}$ and on the propagators in a way described in section 3.4.3. Knowing these actions we can restrict the possible constants in the rational ambiguity f_g to roughly one fourth. Using this information, the conifold gap and the regularity at the other points in the moduli space, especially the orbifold point we can solve the topological string for all classes to genus 9 on an elliptic fibration over \mathbb{P}^2 , with one section. This is already the strongest available result for compact multiparameter Calabi-Yau manifolds and needs only very mild results on the vanishing of the BPS numbers.

Our main result is however as follows. Consider the topological string partition function $Z = \exp(F) = \exp(\sum_{g=0}^{\infty} \lambda^{2g-2} F_g)$ expanded in the base degree(s). E.g. for the \mathbb{P}^2 case one has⁵

$$Z = Z_0 \left(1 + \sum_{d_B=1}^{\infty} Z_{d_B}(\tau, z) Q^{d_B} \right), \tag{1.1}$$

where τ is the complexified Kähler class of the fiber, and $Q = e^{2\pi i t_B}$ with t_B the suitable defined complexified Kähler class of the base, see (4.1). Z_{d_B} is a quotient of even weak Jacobi forms of the following form⁶

$$Z_{d_B}(\tau, z) = \frac{\varphi_{d_B}(\tau, z)}{\eta^{36d_B}(\tau) \prod_{k=1}^{d_B} \varphi_{-2,1}(\tau, kz)}, \tag{1.2}$$

where $\eta(\tau)$ is the Dedekind function and $\varphi_{d_B}(\tau, z)$ is an even weak Jacobi form of index $\frac{1}{3}d_B(d_B - 1)(d_B + 4)$ and weight $16d_B$. Note that the elliptic argument z is identified by $2\pi z = \lambda$ with the topological string coupling λ and as a consequence the topological string amplitudes are modular (4.15) quasiperiodic (4.16) in the string coupling!

The number of coefficients of $\varphi_{d_B}(\tau, z)$ expressed in the basis of the ring of even weak Jacobi forms $Q := E_4(\tau), R := E_6(\tau), A := \varphi_{-2,1}(\tau, z)$ and $B := \varphi_{0,1}(\tau, z)$ still grows very fast with d_B^6 , however we can argue that the Castelnuovo bounds discussed in section 5.2 bring down the growth to d_B^4 . Combined with the restriction from the constraints from the conifold gap and the regularity of the F_g in the interior of the moduli space and with

⁵The base degree zero contribution Z_0 , is rather trivial and discussed separately.

⁶Here we use the standard notation of the elliptic argument z . We trust that there will be no confusion with the B-model moduli, which are also traditionally denoted by z .

the independence of these conditions assumed,⁷ this allows us to solve the model up to genus $g = 189$ for arbitrary base and fibre classes, or up to $d_B \leq 20$ for arbitrary genus and fibre classes.

We will show in section 4.4 that the Jacobi forms satisfy a heat equation form of the holomorphic anomaly equation for Z_{d_B} that is equivalent to a limit of the wave function equation proposed by Witten [73] for the topological string partition function in interpreting the holomorphic anomaly equation of [7] as a consequence of a background independent quantization of $H_3(M, \mathbb{R})$. This heat equation form of the holomorphic anomaly equation further summarizes all the holomorphic anomaly equations for the fibre modulus, which were proven in [54], i.e. the ansatz (1.2) identically satisfies the holomorphic anomaly equation for the fibre modulus. The relation of the wave function equation and the heat equation was pointed out in [73] and applied to the case in which the moduli space is a symmetric space [26], which however is not the case for generic $SU(3)$ holonomy manifolds.

We argue that the special form of the denominator in our ansatz naturally is crucial to satisfying the Castelnuovo bounds. The scaling of the z argument in the denominator can be interpreted as coming from multiple string windings of the base similar as in the elliptic genus of E-strings [11, 30, 51] or more general strings of 6d SCFTs [28] with gauge symmetries. On the other hand, the paper [51] constructs a 2d quiver gauge theory for E-strings and [28] for the \hat{D}_4 string that can in principle compute the elliptic genus of any finite number of E-strings with the techniques of [5].⁸

The prediction from the B-model calculations and from (1.1), (1.2) are successfully checked in section 5.2 against results from enumerative geometry. Since the moduli space of the BPS states tends to become smoother for fixed degree and high g we can make an infinite number of nontrivial checks on the form of Z_{d_B} and the determination of its coefficients following the methods proposed in [50].

A remarkable mathematical characterization of the form (1.2) is that, again due to the form of the denominator, the poles of Z_{d_B} lie only at the points $z \in \mathbb{Q}\tau + \mathbb{Q}$, i.e. at torsion points of the elliptic curve. As such they have a modified theta expansion that involves mock modular forms [75]. These structures played a role in the understanding of wall-crossing for $N = 4$ BPS states [15] based on the reciprocal of the Igusa cusp form and related meromorphic forms.

The general structure (1.1), (1.2) can be viewed as an extension of the only compact case 3-fold case $K3 \times T^2$ for which all BPS states associated⁹ to moduli space of curves are known [50]. Due to the $SU(2)$ holonomy the Type II compactification has $N = 4$ supersymmetry. The prediction is based on the formula for the elliptic genus of symmetric products of $K3$ [17] and M-theory/Type IIB duality. It has been recently proven in [66]. In this case the BPS generating function Z_{BPS} is related to the reciprocal of the Igusa

⁷This holds at least for as far as we could actually calculate i.e. up to $d_B \leq 5$, d_E, g arbitrary or $g \leq 8$, d_B, d_E arbitrary.

⁸It would be very interesting to see if such 2d quiver theory duals exist also for the compact threefold case, i.e. 6d theories that do include gravity.

⁹In order to make this interpretation one has to lift zero modes that come from the T^2 and make the $N = 2$ invariants trivially zero.

cuspidal form of a genus two curve with weight 10 which has a Fourier expansion in $p = \exp(2\pi i\sigma)$ as¹⁰

$$Z_{BPS}(\Omega) = \frac{1}{\Phi_{10}(\Omega)} = \sum_{m=-1} \psi_{-10,m}(\tau, z) p^m, \text{ with } \Omega = \begin{pmatrix} \tau & z \\ z & \sigma \end{pmatrix}, \quad (1.3)$$

where Ω parametrizes the Siegel upper half space \mathbb{H}_2 and $\psi_{0,m+1} = \eta^{24}(\tau)\varphi_{-2,1}\psi_{-10,m}$ are meromorphic Jacobi-Forms of weight zero and index $m + 1$ in the analogous expansion of $\hat{Z}(\Omega) = \eta^{24}\phi_{-2,1}Z_{BPS}(\Omega)$. The latter can be written in terms of the elliptic genera of symmetric products of K3 as

$$\hat{Z}(\Omega) = \sum_{m=-1} \chi(\text{Sym}^{m+1}(K3), q, y) p^m = \frac{1}{p} \prod_{\substack{n>0, m\geq 0, l \\ l, m, n \in \mathbb{Z}}} \frac{1}{(1 - p^n q^m y^l)^{c(m,n,l)}} \quad (1.4)$$

and summed to the total free energy $F(\lambda, Q) = \log(Z) = \sum_{j=1}^{\infty} \frac{1}{j} F^{(j)}$ by [50]

$$F^{(j)} = \left(2 \sin \frac{j\lambda}{2}\right)^{-2} \prod_{\substack{k>0, n>0 \\ m\geq 0, l \in \mathbb{Z}}} \frac{(1 - q^{jk})^4}{(1 - y^j q^{jk})^2 (1 - y^{-j} q^{jk})^2} \frac{1}{(1 - p^{jn} q^{jm} y^l)^{c(m,n,l)}}, \quad (1.5)$$

where $c(m, r)$ are the expansion coefficients of the elliptic genus of K3 (4.22) and we identified $\lambda = 2\pi z$ and defined $y = \exp(i\lambda)$.

Of course a key question for a full $N = 2$ extension of the discussed structure is whether the formulas (1.1), (1.2) come from an underlying modular object as (1.3), (1.5). One obvious difference is the quadratic versus linear growth of the index in the weak Jacobi forms. In section 3.4.2 we discuss the formulation of projective special Kähler manifolds in $N = 2$ supergravity that might give a hint for the construction of such an object. We hope to report on progress on that question in [39].

In this article we focus for simplicity on elliptic fibrations with only I_1 singular fibers in the Kodaira classification, i.e. no gauge symmetries, but eventually flavor symmetries, in the F theory interpretation. We note however that for the gauge theory case, i.e. elliptic fibrations with higher Kodaira fibre singularities, the work [28] gives already the crucial hint for the generalization of the numerator in (1.2). In this case, we will find additional Jacobi theta functions in the numerator, whose different elliptic arguments correspond to the additional Kähler classes of the resolution. This Jacobi theta function will introduce additional zeros in the numerator, which signal the gauge symmetry enhancements, and should be fixed by this structure. The formulas we find also suggest a possible refinement, which we shortly indicate in (5.1.2).

2 The elliptic Calabi-Yau manifolds and their mirrors

In this section we describe the toric construction of elliptically fibered Calabi-Yau spaces with only I_1 Kodaira fibers. Mirror symmetry is manifest in this formalism. We collect the key data of our main example and describe shortly the general structure of the genus zero sector.

¹⁰We follow the notation of [15].

2.1 Construction of toric hypersurface Calabi-Yau spaces

Our construction of mirror pairs of Calabi-Yau n -folds as hypersurfaces in toric ambient spaces $\mathbb{P}_{\Delta}^{n+1}$ follows Batyrev’s construction which relies on dual pairs of $n + 1$ dimensional reflexive pairs of lattice polyhedra (Δ, Δ^*) . Let (Γ, Γ^*) be dual lattices with pairing $\langle \cdot, \cdot \rangle$ and real completion $(\Gamma_{\mathbb{R}}, \Gamma^*_{\mathbb{R}})$. A lattice polyhedron Δ (Δ^*) is the convex hull of integer points in Γ (Γ^*). The dual polyhedron Δ^* to Δ is defined by $\Delta^* = \{y \in \Gamma^*_{\mathbb{R}} | \langle y, x \rangle \geq -1, \forall x \in \Delta\}$. Δ is reflexive if Δ^* is a lattice polyhedron. Note $(\Delta^*)^* = \Delta$ so if Δ is reflexive (Δ, Δ^*) is a reflexive pair and each polyhedron contains the origin as its unique inner point.

The Calabi-Yau n -fold M_n is given by the canonical hypersurface

$$P = \sum_{\nu_i \in \hat{\Delta} \cap \Gamma} a_i^* \prod_{\nu_j^* \in \hat{\Delta}^* \cap \Gamma^*} y_j^{\langle \nu_i, \nu_j^* \rangle + 1} = 0 \tag{2.1}$$

in the toric ambient space \mathbb{P}_{Δ} with coordinate ring y_j , while the mirror Calabi-Yau n -fold W_n is given by the constraint $P^* = 0$ in \mathbb{P}_{Δ^*} with coordinate ring x_i , where $P^* = 0$ is defined analogously to (2.1) with Δ and Δ^* exchanged. The notation $\hat{\Delta}$ ($\hat{\Delta}^*$) means the polyhedra Δ (Δ^*) with the integer points interior to codimension one faces omitted.

To give the ambient space \mathbb{P}_{Δ} a fibration structure¹¹ such that the embedded Calabi-Yau n -fold defined as hypersurface has a fibration by a Calabi-Yau m -fold, we combine a base polyhedron Δ^{B^*} and a reflexive fibre polyhedron Δ^{F^*} into an $n + 1$ dimensional polyhedron Δ^* as follows

$$\left| \begin{array}{cc|cc} \nu_i^* \in \Delta^* & & \nu_j \in \Delta & \\ & \nu_i^{F^*} & & \nu_j^F \\ \Delta_{n-m}^{B^*} & \vdots & s_{ij} \Delta_{n-m}^B & \vdots \\ & \nu_i^{F^*} & & \nu_j^F \\ 0 \dots 0 & & 0 \dots 0 & \\ \vdots & \Delta_{m+1}^{*F} & \vdots & \Delta_{m+1}^F \\ 0 \dots 0 & & 0 \dots 0 & \end{array} \right|. \tag{2.2}$$

If Δ^{*F} and Δ^{*B} are reflexive then (Δ, Δ^*) , given by the complex hull of the indicated points, is a reflexive pair. Reflexivity of Δ^{*F} is required by the CY condition on the fibre. For Δ_{n-m}^{*B} it is not a necessary condition, see [43] for more details on this construction. We defined $s_{ij} = \langle \nu_i^F, \nu_j^{F^*} \rangle + 1 \in \mathbb{N}$ and scaled $\Delta^B \rightarrow s_{ij} \Delta^B$. Here we indicated the dimensions of some polyhedra by subscripts; elliptic fibrations correspond to $m = 1$.

For $n = 3$ and $m = 1$ we get many examples by choosing any of the 16 reflexive polyhedra in 2d as Δ^{*F} and Δ^{*B} respectively and specifying in addition $\nu_i^{F^*} \in \Delta^{*F}$ as well as twisting parameters [10], which not indicated in (2.2).

2.2 The $X_{18}(1, 1, 1, 6, 9)$ 3-fold, an elliptic fibration over \mathbb{P}^2

Our main example is the smooth elliptic fibration over \mathbb{P}^2 . This case¹² is a two parameter model discussed in [32] and in greater detail in [12] and illustrates the general construction described above.

¹¹See exercise in [22] p. 49, where the statement is made in the language of the fans associated to Δ .

¹²It can be also written as the zero locus of a degree 18 polynomial in the weighted projective space $\mathbb{P}^4(1, 1, 1, 6, 9)$ called $X_{18}(1, 1, 1, 6, 9)$.

Pick for the base \mathbb{P}^2 , whose toric polyhedron is the convex hull of the points $\Delta^{*B} = \text{conv}(\{(1,0), (0,1), (-1,-1)\})$, for the fibre polynomial $\Delta^{*F} = \text{conv}(\{(1,0), (0,1), (-2,-3)\})$ and for $\nu_3^{*F} = (-2,-3)$. Then $\nu_3^F = (-1,-1)$ and $s_{33} = 6$.

We list the points which give rise to the coordinate ring of $\mathbb{P} = \mathbb{P}_\Delta / (\mathbb{Z}_{18} \times \mathbb{Z}_6)$, all points $\nu_i^* \in \hat{\Delta}^*$ and the two vectors of linear relations among them, which correspond to the Mori cone of \mathbb{P}_{Δ^*} , as well the toric divisors $D_{x_i} = \{x_i = 0\}$

Div.	ν_i				$\bar{\nu}_i^*$				$l^{(E)}$	$l^{(B)}$
D_{x_0}	0	0	0	0	0	0	0	0	-6	0
$D_{x_1} = L$	12	-6	-1	-1	1	0	-2	-3	0	1
$D_{x_2} = L$	-6	12	-1	-1	0	1	-2	-3	0	1
$D_{x_3} = L$	-6	-6	-1	-1	-1	-1	-2	-3	0	1
$D_z = E$	0	0	-1	-1	0	0	-2	-3	1	-3
$D_x = 2H$	0	0	2	-1	0	0	1	0	2	0
$D_y = 3H$	0	0	-1	1	0	0	0	1	3	0

(2.3)

The classical topological data of the 3-fold M are easily calculable from the toric construction. The Euler is $\chi(M) = -540$, the two independent Hodge numbers are $h^{1,1}(M) = 2$, $h^{2,1}(M) = 272$, the classical triple intersection numbers are given by¹³

$$C_{111}^0 = H^3 = 9, \quad C_{112}^0 = H^2 \cdot L = 3, \quad C_{122}^0 = H \cdot L^2 = 1, \quad C_{222}^0 = L^3 = 0 \quad (2.4)$$

where H and L are the divisors dual to the curves defined by the Mori vectors $l^{(E)}$ and $l^{(B)}$ and the Kähler classes J_E and J_B . The intersection with the second Chern class c_2 of M are

$$\int_M c_2 \wedge J_E = [c_2] \cdot H = 36, \quad \int_M c_2 \wedge J_B = [c_2] \cdot L = 102. \quad (2.5)$$

The mirror manifold is given by the zero locus

$$P^* = x_0(z^6(x_1^{18} + x_2^{18} + x_3^{18} - b(x_1x_2x_3)^6) - 2^{\frac{1}{3}}\sqrt{3}azx_1x_2x_3x_4x_5 + x^3 + y^2) = 0 \quad (2.6)$$

in the space \mathbb{P} . $z := x_4, x := x_5$ and $y := x_6$ are the conventional names of variables in the Weierstrass form of the elliptic fibre. In \mathbb{P} there are toric \mathbb{C}^* actions on the coordinates $x_i, i = 1, \dots, 6$, which can be used to eliminate all a_i , but the two complex structure variables (a, b) of W . This is because two \mathbb{C}^* actions

$$x_i \rightarrow \mu_r^{l_i^{(r)}} x_i, \quad \text{with } \mu_r \in \mathbb{C}^*, \quad (2.7)$$

are divided out from the coordinate ring of \mathbb{P} . One can introduce manifestly \mathbb{C}^* invariant combinations the a_i as complex structure variables of W , namely

$$z_i = (-1)^{l_0^{(i)}} \prod_{k=1} a_k^{l_k^{(i)}}, \quad i = 1, \dots, h_{21}(W_3) = h_{11}(M_3). \quad (2.8)$$

¹³In the notation of [32] these intersections are encoded in the ring $\mathcal{R} = 9J_E^3 + 3J_E^2J_B + J_EJ_B^2$. H, L are the notations for the divisors used in [12].

In the case at hand $z_1 := z_E = \frac{a_4 a_5^2 a_6^3}{a_0^6}$ corresponds to the elliptic fibre and $z_2 := z_B = \frac{a_1 a_2 a_3}{a_4^3}$ to the base class. Using the \mathbb{C}^* actions on the period integrals $\Pi(z) = \int_{\gamma_3} \Omega$ with $(a = a_0)$

$$\Omega = \oint_{\gamma_\epsilon} \frac{a\mu}{P^*}, \quad (2.9)$$

given by a residuum integral around $P^* = 0$ with the measure $\mu = \sum_{i=1}^5 (-1)^i w_i d\tilde{x}_1 \wedge \dots \wedge \widehat{d\tilde{x}_i} \dots \wedge d\tilde{x}_5$, one can derive two Picard-Fuchs (PF) differential equations [32]

$$\begin{aligned} \mathcal{L}_1 &= \theta_1(\theta_1 - 3\theta_2) - 12z_1(6\theta_1 + 1)(6\theta_1 + 5), \\ \mathcal{L}_2 &= \theta_2^3 + z_2 \prod_{i=0}^2 (3\theta_2 - \theta_1 + i) \end{aligned} \quad (2.10)$$

determining the periods from $\mathcal{L}_i \Pi(z) = 0$, $i = 1, 2$. Here $\theta_i = z_i \frac{\partial}{\partial z_i}$. The discriminants of the operators are

$$\begin{aligned} \Delta_1 &= (1 - 432z_1)^3 - 27z_2(432z_1)^3, \\ \Delta_2 &= 1 + 27z_2 \end{aligned} \quad (2.11)$$

The 3-point couplings can be computed from the PF operators [32]

$$\begin{aligned} C_{111} &= \frac{9}{z_1^3 \Delta_1}, & C_{112} &= C_{121} = C_{211} = \frac{3\Delta_3}{z_1^2 z_2 \Delta_1}, \\ C_{122} &= C_{212} = C_{221} = \frac{\Delta_3^2}{z_1 z_2^2 \Delta_1}, & C_{222} &= \frac{9(\Delta_3^3 + (432z_1)^3)}{z_2^2 \Delta_1 \Delta_2}, \end{aligned} \quad (2.12)$$

where for convenience we can define the factor $\Delta_3 = 1 - 432z_1$.

2.3 Integral symplectic basis and genus zero topological string amplitudes

The PF operators determine the 3-point couplings. Its solutions in a special basis determine the metric on the moduli space, which is a Kähler manifold $G_{i\bar{j}} = \partial_{t_i} \partial_{\bar{t}_j} K$, whose Kähler potential K is given by

$$e^{-K} = i \int_W \Omega \wedge \bar{\Omega} = i \Pi^\dagger \eta \Pi, \quad (2.13)$$

where $(k = h^{1,1}(W) + 1)$

$$\eta = \begin{pmatrix} 0 & \mathbf{1}_{k \times k} \\ -\mathbf{1}_{k \times k} & 0 \end{pmatrix}. \quad (2.14)$$

is the symplectic pairing on $H_3(W, \mathbb{Z})$ and $\Pi(z) = (X^I = \int_{A^I} \Omega, F_I = \int_{B_I} \Omega)^T$ is the period vector w.r.t. to the corresponding symplectic basis (A^I, B_I) , $I = 0, \dots, k-1$ of $H_3(W, \mathbb{Z})$. The two structures are related by special geometry, which implies the existence of a prepotential, the genus zero amplitude $\mathcal{F}^{(0)}$, with

$$\Pi = \begin{pmatrix} X^0 \\ X^i \\ F_0 \\ F_i \end{pmatrix} = X^0 \begin{pmatrix} 1 \\ t^i \\ 2\mathcal{F}^{(0)} - t^i \partial_i \mathcal{F}^{(0)} \\ \frac{\partial \mathcal{F}^{(0)}}{\partial t^i} \end{pmatrix} = X^0 \begin{pmatrix} 1 \\ t^i \\ -\frac{C_{ijk}^0}{3!} t^i t^j t^k + \frac{\int c_2 \wedge J_i}{24} t^i - i \frac{\chi \zeta(3)}{(2\pi)^3} + f(q) \\ \frac{C_{ijk}^0}{2} t^i t^j + n_{ij} t^j + \frac{\int c_2 \wedge J_i}{24} + \partial_i f(\underline{q}) \end{pmatrix}. \quad (2.15)$$

Here

$$\mathcal{F}^{(0)} = \left[-\frac{C_{ijk} t^i t^j t^k}{3!} + n_{ij} \frac{t^i t^j}{2} + \frac{\int c_2 \wedge J_i}{24} t^i - i \frac{\chi \zeta(3)}{2(2\pi)^3} + f(\underline{q}) \right] \quad (2.16)$$

and X^I are homogeneous coordinates and $t^i(z)$ are inhomogeneous coordinates

$$t^i(z) = \frac{X^i(\underline{z})}{X^0(\underline{z})} = \frac{1}{2\pi i} (\log(z_i) + \Sigma^i(\underline{z})), \quad i = 1, \dots, h^{1,1}(W), \quad (2.17)$$

which serve as a mirror map. In particular the complexified volumes of the curves in the Mori cone t_i are the flat coordinates at large radius near $z_i = 0, \forall i = 1, \dots, h^{1,1}(M)$, where the third equal sign in (2.15) is valid, X^0 is the unique holomorphic period normalized to $\omega_0(z) = 1 + \mathcal{O}(z)$. The $C_{ijk}^{(0)}$ are the classical intersection numbers. There is a freedom in choosing the n_{ij} , but because of the odd intersections and the requirement of integer monodromy around $z_i = 0$, they cannot be set to zero as they must be half integral. Following [12] or the topological description the quadratic terms in $\mathcal{F}^{(0)}$ in the appendix of [44] we may take them to $n_{11} = \frac{9}{2}$ and $n_{12} = \frac{3}{2}$ for the $X_{18}(1, 1, 1, 6, 9)$ model. Like for all toric hypersurfaces in this example the Picard-Fuchs equations are of the generalized hyperelliptic type and all logarithmic solutions can be derived from

$$\omega_0(\underline{z}, \underline{\rho}) = \sum_{\underline{n}} c(\underline{n}, \underline{\rho}) z^{\underline{n}+\underline{\rho}} \quad \text{where} \quad c(\underline{n}, \underline{\rho}) = \frac{\prod_j \Gamma(\sum_{\alpha} l_{0j}^{(\alpha)} (n_{\alpha} + \rho_{\alpha}) + 1)}{\prod_i \Gamma(\sum_{\alpha} l_i^{(\alpha)} (n_{\alpha} + \rho_{\alpha}) + 1)}, \quad (2.18)$$

with $X^0(\underline{z}) = \omega_0(\underline{z}, 0)$, by taking derivatives with respect to ρ_{α} , in particular

$$X^i(\underline{z}) = \frac{\partial_{\rho_i}}{2\pi i} \omega_0(\underline{z}, \underline{\rho})|_{\underline{\rho}=\underline{0}}, \quad (2.19)$$

see the appendix in [33] for the explicit structure of the higher derivatives.

3 Involution symmetry and BCOV formalism

In this section we mainly discuss the involution symmetry, its realization on the topological string amplitudes and its consequences. In the course of the exposition we also review the relevant aspects of the BCOV formalism. Many of the actual calculations are quite technical and relegated to the appendices. We comment on the relation of the involution symmetry to the monodromies in subsection 3.2 and on the local limit in subsection 3.2.1. The section 3.4.1 illustrates the idea of the BCOV ring and contains many modular properties needed in section 4. Subsection 3.4.2 is more speculative and tries to give a perspective on a possible global modular object related to the all genus amplitude.

3.1 The involution symmetry

Fibrations with fibre polytope Δ^{*F} have an involution symmetry I acting on the moduli space, which is independent of the chosen base B . The action of I on the $X_{18}(1, 1, 1, 6, 9)$ moduli space that we describe below appeared in [12].

I acts non-trivial on the higher genus amplitudes $F_g(S^{ij}, S^i, S)$ and the anholomorphic propagators S^{ij}, S^j, S . As mentioned in the introduction it follows from the Feynman rules

of [7] that the F_g are inhomogeneous polynomials of degree $3g - 3$ with weights 1, 2, 3 and rational functions in the moduli as coefficients. Moreover the structure of the genus zero three-point vertex within these Feynman rules implies, that the degree of the denominators of these rational functions grows generically with $\alpha(2g - 2)$. Since the amplitudes are finite for large moduli values this is also an upper bound for degree of the numerators, which turns out to be nearly saturated. This implies that for an r parameter model the complexity of the F_g grows¹⁴ growth very fast with $g^{(r+1)(r+2)/2+r}$.

In particular the rational function f_g that does not multiply any propagator is not determined by the holomorphic anomaly equation, that otherwise fix the F_g recursively in g . The function f_g is restricted by the pole behavior of the F_g at the boundary divisors of the moduli space to contain finite number of unknown parameters which grows with g^r . This is called the holomorphic ambiguity and its determination is a main problem to solve the topological string on compact Calabi-Yau manifolds. If we know the action of I on F_g and the propagators we can further restrict f_g . In fact this requirement cuts down the possible parameters in f_g to roughly one fourth. Moreover we find that the symmetry I is equivalent to the constraints that the fibre modularity imposes on f_g .

To see the involution symmetry explicitly define the monomial $m = (\prod_i x_i)^6$, where x_i are the coordinates of the base. Then

$$P^* = g(z, \underline{x}, \underline{a}_B) - bz^6 m^6 - 2^{\frac{1}{3}} \sqrt{3} a z m x y + x^3 + y^2 = 0, \tag{3.1}$$

where $g(z, \underline{x}, \underline{a}_B)$ is a polynomial compatible with the scaling, which does not contain m . Now requiring that

$$\begin{aligned} x &\rightarrow x + c_1 z^2 m^2 \\ y &\rightarrow y + c_2 x z m + c_3 z^3 m^3 \end{aligned} \tag{3.2}$$

leaves P^* invariant fixes $c_1 = 2^{\frac{1}{3}} a^2$, $c_2 = \frac{(1-i)}{2^{\frac{1}{3}}} \sqrt{3} a$ and $c_3 = \sqrt{3} a^3$ and acts on the parameters as

$$I : (a, b) \rightarrow (ia, b + a^6) . \tag{3.3}$$

This involution operation acts on the (3, 0) form Ω given in (2.9) by

$$I : \Omega \rightarrow i\Omega, \tag{3.4}$$

because P^* as well as the measure μ are invariant. Since Ω defines vacuum line bundle \mathcal{L} and the higher genus amplitudes \mathcal{F}_g transforms as section $\mathcal{F}_g \in \mathcal{L}^{2g-2}$ we conclude that the involution symmetry maps $\mathcal{F}^{(g)}$ to $\tilde{\mathcal{F}}^{(g)}$ with

$$\tilde{\mathcal{F}}^{(g)} = (-1)^{g-1} \mathcal{F}^{(g)}. \tag{3.5}$$

In the the z_i coordinates the involution acts as

$$I : (z_1, z_2) \rightarrow (x_1, x_2) = \left(\frac{1}{432} - z_1, -\frac{(432z_1)^3 z_2}{(1 - 432z_1)^3} \right). \tag{3.6}$$

¹⁴This estimate is in good agreement with the byte count for the F_g in the two parameter model, which occupy $6.5 \times 10^4, 1.3 \times 10^6, 1.1 \times 10^7, 6 \times 10^7, 2.4 \times 10^8, 7.8 \times 10^8, 2.2 \times 10^9, 5.5 \times 10^9$ bytes of memory for $g = 2, \dots, 9$.

The PF operators (2.10) are invariant up to some trivial factors under the involution. On the other hand, the involution exchanges the two discriminants up to some factors as

$$I(\Delta_1) = (432z_1)^3 \Delta_2, \quad I(\Delta_2) = \frac{\Delta_1}{(1 - 432z_1)^3}. \quad (3.7)$$

3.2 Monodromy group versus involution symmetry

The involution symmetry multiplies the periods w.r.t. Ω with i , so it is in particular not a symplectic transformation (in fact $M_I^T \eta M_I = -\eta$) and cannot be related to an actual monodromy action.

The monodromy group is most quickly described as follows. There are Neveu-Schwarz B-field shifts that leave the instanton action invariant $t_i \rightarrow t_i + 1$, $i = 1, \dots, h^{1,1}(M)$. Their monodromy is completely fixed by (2.15) after specifying the topological data as in subsections 2.2, 2.3 for the main example. We call these monodromies and the corresponding 6×6 matrices $T_1 (= T_E)$ and $T_2 (= T_B)$.

Further there is a cycle ν_1 , which corresponds to the $B_3 \sim S^3$ base of the Strominger-Yau-Zaslow $T^3 \rightarrow B_3$ fibration [70], that vanishes at the conifold $\Delta_1 = 0$. In the $X_6(1, 2, 3)$ elliptic fibrations such as the $X_{18}(1, 1, 1, 6, 9)$ one has a second conifold discriminant $\Delta_2 = 0$, where Δ_1 and Δ_2 are exchanged, up to irrelevant factors, by the involution symmetry as discussed in the last section. Let us call the corresponding vanishing cycle ν_2 and use the integral symplectic basis of cycles that corresponds to the period vector (2.15). Using some basic analytic continuation one calculates the vanishing cycles in this basis as¹⁵

$$\begin{aligned} \nu_1 &= (0, 0, 0, 1, 0, 0), \\ \nu_2 &= (-1, 0, 0, 0, 1, -3). \end{aligned} \quad (3.8)$$

The Lefschetz monodromy theorem for 3-folds states that the monodromy around a conifold divisor, where by definition an S^3 sphere ν vanishes, on each cycle $\gamma \in H_3(M, \mathbb{Z})$ is given by the symplectic reflection

$$S_\nu(\gamma) = \gamma - \langle \gamma, \nu \rangle \nu, \quad (3.9)$$

where \langle, \rangle denotes the symplectic pairing. By this formula and (3.8) we can calculate the monodromy in the basis (2.15). We call the corresponding monodromies and 6×6 matrices C_1 and C_2 respectively. The T_i and C_i generate the monodromy group of the $X_{18}(1, 1, 1, 6, 9)$ model in fact redundantly, as can be seen by the Van Kampen relations analyzed for this case in great detail in [12]. In particular one has that an order 18 element A is given by¹⁶

$$A^{-1} = C_1 C_2 T_2 \quad (3.10)$$

and the two conifold monodromies C_1 and C_2 are conjugated to each other by the order 6 element A^3

$$C_1 = A^3 C_2 A^{-3}, \quad (3.11)$$

¹⁵In fact ν_1 corresponds always to F_0 , while ν_2 corresponds to $F_E - X_0 - \sum_{i=1}^{h^{1,1}(B)} a^i F_{B,i}$, where the a_i defined above (4.1).

¹⁶We use the same notation A for this element as in [12]. The other notions are related by $C_1 = T$, $C_2 = B$, $T_1 = T_\infty$ and $T_2 = D_\infty$.

which exchanges $\nu_1 \leftrightarrow \nu_2$ and is explicitly

$$A^3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 10 & 3 & -1 & 1 & -3 \\ 0 & 3 & 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.12}$$

One can conclude that the monodromy group is generated by A and C_1 [12]. Note that T_1 and A^3 acts on t_1 as [12]

$$T_1 : t_1 \mapsto t_1 + 1, \quad A^3 : t_1 \mapsto -\frac{1}{t_1 + 1} \tag{3.13}$$

These operations generate an $SL(2, \mathbb{Z})$ action on the elliptic fibre parameter, which we call therefore often τ in the following. As we mentioned already invariance under the involution symmetry I is equivalent to the fibre modularity. This could be guessed from the fact that it is A^3 that really imposes the nontrivial part of fibre modularity — the shift symmetry is present for all Kähler moduli — and conjugates the conifold monodromies just as I exchanges the two conifold discriminants. The equivalence of the restrictions imposed the by invariance under I and the fibre modularity will be strictly proven in appendix D.

3.2.1 The local limit

As we have seen there is an $SL(2, \mathbb{Z})$ action on the elliptic fibre modulus. This symmetry governs the amplitudes order by order in the exponential $Q_B = \exp(2\pi i t_B)$ of the flat coordinate representing the base curve in \mathbb{P}^2 . There is an other action of the congruence subgroup $\Gamma_0(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \mid c = 0 \pmod{3} \right\}$ of $SL(2, \mathbb{Z})$ on the base modulus. To see it we must make the volume of the fibre large, $\lim(t_E) \rightarrow i\infty$, which corresponds to $z_E \sim q = \exp(2\pi i t_E) \sim z_E \sim 0$. This is known as the local limit. In the A -model language it focuses on the $\mathcal{O}(-3) \rightarrow \mathbb{P}^2$ geometry by decompactifying the elliptic fibre of $X_{18}(1, 1, 1, 6, 9)$. The periods of the local geometry are given by integrals of a meromorphic differential λ over cycles on an elliptic curve \mathcal{C}_B that is the the mirror geometry. These periods $\Pi_{loc} = (\int_i \lambda, i = 0, a, b,) = (1, t_B, -3\partial_{t_B} F_{loc}^{(0)})$ fulfill local rigid special geometry [38]. The occurrence of the 1 indicates that the CY (3, 0) form Ω becomes in the local limit the meromorphic one form λ which has a non-vanishing residuum at a pole λ on \mathcal{C}_B . From (2.10) one sees that the periods in the local limit Π_{loc} are governed by a specialization of the second differential operator $\mathcal{L} = \theta_2^3 + z_2 \prod_{i=0}^2 (3\theta_2 + i)$, which is another way to see the constant solution.

However we want to understand precisely, which periods $\Pi = \int_{\Gamma} \Omega$ on the compact CY 3-fold become the local periods in the limit or equivalently how the $\Gamma_0(3)$ action is embedded in the action of $Sp(6, \mathbb{Z})$. To see this notice that we have to make a linear

change in the $\partial_{t_i} F^{(0)}$, $i = 1, 2$ part of the basis Π (2.15) to keep with

$$\begin{aligned}\tilde{F}_1 &= F_B = -\frac{3}{2}t_E^2 - t_E t_B + \frac{3t_E}{2} + \frac{3}{2} \\ \tilde{F}_2 &= 3F_B - F_E = \frac{t_B^2}{2} - \frac{3t_B}{2} + \frac{1}{4}\end{aligned}\tag{3.14}$$

in the limit $z_E = 0$ three fine periods ($X_0 = 1, t_B, \tilde{F}_2$). If we now conjugate the monodromies that we found in the last section by the corresponding element $C = \begin{pmatrix} \mathbf{1}_{4 \times 4} & 0 & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & -\mathbf{1} & 3 \end{pmatrix}$, we obtain the following monodromies $M_i = CT_2C^{-1}$, $M_c = CC_2C^{-1}$ and $M_o = (M_c M_i)^{-1}$

$$M_i = \begin{pmatrix} \underline{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \underline{1} & 0 & \mathbf{1} & 0 & 0 & \mathbf{0} \\ 3 & 2 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ -\underline{1} & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} \end{pmatrix}, M_c = \begin{pmatrix} \underline{1} & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ -\underline{3} & 0 & \mathbf{1} & 0 & 0 & -\mathbf{3} \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \underline{0} & 0 & \mathbf{0} & 0 & 0 & \mathbf{1} \end{pmatrix}, M_o = \begin{pmatrix} \underline{1} & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & -1 \\ \underline{2} & 0 & \mathbf{1} & 0 & 0 & \mathbf{3} \\ -3 & -1 & 0 & 1 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 & -1 \\ -\underline{1} & 0 & -\mathbf{1} & 0 & 0 & -\mathbf{2} \end{pmatrix}.\tag{3.15}$$

Here we have printed the elements $\begin{pmatrix} d & c \\ a & b \end{pmatrix}$ of the $\Gamma_0(3)$ subgroup, see [1], in bold face and underlined the shifts due to the non-vanishing residua of λ . A consequence of this symmetry is the fact that in the strict local limit the amplitudes $F^{(g)}(\hat{E}_2(\tau_B), G_2(\tau_B), G_4(\tau_B), G_6(\tau_B))$ [1], with $\tau_B = \frac{-3\partial_u \partial_{t_B} F^{(0)}}{\partial_u t_B}$, can be expressed in the indicated generators of the ring of almost holomorphic forms of $\Gamma_0(3)$ and has been completely solved [29]. Here u is the complex modulus of the local geometry [29], which is identified with the b parameter in (2.6). Of course this base modularity that holds in the strict large fibre limit, and acts not directly on the periods of the 3-fold geometry, extends much less trivially to the whole two parameter family than the fibre modularity, that we discuss further in section 4.

3.3 Involution symmetry at genus one

First we consider the genus one amplitude $\mathcal{F}^{(1)}$ which fulfills the holomorphic anomaly equation [7]¹⁷

$$\partial_i \bar{\partial}_{\bar{j}} F^{(1)} = \frac{1}{2} \text{Tr} C_i \bar{C}_{\bar{j}} - \frac{\chi(M)}{24} G_{i\bar{j}},\tag{3.16}$$

where $\langle j | \bar{\phi}_{\bar{i}} | k \rangle = \bar{C}_{\bar{i}\bar{j}k} = C_{\bar{i}}^{j'k'} \eta_{j'j} \eta_{kk'}$ and $\langle j | \phi_i | k \rangle = C_{ijk}$ are the 3 point functions discussed above, with the relation $C_{\bar{i}}^{jk} = e^{2K} G^{j\bar{j}} G^{k\bar{k}} \bar{C}_{\bar{i}\bar{j}\bar{k}}$. The solution has the well known form

$$\mathcal{F}^{(1)} = \frac{1}{2} \left(3 + h^{1,1} - \frac{\chi}{12} \right) K + \frac{1}{2} \log \det G^{-1} - \frac{1}{24} \sum_{i=1}^2 s_i \log z_i - \frac{1}{12} \sum_{a=1}^2 \log \Delta_a.\tag{3.17}$$

The topological data for the $X_{18}(1, 1, 1, 6, 9)$ model are given in section 2.2. The leading asymptotic of $\mathcal{F}^{(1)}$ anholomorphic near large volume limit is

$$\mathcal{F}^{(1)} = -\frac{1}{24} \sum_{i=1}^2 t_i \int_M c_2 J_i + \mathcal{O}(Q).\tag{3.18}$$

¹⁷This was derived first in [6]. We use however the conventions of [7].

The values of the second Chern-Classes (2.5) determine the constants $s_1 = 114, s_2 = 48$. In the holomorphic limit, the determinant of the Kahler metric transforms as $\det(\tilde{G}) = \det(\frac{\partial z_i}{\partial x_j}) \det(G)$. A simple calculation shows that the invariance of the genus one amplitude $\mathcal{F}^{(1)}$ under the involution transformation (3.6) impose the constrain on the constants $s_1 = -30 + 3s_2$, which is consistent with the evaluation of the second Chern class on J_i .

3.4 Involution symmetry at higher genus

The higher genus amplitudes are defined recursively from the holomorphic anomaly equations [7]

$$\partial_{\bar{i}} F^{(g)} = \frac{1}{2} \bar{C}_{\bar{i}}^{kl} \left(D_k D_l F^{(g-1)} + \sum_{r=1}^{g-1} D_k F^{(r)} D_l F^{(g-r)} \right). \quad (3.19)$$

The $F^{(g)}$ can be integrated using an anholomorphic potential S for $\bar{C}_{\bar{i}}^{jk}$, whose existence is consequence of special geometry or more generally the tt^* geometry [7]. One has

$$\partial_{\bar{i}} S = G_{\bar{i}j} S^j, \quad \partial_{\bar{i}} S^j = G_{\bar{i}k} S^{jk}, \quad \partial_{\bar{i}} S^{ij} = \bar{C}_{\bar{i}}^{ij}. \quad (3.20)$$

The $F^{(g)}$ were obtained in [7] either using the special geometry commutator $[D_i, \partial_{\bar{j}}]_k^l = G_{k\bar{j}} \delta_i^l + G_{i\bar{j}} \delta_k^l - C_{ikn} \bar{C}_{\bar{j}}^{nl}$ and partial integration or a Feynman graph expansion of a master functional, which fulfills a heat equation type of equation. The most vantage point of view is to find directly a solution for this master integral, which we achieve at least iteratively in the base modulus.

3.4.1 The propagators and rigid special geometry

It was shown in [74] as a consequence of special geometry that the covariant derivatives D_i in (3.19) close on anholomorphic generators, that are closely related to the propagators, up to rational functions of \underline{z} . We call the corresponding ring generated by these anholomorphic generators over the rational functions in \underline{z} the BCOV ring.

This principal structure of the BCOV ring can be understood by the analogy to the ring of almost holomorphic forms $\bigoplus_k \widehat{\mathcal{M}}_k(\Gamma_0) = \mathbb{C}[\hat{E}_2, E_4, E_6]$ of $\Gamma_0 = \text{SL}(2, \mathbb{Z})$ (or congruence subgroups thereof) that is graded by the weight k . Let us denote the modular transformation $\tau \mapsto \tau_\gamma = \frac{a\tau+b}{c\tau+d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z})$. For $k \in 2\mathbb{N}_+$ the normalized Eisenstein series E_k of modular weight are defined as [9]

$$E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{1}{(m\tau + n)^k} = 1 + \frac{(2\pi i)^k}{(k-1)! \zeta(k)} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad (3.21)$$

where the last equal sign holds straightforwardly only for $k > 2$. Here $\sigma_k(n)$ is the sum of the k -th power of the positive divisors of n and $\zeta(k) = \sum_{r \geq 0} \frac{1}{r^k}$ with

$$\zeta(2k) = -\frac{(2\pi i)^{2k} B_{2k}}{4k(2k-1)!}. \quad (3.22)$$

The Bernoulli numbers B_k $k \neq 1$ are defined by the generating function

$$\sum_{m=0}^{\infty} \frac{B_m x^m}{m!} := \frac{x}{e^x - 1}. \quad (3.23)$$

If the sum in (3.21) converges $k > 2$ it is obvious that $E_k(\tau_\gamma) = (c\tau + d)^k E_k(\tau)$ transforms as a weight k form under modular transformations. For $k = 2$ the second equal sign in (3.21) can be viewed as a regularization of the sum on the right [9]. In this case the modular transformation is broken to

$$E_2(\tau_\gamma) = (c\tau + d)^2 E_2(\tau) - \frac{6i}{\pi} c(c\tau + d). \quad (3.24)$$

Since $\frac{1}{\text{Im}(\tau_\gamma)} = \frac{(c\tau + d)^2}{\text{Im}(\tau)} - 2ic(c\tau + d) = \frac{|c\tau + d|^2}{\text{Im}(\tau)}$ one can define an almost holomorphic object

$$\hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi \text{Im}(\tau)}, \quad (3.25)$$

which transforms as a modular form of weight 2.

In the solution of the holomorphic anomaly equation $\hat{E}_2(\tau)$ plays the rôle of the propagator and the Maass derivative $D_\tau : \hat{\mathcal{M}}_k \rightarrow \hat{\mathcal{M}}_{k+2}$

$$D_\tau f_k = \left(d_\tau - \frac{k}{4\pi \text{Im}(\tau)} \right) f_k \quad (3.26)$$

on weight k forms, with $d_\tau = \frac{1}{2\pi i} \frac{d}{d\tau}$, plays the role of the covariant derivative D_i . The an-holomorphic version of the Ramanujan identities

$$D_\tau \hat{E}_2 = \frac{1}{12} (\hat{E}_2^2 - E_4), \quad D_\tau \hat{E}_4 = \frac{1}{3} (\hat{E}_2 E_4 - E_6), \quad D_\tau \hat{E}_6 = \frac{1}{2} (\hat{E}_2 E_6 - E_4^2), \quad (3.27)$$

show that it closes on $\bigoplus_k \widehat{\mathcal{M}}_k(\Gamma_0)$. Note that by the isomorphism ([48] Prop. 1) of the ring $\bigoplus_k \widehat{\mathcal{M}}_k(\Gamma_0)$ and the one quasi modular forms $\bigoplus_k \tilde{\mathcal{M}}_k(\Gamma_0) = \mathbb{C}[E_2, E_4, E_6]$ one can take the holomorphic limit in (3.27) by replacing $\hat{E}_2 \rightarrow E_2$ and $D_\tau \rightarrow d_\tau$ without losing information about the ring structure.

The relation between the BCOV ring of the $X_{18}(1, 1, 1, 6, 9)$ Calabi-Yau manifold and the ring of almost holomorphic forms is more than an analogy. In the limit of large fibre discussed in section 3.2.1 the special Kähler structure of supergravity reduces to local rigid special geometry and the BCOV ring of $X_{18}(1, 1, 1, 6, 9)$ reduces to $\bigoplus_k \widehat{\mathcal{M}}_k(\Gamma_0(3)) = \mathbb{C}[\hat{E}_2, G_2, G_4, G_6]$ [1].

In the limit of large base the relevant ring becomes $\bigoplus_k \widehat{\mathcal{M}}_k(\Gamma_0)$ and the structure of the holomorphic anomaly equation can be even more neatly combined into the ring of weak Jacobi forms, which include all powers of the topological string coupling as we will discuss in section 4.

3.4.2 Projective special Kähler manifolds

The form (1.5) suggests that the all genus amplitude in $K3 \times T2$ is related to a modular form associated to a degenerate Riemann surface of genus 3. In this section we recall some structure of projective special Kähler manifolds, see [21] for a modern review, that relates for N=2 compactifications of type II string the complex structure moduli of Calabi-Yau 3 folds to a family of Riemann surfaces of genus $h_{21} + 1$.

Of course we could first address the simpler question, how the objects of rigid special Kähler geometry described in the last section extend to the moduli space of Riemann

surfaces of higher genus. However this is done in [56], with the result that a holomorphic analog of E_2 does not exist, but a meromorphic one does.

In the formalism of projective special Kähler manifolds the formal description of the formulae for the propagators and the closing of the covariant derivatives on the propagators, i.e the analogs of the relations (3.27), (B.2), as well as the recursive integration w.r.t. to the anholomorphic generators all simplify relative to the formalism [74] and resemble closely the formalism in the rigid case [25].

However the price one has to pay is a very complicated anholomorphic embedding of the complex moduli space of the Calabi-Yau manifold including the string coupling λ , that is related to the projective scaling, into the Siegel upper half plane $\mathbb{H}_{h_{21}+1}$ of Riemann surfaces of genus $h_{21} + 1$, which is given by [16]

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i \frac{\text{Im}(F_{IL})X^L \text{Im}(F_{JK})X^K}{\text{Im}(F_{KL})X^K X^L}, \tag{3.28}$$

see [62] for a concise review.¹⁸ Here upper case letters refer to the projective coordinates of the complex moduli space given by a set of A -cycle periods $X^I(\underline{z})$, $I = 0, \dots, h_{21}(W)$ in the integral symplectic basis discussed in section 2.3. $F_{IJ}(\underline{z}) = \frac{\partial^2 F}{\partial X^I \partial X^J}$ with $F(\underline{z}) = (X^0)^2 \mathcal{F}^{(0)}$ the prepotential in homogeneous coordinates. The positivity, or rather negativity in the supergravity conventions $\text{Im}(\mathcal{N}_{IJ}) < 0$, is discussed in [72]. It is physically enforced by the condition that kinetic terms in vector multiplets and the gauge kinetic terms $\mathcal{L} = -\frac{1}{4}g_{IJ}^{-2}F_{nm}^I F^{J\ nm} - \frac{\theta_{IJ}}{32\pi^2}F^I \tilde{F}^J + \dots$ of the graviphoton F_{nm}^0 and the vector bosons F_{nm}^i , $i = 1, \dots, h_{21}$ in vector multiplets with $g_{IJ}^{-2} = \frac{i}{4}(\mathcal{N}_{IJ} - \bar{\mathcal{N}}_{IJ})$ and $\theta_{IJ} = 2\pi^2(\mathcal{N}_{IJ} + \bar{\mathcal{N}}_{IJ})$ must be positive definite.

Note that \mathcal{N}_{IJ} transforms under real or integer symplectic transformations, w.r.t. to (2.14)

$$\begin{pmatrix} X'^A \\ F'_A \end{pmatrix} = \begin{pmatrix} D & B \\ B & A \end{pmatrix} \begin{pmatrix} X^A \\ F_A \end{pmatrix}, \tag{3.29}$$

with $A^T C = C^T A$, $B^T D = D^T B$ and $A^T D - C^T B = 1$, such as the monodromies discussed in section 3.2, in the canonical way

$$\mathcal{N}' = (A\mathcal{N} + B)(C\mathcal{N} + D)^{-1}. \tag{3.30}$$

This suggests that the special monodromy family of Riemann surface of genus 3 and a better understanding of the map (3.28) might be essential to understand whether the formulae (1.1), (1.2) come from an underlying modular object as in the $N = 4$ case.

3.4.3 Involution symmetry on the propagators

For the concrete calculation we follow [74] and work in the in-homogenous coordinates and use the BCOV propagators with a shift by derivatives of the Kähler potential as the generators [2]. We denote the propagators after this shift (B.1) as S^{ij}, S^i, S . The definition of the propagators involves a choice of rational functions in the moduli \underline{z} [7, 52], which is

¹⁸We would like to thank Jan Louis for a guidance to the literature.

not unique. It has to be accompanied by a consistent choice of rational functions of \underline{z} in the closing relations (B.2) [2] that generalize (3.27). The derivation of these rational functions for non-hypergeometric, but more interesting Apéry like Picard-Fuchs equations that arise in non-abelian gauged linear σ models requires a minimal choice that uses the properties of differential equations [46]. For the $X_{18}(1, 1, 1, 6, 9)$ model, we summarize two canonical choices and the derivation of the action under the involution symmetry in the appendix B.

Let us now state the action of the involution symmetry on the propagators. Suppose $f(z_1, z_2)$ is a rational function of z_1, z_2 , we denote with the tilde symbol $\tilde{f} = f(x_1, x_2)$, the transformed function by replacing the z_i 's in the arguments with x_i 's in the involution (3.6). The \tilde{f} can be considered as a function of either x_i 's or z_i 's using the transformation (3.6). For example under the involution, the three point functions (2.12) transform as a tensor except for a minus sign due to (3.5)

$$\tilde{C}_{ijk} = -\frac{\partial z_l}{\partial x_i} \frac{\partial z_m}{\partial x_j} \frac{\partial z_n}{\partial x_k} C_{lmn}. \tag{3.31}$$

We find in B that the action of the involution on the propagators has the following form

$$\tilde{S}^{ij} = -\frac{\partial x_i}{\partial z_k} \frac{\partial x_j}{\partial z_m} S^{km}, \quad \tilde{S}^i = -\frac{\partial x_i}{\partial z_k} S^k + f^i, \quad \tilde{S} = -S + f^0, \tag{3.32}$$

The precise form of the rational functions f_i, f_0 depends on the choice of the rational functions in the choice of the propagator as well as in (B.2). Fixing this choice as in (B.3) [2] we find the solution

$$\begin{aligned} f^1 &= \frac{5z_1z_2(1 - 1296z_1 + 559872z_1^2)}{12(1 - 432z_1)^2}, \\ f^2 &= \frac{3732480z_1^3z_2[1 - 1296z_1 + 559872z_1^2 - 80621568z_1^3(1 + 27z_2)]}{(1 - 432z_1)^6} \\ f^0 &= 5[45z_2 - 67 + 1296z_1(67 - 135z_2) - 559872z_1^2(67 - 315z_2) \\ &\quad + 80621568z_1^3(67 - 630z_2)] / (23328(1 - 432z_1)^3). \end{aligned} \tag{3.33}$$

3.4.4 Involution symmetry on the higher genus amplitudes

The topological string amplitudes transform under the involution symmetry (3.6) by replacing the propagators with the transformed propagators (3.32) and the complex structure coordinates z_i 's with x_i 's. We would like to understand how the transformed amplitude is related to the original amplitude and how many constraints the symmetry implies.

At higher genus $g \geq 2$, the BCOV holomorphic anomaly equation [7] can be written as partial derivatives with respect to the an-holomorphic generators [2, 25, 74]. This is another way to see the polynomial structure of the amplitudes in the an-holomorphic generators and more importantly to obtain them by direct integration in an algorithm that grows polynomial in complexity in g . Assuming the algebraic independence of K_i 's, we can write the partial derivatives in terms of lower genus amplitudes. Here the K_i appear as in the covariant derivative of higher genus amplitude $D_i \mathcal{F}^{(g)} = \partial_i \mathcal{F}^{(g)} - (2g - 2)K_i \mathcal{F}^{(g)}$. We need

to be a little careful for the genus one case, as the K_i already appear at the ordinary derivative

$$\partial_i \mathcal{F}^{(1)} = \frac{1}{2} C_{ijk} S^{jk} - \left(\frac{\chi}{24} - 1 \right) K_i - \frac{1}{2} s_{ij}^j - \partial_i \left(\frac{1}{24} \sum_{k=1}^2 s_k \log z_k + \frac{1}{12} \sum_{a=1}^2 \log \Delta_a \right), \quad (3.34)$$

where we have computed the derivative of Kahler metric in terms of Christoffel connections and use the first equation in (B.2). The equations for partial derivative in the multi moduli case have been worked out in [2]

$$\begin{aligned} \frac{\partial \mathcal{F}^{(g)}}{\partial S^{ij}} &= \frac{1}{2} \partial_i (\partial_j' \mathcal{F}^{(g-1)}) + \frac{1}{2} (C_{ijl} S^{lk} - s_{ij}^k) \partial_k' \mathcal{F}^{(g-1)} + \frac{1}{2} (C_{ijk} S^k - h_{ij}) c_{g-1} \\ &\quad + \frac{1}{2} \sum_{h=1}^{g-1} \partial_i' \mathcal{F}^{(h)} \partial_j' \mathcal{F}^{(g-h)}, \\ \frac{\partial \mathcal{F}^{(g)}}{\partial S^i} &= (2g-3) \partial_i' \mathcal{F}^{(g-1)} + \sum_{h=1}^{g-1} c_h \partial_i' \mathcal{F}^{(g-h)}, \\ \frac{\partial \mathcal{F}^{(g)}}{\partial S} &= (2g-3) c_{g-1} + \sum_{h=1}^{g-1} c_h c_{g-h}, \end{aligned} \quad (3.35)$$

where the c_g is defined as

$$c_g = \begin{cases} \frac{\chi}{24} - 1, & g = 1; \\ (2g-2) \mathcal{F}^{(g)}, & g > 1. \end{cases} \quad (3.36)$$

We have also used the notation ∂' to denote

$$\partial_i' \mathcal{F}^{(g)} = \begin{cases} \partial_i \mathcal{F}^{(g)} + \left(\frac{\chi}{24} - 1 \right) K_i, & g = 1; \\ \partial_i \mathcal{F}^{(g)}, & g > 1, \end{cases} \quad (3.37)$$

i.e. on the right hand in (3.35), we use the formula (3.34) for $\partial_i \mathcal{F}^{(1)}$ omitting the $-\left(\frac{\chi}{24} - 1\right) K_i$ term.

Since the propagators are symmetric $S^{ij} = S^{ji}$, we can choose to use only the S^{ij} with $i \leq j$. In the case of $i \neq j$, the right hand side of first equation in (3.35) need to be multiplied by an extra factor of 2 to take account of the double contribution.

Let us make some remarks about (3.5). When we combine the amplitude with its involution transformation as in (3.5), we find the dependence on the propagators S^{ij}, S^i, S cancels with our proposal for the transformed propagators (3.32) and the shifts (3.33), which is a consistency check of the formulae (3.32), (3.33).

To make an ansatz for the holomorphic ambiguity $f^{(g)}$ at genus g , one considers a rational function of z_1, z_2 with a pole of $(\Delta_1 \Delta_2)^{2g-2}$. When we combine the correct holomorphic ambiguity with its transformation $\tilde{f}^{(g)} + (-1)^g f^{(g)}$ as in (3.5), we find the degree of the pole at discriminant $\Delta_1 \Delta_2$ is reduced for $g > 2$, and the pole completely cancels

for the genus $g = 2$ case. Meanwhile, a new pole at $\Delta_3 = 1 - 432z_1$ appears due to the involution transformation (3.6).¹⁹

The use of the involution symmetry greatly reduce the number of unknown constants in the holomorphic ambiguity. In the naive ansatz in [3], the holomorphic ambiguity has the form $\frac{p^{(g)}(z_1, z_2)}{(\Delta_1 \Delta_2)^{2g-2}}$, where $p^{(g)}(z_1, z_2)$ is a polynomial of degree $7(g-1)$ in z_1 and degree $5(g-1)$ in z_2 . So naively there are $(7g-6)(5g-4)$ unknown constants. For example, at genus 2 and 3, the naive ansatz has 48 and 165 unknown constants. We find the equation (3.5) imposes 36 and 125 conditions on the naive ansatz, so we are left with only 12 and 40 unknown constants to be fixed by boundary conditions. We present a detailed analysis on the reduction of unknowns in appendix C. Together with the conifold gap discussed in appendix E, this allows us to fix the ambiguity to genus 9.

4 Fiber modularity

In this section we reformulate the holomorphic anomaly equation expressing the fibre modularity [54] in a convenient way that allows us to summarize the structure in terms of even weak Jacobi forms. We exemplify the geometry setting with the elliptic fibration over \mathbb{F}_1 , which exhibits geometric limits that yields the BPS numbers the $K3$ fibre, the BPS numbers on an embedded $\frac{1}{2}K3$ and maybe most remarkably by blowing down the $\frac{1}{2}K3$, the compact elliptic fibration over \mathbb{P}^2 . The even weak Jacobi forms are hence a unified mathematical language to address heterotic type duality [63], the six dimensional superconformal theories [23, 55] and compact elliptic Calabi-Yau spaces.

4.1 The modular anomaly equation

In the simplest case the base B_{n-1} of the elliptic fibration is Fano (or almost Fano),²⁰ the degenerate elliptic fibers are only of Kodaira type I_1 and the elliptic fibre has only one holomorphic section and no rational sections. In the language of F -theory the complex structure is so generic that the $12-2n$ dimensional theory is completely higgsed.

The cohomology and the C.T.C. Wall data of M_3 is then fixed by the cohomology of the base and the fibre type [54]. One key feature of topological string amplitudes that comes from the described geometrical setting is the $\Gamma_F \in \text{SL}(2, \mathbb{Z})$ modularity of the amplitudes. Here Γ_F is the modular group acting projectively on the fibre modulus. In particular for one holomorphic section Γ_F is the full²¹ $\text{SL}(2, \mathbb{Z})$. In this case the $b_2(M_3)$ Kähler classes of M_3 split in $b_2(B_{n-1})$ and one class, which corresponds to the elliptic fibre. Let \tilde{t}_i be the complexified volumes of the curve classes $[C]_i$, $i = 1, \dots, b_2(B_2)$ in the base and τ the complexified volume of the elliptic fibre. One defines with $q = \exp(2\pi i \tau)$ a parameter, which becomes exponentially small, when the volume $\text{Im}(\tau)$ of the elliptic fibre becomes large and similarly $\tilde{Q}^\beta \equiv \prod_{\alpha=1}^{h_2(B_n)} \exp(2\pi i \tilde{t}^\alpha \beta_\alpha)$, with $\beta \in H_2(B_n, \mathbb{Z})$. Let a^i be

¹⁹The genus two and three holomorphic ambiguity formulae in [3] miss a constant $\frac{511}{144}$ and $-\frac{2105053}{1959552}$.

²⁰To construct already many concrete examples we may assume that B_{n-1} is a toric variety $\mathbb{P}_{\Delta_{n-1}^* B}$, i.e. specified by an $n-1$ dimensional reflexive polyhedron Δ_{n-1}^{*B} .

²¹It gets restricted to congruence subgroup $\Gamma_F \in \text{SL}(2, \mathbb{Z})$ if the fibration has more rational sections or multi sections.

the intersections of the curves $[C]_i$ with the canonical class K_B of the base. E.g. for B_2 a \mathbb{P}^2 one has $a^1 = 3$, while for B_2 the Hirzebruch surfaces \mathbb{F}_n one has $(a^1, a^2) = (2, 2 - n)$ etc, see [54] for the other toric bases. In order to make the $\text{SL}(2, \mathbb{Z})$ duality in the fibre direction more manifest it is convenient to redefine

$$t^i = \tilde{t}^i + \frac{\tau a^i}{2}, \tag{4.1}$$

so that the base moduli are summarized in $Q^\beta \equiv \prod_{\alpha=1}^{h_2(B_n)} \exp(2\pi i t^\alpha \beta_\alpha)$.

We can write an expansion of the disconnected topological string amplitudes also known as the free energy in the large volume limit

$$F(\tau, \lambda, Q) = \text{class}(\underline{t}) + \sum_{g=0}^{\infty} \lambda^{2g-2} F^{(g)}(\tau, Q). \tag{4.2}$$

The classical terms

$$\text{class}(\underline{t}) = \lambda^{-2} c_3(\underline{t}) + c_1(\underline{t}) \tag{4.3}$$

are at genus zero a cubic polynomial $c_3(\underline{t})$ in the Kähler parameter t that can be read from (2.16) without the constant term. Similarly $c_1(\underline{t})$ can be read from (3.18). Constant terms are proportional to the Euler number and are included via the 0 in the sum over $H_2(B_n, \mathbb{Z})$ in (4.4). This a formal expansion in the string coupling λ , but each genus g amplitude $F^{(g)}(\tau, Q)$ has a finite radius of convergence in the (q, Q) parameters, determined by the discriminant of the Picard-Fuchs equations. We decompose the genus g amplitudes in fibre q and base moduli Q as

$$F^{(g)}(\tau, Q) = \sum_{\beta \in H_2(B_n, \mathbb{Z})} F_\beta^{(g)}(\tau) Q^\beta. \tag{4.4}$$

There are two closely related properties partially characterizing the $F_\beta^{(g)}(q)$.

- The $F_\beta^{(g)}(\tau)$ are, up to a potential sign transformation, meromorphic modular forms in q of weight $2g - 2$. More precisely the $F_\beta^{(g)}(\tau)$ can be decomposed into

$$F_\beta^{(g)}(\tau) = Z_\beta^{\text{osc}}(\tau) P_\beta^{(g)}(\tau). \tag{4.5}$$

The first factor $Z_\beta^{\text{osc}}(\tau)$ is the partition function of

$$n(\beta) = 12 \sum_i \beta_i a^i \tag{4.6}$$

bosons, i.e. given by $Z_\beta^{\text{osc}} = \frac{1}{\eta^{n(\beta)}}$. Here $i = 1, \dots, h_2(B_{n-1})$. Since $n(\beta)/12$ can be an odd integer (4.5) can take an additional sign under modular transformations.

The second factor $P_\beta^{(g)}(\tau)$ is a polynomial in the generators of almost holomorphic modular forms with respect to the subgroup Γ_F of $\text{SL}(2, \mathbb{Z})$. In particular for our main examples with the $X_6(1, 2, 3)$ elliptic fibre type $\Gamma_F = \text{SL}(2, \mathbb{Z})$ and the generators are given by almost holomorphic weight 2 Eisenstein series \hat{E}_2 and the holomorphic

Eisenstein series E_4 and E_6 . The w weight and hence the degree of the polynomial $P_\beta^{(g)}(q)$ grow linearly with g and β as

$$w = 2g + 6 \sum_i \beta_i a^i - 2 . \tag{4.7}$$

- They satisfy a holomorphic anomaly equation, which for elliptic fibrations one section and only I_1 fibers is completely determined by topological data of the base²² [3, 54]

$$\frac{\partial F_\beta^{(g)}(\tau)}{\partial \hat{E}_2} = \frac{(-1)^{n+1}}{24} \left(\sum_{h=0}^g \sum_{\beta'+\beta''=\beta} (\beta' \cdot \beta'') F_{\beta'}^{(h)} F_{\beta''}^{(g-h)} + \beta \cdot (\beta - K_B) F_\beta^{(g-1)} \right) . \tag{4.8}$$

In particular n is the dimension of the base,²³ K_B is the canonical class of the base and the dot \cdot is the intersection form on the base. Z_β^{osc} can be factored out and (4.8) holds equivalently for $P_\beta^{(g)}$. Clearly (4.8) can determine the $P_\beta^{(g)}$ only up to a polynomial $P_\beta^{(g),hol}$, which depends only on the holomorphic generators of Γ_F , i.e. in particular not on \hat{E}_2 , and is called the holomorphic or modular ambiguity.

- We define a similar split expansion in fibre and base degree for the partition function $Z = \exp(F(\tau, \lambda, Q))$

$$Z = 1 + \sum_{\beta \in H_2(B_2, \mathbb{Z})} Z_\beta(\tau, \lambda) Q^\beta . \tag{4.9}$$

We identify now $\lambda = z$ with the elliptic parameter of the weak Jacobi forms²⁴ and assign modular weight -1 to it so that Z and $Z_\beta(\tau, \lambda)$ have modular weight zero. From (4.8) follows

$$\left(\partial_{\hat{E}_2} + \frac{\beta \cdot (\beta - K_B)}{24} z^2 \right) Z_\beta(\tau, z) = 0 \tag{4.10}$$

Basic properties of the weak Jacobi forms are summarized in the next section, see equation (4.26), yield then that the index of $Z_\beta(\tau, z)$ is $\frac{\beta \cdot (\beta - K_B)}{2}$.

As a corollary to the holomorphic anomaly equation (4.8) we have the following. If the base has itself a fibration $B_{n-2} \cdot F = 1$ and $F^2 = 0$, we can decompactify the fibre and consider the topological string on the local geometry of a rational fibration over the B_{n-2} .

4.2 The Hirzebruch surface \mathbb{F}_1 as base

In this section we present the base $\mathbb{F}_\alpha = \mathbb{F}_1$ as an explicit example and discuss the relation between the holomorphic anomaly equation in various dimensions. Our base has two divisor classes a minus $-\alpha$ curve which we call S and the fibre class F . The canonical class of the base is $K_B = 2S + (2 + \alpha)F$. One has the intersections $S^2 = -\alpha$, $FS = 1$ and $F^2 = 0$. We

²²A similar form of the holomorphic anomaly equation appeared for rational elliptic surfaces in [34] and for a compact 3-fold with an elliptic and a Enriques fibration in [25].

²³Since only one dimension was considered the alternating factor was absorbed in the definitions in [54]. Here we exhibit it to specialize to from the threefold to the surface case.

²⁴We also use the standard arguments of these forms (τ, z) .

denote $\beta = mF + nS$ and a short form $F_{mF+nS}^{(g)} = F_{m,n}^{(g)}$ so that our anomaly formula (4.8) reads

$$\frac{\partial F_{m,n}^{(g)}(q)}{\partial \hat{E}_2} = -\frac{1}{24} \left(\sum_{h=0}^g \sum_{k=0}^m \sum_{l=0}^n [k(n-l) + l(m-k) - \alpha l(n-l)] F_{k,l}^{(h)} F_{m-k,n-l}^{(g-h)} + [2m(n-1) - n(\alpha(n-1) + 2)] F_{m,n}^{(g-1)} \right). \quad (4.11)$$

This formula has interesting specialization, e.g. for the $\alpha = 1$ case we get

- In the $n = 0$ case corresponds to the corollary mentioned above. The two dimensional surface is a specially polarized $K3$. In this case we get a very simple recursion in the genus

$$\frac{\partial F_m^{(g)}(q)}{\partial \hat{E}_2} = -\frac{m}{12} F_m^{(g-1)}, \quad (4.12)$$

but no recursion in the base degree. An all genus result can be obtained for all m using the heterotic one loop result.

- The $m = 0$ case corresponds likewise to the corollary. The rational elliptic surface is the $\frac{1}{2}K3$. The solution has been discussed first in [55] and in the context of the (refined) holomorphic anomaly equation in [11, 30, 43]. A quiver description has been found in [51].
- $m = n$ corresponds to the blow down of the Hirzebruch surface \mathbb{F}_1 to \mathbb{P}^2 . For this geometry we get hence

$$\frac{\partial F_m^{(g)}(q)}{\partial \hat{E}_2} = -\frac{1}{24} \left(\sum_{h=0}^g \sum_{k=0}^m [k(m-k)] F_k^{(h)} F_{m-k}^{(g-h)} + [m(m-3)] F_m^{(g-1)} \right). \quad (4.13)$$

The solution will be discussed in section 4.6.

4.3 The ring of weak Jacobi forms

Jacobi forms $\varphi : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ depend on a modular parameter $\tau \in \mathbb{H}$ and an elliptic parameter $z \in \mathbb{C}$. They transform under the modular group [19]

$$\tau \mapsto \tau_\gamma = \frac{a\tau + b}{c\tau + d}, \quad z \mapsto z_\gamma = \frac{z}{c\tau + d} \quad \text{with} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2; \mathbb{Z}) \quad (4.14)$$

as

$$\varphi(\tau_\gamma, z_\gamma) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau + d}} \varphi(\tau, z) \quad (4.15)$$

and under quasi periodicity in the elliptic parameter as

$$\varphi(\tau, z + \lambda\tau + \mu) = e^{-2\pi i m(\lambda^2\tau + 2\lambda z)} \varphi(\tau, z), \quad \forall \lambda, \mu \in \mathbb{Z}. \quad (4.16)$$

Here $k \in \mathbb{Z}$ is called the *weight* and $m \in \mathbb{Z}_{>0}$ is called the *index* of the Jacobi form.

	$Q = E_4$	$R = E_6$	$A = \varphi_{-2,1}$	$B = \varphi_{0,1}$	φ_{d_B}	$Z_{d_B}(\tau, z)$
weight k :	4	6	-2	0	$16d_B$	0
index m :	0	0	1	1	$\frac{1}{3}d_B(d_B - 1)(d_B + 4)$	$\frac{d_B(d_B - 3)}{2}$

Table 1. $E_4, E_6, \varphi_{0,1}, \varphi_{-2,1}$ are generators of the ring of weak Jacobi forms with even weights, $\varphi_{d_B}(\tau, z)$ captures the all genus amplitudes for the $X_{18}(1, 1, 1, 6, 9)$ CY 3-fold and $Z_{d_B}(\tau, z)$ is the ratio of weak Jacobi forms.

The Jacobi forms have a Fourier expansion

$$\varphi(\tau, z) = \sum_{n,r} c(n,r)q^n y^r, \quad \text{where } q = e^{2\pi i\tau}, \quad y = e^{2\pi iz} \quad (4.17)$$

Because of the translation symmetry one has $c(n, r) =: C(4nm - r^2, r)$, which depends on r only modulo $2m$. For a holomorphic Jacobi form $c(n, r) = 0$ unless $4mn \geq r^2$, for cusp forms $c(n, r) = 0$ unless $4mn > r^2$, while for weak Jacobi forms one has only the condition $c(n, r) = 0$ unless $n \geq 0$.

According to [19],²⁵ a weak Jacobi form of given index m and even modular weight k is freely generated over the ring of modular forms of level one, i.e. polynomials in $R = E_4(\tau), P = E_6(\tau), A = \varphi_{0,1}(\tau, z), B = \varphi_{-2,1}(\tau, z)$ as

$$J_{k,m}^{weak} = \bigoplus_{j=0}^m M_{k+2j}(\text{SL}(2, \mathbb{Z})) \varphi_{-2,1}^j \varphi_{0,1}^{m-j}. \quad (4.18)$$

We summarize the weights and index of some important forms in the table 1. The first generators are defined in (3.21). Our conventions for the elliptic theta function and weak Jacobi forms A and B are the following²⁶

$$\begin{aligned} \theta_1(\tau, z) &= z \cdot \eta(\tau)^3 \exp \left[\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k)!} (iz)^{2k} E_{2k}(\tau) \right], \\ A = \varphi_{-2,1}(\tau, z) &= -\frac{\theta_1(\tau, z)^2}{\eta^6(\tau)}, \\ B = \varphi_{0,1}(\tau, z) &= 4 \left[\frac{\theta_2(\tau, z)^2}{\theta_2(0, \tau)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(0, \tau)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(0, \tau)^2} \right]. \end{aligned} \quad (4.20)$$

The weak Jacobi form of index $\varphi_{-2,1}$ has simple product form using the Jacobi triple product for θ_1 and $x = \left(2 \sin \left(\frac{\lambda}{2}\right)\right)^2 = -(y^{\frac{1}{2}} - y^{\frac{1}{2}})^2$

$$A = -x \prod_{n=1}^{\infty} \frac{(1 - yq^n)^2 (1 - y^{-1}q^n)^2}{(1 - q^n)^4}. \quad (4.21)$$

²⁵A review of the theory can be found in [15]. We try to follow the notation used there.

²⁶Our conventions for the θ functions associated to the spin structure on the torus are

$$\Theta \begin{bmatrix} a \\ b \end{bmatrix} (\tau, z) = \sum_{n \in \mathbb{Z}} e^{\pi i(n+a)^2 \tau + 2\pi i z(n+a) + 2\pi i b n} \quad (4.19)$$

and the Jacobi theta functions $\theta_1 = i\Theta \left[\frac{1}{2}\right], \theta_2 = \Theta \left[\frac{1}{0}\right], \theta_3 = \Theta \left[0\right]$ and $\theta_4 = \Theta \left[\frac{0}{\frac{1}{2}}\right]$.

and the weight zero index one form is one half of the elliptic genus of the $K3$

$$\chi(K3; q, y) = 2\varphi_{0,1}(\tau, z) = \left(2y + 20 + \frac{2}{y}\right) + \left(\frac{20}{y^2} - \frac{128}{y} + 216 - 128y + 20y^2\right)q + \mathcal{O}(q^2) \tag{4.22}$$

4.4 Weak Jacobi Forms and holomorphic anomaly equation

There is a ring isomorphism between the ring of quasi modular forms and the ring of almost holomorphic forms [48], which are crucial in the solution of the holomorphic anomaly equations in rank one Seiberg Witten theories [40] and local Calabi-Yau spaces [1].

In this section we will show by a related argument that the master holomorphic anomaly equation (4.10) for fibre modularity, which reads for the main example $X_{18}(1, 1, 1, 6, 9)$

$$\left(\partial_{\hat{E}_2} + \frac{d_B(d_B - 3)}{24}\lambda^2\right) Z_{d_B}(\tau, z) = 0, \tag{4.23}$$

is solved by a weak Jacobi Form of index $m = \frac{d_B(d_B-3)}{2}$.

Because of (4.15) and (3.24), given a weak Jacobi form $\varphi_{k,m}(\tau, z)$ one can always define an honest modular form of weight k as follows

$$\tilde{\varphi}_k(\tau, z) = e^{\frac{\pi^2}{3}mz^2E_2(\tau)}\varphi_{k,m}(\tau, z). \tag{4.24}$$

It follows that the weak Jacobi forms $\varphi_{k,m}(\tau, z)$ have a Taylor expansion in z with coefficients that are quasi-modular forms as [15, 19]

$$\begin{aligned} \varphi_{k,m} &= \xi_0(\tau) + \left(\frac{\xi_1(\tau)}{2} + \frac{m\xi'_0(\tau)}{k}\right)(2\pi iz)^2 \\ &+ \left(\frac{\xi_2(\tau)}{24} + \frac{m\xi'_1(\tau)}{2(k+2)} + \frac{m^2\xi''_0(\tau)}{2k(k+1)}\right)(2\pi iz)^4 + \mathcal{O}(z^6). \end{aligned} \tag{4.25}$$

Here the $' = d_\tau = \frac{d}{2\pi i d\tau}$ and $\xi_\nu \in \tilde{\mathcal{M}}_{k+2\nu}(\Gamma_0)$, i.e. the coefficients of can be expressed as polynomials of Eisenstein series $E_2(\tau)$, $E_4(\tau)$ and $E_6(\tau)$. Moreover from (4.24) one has

$$\left(\partial_{E_2} + \frac{m\lambda^2}{12}\right)\varphi_{k,m}(\tau, z) = 0. \tag{4.26}$$

Prop. 1 of [48]²⁷ implies the claim (4.23). Moreover by (3.25) we can write this as holomorphic anomaly equation

$$\left(2\pi i \text{Im}^2(\tau)\bar{\partial}_\tau - \frac{m\lambda^2}{4}\right)\hat{\varphi}_{k,m}(\tau, z) = 0. \tag{4.27}$$

We note as examples of (4.25) the first coefficients in the expansion of $\phi_{-2,1}(\tau, z)$ and $\phi_{0,1}(\tau, z)$ are

$$\begin{aligned} \varphi_{-2,1}(\tau, z) &= -z^2 + \frac{E_2z^4}{12} + \frac{-5E_2^2 + E_4}{1440}z^6 + \frac{35E_2^3 - 21E_2E_4 + 4E_6}{362880}z^8 + \mathcal{O}(z^{10}), \\ \varphi_{0,1}(\tau, z) &= 12 - E_2z^2 + \frac{E_2^2 + E_4}{24}z^4 + \frac{-5E_2^3 - 15E_2E_4 + 8E_6}{4320}z^6 + \mathcal{O}(z^8). \end{aligned} \tag{4.28}$$

²⁷See also <http://people.mpim-bonn.mpg.de/zagier/>.

Therefore $\phi_{-2,1}(\tau, z)$ and $\phi_{0,1}(\tau, z)$ can be thought of as quasi-modular forms and we can see that they satisfy the modular anomaly equation

$$\partial_{E_2}\varphi_{-2,1}(\tau, z) = -\frac{z^2}{12}\varphi_{-2,1}(\tau, z), \quad \partial_{E_2}\varphi_{0,1}(\tau, z) = -\frac{z^2}{12}\varphi_{0,1}(\tau, z). \quad (4.29)$$

Let us finish the section with a comparison of (4.23), (4.27) with Witten's wave equation for the the topological string partition function that reads [73]

$$\left(\frac{\partial}{\partial(t')^{\bar{a}}} + \frac{i}{2}\lambda^2 C_{\bar{a}\bar{b}\bar{c}} g^{\bar{b}\bar{b}} g^{c\bar{c}} \frac{D}{Dt^{\bar{b}}} \frac{D}{Dt^c} \right) Z(\lambda, \tau, t_B) = 0. \quad (4.30)$$

If we apply this equation to Z defined in (4.9) with $(t')^{\bar{a}} = \bar{\tau}$ and $Q^\beta = e^{2\pi i d_B t_B}$, we get in the large base because of the special form of the intersection matrix of elliptically fibered Calabi-Yau 3 folds only derivatives in the base direction t_B for $t^{\bar{b}}$ and t^c . Identifying λ with z we see already that the index will grow quadratically in d_B . A more detailed analysis as in [54] shows also the shift by K_B so that the large base limit of (4.30) becomes equivalent to all equations (4.10).

4.5 Exact formulae for base degree zero

For base degree $d_B = 0$, since the BPS numbers $n_{d_E, 0}^{(g)} = 0$ for genus $g \geq 2$, we get from (5.1) the formula

$$P_0^{(g)} = -\chi \frac{B_{2g}}{2g(2g-2)!} \left(\frac{B_{2g-2}}{2(2g-2)} - \sum_{d=1}^{\infty} \text{Li}_{3-2g}(q^d) \right), \quad g \geq 2, \quad (4.31)$$

where χ is the Euler number. This formula holds not just for the cases of elliptic fibrations with only I_1 Kodaira fibers [54], whose toric description is reviewed in section 2, but also for the case discussed in [28] with higher Kodaira fibre and non-abelian gauge symmetries.

We recognize in the multi-cover formula (4.31) the well-known formula for Eisenstein series

$$P_0^{(g)} = -\chi \frac{B_{2g} B_{2g-2}}{8g(g-1)(2g-2)!} E_{2g-2}(q), \quad g \geq 2. \quad (4.32)$$

So we see for $\chi = -540$ that $P_0^{(2)} = -\frac{3}{32} E_2$, and in general $P_0^{(g)}$ is a $\text{SL}(2, \mathbb{Z})$ modular form of weight $2g - 2$ for $g > 2$.

For the $d_B = 0$ genus one amplitude for our case, we can compute the base degree zero amplitude $P_0^{(1)}$ from the formula (3.17) and modularity formulas proven in [39]. Taking $z_2 \rightarrow 0$ of theses, we compute the determinant of Kähler metric as $\det(G) \sim (\partial_{z_1} t_E)(\partial_{z_2} t_B) \sim \frac{E_4(q_E)^{\frac{5}{2}}}{\eta(q_E)^{24}} \frac{1}{z_2}$, with $\partial_{z_1} t_E = \frac{E_4(q_E)^{\frac{5}{2}}}{\eta(q_E)^{24}} + \mathcal{O}(z_2)$. We can hence write the genus one amplitude in the $z_2 \rightarrow 0$ limit as

$$\mathcal{F}^{(1)} = -\frac{25}{4} \log(E_4) - \frac{1}{2} \log \left(\frac{E_4(q_E)^{\frac{5}{2}}}{\eta(q_E)^{24}} \right) - \frac{19}{4} \log(z_1) - \frac{3}{2} \log(z_2) - \frac{1}{12} \log(\Delta_1 \Delta_2) + \mathcal{O}(z_2). \quad (4.33)$$

A careful calculation show that

$$P_0^{(1)} = -48 \log(\eta(q_E)) - \frac{3}{2} \log(q_B q_E^{\frac{3}{2}}). \tag{4.34}$$

For the $d_B = 0$ genus the zero amplitude $P_0^{(0)}$ has formally weight -2 . We therefore consider the three point coupling which is the well defined observable after fixing the conformal Killing symmetries on the world-sheet. The classical contribution is a cubic polynomial of the flat coordinates (2.16), which encode the classical triple intersection numbers. Similar to genus one formula (4.34), we should shift the base flat coordinate according to (4.1) and find

$$\partial_{t_E}^3 P_0^{(0)} = \frac{9}{4} E_4(q_E), \tag{4.35}$$

which confirms the effective modular weight of -2 since each derivative increases modular weight by 2. The modularity at genus zero in this limit is a consequence of (3.13) and equation (4.35) has been observed in [12].

4.6 Exact formulae for higher base degrees

Our main claim is that the all genus partition function for the topological string elliptic fibration over \mathbb{P}^2 realized e.g. as degree 18 hypersurface in the weighted projective space is given by

$$Z_{d_B}(\tau, z) = \frac{\varphi_{d_B}(\tau, z)}{\eta(\tau)^{36d_B} \prod_{s=1}^{d_B} \varphi_{-2,1}(\tau, sz)}, \tag{4.36}$$

where φ_{d_B} is an weak Jacobi form of weight $k = 16d_B$ and index $m = \frac{d_B}{3}(d_B - 1)(d_B + 4)$. Scaling z in $\varphi_{k,m}(\tau, z) \in J_{k,m}$ by s corresponds to a Hecke-like operator $U_s : J_{k,m}^* \rightarrow J_{k,s^2m}^*$. The weights and indices of the relevant weak Jacobi forms are summarized in table 1. Since

$$\dim M_k(\text{SL}(2, \mathbb{Z})) = \begin{cases} [k/12] + 1 & \text{if } k \not\equiv 2 \pmod{12} \\ [k/12] & \text{if } k \equiv 2 \pmod{12} \end{cases} \tag{4.37}$$

the number of coefficients in $\varphi_{d_B}^{ts}$ grows like $\frac{d_B^6}{108}$ for large d_B . The first few numbers for $d_b = 1, 2, \dots$ are 2, 17, 84, 278, 737, 1692, 3501, \dots

For the case of $d_B = 1$ the two coefficients are fixed by two genus 0 BPS numbers

$$\varphi_1 = -\frac{Q(31Q^3 + 113R^2)}{48}. \tag{4.38}$$

This determines for base degree 1 all genus BPS invariants by (4.36), (4.9), (4.5) and the multi-covering formula (5.1). Up to $g = 6$ and $d_E = 6$ we list them in the table (2). We notice that they match all the numbers that have been obtained to genus 8 using the direct integration of the holomorphic anomaly condition, the involution symmetry and the conifold gap condition. A detailed discussion on the perfect matching with the accessible enumerative invariants can be found in section 5.2.1

$g \backslash d_E$	$d_E = 0$	1	2	3	4	5	6
$g = 0$	3	-1080	143370	204071184	21772947555	1076518252152	33381348217290
1	0	-6	2142	-280284	-408993990	-44771454090	-2285308753398
2	0	0	9	-3192	412965	614459160	68590330119
3	0	0	0	-12	4230	-541440	-820457286
4	0	0	0	0	15	-5256	665745
5	0	0	0	0	0	-18	6270
6	0	0	0	0	0	0	21

Table 2. Some BPS invariants $n_{(d_E,1)}^g$ for base degree $d_B = 1$ and $g, d_E \leq 6$ as determined by (4.38) for all g, d_E .

Note that (4.36) implies Hilbert-scheme like infinite product formulae for the generating functions. E.g. for $d_B = 1$ one gets

$$Z_1(\tau, z) = \left[\frac{1}{2i \sin(\frac{z}{2}) \eta(\tau)^{18}} \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - e^{iz} q^n)(1 - e^{-iz} q^n)} \right]^2 \varphi_1. \quad (4.39)$$

where B_{2k} are the Bernoulli numbers. This infinite product formula alone eliminates the subspace $V_{\pm}^{(1, [\frac{g-1}{3}], 1)}$ in the holomorphic ambiguity (E.8) and allows already to topological string amplitudes up to $g \leq 18$ for this model.

For the $d_B = 2$ case three of 17 coefficients can be already fixed by demanding that there is no pole $z^{-4} = \lambda^{-4}$ in $P_2(\tau, z)$. Note that this pole has to be canceled by the $(Z_1)^2$ contribution in $P_2(\tau, z)$. This explains the first term in (4.42). The vanishing bound which we will demonstrate in section 5.2.1 is a Castelnuovo-like criterium, namely that

$$n_{d_E,2}^g = 0, \quad \text{for } d_E \geq 5, g \geq 2d_E - 3 \quad (4.40)$$

fixes eleven other constants and implies all vanishing on the right from the edge. The actual nonzero values of the BPS numbers on the generic part of the edge for $d_E \geq 5$

$$n_{d_E,2}^{2d_E-4} = 12 - 6d_E \quad (4.41)$$

are then calculated geometrically and yield no further constraints. In order to fix the remaining three constants one needs the information of any three nonzero numbers in the same row or the same column way from the numbers (4.41). Notice that three numbers $n_{d_E,2}^{2d_E-5}$, $d_E = 5 + i$, $i = 0, 1, 2$ do yield only one constraint. Random patterns of nonzero numbers away from (4.41) will lead in general to independent equations. The result e.g. fixed from three genus zero numbers gives

$$\begin{aligned} \varphi_2 = & \frac{B^4 Q^2 (31Q^3 + 113R^2)^2}{23887872} + \frac{1}{1146617856} [2507892B^3 A Q^7 R + 9070872B^3 A Q^4 R^3 \\ & + 2355828B^3 A Q R^5 + 36469B^2 A^2 Q^9 + 764613B^2 A^2 Q^6 R^2 - 823017B^2 A^2 Q^3 R^4 \\ & + 21935B^2 A^2 R^6 - 9004644BA^3 Q^8 R - 30250296BA^3 Q^5 R^3 - 6530148BA^3 Q^2 R^5 \\ & + 31A^4 Q^{10} + 5986623A^4 Q^7 R^2 + 19960101A^4 Q^4 R^4 + 4908413A^4 Q R^6], \quad (4.42) \end{aligned}$$

which predicts the BPS numbers in all genus and fibre classes for $d_B = 2$.

$g \backslash d_E$	$d_E = 0$	1	2	3	4	5	6
$g = 0$	-6	2700	-574560	74810520	-49933059660	7772494870800	31128163315047072
1	0	15	-8574	2126358	521856996	1122213103092	879831736511916
2	0	0	-36	20826	-5904756	-47646003780	-80065270602672
3	0	0	0	66	-45729	627574428	3776946955338
4	0	0	0	0	-132	-453960	-95306132778
5	0	0	0	0	0	-5031	1028427996
6	0	0	0	0	0	-18	-771642
7	0	0	0	0	0	0	-7224
8	0	0	0	0	0	0	-24

Table 3. Some BPS invariants for $n_{(d_E,2)}^g$.

For $d_B = 3$ the vanishing, given by

$$n_{d_E,3}^g = 0, \quad \text{for } d_E \geq 8, g \geq 3d_E - 10 \quad (4.43)$$

fixes 74 other constants. These conditions are not independent from the conditions that eliminate the z^{-6} and z^{-4} poles and implies all vanishing on the right from the edge. If we impose the conditions successively we get $\{25, 47, 58, 63, 67, 70, 71, 72, 73, \overline{74}\}$ conditions for $d_E = 8, 9, \dots$. The genus zero and one invariants yield 5 and 5 conditions fixing all coefficients of φ_3 .

As we will argue in section 5.2.1 the vanishing conditions for general d_B is

$$n_{d_E, d_B}^g = 0, \quad \text{for } d_E \geq 3d_B - 1, g \geq d_B d_E - \frac{1}{2}(3d_B^2 - d_B - 4). \quad (4.44)$$

Together with the data up to genus eight tabulated in appendix A, this allows to fix φ_4 with 20 and φ_5 with 2 nontrivial checks respectively.²⁸ The tables of base degree $d_B = 4$ and $d_B = 5$ to genus 27 and 41 respectively can be found likewise in the appendix. We appended the expression for φ_d , $d = 1, \dots, 5$ in a for algebraic programs readable form to this TeX file.

One can make further checks on the results based on considerations in section (5.2.1). For examples we see that for each d_B there is a critical value

$$(g^c(d_B) = \frac{1}{2}(3d_B^2 - d_B + 2), d_E^c(d_B) = 3d_B - 1) \quad (4.45)$$

in the (g, d_E) plane so that the BPS states on the line in the (g, d_E) takes the value

$$n_{d_E, d_B}^{d_B d_E - g^c(d_B) + 2} = (-1)^{\frac{2d_E d_B + d_B(d_B - 1)}{2}} 3(d_B d_E - (3d_B^2 + d_B - 6)/2) \text{ for } d_E \geq 3d_B - 1, \quad (4.46)$$

which can be confirmed as well as many other predictions by direct curve counting in section 5.2.1.

These formulas (4.38), (4.42) are valid for all genera, so we can consider their implications for Gopakumar-Vafa (GV) invariants at large genus. We use the Mathematica

²⁸We have checked the results now at genus nine, which gives 18 new independent checks for φ_4 and 20 for φ_5 . The data are available on request.

$g \backslash d_E$	0	1	2	3	4	5	6	7	8	9
0	27	-17280	5051970	-91(5)00	22(8)00	-42(10)68	40(12)60	-16(16)20	55(17)80	12(22)00
1	-10	4764	-1079298	15(5)86	-16(7)80	-33(9)68	12(12)88	-55(14)44	10(18)24	37(21)76
2	0	27	-16884	4768830	-81(5)36	28(8)63	67(10)51	45(13)02	36(16)60	54(19)78
3	0	0	-72	48036	-14(4)90	29(5)64	-79(9)12	-63(12)64	-94(15)98	-11(19)86
4	0	0	0	154	-110574	38(4)41	21(8)04	50(11)76	12(15)04	20(18)021
5	0	0	0	0	-306	247014	-25(6)52	-21(10)36	-11(14)06	-30(17)40
6	0	0	0	0	0	612	1401468	50(8)62	74(12)04	34(16)43
7	0	0	0	0	0	0	17386	-49(6)70	-35(11)70	-32(15)78
8	0	0	0	0	0	0	63	3396663	11(10)79	23(14)91
9	0	0	0	0	0	0	0	32418	-21(8)14	-13(13)22
10	0	0	0	0	0	0	0	108	18(6)07	57(11)85
11	0	0	0	0	0	0	0	0	-1151442	-16(10)36
12	0	0	0	0	0	0	0	0	-10917	29(8)80
13	0	0	0	0	0	0	0	0	-36	-24(6)86
14	0	0	0	0	0	0	0	0	0	1458792
15	0	0	0	0	0	0	0	0	0	13770
16	0	0	0	0	0	0	0	0	0	45

Table 4. Some BPS invariants for $n_{(d_E,3)}^g$. To save space we only give the first and the last two significant digits and the number of omitted digits in brackets.

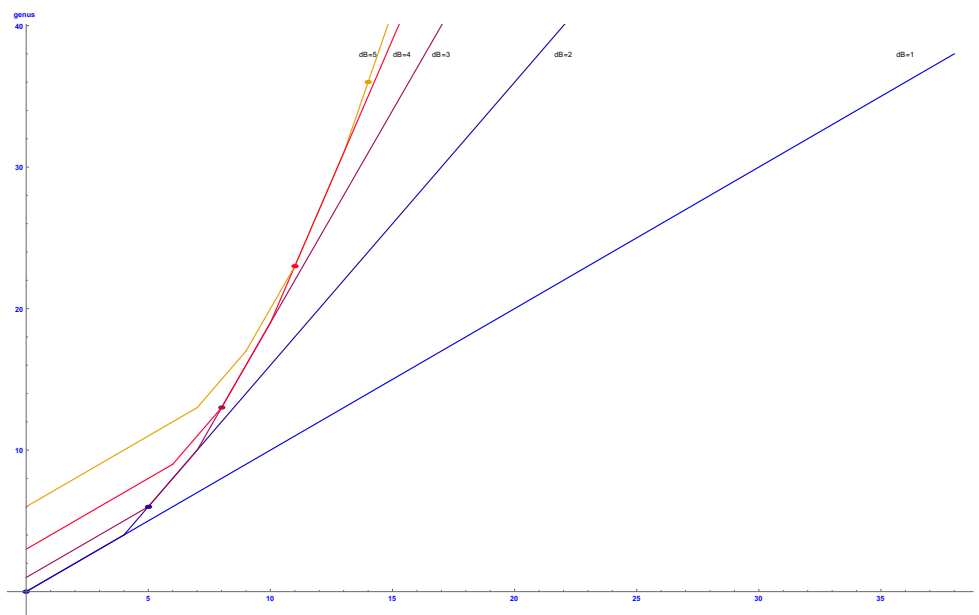


Figure 1. The figure shows the boundary of non-vanishing curves for the values of $d_B = 1, 2, 3, 4, 5$. The dot on each d_B line at $(g^c(d_B) = \frac{1}{2}(3d_B^2 - d_B + 2), d_E^c(d_B) = 3d_B - 1)$ indicates the value at which the boundary slope becomes generic and the numbers of BPS states on the line are given by (4.46).

program to compute the GV invariants to genus 8, and the results are listed in the tables in the appendix. A well-known feature of the GV invariants from algebraic geometric arguments [50] is that, for a given 2nd-homology class, in our case a pair of fiber and base degrees (d_E, d_B) , the GV invariant $n_{(d_E, d_B)}^g$ vanishes for sufficiently large genus g . The largest genus with non-vanishing GV invariant is called the top genus for the given degrees (d_E, d_B) . For example, from the GV tables in the appendix A we can see that the top genus for the degrees $(d_E, 1)$ is always d_E , while the top genus for $(d_E, 2)$ is d_E for the cases of $d_E \leq 4$, and is 6, 8 for $d_E = 5, 6$ respectively. The situation is summarized in figure 1 and the bounds are verified from algebraic geometry in section 5.2.

We consider whether such vanishing conditions help to fix the unknown coefficients in the ansatz of the weak Jacobi form. It turns out for the first case of $d_B = 1$, a general ansatz (4.36) ensures the vanishings of $n_{(d_E, 1)}^g = 0$ for $g > d_E$, therefore this does not help to fix the two coefficients in the numerator in the formula (4.38). To understand this structure, let us analyze the formula for $\phi_{-2,1}(\tau, z)$ in (4.20), which can be also written in the infinite product form. For comparing with GV invariants, we define the variable $x = (2 \sin(\frac{z}{2}))^2$, then we find

$$\phi_{-2,1}(\tau, z) = -x \prod_{n=1}^{\infty} \frac{(1 + xq^n - 2q^n + q^{2n})^2}{(1 - q^n)^4}. \tag{4.47}$$

So we see that excluding the total factor of x , in the power series expansion of the infinite product, the power of q is always no less than that of x . Using the series expansion $(1 + x)^{-2} = \sum_{n=0}^{\infty} (-1)^n (n + 1) x^n$, we find the inverse can be written as

$$\frac{1}{\phi_{-2,1}(\tau, z)} = -\frac{1}{x} \sum_{n=0}^{\infty} (-1)^n (n + 1) q^n [x^n + f_{n-1}(x)], \tag{4.48}$$

where $f_{n-1}(x)$ is a polynomial of x of degree $n - 1$. Topological string free energy in the $d_B = 1$ case has no multi-cover contributions from lower degrees, and according to the general ansatz (4.36) is simply $\phi_{-2,1}(\tau, z)^{-1}$ multiplying a modular form of weight 16 and the factor of $(\frac{q}{\eta(q)^{24}})^{\frac{3}{2}}$, which only increase the q power in the series expansion. Here the factor $q^{\frac{3}{2}}$ comes from the shift of base Kahler parameter such that the partition function has a nice modular form. From the well-known GV expansion of the topological free energy, the coefficient of $q^{d_E} x^{g-1}$ is exactly the GV invariant $n_{(d_E, 1)}^g$. Therefore we see from the expansion (4.48) that indeed the GV invariant $n_{(d_E, 1)}^g$ vanishes for $g > d_E$. Furthermore, using the formula (4.38) we can easily calculate the top genus numbers $n_{(g, 1)}^g = 3(-1)^g (g + 1)$. This agrees with the numbers in the tables of GV invariants and can be also derived from algebraic geometric arguments in section 5.2.1.

Now we consider the general cases of higher base degrees. Similar to the case of $\phi_{-2,1}(\tau, z)$ in (4.47), we can also expand $\phi_{0,1}(\tau, z)$ as power series of x, q

$$\phi_{0,1}(\tau, z) = -x(1 - 10xq + x^2q^2) + \sum_{n=0}^{\infty} q^n f_n(x), \tag{4.49}$$

where $f_n(x)$ symbolizes a degree n polynomial of x . Overall the difference of the powers of x and q in both the series $\phi_{-2,1}(\tau, z)$ and $\phi_{0,1}(\tau, z)$ is always no larger than 1. So we can consider the sub-family of ansatz with only $\phi_{-2,1}(\tau, z)$ in the denominator, namely

$$\sum_{k=-1}^{\frac{d_B(d_B-3)}{2}} \frac{f_{18d_B+2k}(E_4, E_6)}{\eta(\tau)^{36d_B}} \phi_{-2,1}(\tau, z)^k \phi_{0,1}(\tau, z)^{\frac{d_B(d_B-3)}{2}-k}, \quad (4.50)$$

where $f_{18d_B+2k}(E_4, E_6)$ symbolizes a modular form of weight $18d_B+2k$, which is a polynomials of E_4 and E_6 with many unknown coefficients, so that the total modular weight and index are the same as the more general ansatz (4.36) with the product $\prod_{k=1}^{d_B} \phi_{-2,1}(\tau, kz)$ in the denominator. From the power series of $\phi_{-2,1}(\tau, z)$ and $\phi_{0,1}(\tau, z)$ in x, q and including the factor of $q^{\frac{3d_B}{2}}$ from the shift of base Kahler parameter, we deduce that the sub-family of ansatz (4.50) contribute only to the GV invariants $n_{(d_E, d_B)}^g$ with $g \leq d_E + \frac{d_B(d_B-3)}{2} + 1$. On the other hand, from the algebro-geometric arguments in section 5.2.1, the top genus with non-vanishing GV invariants for the case of $d_E = 0$ is always $\frac{d_B(d_B-3)}{2} + 1$. For a given d_B , the top genus as a function of d_E is strictly monotonically increasing, i.e. the top genus for $(d_E + 1, d_B)$ is always strictly bigger than that of (d_E, d_B) . Therefore we deduce that the top genus with non-vanishing GV invariants for degrees (d_E, d_B) is always no less than $d_E + \frac{d_B(d_B-3)}{2} + 1$. Furthermore, when d_E is sufficiently large, from the algebraic geometric arguments, the top genus is given by a formula $d_B d_E + \frac{1}{2}(-3d_B^2 + d_B + 2)$, which is certainly much larger than $d_E + \frac{d_B(d_B-3)}{2} + 1$. We can choose a particular solution from the general ansatz (4.36), so that the GV invariants vanish at sufficiently large genus, then the addition of extra ansatz of the form (4.50) does not affect the conditions. So we conclude that the sub-family of ansatz (4.50) can not be fixed by vanishing GV conditions.

The reverse is also true, namely, any remaining ambiguities which can not be fixed by vanishing GV conditions are necessarily of the form (4.50). To see this, we note that $(2 \sin(\frac{kz}{2}))^2$ can be written as a polynomial of $x \equiv (2 \sin(\frac{z}{2}))^2$ of degree k , more explicitly

$$\left(2 \sin\left(\frac{kz}{2}\right)\right)^2 = x f_{k-1}(x), \quad (4.51)$$

where $f_{k-1}(x)$ is a degree $k-1$ polynomial, related to the Chebyshev polynomials. For example, for $k=2$, we have $(2 \sin z)^2 = x(4-x)$. The constant term $f_{k-1}(0)$ in the polynomial is always k^2 since we have $\frac{(2 \sin(\frac{kz}{2}))^2}{x} \sim k^2$ in the limit $x \sim z \sim 0$. When we expand $\frac{1}{\phi_{-2,1}(\tau, kz)}$ for $k > 1$ as power series of x, q , we see that the power of x has no upper bound for a given power of q , unlike the case of $k=1$. So if the factor $f_{k-1}(x)$ is not cancelled in the denominator, the ansatz would contribute to the GV invariants $n_{(d_E, d_B)}^g$ of arbitrarily large genus g for the given degrees (d_E, d_B) , and we can always fix them with vanishing GV invariants, even without the precise knowledge of the top genus. Furthermore, we require that the power of x in the generating function of the GV invariants is always no less than -1 , since the lowest genus is zero. Those ansatz that contribute to x^{-n} with $n > 1$ can be always fixed by this requirement. Overall we have at most one factor x in the denominator for those ansatz that are not affected by these considerations. Thus all ansatz except for those of the form (4.50) can be fixed in this way.

This is checked by actual calculations. One can easily count that the sub-family of ansatz (4.50) has 2, 3, 10, 26 unknown coefficients for base degrees $d_B = 1, 2, 3, 4$ respectively. After we impose the conditions that the generating function of GV invariants start from genus zero and vanish at sufficiently large genus for a given fiber degree, these are the indeed the remaining numbers of unfixed coefficients in the more general ansatz (4.36) with the denominator $\prod_{k=1}^{d_B} \phi_{-2,1}(\tau, kz)$.

We can also see the scaling of the top genus for a given base degree d_B and for large fiber degree d_E . We can expand more explicitly

$$\frac{1}{\phi_{-2,1}(\tau, kz)} = -\frac{1}{xf_{k-1}(x)} + \sum_{n=1}^{\infty} q^n g_{n-1}(xf_{k-1}(x)), \tag{4.52}$$

where $f_{k-1}(x)$ is the polynomial in (4.51) and g_{n-1} is a degree $n-1$ polynomial. There are also multi-cover contributions from lower degrees. For example, we can consider k -cover contributions from the base degree one formula. The denominator can be expanded as

$$\frac{1}{\phi_{-2,1}(k\tau, kz)} = -\frac{1}{xf_{k-1}(x)} + \sum_{n=1}^{\infty} q^{kn} g_{n-1}(xf_{k-1}(x)), \tag{4.53}$$

where the polynomials f_{k-1}, g_{n-1} are the same as in (4.52). We see that the first terms, which would have unbounded powers of x in series expansion, are the same. At the end, the polynomial $f_{k-1}(x)$ must be cancelled in the denominator, so that the resulting GV invariants have a top genus. In this way we see that the minimal requirements for the denominator of the ansatz at base degree d_B is indeed $\prod_{k=1}^{d_B} \phi_{-2,1}(\tau, kz)$, otherwise there will be non-vanishing GV invariants at arbitrarily large genus for fixed degrees from the multi-cover contributions. Since the factor $f_{k-1}(x)$ in the denominator of the first term in (4.52) is eventually canceled, this term contributes a finite polynomial of x in the generating function of GV invariants and does not affect the leading scaling behavior of the top genus for large fiber degree. At the base degree d_B , the leading contribution to the top genus comes from the second term in (4.52) for $k = d_B$ with large n , and scales like $g_{\text{top}} \sim d_B d_E$. However it is more tricky to determine the exact formula in this way for the cases of $d_B > 1$. We find that the exact top genus numbers are not universal for the general ansatz (4.36), and the top genus formula $d_B d_E + \frac{1}{2}(-3d_B^2 + d_B + 2)$ for large d_E derived from algebraic arguments only appears for the particular solution in topological string theory.

This approach can be combined with the B-model holomorphic anomaly approach to compute higher genus topological string amplitudes. Using the involution symmetry and the boundary conditions at the conifold point, we find that the exact formula at base degree d_B can provide sufficient boundary data to fix the B-model formula at genus $9(d_B + 1)$, valid for all base and fiber degrees, see the appendixes for more details. On the other hand, in order to fix the exact formula at base degree d_B , we need to fix the ansatz (4.50) with some non-vanishing GV invariants. In particular, consider the term in (4.50) with $k = \frac{d_B(d_B-3)}{2}$ in the sum. Since $\phi_{-2,1}(\tau, z) \sim x$ in the power series expansion of x, q , this term contributes only to and can be only fixed by topological string free energy or some non-vanishing GV invariants of genus no less than $\frac{d_B(d_B-3)}{2} + 1$. The contributions of the other

terms in (4.50) start from lower genus. On the other hand, the modular form coefficients in the ansatz (4.50) have different modular weights for different k , and their contributions can not cancel when we extract the contributions at a given genus by expanding for small z . Therefore if we only know the B-model formula at any genus no less than $\frac{d_B(d_B-3)}{2} + 1$, it is also already sufficient to fix the ansatz (4.50) at base degree d_B . This is checked by actual low base degrees calculations. Thus, as long as $9(d_B + 1) \geq \frac{(d_B+1)(d_B-2)}{2} + 1$, we can repeat this procedure to fix the exact formula with increasing base degrees. If no other obstacle arises, we foresee the eventual termination of this recursive process only at $d_B = 20$. In this way we can in principle determine the exact formula up to base degree $d_B = 20$ (for all genera and fiber degrees), and the topological string free energy up to genus 189 (for all base and fiber degrees). In practice we provide here the solution of the topological strings up to $d_B = 5$ for all genera and genus $g = 8$ for all degrees.

5 BPS invariants

After a technical recapitulating of the the physical definition of the BPS invariants in section 5.1, the main part of the section is devoted to the direct geometrical calculation of the unrefined BPS invariants or Gopakumar-Vafa invariants for the $X_{18}(1, 1, 1, 6, 9)$ Calabi-Yau 3-fold in section 5.2.

5.1 Physical definition of the BPS invariants

In this section we recall the definition of the integer BPS invariants from $\mathcal{F}(\lambda, \underline{t})$ or $\mathcal{Z}(\lambda, \underline{t})$. In view of the similarity of the all genus expansions discussed in section 4.6 and the product expansion of (1.4) based on the Borchers lifting of automorphic forms of $O(2, n, \mathbb{Z})$, see [8] and [59] the formulae (5.5) and (5.11) should give further hints how to characterize \mathcal{Z} . We comment on possible refinements in 5.1.2 in analogy to the E -string.

5.1.1 Unrefined BPS invariants

The unrefined BPS states multiplicities $n_\beta^g \in \mathbb{Z}$ are defined by their contribution to a Schwinger-Loop integral in $5d$ $N = 2$ supergravity theory [24]²⁹

$$\mathcal{F}(\lambda, \underline{t}) = \text{class}(\underline{t}) + \sum_{\beta \in H_2(M_3, \mathbb{Z})} \sum_{g=0}^{\infty} \sum_{m=1}^{\infty} \frac{n_\beta^g}{m} \left(2 \sin \left(\frac{m\lambda}{2} \right) \right)^{2g-2} q^{\beta m}. \quad (5.1)$$

Note that $q^\beta = \exp(2\pi i \sum_{\alpha=1}^{b_2(M_3)} t^\alpha \beta_\alpha)$ and $y^{\frac{m}{2}} - y^{-\frac{m}{2}} = 2i \sin \left(\frac{m\lambda}{2} \right)$. One can interpret β as BPS charge and $\sum_\alpha t^\alpha \beta_\alpha$ as the central charge mass term. In the 5-field theory the fugacity λ traces the left spin of the BPS states.

The $n_\beta^g \in \mathbb{Z}$ may be calculated by identifying the above expression to the perturbative topological string expansion on M

$$\mathcal{F}(\lambda, \underline{t}) = \text{class}(\underline{t}) + \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}^g(\underline{t}) = \sum_{g=0}^{\infty} \sum_{\beta \in H_2(M, \mathbb{Z})} \lambda^{2g-2} r_\beta^g q^\beta, \quad (5.2)$$

²⁹Caligraphic letters should refer to quantities in the holomorphic limit.

where λ is now the topological string coupling constant. The Gromov-Witten invariants $r_\beta^g \in \mathbb{Q}$ can be mathematically defined and in suitable cases calculated using localization. In this article we use mirror symmetry, modularity, the holomorphic anomaly equation and direct curve counting to obtain the $F^{(g)}(\underline{t})$. With

$$\sum_{m=1}^{\infty} \frac{1}{m} \frac{q^m}{\left(2 \sin\left(\frac{m\lambda}{2}\right)\right)^2} = \sum_{g=0}^{\infty} \lambda^{2g-2} (-1)^{g+1} \frac{B_{2g}}{2g(2g-2)!} \text{Li}_{3-2g}(q) \quad (5.3)$$

where $\text{Li}_n(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n}$ and the Bernoulli numbers are defined in (3.23). Using further $\zeta(-n) = -\frac{B_{n+1}}{n+1}$ for $n \in \mathbb{N}_+$ we get

$$F_g(\underline{t}) = (-1)^{g+1} n_0^0 \frac{|B_{2g} B_{2g-2}|}{2g(2g-2)(2g-2)!} + \mathcal{O}(q), \quad \text{for } n > 1, \quad (5.4)$$

which matches the constant map contribution [20] for $n_0^0 = -\frac{\chi}{2}$. Exponentiating (5.1) one gets [49, 52]

$$\mathcal{Z}(\lambda, \underline{t}) = \prod_{\beta} \left[\left(\prod_{k=1}^{\infty} (1 - y^k q^\beta)^{kn_0^0} \right) \prod_{g=1}^{\infty} \prod_{l=0}^{2g-2} (1 - y^{g-l-1} q^\beta)^{(-1)^{g+l} \binom{2g-2}{l} n_\beta^g} \right]. \quad (5.5)$$

5.1.2 Refined BPS invariants

Let us define as in [42] in analogy with (4.2), (4.4)

$$F = \sum_{g=0}^{\infty} (\epsilon_1 + \epsilon_2)^{2h} (\epsilon_1 \epsilon_2)^{g-1} F^{(g,h)}(q, Q). \quad (5.6)$$

We decompose the genus g amplitude as

$$F^{(g,h)} = \sum_{\beta \in H_2(B_n, \mathbb{Z})} F_\beta^{(g,h)}(q) Q^\beta. \quad (5.7)$$

then holomorphic anomaly equation for E-strings [43] becomes

$$\frac{\partial F_n^{(g,h)}}{\partial \hat{E}_2} = \frac{1}{24} \sum_{h_1=0}^h \sum_{g_1=0}^g \sum_{s=1}^{n-1} s(n-s) F_s^{(g_1, h_1)} F_{n-s}^{(g-g_1, h-h_1)} + \frac{n(n+1)}{24} F_n^{(g-1, h)} - \frac{n}{24} F_n^{(g, h-1)}. \quad (5.8)$$

Making an analogous fibre and base separation as in (4.9) of the E -string geometry $\frac{1}{2}K3$ surface with the rational elliptic fibration $\mathcal{E} \rightarrow \mathbb{P}^1$, one finds [30] for the massless E -string the analogous equation to (4.10), compare [28]

$$\left(\partial_{E_2} + \frac{1}{24} [\epsilon_1 \epsilon_2 (n^2 + n) - (\epsilon_1 + \epsilon_2)^2 n] \right) Z_n^{E-string}(\tau, \epsilon_1, \epsilon_2) = 0. \quad (5.9)$$

We note that in the Nekrasov-Shatashvili limit the solution will be a weak Jacobi form. Indeed the index grows linearly as in the $N = 4$ case discussed in the introduction. In the general case we expect as solution a meromorphic weak Jacobi theta function. The refined BPS invariants

$$N_{j_L, j_R}^\beta \in \mathbb{N}_0.$$

are labelled by $\beta \in H_2(M, \mathbb{Z})$ and j_L, j_R , the 5d left and right spin

$$j_L, j_R \in \frac{1}{2}\mathbb{Z}_{\geq 0} .$$

If we define

$$[j]_x = x^{2j} + x^{-2j+2} + \dots + x^{2j-2} + x^{2j}$$

then (5.1) generalizes with $u = e^{i\epsilon_1}$ and $v = e^{i\epsilon_2}$ to

$$\mathcal{F}(\epsilon_1, \epsilon_2, t) = \sum_{\beta \in \mathbb{Z}^r} \sum_{j_L, j_R} \sum_{m=1}^{\infty} \frac{(-1)^{2(j_L+j_R)} N_{\beta}^{j_L, j_R} [mj_L]_{(uv)} [mj_R]_{(\frac{u}{v})}}{m(u^{\frac{m}{2}} - u^{-\frac{m}{2}})(v^{\frac{m}{2}} - v^{-\frac{m}{2}})} q^{m\beta} , \quad (5.10)$$

while (5.5) generalizes to [14, 45]³⁰

$$\mathcal{Z} = \prod_{\beta} \prod_{j_{L/R}=0}^{\infty} \prod_{\substack{j_{L/R} \\ m_{L/R}=-j_{L/R}}} \prod_{m_1, m_2=1}^{\infty} \left(1 - \left(\frac{u}{v}\right)^{m_L} (uv)^{m_R} u^{(m_1-\frac{1}{2})} v^{(m_2-\frac{1}{2})} q^{\beta}\right)^{(-1)^{2(j_L+j_R)} N_{\beta}^{j_L, j_R}} , \quad (5.11)$$

For the E-string (4.39) generalizes to

$$\mathcal{Z}_1 = \sum_{j_L, j_R} \sum_{n_e=0}^{\infty} N_{j_L, j_R}^{n_e, 1} [j_L]_u [j_R]_v q^{n_e} = \frac{E_4(q)}{\eta(q)^8 \prod_{n=1}^{\infty} (1 - uvq^n)(1 - \frac{uq^n}{v})(1 - \frac{q^n}{uv})(1 - \frac{vq^n}{u})} . \quad (5.12)$$

This can easily be seen to be compatible with (5.8). Eqs. (5.8), (5.9) and (5.12) suggest that a refinement of (4.13), (4.10) and (4.39). We will discuss this further in [39].

5.2 Gopakumar-Vafa invariants from geometry

5.2.1 Geometry of curves

In this section, we describe the geometry of a number of families of curves in the $X_{18}(1, 1, 1, 6, 9)$ model, from which a good number of GV invariants can be extracted. For simplicity, we will simply refer to this model as X in this section.

We begin by reviewing some results and notation from [12], which will be used to describe the geometry.

The Calabi-Yau X is described as a blowup of a degree 18 weighted hypersurface in $\mathbb{P}(1, 1, 1, 6, 9)$ at its (unique) singular point $x_1 = x_2 = x_3 = 0$. As noted earlier, X has Hodge numbers $h^{1,1} = 2$ and $h^{2,1} = 272$, and Euler characteristic -540 .

The exceptional divisor E is isomorphic to \mathbb{P}^2 . Projection to the first three coordinates of the weighted projective space presents X as a Weierstrass elliptic fibration $\pi : X \rightarrow \mathbb{P}^2$. Let $L = \pi^{-1}(\ell)$ be the pullback of a line $\ell \subset \mathbb{P}^2$ to X , and let $H = 3L + E$. Then

³⁰See [14] for a mathematical definition of the N_{j_L, j_R}^{β} in the local case using the virtual Bialynicki-Birula decomposition of the moduli space Pandharipande-Thomas invariants.

$\pi^*(x_1)$, $\pi^*(x_2)$, and $\pi^*(x_3)$ are sections of L , $\pi^*(x_4)$ is a section of $2H$, and $\pi^*(x_5)$ is a section of $3H$.

For the benefit of those readers more familiar with the toric description, we refer to (2.3) where the two descriptions of the divisors are related: the toric divisor D_{x_0} is identified with E , the divisor classes of the toric divisors D_{x_i} are all equal to L for $i = 1, 2, 3$, the toric divisor D_x is in the class $2H$, and the toric divisor D_y is in the class $3H$.

The Kähler cone of X is generated by H and L . The triple intersections are

$$H^3 = 9, \quad H^2L = 3, \quad HL^2 = 1, \quad L^3 = 0. \quad (5.13)$$

Dually, the Mori cone is generated by the class of an elliptic fiber f and the class of a line $\tilde{\ell} \subset E \simeq \mathbb{P}^2$.

Now, let $C \subset X$ be a connected curve (not necessarily reduced or irreducible, but having no embedded points). The invariants of C are

$$d_E = C \cdot H, \quad d_B = C \cdot L, \quad g = p_a(C), \quad (5.14)$$

where the last is the arithmetic genus of C . The curves of fixed $(g; d_E, d_B)$ are parametrized by a Hilbert scheme, and we will be able to describe many of these as well as the corresponding GV invariants.

As a comment on the mathematical rigor of our calculations, we are actually studying PT moduli spaces and computing PT invariants, which are mathematically equivalent to the GV invariants invariants by [67]. However, we will frequently simplify the exposition by using the language of Hilbert schemes as in [50] and then describing corrections required by the use of stable pairs. The reader is referred to [67] for more details, and to [14] for an amplification of the comparison between the methods of [14] and [50].

We say that (d_E, d_B) is the *degree* of C , and sometimes use the notation C_{d_E, d_B} for a curve of degree (d_E, d_B) . A fiber f has degree $(1, 0)$ and a line $\tilde{\ell} \subset E$ has degree $(0, 1)$.

We are able to completely describe the moduli space of connected curves of degree (d_E, d_B) and genus g for various d_E , d_B , and g . In this section we describe the geometric principles that these descriptions are based on. In the next section, we will apply these principles to compute various moduli spaces with $d_B \leq 5$ and use this description to compute the associated Gopakumar-Vafa invariants, and make some comments about general d_B .

Lemma 1. *Let C have degree (d_E, d_B) . Then $\pi(C)$ is a plane curve of degree d_B , including multiplicity.*

Proof. We compute the degree of $\pi(C)$ as

$$\pi(C) \cdot \ell = C \cdot \pi^{-1}(\ell) = C \cdot L = d_B, \quad (5.15)$$

where $\ell \subset \mathbb{P}^2$ is a line. □

Corollary 1. *A curve of degree $(d_E, 0)$ is a union of d_E fibers (including multiplicity).*

Proof. By Lemma 1, $\pi(C)$ is a finite point set, hence C is a union of fibers. Since each fiber has degree $(1, 0)$, the result follows. □

Lemma 2. *If $d_E < 3d_B$, then for any curve C of degree (d_E, d_B) at least one component of C (with its reduced structure) is contained in E . In particular, if in addition C is irreducible, then $d_E = 0$ and $C_{\text{red}} \subset E$, where C_{red} is the reduced (multiplicity 1) structure on C .*

Proof. To see this, we first note that if a curve C' is not contained in E , then $C' \cdot E \geq 0$, as the intersection number is just a count of the intersection points of C' and E with multiplicity. Then we compute

$$C \cdot E = C \cdot (H - 3L) = d_E - 3d_B. \tag{5.16}$$

If this is negative, it follows immediately that some component C' of C must be contained in E . If in addition C is irreducible, it follows that $C_{\text{red}} \subset E$. Since $H \cdot E = 0$, it follows that $d_E = C \cdot H = 0$ as claimed. \square

Remark. If $C_{\text{red}} \subset E$, it is possible for C itself to not be contained in E due to thickenings in a direction transverse to E .

Proposition 1. *If C has degree $(0, d_B)$, then its genus satisfies*

$$g \leq \frac{1}{2} (d_B - 1) (d_B - 2),$$

with equality holding if and only if C is a plane curve of degree d_B after identifying E with \mathbb{P}^2 .

The second half of the proposition says that if we try to thicken components of C outside of E , the genus would be less than if we thicken inside E .

Proof. We have $C \cdot H = d_E = 0$. Since H is ample on the weighted hypersurface in $\mathbb{P}(1, 1, 1, 6, 9)$ before the blowup, the only way for the intersection $C \cdot H$ to be 0 is for C_{red} to be contained in E . Since any \mathbb{P}^2 in a Calabi-Yau threefold has a neighborhood which is isomorphic to a neighborhood of \mathbb{P}^2 inside local \mathbb{P}^2 , the result follows from the corresponding result for local \mathbb{P}^2 . \square

Remark. We will see in the next subsection that for each $d_B \geq 2$, non-reduced curves occur generically in moduli spaces of curves with particular d_E and g .

We now set out to find the largest possible genus of a connected curve of degree (d_E, d_B) . Part of our strategy in studying curves is to first study irreducible curves and then study how the components can glue together.

By Lemma 1, a curve C of degree (d_E, d_B) is contained in a surface $\pi^{-1}(D)$ where $D \subset E \simeq \mathbb{P}^2$ is a plane curve of degree less than or equal to d_B . The degree will be strictly less than d_B if and only if some component of $\pi(C)$ has multiplicity greater than 1.

By analogy with the case of curves in \mathbb{P}^3 , where the genus of a curve of degree d is maximized when the curve is contained in a plane, one might expect that in our situation the genus is maximized if the degree of D is one, i.e. when $C \subset \pi^{-1}(\ell)$ for some line $\ell \subset E$. This turns out to be the case for irreducible curves with $d_E > 0$.

Proposition 2. *Suppose C is an irreducible curve of degree (d_E, d_B) with $d_E > 0$. Then*

$$g \leq d_E d_B - \frac{1}{2} (3d_B^2 - d_B - 2),$$

with equality holding if and only if $C \subset \pi^{-1}(\ell)$ for some ℓ . In that case, $C \subset S$ is the zero locus of a section of $\mathcal{O}_S(d_E L + d_B E)$. The moduli space of all curves C (not necessarily irreducible) given by all ℓ and all sections of $\mathcal{O}_S(d_E L + d_B E)$ is a $\mathbb{P}^{d_E d_B - (1/2)(3d_B^2 + d_B - 4)}$ -bundle over \mathbb{P}^2 .

Part of the proof involves an analysis of certain curves in surfaces of the form $S = \pi^{-1}(\ell)$ which will be given below. Another part of the proof is analogous to the proof of the Castelnuovo bound for the genus of a curve of fixed degree in \mathbb{P}^3 . The proof itself is omitted.

We now describe curves in a smooth $S = \pi^{-1}(\ell)$.

Suppose that $S \in |L|$, i.e. $S = \pi^{-1}(\ell)$ for some line $\ell \subset \mathbb{P}^2$. Then for $C \subset S$, $\pi(C)$ is either a point p (in which case C is an elliptic fiber of π , hence completely understood), or $\pi(C) = \ell$. Conversely, if C is any curve in X for which $\pi(C) = \ell$, then $C \subset S = \pi^{-1}(\ell)$.

We now describe all divisors on S associated to line bundles on S which are restrictions of line bundles on X . We denote restrictions to S by a subscript, so that the basic divisor classes on S are E_S and L_S . For divisors D and D' on X we have for the intersection on S of their restrictions

$$D_S \cdot D'_S = D \cdot D' \cdot L, \tag{5.17}$$

where the intersection on the right hand side is taken in X .

From (5.17), (5.13), and $E = H - 3L$ we get

$$E_S^2 = -3, \quad E_S H_S = 0, \quad H_S^2 = 3, \quad H_S L_S = 1, \quad E_S L_S = 1, \quad L_S^2 = 0. \tag{5.18}$$

Let $C \subset S$ be in the class $[C] = d_E L_S + d_B E_S$. Then by (5.18) we see that $C \cdot L = d_B$ and $C \cdot H = d_E$, consistent with our earlier conventions. The moduli of such C in fixed S is a projective space of dimension depending on (d_E, d_B) of a genus also depending on (d_E, d_B) . We now turn to computing the genus and dimension.

For simplicity of exposition, we assume that S is smooth. The adjunction formula says

$$K_S = \mathcal{O}_S(L_S). \tag{5.19}$$

By the projection formula we get $H^0(S, K_S) = H^0(S, \mathcal{O}_S(L_S)) = H^0(\ell, \mathcal{O}_\ell(1))$ and so $p_g = \dim H^0(S, K_S) = 2$. Here we are using the language of classical algebraic geometry, where $p_g = \dim H^0(S, K_S)$ is the geometric genus of a complex surface.

It is not hard to see that $q := h^1(S, \mathcal{O}_S) = 0$. So

$$\chi(\mathcal{O}_S) = 1 - q + p_g = 3. \tag{5.20}$$

The genus is given by

$$g = \frac{1}{2} C \cdot (C + K_S) + 1 = \frac{1}{2} (d_E L_S + d_B E_S) \cdot ((d_E + 1) L_S + d_B E_S) + 1 \tag{5.21}$$

which simplifies to

$$g = d_E d_B + \frac{1}{2} (-3d_B^2 + d_B + 2). \quad (5.22)$$

The curves $C \subset S$ are a projective space of dimension $h^0(S, \mathcal{O}(C)) - 1$. If $d_E > 0$ and $d_B > 1$ then $C - K_S = C - L_S$ is ample, so by Kodaira vanishing we have $h^i(S, \mathcal{O}(C)) = 0$ for $i > 0$. So we only need to compute $\chi(\mathcal{O}(C))$:

$$\chi(\mathcal{O}(C)) = \frac{1}{2} C(C - K_S) + \chi(\mathcal{O}_S) = d_E d_B - \frac{1}{2} (3d_B^2 + d_B - 6). \quad (5.23)$$

Since the moduli of these S is just \mathbb{P}^2 corresponding to the family of lines in \mathbb{P}^2 , the moduli space of such curves C is a $\mathbb{P}^{\chi(\mathcal{O}(C)) - 1}$ -bundle over \mathbb{P}^2 , with $\chi(\mathcal{O}(C))$ given by (5.23).

5.2.2 Examples and computations

In this section, we apply the results of the previous section to describe many moduli spaces of curves and compute the associated Gopakumar-Vafa invariants. We denote the moduli space of curves of degree (d_E, d_B) and genus g by \mathcal{M}_{d_E, d_B}^g . If \mathcal{M}_{d_E, d_B}^g is smooth and there are no curves of degree (d_E, d_B) and genus strictly greater than g , we have

$$n_{d_E, d_B}^g = (-1)^{\dim \mathcal{M}_{d_E, d_B}^g} \chi(\mathcal{M}_{d_E, d_B}^g), \quad (5.24)$$

a formula that may need correction as discussed in section 5.2.1.

Now let \mathcal{C}_{d_E, d_B}^g be the universal curve, also supposed smooth, and suppose that there are no other curves of degree (d_E, d_B) and genus greater than or equal to $g - 1$. Then we have, by [50]

$$n_{d_E, d_B}^{g-1} = (-1)^{\dim \mathcal{M}_{d_E, d_B}^g + 1} \left(\chi(\mathcal{M}_{d_E, d_B}^g) + (2g - 2) \chi(\mathcal{M}_{d_E, d_B}^g) \right). \quad (5.25)$$

This formula may also need correction.

Examples: we begin with $d_E = 0$ and any $d_B > 0$. Then we assume $d_E > 0$ and treat $0 \leq d_B \leq 5$ in turn.

$\mathbf{d_E = 0}$. If $g = (d_B - 1)(d_B - 2)/2$, then Proposition 1 says that \mathcal{M}_{0, d_B}^g is identical to the moduli space of plane curves of degree d_E in local \mathbb{P}^2 . In particular, the GV invariants n_{0, d_B}^g for $g \leq p_a(d_B)$ are identical with the well-known GV invariants of local \mathbb{P}^2 . This observation provides the geometric verification of the GV invariants in the entire first columns of tables 5–13.

We will see later that a similar argument can be used to compute or verify many of the GV invariants in some of the subsequent columns.

Now let C have degree (d_E, d_B) with $d_E > 0$ and we set out to bound the genus.

$\mathbf{d_B = 0}$. If $d_B = 0$, then by Corollary 1 any curve with $d_B = 0$ is a union of fibers.

It is well known that for an elliptically fibered Calabi-Yau threefold X over a base B , the only nonzero Gopakumar-Vafa invariants in multiples of the fiber class f are

$$\begin{aligned} n_{df}^1 &= \chi(B), & d > 0 \\ n_f^0 &= -\chi(X). \end{aligned} \quad (5.26)$$

We therefore get

$$n_{d_E,0}^0 = 540 \ (d_E > 0), \quad n_{1,0}^1 = 3, \quad (5.27)$$

in agreement with the $d_B = 0$ rows of tables 5–13.

$d_B = 1$. We compute some Gopakumar-Vafa invariants for $d_B = 1$ by geometry and always find complete agreement with table 2 and additional calculations we have done beyond the table.

We put $D = \pi(C)$ and note that $D = \ell$ is a line, by Lemma 1. Then C has a unique irreducible component C' mapping by π onto D . Since any other components of C map to points, they must be unions of fibers, which have $d_B = 0$. Thus C has $d_B = 1$. By Proposition 1 we have $g(C) \leq 0$. Hence $g(C) = 0$ and C is a line in E , again by Proposition 1. Since C has degree $(0, 1)$, we see that the remaining components have degree $(d_E, 0)$, hence consists of d_E fibers by the $d_B = 0$ case. Thus C is the union of a line in E and d_E fibers, and necessarily has genus d_E . From this description, we see that $\mathcal{M}_{d_E,1}^{d_E}$ is the relative Hilbert scheme of d_E points on lines in the plane, a \mathbb{P}^{d_E} -bundle over \mathbb{P}^2 . Hence by (5.24) we have

$$\begin{aligned} n_{d_E,1}^{d_E} &= (-1)^{d_E} 3(d_E + 1) \\ n_{d,1}^{d_E} &= 0 \ (d < d_E). \end{aligned} \quad (5.28)$$

The second equation follows since the same description shows that all curves of degree $(d, 1)$ have genus $d < d_E$.

These results agree with those presented in the $d_B = 1$ row of tables 5–13. We have also checked these results for much larger values of d_E , and always find agreement.

$d_B = 2$. We compute some Gopakumar-Vafa invariants for $d_B = 2$ by geometry and always find complete agreement with table 3 and additional calculations we have done beyond the table.

We $D = \pi(C)$ and note that D has degree 1 or 2.

If D is a line ℓ , then $C \subset \pi^{-1}(\ell)$. By Proposition 2 we get $g = 2d_E - 4$.

If $D = \ell_1 \cup \ell_2$ is union of lines, then just as in the $d_B = 1$ case we have that C contains irreducible components C_1 and C_2 mapping isomorphically by π to ℓ_1 and ℓ_2 respectively. Since each C_i has degree $(0, 1)$, the remaining components of C have degree $(d_E, 0)$, hence consist of d_E fibers. Thus C consists of the reducible conic $C_1 \cup C_2$ in E with d_E fibers attached.

If D is an irreducible conic, then C has a unique component C mapping to C with degree 1. Hence by smoothness of C we see that C is isomorphic to C . By Proposition 1, C is a conic in E . Since C has degree $(0, 2)$, we see just as in the last case that C is the union of C and d_E fibers. The last two cases combine into one moduli space of curves, parametrizing conics in E (both smooth and singular) with d_E fibers attached. This description shows that moduli space is isomorphic to the relative Hilbert scheme of d_E points on plane conics. This space is smooth for $d_E \leq 3$ [50].

Thus the moduli space of all curves of degree $(d_E, 2)$ has two components: the one just described as the relative Hilbert scheme of d_E points on plane conics, and the other component parametrizing curves in a surface $S = \pi^{-1}(\ell)$ as in Proposition 2.

The curves of the first component have genus d_E , and the curves of the second component have genus $2d_E - 4$. Thus we have $g \leq \max(d_E, 2d_E - 4)$. The maximum is realized on the first component for $d_E \leq 4$ and on the second component for $d_E \geq 4$.

So for $d_E \leq 3$, all curves of genus $g = d_E$ are unions of plane conics in E with d_E fibers attached, so this moduli space is isomorphic to the relative Hilbert scheme of d_E points on plane conics, and this Hilbert scheme is smooth. Computing Euler characteristics exactly as in [50] or [14], we have by (5.24)

$$n_{0,2}^0 = -6, \quad n_{1,2}^1 = 15, \quad n_{2,2}^2 = -36, \quad n_{3,2}^3 = 66 \tag{5.29}$$

in agreement with the numbers in tables 5–8. The same argument also shows that all numbers to the left of these numbers are zero.

For $d_E \geq 5$, the above discussion shows that curves of genus $g = 2d_E - 4$ are all contained in surfaces $S = \pi^{-1}(\ell)$ and are described by Proposition 2. So $\mathcal{M}_{d_E,2}^{2d_E-4}$ is a \mathbb{P}^{2d_E-5} bundle over \mathbb{P}^2 , hence by (5.24) we get

$$n_{d_E,2}^{2d_E-4} = -3(2d_E - 4) \tag{5.30}$$

and

$$n_{d,2}^{2d_E-4} = 0, \quad d < d_E, \tag{5.31}$$

in agreement with the $d_B = 2$ lines of tables 11–13 for $d_E \geq 5$. We have also checked these results for much larger values of d_E and always find agreement.

Our last set of checks come from the application of (5.25) to curves of type $((1, 2))$. From the genus formula $\max(d_E, 2d_E - 4)$ we see that the hypotheses of (5.25) hold if $d_E < 2d_E - 5$, so that $d_E \geq 6$. We need to describe the universal curve $\mathcal{C}_{d_E,2}^{2d_E-4}$. Given a point of the universal curve, which we denote by (C, p) , we have a natural map $\mathcal{C}_{d_E,2}^{2d_E-4} \rightarrow X$ obtained by forgetting C . We claim that the fiber is a \mathbb{P}^{2d_E-6} -bundle over \mathbb{P}^1 , hence the universal curve is smooth.

To see this, note that the line ℓ for which $C \subset S = \pi^{-1}(\ell)$ must contain the point $q = \pi(p)$. Therefore the lines ℓ used to describe curves in the fiber over p form a \mathbb{P}^1 rather than a \mathbb{P}^2 . The fiber is itself fibered over this \mathbb{P}^1 , and it only remains to find the fiber of this last fibration. Now fixing ℓ , hence fixing S , the possible curves C are those corresponding to sections of $H^0(S, \mathcal{O}(d_E L_S + d_B E_S))$ which vanish at p , which is a hyperplane in $H^0(S, \mathcal{O}(d_E L_S + d_B E_S))$. So the fiber is a \mathbb{P}^{2d_E-6} rather than a \mathbb{P}^{2d_E-5} , as claimed.

We can now apply (5.25) to compute

$$n_{d_E,2}^{2d_E-5} = (-540)(2)(2d_E - 5) + (4d_E - 10)3(2d_E - 4), \quad d_E \geq 6,$$

which simplifies to

$$n_{d_E,2}^{2d_E-5} = 12(2d_B - 5)(d_B - 92), \quad d_E \geq 6, \tag{5.32}$$

which agrees with the results in table 3 as well as larger values of d_E . Furthermore, all numbers to the left of these numbers are zero, which also checks.

The general procedure should now be clear. We let $D = \pi(C)$ and we study the cases where D has irreducible components of each possible degree and multiplicity. In other words the degree d_B curve D can split up into components D_j of degree d_j and multiplicity m_j . So the possible components of moduli can be indexed by partitions of d_B into unordered lists of pairs $((d_j, m_j))$ with repetition allowed satisfying

$$d_B = \sum_j m_j d_j \tag{5.33}$$

Correspondingly, C splits up into a union of components C_j , mapping by π to D_j with multiplicity m_j , possibly together with fibers. We investigate the maximum genus of each C_j . The genus of C can then be maximized by configuring the curves C_j and any fibers so that they intersect in as many points (including multiplicity) as possible.

For $d_B = 1$, only the partition $((1, 1))$ is possible. For $d_B = 2$ we could have either $((2, 1))$, $((1, 2))$, or $((1, 1), (1, 1))$, and all cases occurred in our discussion above.

The analysis gets more intricate for each degree. Rather than carry out this program for $d_E \leq 5$, we content ourselves with presenting some Gopakumar-Vafa invariants that can be computed by these methods, giving the corresponding $((d_j, m_j))$, and matching to the tables.

For any $d_B \geq 2$ some strata combine naturally, as we already saw for $d_B = 2$. Let $d_B = \sum_{j=1}^k m_j d_j$ be a partition of d_B and consider the list $((d_j, 1)_{j=1}^k)$. These strata combine to describe the Hilbert scheme of d_E points on plane curves of degree d_B . The stratification is just an artifact of our method, corresponding to splitting up the moduli space according to the degrees of the components of the plane curve.

$d_B = 3$. We compute some Gopakumar-Vafa invariants for $d_B = 3$ by geometry and always find complete agreement with table 4 and additional calculations we have done beyond the table.

In this case, the possible lists are $((1, 1), (1, 1), (1, 1))$, $((3, 1))$, $((1, 3))$, $((2, 1), (1, 1))$, $((1, 2), (1, 1))$,

- $((3, 1))$, $((2, 1), (1, 1))$, and $((1, 1), (1, 1), (1, 1))$ combine to give plane cubics union d_E fibers, with moduli space the relative Hilbert scheme of d_E points on plane cubics, which are curves of genus $d_E + 1$.
- $((1, 3))$ are curves in a surface of the form $S = \pi^{-1}(\ell)$, which have genus $3d_E - 11$ by Proposition 2.
- $((1, 2), (1, 1))$ is the union of a curve in a surface of the form $S = \pi^{-1}(\ell)$, a line in E , and possibly some fibers. It is not hard to see by extending the analysis below that the maximum genus is attained when there are no additional fibers. So C is the union of a curve $C_{d_E, 2}$ in S and a line $C_{0, 1} \subset E$ (which has degree $(0, 1)$). Here and in the sequel, a subscript denotes the degree of a curve. We have already seen that $C_{d_E, 2}$ has genus $2d_E - 4$.

The genus of C is therefore $(2d_E - 4) + m - 1$, where m is the number of intersection points Z of $C_{d_E,2}$ and ℓ . We claim that either $m = 1$ or we are actually in the $((1, 3))$ case. Since $C_{0,1} \subset E$ we see that $Z \subset C_{d_E,2} \cap E$. Since $S \cap E$ is a line, it follows that if $m \geq 2$, then $C_{0,1}$ must be equal to that line. In that case $C \subset S$, and we are really in the $((1, 3))$ case. So $m = 1$ and C has genus $2d_E - 4$.

Combining the three cases, we have $g \leq \max(d_E + 1, 3d_E - 11, 2d_E - 4)$.

The maximum genus is attained in the first case for $d_E \leq 5$, in the second case for $d_E \geq 7$, and in the third case for $5 \leq d_E \leq 7$.

If $d_E \leq 4$, then the maximum genus $d_E + 1$ can only be attained in the first case, and $\mathcal{M}_{d_E,3}^{d_E+1}$ is the relative Hilbert scheme of d_E points on plane cubics, which is smooth by [50]. So we compute from the relative Hilbert scheme

$$n_{0,3}^1 = -10, \quad n_{1,3}^2 = 27, \quad n_{2,3}^3 = -72, \quad n_{3,3}^4 = 154, \quad n_{4,3}^5 = -306. \quad (5.34)$$

This agrees with tables 6–10. In addition, our now-familiar argument shows that all numbers to the left of these numbers in the tables are zero, in agreement with the tables.

If $d_E \geq 8$, then the maximum genus $3d_E - 11$ can only be attained in the second case $((1, 3))$, and $\mathcal{M}_{d_E,3}^{3d_E-11}$ is a \mathbb{P}^{3d_E-13} -bundle over \mathbb{P}^2 by Proposition 2. Hence, for $d_E \geq 8$ we have

$$n_{d_E,3}^{3d_E-11} = (-1)^{d_E+1} 9(d_E - 4). \quad (5.35)$$

For these d_E we also have $n_{d,3}^{3d_E-11} = 0$ for $d < d_E$. We have checked these results for large values of d_E and always find agreement.

For $d_E = 6$, the maximum genus $2d_E - 4 = 8$ can only be realized in the third case $((1, 2), (1, 1))$. By the discussion above, we have $C = C_{6,2} \cup C_{0,1}$, where $C_{6,2}$ lies in a surface $S = \pi^{-1}(\ell)$ and the line $C_{0,1}$ intersects $C_{6,2}$ in exactly one point. But $C_{6,2} \cdot E = 0$ by (5.16). On the other hand, $C_{6,2}$ cannot be disjoint from E , since it intersects $C_{0,1} \subset E$ nontrivially. Therefore $C_{6,2}$ has a component which lies entirely in E , necessarily a line or a conic. If it contains a conic plane curve, then since it also contains the line $C_{0,1}$ then it contains a (reducible) cubic plane curve. So we are really in the case of a cubic with d_E fibers already considered. So $C_{6,2}$ contains a line. The remaining components are a curve $C_{6,1}$, which by the $d_B = 1$ case are a line and 6 fibers. If the two lines are distinct, then they form a conic and we are in the previous case. Otherwise the line has multiplicity 2, and the curve $C_{6,2}$ is the union of the line E_S with multiplicity 2 (in S) and 6 fibers. The line $C_{0,1}$ is arbitrary. So $\mathcal{M}_{6,3}^8$ is the product of \mathbb{P}^2 and the relative Hilbert scheme of 6 points in lines in \mathbb{P}^2 . Since this relative Hilbert scheme is a \mathbb{P}^6 -bundle over \mathbb{P}^2 , this gives

$$n_{6,3}^8 = 63, \quad (5.36)$$

in agreement with table 13. Furthermore, all of the numbers to the left are zero, again in agreement.

We can also apply (5.25) for curves of type $((1, 3))$, which have genus $3d_E - 11$. The hypotheses hold if $d_E \geq 9$ and then we get

$$n_{d_E,3}^{3d_E-12} = (-1)^{d_E} ((-540)(2)(3d_E - 13) + (6d_E - 24)(3)(3d_E - 12)),$$

which simplifies to

$$n_{d_E,3}^{3d_E-12} = (-1)^{d_E}(54)(d^2 - 68d + 276), \quad d_E \geq 9, \quad (5.37)$$

which agrees with the results in table 4 as well as larger values of d_E . Furthermore, all numbers to the left of these numbers are zero, which also checks.

$d_B = 4$. We compute some Gopakumar-Vafa invariants for $d_B = 4$ by geometry and always find complete agreement with table 14 and additional calculations we have done beyond the table. By now we have provided enough examples that we can simply present our results and the reader will be able to check details.

Combining the cases $((4, 1))$, $((3, 1), (1, 1))$, $((2, 1), (2, 1))$, $((2, 1), (1, 1), (1, 1))$, and $((1, 1), (1, 1), (1, 1), (1, 1))$, we consider plane quartic curves in E with d_E fibers attached. These curves have genus $d_E + 3$. If $d_E \leq 6$, any curve of degree $(d_E, 4)$ and genus $d_E + 3$ can be seen to be of this type, and furthermore there are no curves of that degree and higher genus. We conclude that $\mathcal{M}_{d_E,4}^{d_E+3}$ is the relative Hilbert scheme of d_E points on plane quartic curves, a \mathbb{P}^{14-d_E} bundle over $\text{Hilb}^{d_E}(\mathbb{P}^2)$, and is smooth for $d_E \leq 5$ [50]. We therefore get

$$n_{d_E,4}^{d_E+3} = (-1)^{d_E}(15 - d_E)\chi(\text{Hilb}^{d_E}(\mathbb{P}^2)), \quad (5.38)$$

hence by computing Euler characteristics

$$n_{0,4}^3 = 15, \quad n_{1,4}^4 = -42, \quad n_{2,4}^5 = 117, \quad n_{3,4}^6 = -264, \quad n_{4,4}^7 = 561, \quad n_{5,4}^8 = -1080. \quad (5.39)$$

The same argument confirms in the usual way that all numbers to the left of these are zero, in agreement with the tables.

For curves of type $((1, 4))$, i.e. curves contained in a surface $S = \pi^{-1}(\ell)$, we have $g = 4d_E - 21$, and $\mathcal{M}_{d_E,4}^{4d_E-21}$ is a \mathbb{P}^{4d_E-24} -bundle over \mathbb{P}^2 by Proposition 2. If $d_E \geq 11$, then any curve of this degree and genus is of this type, and there are no curves of this degree and higher genus. This gives

$$n_{d_E,4}^{4d_E-21} = 3(4d_E - 23) = 12d_E - 69, \quad d_E \geq 11 \quad (5.40)$$

in complete agreement with our calculations. Also $n_{d,4}^{4d-21} = 0$ for $d < d_E$.

For $d_E = 7$ we have curves of type $((1, 2), (2, 1))$, which consist of a curve $C_{7,2}$ of degree $(7, 2)$ union a conic $C_{0,2} \subset E$. The curve $C_{7,2}$ lies in a surface $S = \pi^{-1}(\ell)$ and has genus $2d_E - 4 = 10$. Since $C_{0,2}$ is to meet $C_{7,2}$ in two points but $C_{7,2} \cdot E = 1$, we see that $C_{7,2}$ must contain a line in E . Repeating an argument from the $d_B = 3$ case, we further see that $C_{7,2}$ must contain a line doubled in S together with d_E fibers. Our geometric description shows that the moduli space of $C_{7,2}$ is the relative Hilbert scheme of 7 points on lines, a \mathbb{P}^7 -bundle over \mathbb{P}^2 . There are no curves of degree $(7, 2)$ and higher genus, so we can now multiply by the \mathbb{P}^5 moduli of $C_{0,2}$ and compute the Euler characteristic, giving

$$n_{7,4}^{16} = 144, \quad (5.41)$$

in agreement with our other calculations. In addition we have $n_{d,4}^{16} = 0$ for $d < 7$, also in agreement.

We can also apply (5.25) for curves of type $((1, 4))$, which have genus $4d_E - 21$. The hypotheses hold if $d_E \geq 12$ and then we get

$$n_{d_E,4}^{4d_E-22} = -((-540)(2)(4d_E - 24) + (8d_E - 44)(3)(4d_E - 23)),$$

which simplifies to

$$n_{d_E,4}^{4d_E-22} = -54(d^2 - 68d + 276), \quad d_E \geq 12, \quad (5.42)$$

which agrees with the results in table 14 as well as larger values of d_E . Furthermore, all numbers to the left of these numbers are zero, which also checks.

d_B = 5. We compute some Gopakumar-Vafa invariants for $d_B = 5$ by geometry and always find complete agreement with table 15 and additional calculations we have done beyond the table.

We first consider the curves which are unions of plane quintics and d_E fibers, which have genus $d_E + 6$. We also compute that for $d_E \leq 7$, all curves of degree $(d_E, 5)$ and genus $d_E + 6$ are of this type. Therefore $\mathcal{M}_{d_E,5}^{d_E+6}$ is the relative Hilbert scheme of d_E points on plane quintics, which is smooth for $d_E \leq 6$. We also compute that there are no curves of that degree and higher genus. Since $\mathcal{M}_{d_E,5}^{d_E+6}$ is a \mathbb{P}^{20-d_E} -bundle over $\text{Hilb}^{d_E}(\mathbb{P}^2)$, we get

$$n_{d_E,5}^{d_E+6} = (-1)^{d_E} (21 - d_E) \chi(\text{Hilb}^{d_E}(\mathbb{P}^2)) \quad d_E \leq 6, \quad (5.43)$$

or

$$n_{0,5}^6 = 21, \quad n_{1,5}^7 = -60, \quad n_{2,5}^8 = 171, \quad n_{3,5}^9 = -396, \quad n_{4,5}^{10} = 867, \quad n_{5,5}^{11} = -1728, \quad n_{6,5}^{12} = 3315. \quad (5.44)$$

in complete agreement with our calculations. Also the Gopakumar-Vafa invariants to the left of these numbers are all zero, also in agreement.

For curves of type $((1, 5))$, i.e. those contained in a surface $S = \pi^{-1}(\ell)$ we have $g = 5d_E - 34$ and moduli space a \mathbb{P}^{5d_E-38} -bundle over \mathbb{P}^2 . For $d_E \geq 14$ we compute that all curves of degree $(d_E, 5)$ are of this type, and that there are no curves of higher genus. This gives

$$n_{5,d_E}^{5d_E-34} = (-1)^{d_E} 3(5d_E - 37) \quad d_E \geq 14. \quad (5.45)$$

in complete agreement with our calculations. Also the Gopakumar-Vafa invariants to the left of these numbers are all zero, also in agreement.

We can also apply (5.25) for curves of type $((1, 5))$, which have genus $5d_E - 34$. The hypotheses hold if $d_E \geq 15$ and then we get

$$n_{d_E,5}^{5d_E-35} = (-1)^{d_E+1} ((-540)(2)(5d_E - 38) + (10d_E - 70)(3)(5d_E - 37)),$$

which simplifies to

$$n_{d_E,5}^{5d_E-35} = (-1)^{d_E+1} (30)(5d_B^2 - 252d_B + 1627), \quad d_E \geq 15, \quad (5.46)$$

which agrees with the results in table 14 as well as larger values of d_E . Furthermore, all numbers to the left of these numbers are zero, which also checks.

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A Gopakumar-Vafa invariants

The involution symmetry of the topological strings on elliptic Calabi-Yau manifolds restricts the number of the coefficients in the ambiguity to roughly one fourth. This allows us to extract the higher genus invariants to sufficient genus $g \leq 8$ to test the conjectures about the even weak Jacobi-Forms in reasonable detail³¹

$d_B \backslash d_E$	0	1	2	3	4	5	6
0		540	540	540	540	540	540
1	3	-1080	143370	204071184	21772947555	1076518252152	33381348217290
2	-6	2700	-574560	74810520	-49933059660	7772494870800	31128163315047072
3	27	-17280	5051970	-913383000	224108858700	-42712135606368	4047949393968960
4	-192	154440	-57879900	13593850920	-2953943334360	603778002921828	-90433961251273800
5	1695	-1640520	751684050	-218032516800	51350781706785	-11035406089270080	2000248139674298880

Table 5. The GV invariants $n_{(d_E, d_B)}^g$ for genus $g = 0$ for the elliptic Calabi-Yau $X(1, 1, 1, 6, 9)$.

$d_B \backslash d_E$	0	1	2	3	4	5	6
0		3	3	3	3	3	3
1	0	-6	2142	-280284	-408993990	-44771454090	-2285308753398
2	0	15	-8574	2126358	521856996	1122213103092	879831736511916
3	-10	4764	-1079298	152278986	-16704086880	-3328467399468	1252978673852088
4	231	-154662	48907815	-9759419622	1591062421074	-186415241060547	8624795298947118
5	-4452	3762246	-1510850250	385304916960	-76672173887766	12768215950604064	-1663415916220743876

Table 6. The GV invariants $n_{(d_E, d_B)}^g$ for genus $g = 1$ for the elliptic Calabi-Yau $X(1, 1, 1, 6, 9)$.

³¹As we explained it is in principle possible to evaluate them to genus 189. The genus $g = 0$ invariants agree with [32], the $g \leq 1$ invariants with [12] and the invariants $g \leq 3$ invariants with [3].

$d_B \backslash d_E$	0	1	2	3	4	5	6
0		0	0	0	0	0	0
1	0	0	9	-3192	412965	614459160	68590330119
2	0	0	-36	20826	-5904756	-47646003780	-80065270602672
3	0	27	-16884	4768830	-818096436	288137120463	67873415627151
4	-102	57456	-15452514	2632083714	-320511624876	18550698291252	780000198300540
5	5430	-4032288	1430896428	-323858122812	55058565096630	-7249216518163620	691264676523200805

Table 7. The GV invariants $n_{(d_E, d_B)}^2$ for the elliptic Calabi-Yau $X(1, 1, 1, 6, 9)$. Note that $n_{g>1}^{(0, d_2)} = 0, \forall d_2 \in \mathbb{N}$. We therefore omit the $(0, d_2)$ line below.

$d_B \backslash d_E$	0	1	2	3	4	5	6
1	0	0	0	-12	4230	-541440	-820457286
2	0	0	0	66	-45729	627574428	3776946955338
3	0	0	-72	48036	-14756490	297044064	-7900517344212
4	15	-7236	1638918	-226431351	20419274259	-719284158099	236091664016826
5	-3672	2417742	-764921214	154856849136	-22866882491772	2493418732350750	-194361733345447458

Table 8. The GV invariants $n_{(d_E, d_B)}^3$ for the elliptic Calabi-Yau $X(1, 1, 1, 6, 9)$.

$d_B \backslash d_E$	0	1	2	3	4	5	6
1	0	0	0	0	15	-5256	665745
2	0	0	0	0	-132	-453960	-95306132778
3	0	0	0	154	-110574	38259441	218140445904
4	0	-42	26946	-7824888	1386011568	-172919782116	-4345528029372
5	1386	-819123	232934157	-42321589218	5500907292240	-520718843839590	38245592568676608

Table 9. The GV invariants $n_{(d_E, d_B)}^4$ for the elliptic Calabi-Yau $X(1, 1, 1, 6, 9)$.

$d_B \backslash d_E$	0	1	2	3	4	5	6
1	0	0	0	0	0	-18	6270
2	0	0	0	0	0	-5031	1028427996
3	0	0	0	0	-306	247014	-2562122952
4	0	0	117	-81225	26211942	-5223900087	1263109811373
5	-270	144414	-36870264	5929743618	-665294451264	53375661928620	-3651471177372864

Table 10. The GV invariants $n_{(d_E, d_B)}^5$ for the elliptic Calabi-Yau $X(1, 1, 1, 6, 9)$.

$d_B \backslash d_E$	0	1	2	3	4	5	6	7
1	0	0	0	0	0	0	21	-7272
2	0	0	0	0	0	-18	-771642	-147864402162
3	0	0	0	0	0	612	1401468	502063861662
4	0	0	0	-264	200430	-70438068	9510828972	-29413672557570
5	21	-9972	2156373	-268703481	18682746903	-182455706016	-81900631565910	22514515679407491

Table 11. The GV invariants $n_{(d_E, d_B)}^6$ for the elliptic Calabi-Yau $X(1, 1, 1, 6, 9)$.

$d_B \backslash d_E$	0	1	2	3	4	5	6	7
1	0	0	0	0	0	0	0	-24
2	0	0	0	0	0	0	-7224	1443561648
3	0	0	0	0	0	0	17386	-4962183570
4	0	0	0	0	561	-447903	170978160	605964021294
5	0	-60	38340	-10994520	1895073858	-217773585972	16072935664050	-1937578925283840
6	27538	-16386600	4710791727	-868872423987	115076024047737	-11554540079426667	915758222342784613	-59159328867116232828

Table 12. The GV invariants $n_{(d_E, d_B)}^7$ for $X(1, 1, 1, 6, 9)$.

$d_B \backslash d_E$	0	1	2	3	4	5	6	7	8
1	0	0	0	0	0	0	0	0	27
2	0	0	0	0	0	0	-24	-995490	-203754011670
3	0	0	0	0	0	0	63	3396663	11118565777779
4	0	0	0	0	0	-1080	951204	-6600292956	-74890630203552
5	0	0	171	-119415	38611944	-7672460076	1056853387755	-49874149196514	29707605109699254
6	-5310	2949516	-785916540	$132969 \cdot 10^7$	$-159045 \cdot 10^8$	$14105 \cdot 10^{11}$	$-952701 \cdot 10^{11}$	$471505 \cdot 10^{13}$	$-112148 \cdot 10^{15}$

Table 13. The GV invariants $n_{(d_E, d_B)}^8$ for $X(1, 1, 1, 6, 9)$. For last 6 numbers we give only 6 significant digits.

B Derivation of the involution symmetry on the propagators

For the compact models we have the an-holomorphic propagators S^{ij}, S^i, S , where i, j runs over the complex structure moduli. It is convenient to make a change of variables with the derivative of the Kähler potential $K_i = \partial_i K$ by the following

$$S^{ij} \rightarrow S^{ij}, \quad S^i \rightarrow S^i - S^{ij} K_j, \quad S \rightarrow S - S^i K_i + \frac{1}{2} S^{ij} K_i K_j, \quad (\text{B.1})$$

In the following we refer to the propagators S^{ij}, S^i, S as the ones after the change of variables in [2].

The propagators are defined by relating their anti-holomorphic derivatives to three point couplings. One can integrate these relations and also special geometry relation, and gets [2]

$$\begin{aligned} \Gamma_{ij}^k &= \delta_i^k K_j + \delta_j^k K_i - C_{ijl} S^{kl} + s_{ij}^k, \\ \partial_i S^{jk} &= C_{imn} S^{mj} S^{nk} + \delta_i^j S^k + \delta_i^k S^j - s_{im}^j S^{mk} - s_{im}^k S^{mj} + h_i^{jk}, \\ \partial_i S^j &= C_{imn} S^{mj} S^n + 2\delta_i^j S - s_{im}^j S^m - h_{ik} S^{kj} + h_i^j, \\ \partial_i S &= \frac{1}{2} C_{imn} S^m S^n - h_{ij} S^j + h_i, \\ \partial_i K_j &= K_i K_j - C_{ijn} S^{mn} K_m + s_{ij}^m K_m - C_{ijk} S^k + h_{ij}. \end{aligned} \quad (\text{B.2})$$

Here the holomorphic ambiguities $s_{ij}^k, h_i^{jk}, h_{ij}, h_i^j, h_i$ are some rational functions from the integration constants of the anti-holomorphic derivatives. We can compute the Kahler potential K and Christoffel connections Γ_{ij}^k in the holomorphic limit from Picard-Fuchs equations. There is freedom to choose some of the ambiguities such that the set of equations has a consistent solution for the other ambiguities and the propagators S^{ij}, S^i, S .

$g \setminus d_E$	0	1	2	3	4	5	6	7	8	9	10	11	12
0	-192	154440	-57879900	13(7)20	-29(9)60	60(11)28	-90(13)00	50(16)00	-48(18)00	-82(20)80	-10(25)00	-13(27)20	67(30)72
1	231	-154662	48907815	-97(6)22	15(9)74	-18(11)47	86(12)18	20(15)20	-45(18)44	29(20)28	-37(24)96	23(26)82	68(30)88
2	-102	57456	-15452514	26(6)14	-32(8)76	18(10)52	78(11)40	-25(14)44	-15(17)96	27(20)32	-95(22)96	44(26)98	22(30)64
3	15	-7236	1638918	-226431351	20(7)59	-71(8)99	23(11)26	-65(13)34	46(16)88	79(18)02	19(21)16	16(25)73	16(29)30
4	0	-42	26946	-7824888	13(6)68	-17(8)16	-43(9)72	-32(12)50	-32(15)74	-41(18)06	-30(21)82	-33(24)50	-26(27)20
5	0	0	117	-81225	26211942	-52(6)87	12(9)73	45(11)41	38(14)26	69(17)56	80(20)68	10(24)95	12(27)02
6	0	0	0	-264	200430	-70438068	95(6)72	-29(10)70	-34(13)96	-89(16)70	-15(20)66	-25(23)86	-39(26)80
7	0	0	0	0	561	-447903	170978160	60(8)94	20(12)96	89(15)66	23(19)94	55(22)47	10(26)89
8	0	0	0	0	0	-1080	951204	-66(6)56	-74(10)52	-68(14)40	-29(18)10	-10(22)00	-27(25)42
9	0	0	0	0	0	0	2136	2312838	15(9)21	39(13)66	30(17)24	16(21)56	61(24)68
10	0	0	0	0	0	0	0	38160	-13(7)34	-16(12)88	-25(16)96	-22(20)48	-12(24)34
11	0	0	0	0	0	0	0	144	8502057	47(10)15	16(15)89	26(19)83	21(23)35
12	0	0	0	0	0	0	0	0	83124	-83(8)68	-85(13)84	-26(18)64	-33(22)98
13	0	0	0	0	0	0	0	0	279	68(6)48	33(12)40	22(17)46	46(21)01
14	0	0	0	0	0	0	0	0	0	-3921582	-89(10)36	-16(16)08	-56(20)92
15	0	0	0	0	0	0	0	0	0	-37968	14(9)23	91(14)86	59(19)08
16	0	0	0	0	0	0	0	0	0	-126	-11(7)74	-41(13)20	-54(18)60
17	0	0	0	0	0	0	0	0	0	0	6330231	14(12)70	42(17)12
18	0	0	0	0	0	0	0	0	0	0	61632	-34(10)28	-27(16)82
19	0	0	0	0	0	0	0	0	0	0	204	52(8)13	14(15)85
20	0	0	0	0	0	0	0	0	0	0	0	-37(6)20	-62(13)82
21	0	0	0	0	0	0	0	0	0	0	0	1819827	20(12)15
22	0	0	0	0	0	0	0	0	0	0	0	18648	-46(10)04
23	0	0	0	0	0	0	0	0	0	0	0	63	66(8)95
24	0	0	0	0	0	0	0	0	0	0	0	0	-46(6)82
25	0	0	0	0	0	0	0	0	0	0	0	0	2097267
26	0	0	0	0	0	0	0	0	0	0	0	0	22020
27	0	0	0	0	0	0	0	0	0	0	0	0	75

Table 14. Some BPS invariants for $n_{(d_E,4)}^g$. To save space we only give the first and the last two significant digits and the number of omitted digits in brackets.

$g \setminus d_E$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1695	-1640520	751684050	-21(8)00	51(10)85	-11(13)80	20(15)80	-54(17)60	10(20)65	-70(21)20	25(25)20	31(27)60	-99(29)70	-81(33)40	-23(36)50	43(39)00
1	-4452	3762246	-15(6)50	38(8)60	-76(10)66	12(13)64	-16(15)76	22(17)36	10(19)98	-74(21)06	99(24)52	-18(27)56	-99(29)22	-94(33)24	-25(36)62	10(40)68
2	5430	-4032288	14(6)28	-32(8)12	55(10)30	-72(12)20	69(14)05	-39(16)24	-66(18)64	-10(21)92	70(23)20	-18(27)72	-10(29)93	-35(33)00	-28(35)72	10(40)80
3	-3672	2417742	-764921214	15(8)36	-22(10)72	24(12)50	-19(14)58	11(16)92	-12(18)12	68(20)02	-70(21)40	-84(25)68	19(29)60	-36(32)16	44(35)14	46(39)96
4	1386	-819123	232934157	-42(7)18	55(9)40	-52(11)90	38(13)08	-28(15)92	34(17)58	23(19)08	-26(22)61	13(25)28	11(28)61	-16(31)69	87(34)33	93(38)50
5	-270	144414	-36870264	59(6)18	-66(8)64	53(10)20	-36(12)64	25(14)44	-17(16)10	-15(10)46	-27(21)42	-22(24)32	-31(27)40	-28(30)90	11(33)48	71(37)58
6	21	-9972	2156373	-268703481	18(7)03	-81(10)10	-81(10)10	22(13)91	-37(15)35	26(18)12	12(21)14	92(23)67	99(26)51	10(33)53	28(36)36	28(36)36
7	0	-60	38840	-10994520	18(6)58	-21(8)72	16(10)50	-19(12)40	-21(12)98	-21(17)94	-27(20)58	-24(23)38	-31(26)52	-38(29)24	-46(32)28	-54(35)46
8	0	0	171	-119415	38611944	-109(6)58	10(9)55	-49(10)14	29(13)54	20(16)13	42(19)53	52(22)17	84(25)73	13(29)75	19(32)23	26(35)66
9	0	0	0	-396	304434	-108352800	23(7)36	-48(9)02	-14(12)90	-19(15)44	-52(18)12	-93(21)32	-19(25)94	-39(28)26	-71(31)74	-11(35)70
10	0	0	0	0	867	-706221	268652163	-52(7)55	81(10)13	13(14)92	53(17)38	13(21)73	40(24)22	10(28)60	24(31)74	50(34)08
11	0	0	0	0	0	-1728	1494120	-610535214	-12(9)20	-63(12)68	-43(16)74	-17(20)12	-74(23)92	-25(27)14	-76(30)86	-19(34)82
12	0	0	0	0	0	0	3315	-3061881	13(7)53	19(11)21	28(15)59	18(19)44	11(23)85	56(26)96	21(30)18	70(33)41
13	0	0	0	0	0	0	0	-6270	-1610226	-35(9)86	-14(14)78	-16(18)92	-16(22)22	-10(26)54	-57(29)14	-23(33)82
14	0	0	0	0	0	0	0	0	-66573	29(7)54	52(12)81	11(17)40	19(21)11	19(25)17	13(29)90	72(32)56
15	0	0	0	0	0	0	0	0	0	-165507	-16(7)16	32(14)83	18(19)79	42(23)06	60(27)28	56(31)57
16	0	0	0	0	0	0	0	0	0	-564	8362953	27(11)43	89(16)62	57(21)90	18(26)12	31(30)32
17	0	0	0	0	0	0	0	0	0	0	85002	-42(9)80	-46(15)04	-55(20)28	-27(25)04	-67(29)34
18	0	0	0	0	0	0	0	0	0	0	288	-15047856	-14(18)92	-30(24)36	-30(28)78	-21(32)64
19	0	0	0	0	0	0	0	0	0	0	0	15047856	19(14)10	46(19)61	37(24)90	13(29)42
20	0	0	0	0	0	0	0	0	0	0	0	-61(12)40	-32(18)66	-45(23)04	-23(28)60	0
21	0	0	0	0	0	0	0	0	0	0	0	13(11)69	19(17)96	49(22)11	39(27)25	0
22	0	0	0	0	0	0	0	0	0	0	0	-154206	-19(9)28	-95(15)98	-48(21)02	-58(26)82
23	0	0	0	0	0	0	0	0	0	0	0	-522	13(7)15	37(14)01	40(20)15	80(25)15
24	0	0	0	0	0	0	0	0	0	0	0	0	-5815272	-11(13)26	-30(19)88	-10(25)94
25	0	0	0	0	0	0	0	0	0	0	0	0	-62616	23(11)81	19(18)58	11(24)73
26	0	0	0	0	0	0	0	0	0	0	0	0	-216	-10(17)16	-10(17)16	-11(23)02
27	0	0	0	0	0	0	0	0	0	0	0	0	0	45(15)55	10(22)70	0
28	0	0	0	0	0	0	0	0	0	0	0	0	0	-8650458	-15(14)42	-80(20)54
29	0	0	0	0	0	0	0	0	0	0	0	0	0	-96237	43(12)14	55(19)26
30	0	0	0	0	0	0	0	0	0	0	0	0	0	-336	-84(10)36	-32(18)78
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	10(9)61	16(17)50
32	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-63(6)14	-67(15)76
33	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2245968	22(14)53
34	0	0	0	0	0	0	0	0	0	0	0	0	0	0	27342	-58(12)80
35	0	0	0	0	0	0	0	0	0	0	0	0	0	0	99	10(11)24
36	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-12(9)22
38	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	74(6)99
39	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2395884
41	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-30840
42	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-114

Table 15. Some BPS invariants for $n_{(d_E, 5)}^g$. To save space we only give the first and the last two significant digits and the number of omitted digits in brackets.

We first discuss the gauge choice made in [3], which are the following

$$\begin{aligned}
s_{11}^1 &= -\frac{2}{z_1}, & s_{12}^1 &= -\frac{1}{3z_2}, & s_{22}^1 &= 0, \\
s_{11}^2 &= 0, & s_{12}^2 &= 0, & s_{22}^2 &= -\frac{4}{3z_2}, \\
h_1^{11} &= z_1 \left[\frac{1}{9} - 48z_1 + \frac{5}{6}z_2 - 540z_1z_2 \right], & h_1^{12} &= z_2 \left[-\frac{5}{108} - \frac{5}{4}z_2 + 20z_1 + 540z_1z_2 \right], \\
h_1^{22} &= -60z_2^2(1 + 27z_2), & h_2^{11} &= -60z_1^3, \\
h_2^{12} &= z_1 \left[\frac{1}{9} + \frac{5}{12}z_2 - 48z_1 \right], & h_2^{22} &= z_2 \left[-\frac{23}{54} + 40z_1 - \frac{5}{2}z_2 - 540z_1z_2 \right], \\
h_1^1 &= \frac{155}{27}z_1 - \frac{25}{1296}z_2 + 50z_1z_2, & h_1^2 &= 0, \\
h_2^1 &= -\frac{5}{18}z_1 + 120z_1^2, & h_2^2 &= \frac{155}{27}z_1 + \frac{1055}{1296}z_2 + 50z_1z_2, \\
h_1 &= \frac{25}{23328z_1}, & h_2 &= -\frac{50}{3}z_1, \\
h_{11} &= \frac{5}{36z_1z_2}, & h_{12} &= \frac{5}{108z_1z_2}, & h_{22} &= 0.
\end{aligned} \tag{B.3}$$

We will need to know how the propagators transform under the involution. Our guiding principle is the following. We replace the coordinates z_i 's with x_i 's and all quantities with their transformations in the equations (B.2). We then require that the resulting equations are equivalent to the original ones. Here for the holomorphic ambiguities we can simply replace the z_i coordinates with x_i 's in the expressions (B.3). Since the Picard-Fuchs operators are invariant under the involution, we argue that the Kahler potential is invariant $\tilde{K} = K$, and the Christoffel symbols transform as

$$\tilde{\Gamma}_{ij}^k = \frac{\partial x_k}{\partial z_l} \frac{\partial z_m}{\partial x_i} \frac{\partial z_n}{\partial x_j} \Gamma_{mn}^l + \frac{\partial^2 z_m}{\partial x_i \partial x_j} \frac{\partial x_k}{\partial z_m} \tag{B.4}$$

We find that it is possible to solve for the transformed propagators $\tilde{S}^{ij}, \tilde{S}^i, \tilde{S}$ such that the transformed equations of (B.2) are equivalent to the original ones.

It is easy to check that with the choices of the ambiguities s_{jk}^i in (B.3), the transformations are

$$\tilde{s}_{ij}^k = \frac{\partial x_k}{\partial z_l} \frac{\partial z_m}{\partial x_i} \frac{\partial z_n}{\partial x_j} s_{mn}^l + \frac{\partial^2 z_m}{\partial x_i \partial x_j} \frac{\partial x_k}{\partial z_m}. \tag{B.5}$$

The shift exactly cancels that of the Christoffel symbols in the first equation in (B.2), therefore if S^{ij} transform as tensor with a minus sign, the transformed equation is equivalent to the original one by a coordinate transformation.

To solve for the shifts in the transformed propagators \tilde{S}^i , we combine the involution transformation with the coordinate transformation of the second equation in (B.2). We find the shift in the gauge s_{ij}^k exactly cancels the one from the derivative of S^{jk} propagator and all propagators cancel out, so the shifts f^i ($i = 1, 2$) are determined by the equations

$$\delta_i^j f^k + \delta_i^k f^j + \tilde{h}_i^{jk} + \frac{\partial x_j}{\partial z_l} \frac{\partial x_k}{\partial z_m} \frac{\partial z_n}{\partial x_i} h_n^{lm} = 0. \tag{B.6}$$

We see that if h_i^{jk} transformed as a tensor with a minus sign, the shifts would have vanished. The shifts are results of the non-tensorial transformation of the gauge choice h_i^{jk} .

Similarly we combine the involution transformation with the coordinate transformation of the third equation in (B.2). Utilizing the solution (3.33), we again find all dependence on propagators cancel, and the shift f^0 is determined by

$$\frac{\partial z_k}{\partial x_i}(\partial_k f^j) = 2\delta_i^j f^0 - \tilde{s}_{im}^j f^m + \tilde{h}_i^j + \frac{\partial x_j}{\partial z_l} \frac{\partial z_n}{\partial x_i} h_n^l \tag{B.7}$$

and we find the the last equation in (3.33)

With the solutions for the shifts (3.33), one can further check that the involution transformation of the fourth and fifth equations in (B.2) are equivalent to the original ones by coordinate transformations.

It is possible to find a gauge choice different from (B.3), such that the shifts in propagators f^0, f^i vanish. We can keep the propagators S^{ij} and ambiguity s_{jk}^i , and shift the gauge choice for h_i^{jk} in (B.3) by

$$h_i^{jk} \rightarrow h_i^{jk} + \delta_i^j h^k + \delta_i^k h^j \tag{B.8}$$

where as an example we choose

$$h^1 = \frac{5z_1}{12}(432z_1 - 1), \quad h^2 = \frac{5z_2}{216}(1 + 54z_2). \tag{B.9}$$

One can easily check that after the shifts, the ambiguity h_i^{jk} transforms as a tensor with a minus sign under the involution transformation. So according to (B.6) this new gauge choice will eliminate the shifts f^1, f^2 in the involution transformation of the propagators S^i . The shift in h_i^{jk} will change the propagators S^i, S and the other ambiguities h_i^j, h_i, h_{ij} as well, which can be straightforwardly determined.

Similarly we can further shift the ambiguity h_i^j to cancel the shift f^0 in the involution transformation of the propagator S .

The gauge choice which gives no shifts for the involution transformation of the propagators is conceptually simple, and gives a much better understanding of the involution symmetry as an autoequivalence of the underlying derived categories. However, as it turns out such a gauge choice would introduce pole of $1 - 432z_1$ in the ambiguities h_i^j, h_i . Without detailed calculations, this point can be seen by noting that the shifts (3.33) have poles in $1 - 432z_1$ which does not appear in the gauge choice in (B.3), so in order to cancel the shifts, we would likely need to introduce poles of $1 - 432z_1$ in the ambiguities. As a consequence, the holomorphic ambiguities at higher genus would have also poles at $1 - 432z_1$. The total topological string amplitudes at higher genus should be regular at $1 - 432z_1$, so the singular part of the holomorphic ambiguities at the pole $1 - 432z_1$ can be fixed and the gauge choice does not really enlarge the space of holomorphic ambiguities at higher genus but complicates the calculations.

In practical calculations it is simpler to use the gauge (B.3), for which it turns out the holomorphic ambiguities at higher genus have no pole at $1 - 432z_1$, but only at the conifold divisors Δ_1 and Δ_2 . In the following we will still use the gauge choice (B.3).

C Reducing the ambiguity with the involution symmetry

We can compute the number of remaining unknown constants for general genus g after imposing the condition (3.5) more precisely. First we can find a particular holomorphic ambiguity such that the total amplitude with the propagator dependent part satisfies the constraint (3.5). Then there is the freedom to add an additional piece $f^{(g)}$ to the holomorphic ambiguity which satisfies the same symmetry condition

$$\tilde{f}^{(g)} = (-1)^{g-1} f^{(g)}. \quad (\text{C.1})$$

The space of such $f^{(g)}$ form a linear space. The number of remaining unknown constants is the dimension of this linear space.

We can define the following linear vector spaces of holomorphic ambiguities

$$\begin{aligned} V_0^{(g,m,n)} &:= \left\{ f \mid f = \frac{p(z_1)z_2^n}{(\Delta_1\Delta_2)^{2g-2}}, \text{ where } p(z_1) \text{ is a polynomial degree } m \text{ in } z_1 \right\}. \\ V_1^{(g,m,n)} &:= \{f \mid f \in V_0^{(g,m,n)} \text{ and } \tilde{f} \in V_0^{(g,m,n)}\}. \\ V_2^{(g,m,n)} &:= \{f \mid f \in V_1^{(g,m,n)} \text{ and } \tilde{f} = (-1)^{g-1} f\}. \\ V_3^{(g,m,n)} &:= \{f \mid f \in V_1^{(g,m,n)} \text{ and } \tilde{f} = (-1)^g f\}. \end{aligned} \quad (\text{C.2})$$

It is easy to check they are indeed linear vector spaces, i.e., any linear combinations of elements of the space is also an element in the space. We would like to construct the explicit linear basis for these spaces.

Clearly $\dim(V_0^{(g,m,n)}) = m + 1$. A convenient linearly independent basis is

$$f(z_1, z_2) = \frac{z_1^k \left(\frac{1}{432} - z_1\right)^{m-k} z_2^n}{(\Delta_1\Delta_2)^{2g-2}}, \quad \text{where } k = 0, 1, \dots, m \quad (\text{C.3})$$

We can work out the involution transformation

$$\frac{z_1^k \left(\frac{1}{432} - z_1\right)^{m-k} z_2^n}{(\Delta_1\Delta_2)^{2g-2}} \rightarrow \frac{(-1)^n z_2^n}{(\Delta_1\Delta_2)^{2g-2}} z_1^{m-k+3n-6g+6} \left(\frac{1}{432} - z_1\right)^{k-3n+6g-6}. \quad (\text{C.4})$$

We discuss two cases

1. The case $n \geq 2g - 2$. The holomorphic ambiguity in (C.3) $f \in V_1^{(g,m,n)}$ for $k \geq 3n - (6g - 6)$. On the other hand, any linear combinations of the basis with $k < 3n - (6g - 6)$ in (C.3) is not in the space $V_1^{(g,m,n)}$ since there will be poles of $\left(\frac{1}{432} - z_1\right)$ in the numerator. Therefore the space $V_1^{(g,m,n)}$ is generated by the linear combinations of $\frac{z_1^k \left(\frac{1}{432} - z_1\right)^{m-k} z_2^n}{(\Delta_1\Delta_2)^{2g-2}}$, where $k = 3n - (6g - 6), \dots, m$. In this case the dimension of $V_1^{(g,m,n)}$ is

$$\dim(V_1^{(g,m,n)}) = \begin{cases} m + 6g - 6 - 3n + 1, & \text{if } m + 6g - 6 - 3n \geq 0; \\ 0, & \text{if } m < 3n - (6g - 6). \end{cases} \quad (\text{C.5})$$

2. The case $n < 2g - 2$. A similar discussion shows that the space $V_1^{(g,m,n)}$ is generated by the linear combinations of $\frac{z_1^k (\frac{1}{432} - z_1)^{m-k} z_2^n}{(\Delta_1 \Delta_2)^{2g-2}}$, where $k = 0, 1, \dots, m + 3n - 6g + 6$. In this case the dimension of $V_1^{(g,m,n)}$ is

$$\dim(V_1^{(g,m,n)}) = \begin{cases} m + 3n - 6g + 6 + 1, & \text{if } m + 3n - 6g + 6 \geq 0; \\ 0, & \text{if } m < 3n - (6g - 6). \end{cases} \quad (\text{C.6})$$

In both cases we can write the formula

$$\dim(V_1^{(g,m,n)}) = \begin{cases} m - |3n - 6g + 6| + 1, & \text{if } m \geq |3n - 6g + 6|; \\ 0, & \text{if } m < |3n - 6g + 6|. \end{cases} \quad (\text{C.7})$$

For any element $f \in V_1^{(g,m,n)}$ we can write

$$f = \frac{1}{2}(f + (-1)^{g-1} \tilde{f}) + \frac{1}{2}(f + (-1)^g \tilde{f}). \quad (\text{C.8})$$

Clearly the first term $\frac{1}{2}(f + (-1)^{g-1} \tilde{f}) \in V_2^{(g,m,n)}$ and the second term $\frac{1}{2}(f + (-1)^g \tilde{f}) \in V_3^{(g,m,n)}$. We also note that $V_2^{(g,m,n)} \cap V_3^{(g,m,n)} = \{0\}$. So we have the decomposition

$$V_1^{(g,m,n)} = V_2^{(g,m,n)} \oplus V_3^{(g,m,n)} \quad (\text{C.9})$$

To construct the linear basis for $V_2^{(g,m,n)}$ and $V_3^{(g,m,n)}$, we further define two spaces $V_+^{(g,m,n)}$ and $V_-^{(g,m,n)}$ generated by linearly independent basis

$$V_+^{(g,m,n)} := \left\{ \begin{array}{l} \text{linear space generated by } \frac{z_1^{k_1} (\frac{1}{432} - z_1)^{k_2} z_2^n}{(\Delta_1 \Delta_2)^{2g-2}} \in V_0^{(g,m,n)}, \\ \text{with } k_1 - k_2 = 3n - 6g + 6. \end{array} \right\}, \quad (\text{C.10})$$

$$V_-^{(g,m,n)} := \left\{ \begin{array}{l} \text{linear space generated by } \frac{(\frac{1}{432} - 2z_1) z_1^{k_1} (\frac{1}{432} - z_1)^{k_2} z_2^n}{(\Delta_1 \Delta_2)^{2g-2}} \in V_0^{(g,m,n)}, \\ \text{with } k_1 - k_2 = 3n - 6g + 6. \end{array} \right\} \quad (\text{C.11})$$

From the transformation (C.4), we can check that for $f \in V_+^{(g,m,n)}$ we have $\tilde{f} = (-1)^n f$, while for $f \in V_-^{(g,m,n)}$ we have $\tilde{f} = (-1)^{n+1} f$. Therefore if $n + g$ is an odd integer, we have $V_+^{(g,m,n)} \subseteq V_2^{(g,m,n)}$ and $V_-^{(g,m,n)} \subseteq V_3^{(g,m,n)}$, while if $n + g$ is an even integer, we have $V_+^{(g,m,n)} \subseteq V_3^{(g,m,n)}$ and $V_-^{(g,m,n)} \subseteq V_2^{(g,m,n)}$. So we find

$$\dim(V_+^{(g,m,n)}) + \dim(V_-^{(g,m,n)}) \leq \dim(V_2^{(g,m,n)}) + \dim(V_3^{(g,m,n)}) = \dim(V_1^{(g,m,n)}). \quad (\text{C.12})$$

We can compute the dimensions of $V_+^{(g,m,n)}$ and $V_-^{(g,m,n)}$ easily from the explicit linearly independent basis in (C.10) and (C.11). Similar to the case of $V_1^{(g,m,n)}$, one also need to

consider two cases $n \geq 2g - 2$ and $n < 2g - 2$, and find a universal formulae

$$\begin{aligned} \dim(V_+^{(g,m,n)}) &= \begin{cases} \lfloor \frac{m - |3n - 6g + 6|}{2} \rfloor + 1, & \text{if } m \geq |3n - 6g + 6|; \\ 0, & \text{if } m < |3n - 6g + 6|. \end{cases} \\ \dim(V_-^{(g,m,n)}) &= \begin{cases} \lfloor \frac{m - 1 - |3n - 6g + 6|}{2} \rfloor + 1, & \text{if } m - 1 \geq |3n - 6g + 6|; \\ 0, & \text{if } m - 1 < |3n - 6g + 6|. \end{cases} \end{aligned} \quad (\text{C.13})$$

Comparing with (C.7) we find actually the inequality in (C.12) is saturated. So the spaces $V_+^{(g,m,n)}$ and $V_-^{(g,m,n)}$ provide the complete linear basis, we have $V_+^{(g,m,n)} = V_2^{(g,m,n)}$, $V_-^{(g,m,n)} = V_3^{(g,m,n)}$ for the case of $n + g$ an odd integer, and $V_+^{(g,m,n)} = V_3^{(g,m,n)}$, $V_-^{(g,m,n)} = V_2^{(g,m,n)}$ for the case of $n + g$ an even integer.

We can estimate the number of unknown constants for the holomorphic ambiguity at large genus g after imposing the involution symmetry, assuming the degree of z_1 is $7(g - 1)$ in the numerator

$$\sum_{n=0}^{\infty} \dim(V_2^{(g,7(g-1),n)}) = \sum_{n=0}^{\frac{13(g-1)}{3}} \frac{7(g-1) - |3n - 6(g-1)|}{2} \sim \frac{97}{12}g^2, \quad \text{for large } g. \quad (\text{C.14})$$

This is about one quarter of the $(7g - 6)(5g - 4)$ unknown constants in the naive ansatz at large genus.

D Fibre modularity versus involution symmetry

In this appendix we will prove that the constraints imposed on the amplitudes and especially on the holomorphic ambiguity from the involution symmetry is equivalent to the constraints from the modularity in the fibre direction discussed in section 4. In addition the appendix provides many identities that are useful for further discussion of the model [39].

From the solutions³² of the (2.10) around the large volume point $(z_1, z_2) \sim (0, 0)$,

$$\begin{aligned} X_0 &= 1 + 60z_1 + 13860z_1^2 + \dots, \\ X_1 &= X_0 \log(z_1) + 312z_1 + 2z_2 + \dots, \\ X_2 &= X_0 \log(z_2) + 180z_1 - 6z_2 + \dots, \end{aligned} \quad (\text{D.1})$$

we find

$$\begin{aligned} q_E := q_1 &= \exp\left(\frac{X_1}{X_0}\right) = z_1 + 312z_1^2 + 2z_1z_2 + \dots, \\ q_B := q_2 &= \exp\left(\frac{X_2}{X_0}\right) = z_2 + 180z_1z_2 - 6z_2^2 + \dots. \end{aligned} \quad (\text{D.2})$$

The above relations can be readily inverted and we can write $z_{1,2}$ as power series of $q_{1,2}$.

³²We lower the index on the A-periods X^i in order to avoid cluttering with exponentials.

If we keep the gauge X_0^{2g-2} explicit and note that q_B is defined without the shift (4.1) then (4.4), (4.5) read

$$X_0^{2g-2} \mathcal{F}^{(g)} = \sum_{k=0}^{\infty} P_k^{(g)}(q_E) \left(\frac{q_E}{\eta(q_E)^{24}} \right)^{\frac{3k}{2}} q_B^k, \quad g \geq 1, \quad (\text{D.3})$$

while for the case of $g = 0$ we do not need the factor of X_0^{-2} in the conventional definition of the prepotential.

From section 4 it follows for the $X_{18}(1, 1, 1, 6, 9)$ model that the $P_k^{(g)}(q_E)$ are quasi-modular forms of $\text{SL}(2, \mathbb{Z})$, of modular weight $18k + 2g - 2$ and satisfy (4.13) as a specialization of (4.8). We would like to understand whether and to what extent this constraint from the modularity of the fiber fixes the holomorphic ambiguity at higher genus. Suppose at genus $g \geq 2$ we have found a particular holomorphic ambiguity such that the topological string amplitude $\mathcal{F}^{(g)}$ satisfies the fiber modularity constrain. Then we may add an additional holomorphic ambiguity $f^{(g)}$, whose expansion is

$$X_0^{2g-2} f^{(g)} = \sum_{k=0}^{\infty} p_k^{(g)}(q_E) \left(\frac{q_E}{\eta(q_E)^{24}} \right)^{\frac{3k}{2}} q_B^k, \quad (\text{D.4})$$

we require $p_k^{(g)}(q_E)$ to be also a quasi-modular form of weight $18k + 2g - 2$. Furthermore, since the sum $P_k^{(g)}(q_E) + p_k^{(g)}(q_E)$ also need to satisfy the equation (4.13), we find

$$\frac{\partial p_k^{(g)}}{\partial E_2} = -\frac{1}{12} \sum_{s=1}^{k-1} s(k-s) p_s^{(g)} P_{k-s}^{(0)}. \quad (\text{D.5})$$

In particular for the case $k = 0, 1$, this equation is understood as $\frac{\partial p_k^{(g)}}{\partial E_2} = 0$, i.e. $p_0^{(g)}$ and $p_1^{(g)}$ are always modular forms.

It turns out that not all rational functions $f^{(g)}$ satisfy the fiber modularity constraint (D.4), (D.5), so the fiber modularity does impose some conditions on holomorphic ambiguities. However we find that there exist some non-vanishing rational functions $f^{(g)}$ satisfying these conditions, so the fiber modularity does not completely fix the holomorphic ambiguity. We see that the BCOV holomorphic anomaly equation alone does not implies the modularity of the fiber at higher genus.

However it turns out that the modularity of the fiber is equivalent to the involution symmetry. It gives more precise bounds for the degrees in the polynomials in the numerators of the holomorphic ambiguity and further reduces the number of unknown constants in the holomorphic ambiguity. To understand this equivalence, we consider a holomorphic ambiguity $f \in V_0^{(g, +\infty, n)}$. Since $f \sim z_2^n \sim q_B^n$ in the small z limit, it is easy to see that the first non-vanish coefficient in the expansion in base degree (D.4) is that of q_B^n

$$X_0^{2g-2} f = \sum_{k=n}^{\infty} p_k(q_E) \left(\frac{q_E}{\eta(q_E)^{24}} \right)^{\frac{3k}{2}} q_B^k, \quad (\text{D.6})$$

According to (D.5) the first coefficient $p_n(q_E)$ has no E_2 dependence and is purely a modular form, i.e. polynomials of E_4 and E_6 .

We define the linear space

$$W_0^{(g,n)} := \{f \mid f \in V_0^{(g,+\infty,n)}, \text{ and the first non-vanishing coefficient } p_n(q_E) \text{ in the expansion (D.6) is a } \text{SL}(2, \mathbb{Z}) \text{ modular form of weight } 18n + 2g - 2\}. \quad (\text{D.7})$$

We will prove the following proposition

Proposition 3.

$$W_0^{(g,n)} = V_2^{(g, [\frac{19}{3}(g-1)], n)}$$

Proof. The proof will proceed in two steps. We shall first show $V_2^{(g, [\frac{19}{3}(g-1)], n)} \subseteq W_0^{(g,n)}$, and then $\dim(V_2^{(g, [\frac{19}{3}(g-1)], n)}) \geq \dim(W_0^{(g,n)})$. We will see how the number $[\frac{19}{3}(g-1)]$ appears in the process.

In order to prove the Proposition 3, we only need the leading order term in the small z_2 expansion. The first Picard-Fuchs equation in (2.10) in the $z_2 \rightarrow 0$ limit has been studied before in [34, 54]. We find the power series solution and A-model parameter q_E are determined by the exact relations

$$E_4(q_E) = X_0^4, \quad z_1(1 - 432z_1) = \frac{\eta(q_E)^{24}}{E_4(q_E)^3}, \quad (\text{D.8})$$

where we have taken the $z_2 \rightarrow 0$ limit for X_0 and q_E . We will drop the argument $q_E = q_1$ in the modular forms and η function for simplicity.

For the other logarithmic solution we need to keep the log term in $z_2 \rightarrow 0$ limit

$$X_2 = X_0 \log(z_2) + \xi(z_1) + \mathcal{O}(z_2), \quad (\text{D.9})$$

where $\xi(z_1) = 180z_1 + \mathcal{O}(z_1^2)$ is a power series. Using the first Picard-Fuchs equation in (2.10) we find

$$[\theta_1^2 - 12z_1(6\theta_1 + 1)(6\theta_1 + 5)]\xi(z_1) = 3\theta_1 X_0 \quad (\text{D.10})$$

There is an exact solution for $\xi(z_1)$ using (D.8) and the well-known Ramanujan derivative relations for quasi-modular forms

$$\xi(z_1) = -\frac{3}{2}X_0 \log \left[\frac{q_1(1 - 432z_1)}{z_1} \right]. \quad (\text{D.11})$$

Of course one can add any linear combination of two solutions of PF equation in the $z_2 \rightarrow 0$ limit, which are $E_4(q_1)^{\frac{1}{4}}$ and $\log(q_1)E_4(q_1)^{\frac{1}{4}}$. The above solution is the only one with the correct asymptotic behavior $\xi(z_1) \sim z_1$ for small z_1 . So the A-model parameter q_B is

$$q_B = z_2 \exp \left(\frac{\xi(z_1)}{X_0} \right) = z_2 \left(\frac{q_1(1 - 432z_1)}{z_1} \right)^{-\frac{3}{2}} + \mathcal{O}(z_2^2) \quad (\text{D.12})$$

We first discuss the case that $n + g$ is an odd integer. Suppose $f \in V_2^{(g, [\frac{19}{3}(g-1)], n)} = V_+^{(g, [\frac{19}{3}(g-1)], n)}$ is a base vector in (C.10)

$$f = \frac{z_1^{k_1} (\frac{1}{432} - z_1)^{k_2} z_2^n}{(\Delta_1 \Delta_2)^{2g-2}},$$

with $k_1 \geq 0, k_2 \geq 0, k_1 + k_2 \leq \frac{19}{3}(g-1), k_1 - k_2 = 3n - 6g + 6,$ (D.13)

then the leading term in the expansion (D.6) is

$$\begin{aligned} X_0^{2g-2} f &\sim E_4^{\frac{g-1}{2}} z_1^{k_1 - \frac{3n}{2}} (1 - 432z_1)^{k_2 + \frac{3n}{2} - 6g + 6} \eta^{36n} \left(\frac{qE}{\eta^{24}} \right)^{\frac{3n}{2}} q_B^n \\ &\sim E_4^{\frac{g-1}{2} + \frac{9n}{2} - 3k_1} \eta^{24k_1} \left(\frac{qE}{\eta^{24}} \right)^{\frac{3n}{2}} q_B^n, \end{aligned} \tag{D.14}$$

where we have used the relations (D.8) and ignore constant factors. Since $n + g$ is an odd integer, the exponent of E_4 is an integer. Furthermore

$$\frac{g-1}{2} + \frac{9n}{2} - 3k_1 = \frac{19}{2}(g-1) - \frac{3}{2}(k_1 + k_2) \geq 0, \tag{D.15}$$

so the coefficient $E_4^{\frac{g-1}{2} + \frac{9n}{2} - 3k_1} \eta^{24k_1}$ is a modular form of weight $18n + 2g - 2$.

The case of $n + g$ an even integer is similar. We need to use the formula $1 - 864z_1 = E_6/E_4^{\frac{3}{2}}$. In both cases we find $V_2^{(g, [\frac{19}{3}(g-1)], n)} \subseteq W_0^{(g, n)}$.

Now we compute $\dim(W_0^{(g, n)})$. We discuss two cases.

1. The case of $n \leq 2g - 2$. We consider the dimension of the space $\mathcal{M}_{18n+2g-2}$ of modular forms of weight $18n + 2g - 2$. It is clear $\dim(\mathcal{M}_{18n+2g-2}) = [\frac{9n+g-1}{6}] + 1$ if $n + g$ is odd, and $\dim(\mathcal{M}_{18n+2g-2}) = [\frac{9n+g-4}{6}] + 1$ if $n + g$ is even. This is exactly the dimension $\dim(V_2^{(g, [\frac{19}{3}(g-1)], n)})$ according to (C.13). The expansion (D.6) is a linear map from $W_0^{(g, n)}$ to $\mathcal{M}_{18n+2g-2}$ and the kernel is the zero element. So in this case we find

$$\dim(W_0^{(g, n)}) \leq \dim(\mathcal{M}_{18n+2g-2}) = \dim(V_2^{(g, [\frac{19}{3}(g-1)], n)}) \tag{D.16}$$

2. The case of $n > 2g - 2$. Suppose $f = \frac{h(z_1)z_2^n}{(\Delta_1 \Delta_2)^{2g-2}}$ where $h(z_1)$ is a polynomial of z_1 . We find the leading term in the expansion (D.6)

$$X_0^{2g-2} f \sim E_4^{\frac{g-1}{2}} \frac{h(z_1)}{z_1^{\frac{3n}{2}} (1 - 432z_1)^{6g-6-\frac{3n}{2}}} \eta^{36n} \left(\frac{qE}{\eta^{24}} \right)^{\frac{3n}{2}} q_B^n, \tag{D.17}$$

We consider the zero at $1 - 432z_1 \sim 0$, which correspond to $E_6 + E_4^{\frac{3}{2}} \sim 0$. From see (D.8) we see $\eta^{24} \sim (1 - 432z_1)$. So the leading coefficient in (D.6) $p_n(qE) \sim (1 - 432z_1)^{3n-6g+6}$. If $p_n(qE)$ is a modular form of weight $18n + 2g - 2$, then $\frac{p_n(qE)}{\eta^{72(n-2g+2)}}$ should be also a modular form of weight $74(g-1) - 18n$. So we find

$$\dim(W_0^{(g, n)}) \leq \dim(\mathcal{M}_{74(g-1)-18n}) = \dim(V_2^{(g, [\frac{19}{3}(g-1)], n)}) \tag{D.18}$$

□

So we have proven $W_0^{(g, n)} = V_2^{(g, [\frac{19}{3}(g-1)], n)}$, and in the proof we see that they are also isomorphic to the space of modular forms $\mathcal{M}_{18n+2g-2}$ for the $n \leq 2g - 2$ case, and $\mathcal{M}_{74(g-1)-18n}$ for the $n > 2g - 2$ case. We further conjecture this is sufficient for the full fiber modularity constrain of the higher terms.

Conjecture 1. For $f \in W_0^{(g,n)}$ with the expansion (D.6), the higher order terms $p_k(q_E)$ with $k > n$ are quasi-modular forms of weight $18k + 2g - 2$ and satisfy the equation (D.5).

If our conjecture is correct, then the space of holomorphic ambiguities at genus g satisfying the modularity constraint of the elliptic fiber is linearly generated by $\sum_{n=0}^{\infty} \oplus W_0^{(g,n)} = \sum_{n=0}^{\infty} \oplus V_2^{(g, \lfloor \frac{19}{3}(g-1) \rfloor, n)}$. For example, the explicit basis at genus $g = 2$ are the the following 7 elements

$$\begin{aligned} & \frac{(1 - 432z_1)^3 z_2}{(\Delta_1 \Delta_2)^2}, \quad \frac{(1 - 432z_1)^4 z_1 z_2}{(\Delta_1 \Delta_2)^2}, \quad \frac{(1 - 864z_1) z_2^2}{(\Delta_1 \Delta_2)^2}, \quad \frac{(1 - 864z_1)(1 - 432z_1) z_1 z_2^2}{(\Delta_1 \Delta_2)^2}, \\ & \frac{(1 - 864z_1)(1 - 432z_1)^2 z_1^2 z_2^2}{(\Delta_1 \Delta_2)^2}, \quad \frac{z_1^3 z_2^3}{(\Delta_1 \Delta_2)^2}, \quad \frac{(1 - 432z_1) z_1^4 z_2^3}{(\Delta_1 \Delta_2)^2}. \end{aligned} \tag{D.19}$$

The numbers of unknown constants at genus 2, 3, 4, 5, 6 after using the fiber modularity constrain are 7, 31, 70, 109, 176 respectively.

We can compute the number of unknown constants in the holomorphic ambiguity at genus g with the formula (C.13), and we only need to change the bound for degree of z_1 from $7(g - 1)$ to $\frac{19}{3}(g - 1)$ in the calculations (C.14)

$$\sum_{n=0}^{\infty} \dim(V_2^{(g, \frac{19}{3}(g-1), n)}) = \sum_{n=0}^{\frac{37}{9}(g-1)} \frac{\frac{19}{3}(g-1) - |3n - 6(g-1)|}{2} \sim \frac{721}{108} g^2, \quad \text{for large } g, \tag{D.20}$$

which is a better estimate than (C.14).

The degree $\lfloor \frac{19}{3}(g - 1) \rfloor$ of z_1 from fiber modularity (3) can be also derived from the regularity condition at $z_1 \sim \infty$. We can use the local coordinates $(\frac{1}{z_1}, z_2)$ around the point $(z_1, z_2) = (\infty, 0)$. It is easy to check that the power series solutions to the PF equations have the asymptotic behavior $w \sim z_1^{-\frac{1}{6}}$ or $w \sim z_1^{-\frac{5}{6}}$. For the scaling factor X_0 in the topological string amplitude we take the one with the lowest exponent $X_0 \sim z_1^{-\frac{1}{6}}$. So the scaling exponent including the conifold divisors is

$$\left(\frac{X_0}{\Delta_1 \Delta_2} \right)^{2g-2} \sim z_1^{-\frac{19}{3}(g-1)}, \tag{D.21}$$

and the regularity of $X_0^{2g-2} \mathcal{F}^{(g)}$ at $z_1 \sim \infty$ requires the degree of z_1 at the numerator in the holomorphic ambiguity be no bigger than $\frac{19}{3}(g - 1)$. The non-zero scaling exponent in X_0 around $z_1 \sim \infty$ is characteristic of compact Calabi-Yau models, and we will see that it also is responsible for preventing the solution of the model at large genus as in one-parameter models like the quintic.

D.1 Some (quasi-)modularity formulae

In order to understand Conjecture 1, we provide some more formulae relating the complex structure coordinates $z_{1,2}$ to (quasi)modular forms of the fiber parameter. We note that for $f \in W_0^{(g,n)}$ in the case of $n + g$ odd, we can factor (D.13) with $k_2 = k_1 - 3n + 6g - 6$ as the following

$$f = [z_1(1 - 432z_1)]^{k_1} \left(\frac{z_2}{(1 - 432z_1)^3} \right)^n \left(\frac{\Delta_1 \Delta_2}{(1 - 432z_1)^3} \right)^{2-2g}. \tag{D.22}$$

So we need to understand the expansion of $z_1(1 - 432z_1)$, $\frac{z_2}{(1-432z_1)^3}$, and $\frac{\Delta_1\Delta_2}{(1-432z_1)^3}$ in terms of exponential of flat coordinates q_E, q_B . There is an extra factor of $1 - 864z_1$ in the case of $n + g$ an even integer. Furthermore, we also need to consider the A-model scaling factor X_0 which is the power series solution to the Picard-Fuchs equation.

We define the parameter

$$Z := \frac{z_2}{(1 - 432z_1)^3 E_4(q_E)^{\frac{9}{2}}}, \quad (\text{D.23})$$

and as a first step we expand some components as asymptotic series of Z , and we find that the coefficients are modular forms of fiber parameter q_E as the followings

$$z_1(1 - 432z_1) = \frac{\eta^{24}}{E_4^3} \left[1 - \frac{77E_4^3 E_6 + 211E_6^3}{144} Z + \frac{1}{165888} (-11858E_4^9 + 330395E_4^6 E_6^2 + 1434560E_4^3 E_6^4 + 1066999E_6^6) Z^2 + \mathcal{O}(Z^3) \right], \quad (\text{D.24})$$

$$\frac{\Delta_1\Delta_2}{(1 - 432z_1)^3} = 1 + \frac{27}{4} (3E_4^3 E_6 + E_6^3) Z - \frac{9}{128} (E_4^3 - E_6^2) (85E_4^6 - 612E_4^3 E_6^2 - 49E_6^4) Z^2 + \mathcal{O}(Z^3), \quad (\text{D.25})$$

$$1 - 864z_1 = \frac{1}{E_4^{\frac{3}{2}}} \left[E_6 + \frac{1}{288} (E_4^3 - E_6^2) (77E_4^3 + 211E_6^2) Z - \frac{1}{331776} E_6 (E_4^3 - E_6^2) \cdot (383525E_4^6 + 1458614E_4^3 E_6^2 + 977957E_6^4) Z^2 + \mathcal{O}(Z^3) \right], \quad (\text{D.26})$$

$$X_0 = E_4^{\frac{1}{4}} \left[1 - \frac{5}{72} E_6 (E_4^3 - E_6^2) Z + \frac{5}{7962624} (E_4^3 - E_6^2) \cdot (15935E_4^6 + 488258E_4^3 E_6^2 + 435839E_6^4) Z^2 + \mathcal{O}(Z^3) \right], \quad (\text{D.27})$$

Here we fix the modular forms in low orders by perturbative calculations. In general we conjecture the following

Conjecture 2. *With the appropriate normalization factors as in (D.24), (D.25), (D.26), (D.27), the coefficients of Z^n in the expansion are modular forms, and the modular weights are $18n$ for the cases of $z_1(1 - 432z_1)$, $\frac{\Delta_1\Delta_2}{(1-432z_1)^3}$, X_0 , and $18n + 6$ for the case of $1 - 864z_1$.*

In the next subsection D.2, we shall prove this modularity conjecture. We note that we only need to prove the modularity of the coefficients of the expansion of $z_1(1 - 432z_1)$ and $1 - 864z_1$, which immediately imply that of $\frac{\Delta_1\Delta_2}{(1-432z_1)^3}$, since we can write

$$\begin{aligned} \frac{\Delta_1\Delta_2}{(1 - 432z_1)^3} &= 1 - \frac{27z_2}{(1 - 432z_1)^3} (1 - 864z_1) [(1 - 864z_1)^2 + 1296z_1(1 - 432z_1)] \\ &\quad - \frac{(27z_2)^2}{(1 - 432z_1)^6} [432z_1(1 - 432z_1)]^3. \end{aligned} \quad (\text{D.28})$$

We also notice that the higher order modular form coefficients in the expansion formula (D.26) contain a factor of $\eta^{24} = \frac{E_4^3 - E_6^2}{1728}$. This condition is sufficient to ensure the modularity of expansion (D.24), and therefore that of (D.25) as well. To see this, suppose

$$1 - 864z_1 = \frac{1}{E_4^{\frac{3}{2}}} \left[E_6 + (E_4^3 - E_6^2) \sum_{n=1}^{\infty} f_{18n-6} Z^n \right], \quad (\text{D.29})$$

where f_{18n-6} are some modular forms of weight $18n - 6$. We can compute (D.24) easily

$$z_1(1 - 432z_1) = \frac{\eta^{24}}{E_4^3} \left[1 - 2E_6 \left(\sum_{n=1}^{\infty} f_{18n-6} Z^n \right) + (E_4^3 - E_6^2) \left(\sum_{n=1}^{\infty} f_{18n-6} Z^n \right)^2 \right] \quad (\text{D.30})$$

Now it is easy to see that if Conjecture 2 is correct, then for $f \in W_0^{(g,n)}$, it has the following expansion

$$f = \sum_{n=0}^{+\infty} f_n Z^n, \quad (\text{D.31})$$

where f_n are modular forms of q_E of modular weight $18n + 2g - 2$.

The next step is to expand Z parameter in terms of the base coordinate q_B . For convenience we denote the normalized Kahler coordinate

$$Q := \left(\frac{q_E}{\eta^{24}} \right)^{\frac{3}{2}} q_B, \quad (\text{D.32})$$

and we find the formulae for low orders

$$\begin{aligned} Z = Q &+ \frac{1}{576} [-E_2 E_4 (31E_4^3 + 113E_6^2) + 1759E_4^3 E_6 + 1841E_6^3] Q^2 \\ &+ \frac{1}{15925248} [72E_2^2 E_4^2 (31E_4^3 + 113E_6^2)^2 \\ &\quad - 108E_2 E_4 E_6 (25249E_4^6 + 143422E_4^3 E_6^2 + 163105E_6^4) \\ &\quad + 102919E_4^9 + 33568143E_4^6 E_6^2 + 79676877E_4^3 E_6^4 + 64318109E_6^6] Q^3 + \mathcal{O}(Q^4) \end{aligned} \quad (\text{D.33})$$

We can check that the coefficients c_n of Q^n are quasi-modular forms, and satisfy the following equation as in (D.5)

$$\frac{\partial c_n}{\partial E_2} = -\frac{1}{12} \sum_{s=1}^{n-1} s(n-s) c_{n-s} P_s^{(0)}, \quad (\text{D.34})$$

where the $P_n^{(0)}$ are the quasi-modular forms of the prepotential expanded in terms of base degree, for example

$$\begin{aligned} P_1^{(0)} &= \frac{E_4}{48} (31E_4^3 + 113E_6^2), \\ P_2^{(0)} &= -\frac{1}{221184} [4E_2 E_4^2 (31E_4^3 + 113E_6^2)^2 + 208991E_4^7 E_6 \\ &\quad + 755906E_4^4 E_6^3 + 196319E_4 E_6^5] \end{aligned} \quad (\text{D.35})$$

We define the following space

$$W_1 := \left\{ f \mid f = \sum_{n=1}^{\infty} c_n Q^n, \text{ where } c_n \text{ are quasi-modular forms} \right. \\ \left. \text{and satisfy equation (D.34)} \right\}. \quad (\text{D.36})$$

Here we do not specify the modular weights of c_n , so unlike the previous cases, the space W_1 is not a linear space. Rather, we will see it is a graded module over modular forms. Since we know the modular weight of $P_n^{(0)}$ is $18n - 2$, the modular weights of all c_n 's are actually determined by the first non-vanishing coefficient according to equation (D.34). It is clear if $f \in W_1$ and g is a modular form, then $fg \in W_1$.

The equation (D.34) can be equivalently written as

$$\frac{\partial f}{\partial E_2} = -\frac{1}{12} Q^2 (\partial_Q f) \partial_Q P^{(0)}, \quad (\text{D.37})$$

where we define the generating function $P^{(0)} := \sum_{s=1}^{\infty} P_s^{(0)} Q^s$. Suppose $f \in W_1$ and $g \in W_1$, then we have $fg \in W_1$ since

$$\frac{\partial(fg)}{\partial E_2} = -\frac{1}{12} g Q^2 (\partial_Q f) \partial_Q P^{(0)} - \frac{1}{12} f Q^2 (\partial_Q g) \partial_Q P^{(0)} = -\frac{1}{12} Q^2 (\partial_Q fg) \partial_Q P^{(0)}. \quad (\text{D.38})$$

Furthermore, if $f \in W_1$, then any algebraic function of f would also satisfy the equation (D.37). However, f may not have the structure of Q -expansion defined by W_1 space in (D.36), and the coefficients c_n may be only algebraic functions of quasi-modular forms.

The equation (D.37) can be further written in a simpler way which will be useful later. In the left hand side of (D.37), we regard f as function of $f(Q, E_2, E_4, E_6)$ when taking partial derivative with respect to E_2 . Alternatively, we can also use inverse function and regard Q as function of $Q(f, E_2, E_4, E_6)$. In this perspective we have $\frac{\partial f}{\partial E_2} = 0$ and $\frac{\partial Q}{\partial E_2} \neq 0$. We can take partial derivative of E_2 on $f = \sum_{n=1}^{\infty} c_n Q^n$, and find

$$\sum_{n=1}^{\infty} (\partial_{E_2} c_n) Q^n + (\partial_Q f) (\partial_{E_2} Q) = 0. \quad (\text{D.39})$$

So we can write equation (D.37) as

$$\frac{\partial \log(Q)}{\partial E_2} = \frac{1}{12} Q \partial_Q P^{(0)}. \quad (\text{D.40})$$

Based on the perturbative calculations (D.33), we conjecture

Conjecture 3. $Z \in W_1$.

If our conjecture is correct, then obviously it is also true that $Z^n \in W_1$ for any positive integer n . Using the expansion (D.31) and keeping track of the modular weights, we can see that Conjectures 2 and 3 and calculations in this subsection would imply the validity of the main conjecture, Conjecture 1.

D.2 Proof of Conjecture 2 and Conjecture 3.

In this subsection we shall prove the Conjectures 2 and 3 for our compact model. The proof uses some ingredients in the proof of genus zero modular anomaly equation for the non-compact half K3 model in [34, 54], but it is much more complicated for compact model.

To solve the Picard-Fuchs equations (2.10), it is convenient to define the variable \tilde{q} , which is related to z_1 by

$$J(\tilde{q}) = \frac{1}{z_1(1 - 432z_1)}, \quad (\text{D.41})$$

with the asymptotic behavior $\tilde{q} \sim z_1 \sim 0$ in small z_1 limit. This is the Kahler parameter in local large base limit $z_2 \rightarrow 0$, but in the compact model, it is different from the physical Kähler parameter q_1 which depends on both z_1 and z_2 . To be clear in this subsection we will keep the arguments \tilde{q} and q_1 in the modular functions, and we also define the logarithmic flat coordinates $t_1 = \log(q_1)$ and $\tilde{t} = \log(\tilde{q})$.

We can write the ansatz for the solution to the equations (2.10) in small z_2 power series expansion

$$w = \sum_{n=0}^{\infty} c_n(z_1) z_2^n \quad (\text{D.42})$$

It is well known in the local $z_2 \rightarrow 0$ limit, there are two linearly independent solutions, which can be written as $E_4(\tilde{q})^{\frac{1}{4}}$ and $\log(\tilde{q})E_4(\tilde{q})^{\frac{1}{4}}$, in terms of the modular parameter \tilde{q} defined in (D.41). So the initial function $c_0(z_1)$ in the expansion is $E_4(\tilde{q})^{\frac{1}{4}}$ or $\log(\tilde{q})E_4(\tilde{q})^{\frac{1}{4}}$.

The first and second PF equations in (2.10) become

$$\begin{aligned} 12z_1(6\theta_1 + 1)(6\theta_1 + 5)c_n(z_1) &= \theta_1(\theta_1 - 3n)c_n(z_1), \\ c_{n+1}(z_1) &= \frac{1}{(n+1)^3} \prod_{i=0}^2 (\theta_1 - 3n - i)c_n(z_1) \end{aligned} \quad (\text{D.43})$$

where $\theta_{z_1} = z_1 \partial_{z_1}$. We see that the second equation provides a recursion relation to compute the higher order coefficients $c_n(z_1)$. Furthermore, one can check that the two equations are consistent, i.e. the c_{n+1} from recursion in the second equation also satisfies the first equation for c_n with by replacing n with $n+1$. The recursion implies

$$c_n(z_1) = \frac{1}{(n!)^3} \prod_{k=0}^{3n-1} (\theta_{z_1} - k)c_0(z_1) \quad (\text{D.44})$$

A particular modularity structure for such recursion was derived in [34, 54]. Here we repeat the analysis. From the relation (D.41) it is easy to compute

$$1 - 864z_1 = \frac{E_6(\tilde{q})}{E_4(\tilde{q})^{\frac{3}{2}}}, \quad \theta_{z_1} = \frac{2E_4(\tilde{q})}{E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q})} \partial_{\tilde{t}}. \quad (\text{D.45})$$

We can write some derivative formulae

$$\begin{aligned}
 \partial_{\tilde{t}} \frac{E_4(\tilde{q})^{\frac{3}{2}}}{E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q})} &= \frac{E_4(\tilde{q})^{\frac{1}{2}} E_4(\tilde{q})^{\frac{3}{2}} - E_6(\tilde{q})}{2 E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q})}, \\
 \partial_{\tilde{t}} \frac{E_2(\tilde{q})}{E_4(\tilde{q})^{\frac{1}{2}}} &= -\frac{E_2(\tilde{q})^2 E_4(\tilde{q}) - 2E_2(\tilde{q})E_6(\tilde{q}) + E_4(\tilde{q})^2}{12E_4(\tilde{q})^{\frac{3}{2}}}, \\
 \partial_{\tilde{t}} \frac{E_6(\tilde{q})}{E_4(\tilde{q})^{\frac{3}{2}}} &= -\frac{E_4(\tilde{q})^3 - E_6(\tilde{q})^2}{2E_4(\tilde{q})^{\frac{5}{2}}}, \\
 \partial_{\tilde{t}} E_4(\tilde{q})^{\frac{1}{4}} &= \frac{E_2(\tilde{q})E_4(\tilde{q}) - E_6(\tilde{q})}{12E_4(\tilde{q})^{\frac{3}{4}}}. \tag{D.46}
 \end{aligned}$$

First we consider the case $c_0(z_1) = E_4(\tilde{q})^{\frac{1}{4}}$. From the derivative formulae (D.46) one can deduce inductively that the recursion function

$$\prod_{k=0}^{n-1} (\theta_{z_1} - k) E_4(\tilde{q})^{\frac{1}{4}} \sim \frac{E_4(\tilde{q})^{\frac{1}{4}}}{(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^n} P_{6n}(E_2, E_4, E_6), \tag{D.47}$$

where $P_{6n}(E_2, E_4, E_6)$ is a quasi-modular form of weight $6n$ and is linear in $E_2(\tilde{q})$. We can write

$$\prod_{k=0}^{n-1} (\theta_{z_1} - k) E_4(\tilde{q})^{\frac{1}{4}} = \frac{E_4(\tilde{q})^{\frac{1}{4}}}{(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^n} (a_n E_2(\tilde{q}) + b_n), \tag{D.48}$$

where a_n and b_n are modular forms of weight $6n - 2$ and $6n$, without E_2 dependence. Some low order formulae are the followings

$$\begin{aligned}
 a_0 &= 0, & b_0 &= 1, & a_1 &= \frac{E_4(\tilde{q})}{6}, & b_1 &= -\frac{E_6(\tilde{q})}{6}, \\
 a_2 &= -\frac{E_4(\tilde{q})E_6(\tilde{q})}{3}, & b_2 &= \frac{5E_4(\tilde{q})^3 + 7E_6(\tilde{q})^2}{36}, \\
 a_3 &= \frac{E_4(\tilde{q})[77E_4(\tilde{q})^3 + 211E_6(\tilde{q})^2]}{216}, & b_3 &= -\frac{E_6(\tilde{q})[197E_4(\tilde{q})^3 + 91E_6(\tilde{q})^2]}{216}.
 \end{aligned}$$

The recursion relation for the modular forms a_n and b_n are

$$\begin{aligned}
 a_{n+1} &= -\left(2n - \frac{1}{6}\right) E_6(\tilde{q})a_n + \frac{1}{6} E_4(\tilde{q})b_n + 2E_4(\tilde{q})^{\frac{3n+1}{2}} \partial_{\tilde{t}} \left[\frac{a_n}{E_4(\tilde{q})^{\frac{3n-1}{2}}} \right], \\
 b_{n+1} &= -\left(2n + \frac{1}{6}\right) E_6(\tilde{q})b_n - \frac{1}{6} E_4(\tilde{q})^2 a_n + 2E_4(\tilde{q})^{\frac{3n+2}{2}} \partial_{\tilde{t}} \left[\frac{b_n}{E_4(\tilde{q})^{\frac{3n}{2}}} \right]. \tag{D.49}
 \end{aligned}$$

We then consider the case $c_0(z_1) = \tilde{t}E_4(\tilde{q})^{\frac{1}{4}}$. From the recursion relation (D.49) we can check inductively that

$$\prod_{k=0}^{n-1} (\theta_{z_1} - k) \left[\tilde{t}E_4(\tilde{q})^{\frac{1}{4}} \right] = \frac{E_4(\tilde{q})^{\frac{1}{4}}}{(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^n} \left[\tilde{t}(a_n E_2(\tilde{q}) + b_n) + 12a_n \right] \tag{D.50}$$

So we find the power series solution and one logarithmic $\log(z_1)$ solution to the Picard-Fuchs equations (2.10)

$$\begin{aligned} X_0 &= E_4(\tilde{q})^{\frac{1}{4}} \left[1 + \sum_{n=1}^{\infty} \frac{a_{3n} E_2(\tilde{q}) + b_{3n}}{2^{3n} (n!)^3 E_4(\tilde{q})^{\frac{9n}{2}}} \frac{z_2^n}{(1 - 432z_1)^{3n}} \right], \\ X_1 &= E_4(\tilde{q})^{\frac{1}{4}} \left[\tilde{t} + \sum_{n=1}^{\infty} \frac{\tilde{t}(a_{3n} E_2(\tilde{q}) + b_{3n}) + 12a_{3n}}{2^{3n} (n!)^3 E_4(\tilde{q})^{\frac{9n}{2}}} \frac{z_2^n}{(1 - 432z_1)^{3n}} \right], \end{aligned} \quad (\text{D.51})$$

We compute the flat coordinate of the compact model

$$\begin{aligned} t_1 &= \frac{X_1}{X_0} \\ &= \tilde{t} + \left[\sum_{n=1}^{\infty} \frac{12a_{3n}}{2^{3n} (n!)^3 E_4(\tilde{q})^{\frac{9n}{2}}} \frac{z_2^n}{(1 - 432z_1)^{3n}} \right] / \left[1 + \sum_{n=1}^{\infty} \frac{a_{3n} E_2(\tilde{q}) + b_{3n}}{2^{3n} (n!)^3 E_4(\tilde{q})^{\frac{9n}{2}}} \frac{z_2^n}{(1 - 432z_1)^{3n}} \right]. \end{aligned} \quad (\text{D.52})$$

The expression quantifies the deformation of t_1 away from \tilde{t} as asymptotic expansion in small z_2 , where the coefficients are rational functions of quasi-modular forms of \tilde{t}

$$t_1 - \tilde{t} = \sum_{n=1}^{\infty} \frac{P_{18n-2}(\tilde{q})}{E_4(\tilde{q})^{\frac{9n}{2}}} \frac{z_2^n}{(1 - 432z_1)^{3n}}, \quad (\text{D.53})$$

where $P_{18n-2}(\tilde{q})$ are some quasi-modular forms of weight $18n-2$. We will need the following nice formula

$$\partial_{E_2(\tilde{q})}(t_1 - \tilde{t})^{-1} = \frac{1}{12}. \quad (\text{D.54})$$

Here we regard z_2 as free variable which is independent of \tilde{q} , so that $\partial_{E_2(\tilde{q})} z_2 = 0$.

Before we proceed further, we shall prove a useful general formula on E_2 derivative. Suppose P_k is a rational function of quasi-modular forms, with modular weight k , then we have the formula

$$\partial_{E_2} \partial_t P_k = \partial_t \partial_{E_2} P_k + \frac{k}{12} P_k, \quad (\text{D.55})$$

where we use the notation $t = \log(q) = 2\pi i \tau$. A similar formula was used in [36] to derive of holomorphic anomaly equation in the Nekrasov-Shatashvili limit, and in [37] to studying the relation between holomorphic and modular anomaly equations in the Seiberg-Witten theory. The derivation of (D.55) uses the covariant derivative. It is well known that E_2 is not a modular form, but with an an-holomorphic shift $\hat{E}_2 = E_2 - \frac{6i}{\pi(\tau - \bar{\tau})}$ it becomes a modular form. To preserve the almost holomorphic structure, we shall use the covariant or Maass derivative

$$D_\tau = \partial_\tau + \frac{k}{(\tau - \bar{\tau})}, \quad (\text{D.56})$$

where k is the modular weight. The derivative with respect to E_2 can be related to the anti-holomorphic derivative

$$\bar{\partial}_{\bar{\tau}} = (\bar{\partial}_{\bar{\tau}} \hat{E}_2) \partial_{\hat{E}_2} = \frac{6}{\pi i (\tau - \bar{\tau})^2} \partial_{\hat{E}_2} \quad (\text{D.57})$$

We can compute the derivative

$$\bar{\partial}_{\bar{\tau}} D_{\tau} P_k = \left(\partial_{\tau} + \frac{k}{(\tau - \bar{\tau})} \right) \bar{\partial}_{\bar{\tau}} P_k + \left(\bar{\partial}_{\bar{\tau}} \frac{k}{(\tau - \bar{\tau})} \right) P_k \quad (\text{D.58})$$

Taking the holomorphic limit and cancel out the infinitesimal factor $(\tau - \bar{\tau})^{-2}$ we arrive at the formula (D.55).

The above formula (D.55) is valid for any rational function P_k of quasi-modular forms, with possible fractional powers, which satisfies the homogeneous relation with modular weight

$$P_k(\lambda^2 E_2, \lambda^4 E_4, \lambda^6 E_6) = \lambda^k P_k(E_2, E_4, E_6). \quad (\text{D.59})$$

However it is a little tricky for the logarithmic function, which does not satisfy the above relation (D.59). Naively the logarithmic function has zero modular weight and the E_2 derivative should commute with τ derivative. But this is not the case, as we can easily compute with the formula (D.55),

$$\partial_{E_2} \partial_t \log(P_k) = \partial_t \partial_{E_2} \log(P_k) + \frac{k}{12}. \quad (\text{D.60})$$

We will not need this logarithmic formula in this subsection, but it will be useful later.

We can take derivative n times on a rational function of quasi-modular forms, and use formula (D.55) to compute

$$\partial_{E_2} \partial_t^n P_k = \partial_t^n \partial_{E_2} P_k + \frac{n(k+n-1)}{12} \partial_t^{n-1} P_k \quad (\text{D.61})$$

Suppose $P_k(q_1)$ is a rational function with possible fractional powers of quasi-modular forms of q_1 , with modular weight k . We can expand it as a power series of $\frac{z_2}{(1-432z_2)^3}$ with coefficients as rational functions of (quasi)-modular forms of \tilde{q} , using the expansion (D.52)

$$P_k(q_1) = \sum_{n=0}^{\infty} \frac{(t_1 - \tilde{t})^n}{n!} \partial_{\tilde{t}}^n P_k(\tilde{q}), \quad (\text{D.62})$$

We can compute the derivative of $E_2(\tilde{q})$ and use the two formulae (D.54), (D.61)

$$\begin{aligned} \partial_{E_2(\tilde{q})} P_k(q_1) &= \sum_{n=0}^{\infty} \left\{ \frac{(t_1 - \tilde{t})^{n-1}}{(n-1)!} \left[\partial_{E_2(\tilde{q})} (t_1 - \tilde{t}) \right] \partial_{\tilde{t}}^n P_k(\tilde{q}) + \frac{(t_1 - \tilde{t})^n}{n!} \partial_{E_2(\tilde{q})} \partial_{\tilde{t}}^n P_k(\tilde{q}) \right\} \\ &= \sum_{n=0}^{\infty} \left\{ -\frac{1}{12} \frac{(t_1 - \tilde{t})^{n+1}}{(n-1)!} \partial_{\tilde{t}}^n P_k(\tilde{q}) \right. \\ &\quad \left. + \frac{(t_1 - \tilde{t})^n}{n!} \left[\partial_{\tilde{t}}^n \partial_{E_2(\tilde{q})} P_k(\tilde{q}) + \frac{n(k+n-1)}{12} \partial_{\tilde{t}}^{n-1} P_k(\tilde{q}) \right] \right\}, \\ &= \partial_{E_2(q_1)} P_k(q_1) + \frac{k}{12} (t_1 - \tilde{t}) P_k(q_1). \end{aligned} \quad (\text{D.63})$$

We see the derivative vanishes when $P_k(q_1)$ has zero modular weight $k = 0$ and is modular, i.e. no E_2 dependence.

Now we have the necessary ingredients for proving the conjecture (2). First we consider the case of $1 - 864z_1 = \frac{E_6(\tilde{q})}{E_4(\tilde{q})^{\frac{3}{2}}}$. We shall expand in terms of the modular parameter q_1 of the compact model

$$\frac{E_6(\tilde{q})}{E_4(\tilde{q})^{\frac{3}{2}}} = \frac{1}{E_4(q_1)^{\frac{3}{2}}} \left[E_6(q_1) + \sum_{n=1}^{\infty} \frac{f_n(q_1)}{E_4(q_1)^{\frac{9n}{2}}} \frac{z_2^n}{(1 - 432z_1)^{3n}} \right], \quad (\text{D.64})$$

We can determine the coefficients f_n recursively by expanding the modular forms of q_1 in terms of \tilde{q} and compare with the left hand side. It is clear that $f_n(q_1)$ are rational functions of quasi-modular forms of q_1 of modular weight $18n + 6$, with possible negative powers of E_4 , but no fractional powers. To show it has no E_2 dependence, we take derivative of $E_2(\tilde{q})$ on both sides and use the formula (D.63)

$$0 = \sum_{n=1}^{\infty} \frac{\partial_{E_2(q_1)} f_n(q_1)}{E_4(q_1)^{\frac{9n+3}{2}}} \frac{z_2^n}{(1 - 432z_1)^{3n}}, \quad (\text{D.65})$$

where we have use the fact that the coefficients have zero modular weight. Since z_2 is a free variable, we find $\partial_{E_2(q_1)} f_n(q_1) = 0$, i.e. it is modular.

We also need to check that there is no negative powers of $E_4(q_1)$ in $f_n(q_1)$. From the expansion (D.53) we see that the coefficients of $\frac{z_2^n}{(1 - 432z_1)^{3n}}$ always have the factor of $E_4(q_1)^{-\frac{9n}{2}}$. There could be also negative powers of E_4 from the derivative $\partial_{\tilde{t}}$ in (D.62) since each derivative increases the negative power of E_4 by one. However, we note that since a_n has modular weight is $6n - 2$, it can not be purely a power of $E_6(\tilde{q})$ and always has a factor $E_4(\tilde{q})$. So the $(t_1 - \tilde{t})^n$ factor in (D.62) contributes a factor $E_4(\tilde{q})^n$ which cancel the one from taking derivative. Therefore $f_n(q_1)$ is a modular form.

Furthermore, since the derivative $\partial_{\tilde{t}}^n \left[\frac{E_6(\tilde{q})}{E_4(\tilde{q})^{\frac{3}{2}}} \right]$ always contains a factor of $E_4(\tilde{q})^3 - E_6(\tilde{q})^2$, we see that the coefficient $f_n(q_1)$ always have a factor of $E_4(q_1)^3 - E_6(q_1)^2$, as confirmed by the explicit expansion in (D.26). According to the discussion in equations (D.29), (D.30), we immediately infer the modularity of the expansion coefficients of $z_1(1 - 432z_1)$.

Alternatively, we can also expand $z_1(1 - 432z_1) = \frac{E_4(\tilde{q})^3 - E_6(\tilde{q})^2}{1728E_4(\tilde{q})^3}$ in the same way as $1 - 864z_1$ in (D.64). From the same reasoning as above we can also see the modularity of the expansion (D.24). We note that the derivative $\partial_{\tilde{t}}^n [E_4(\tilde{q})^3 - E_6(\tilde{q})^2]$ also always contains a factor of $E_4(\tilde{q})^3 - E_6(\tilde{q})^2$, and it is factored out in this case in the expansion (D.24), so that the higher order coefficients do not necessarily contain the factor.

Finally we consider the case of X_0 , which is a little trickier than the previous two cases since the modular weight is not zero. Again we make the ansatz in terms of the modular parameter q_1 of the compact model

$$X_0 = E_4(q_1)^{\frac{1}{4}} \left[1 + \sum_{n=1}^{\infty} \frac{f_n(q_1)}{E_4(q_1)^{\frac{9n}{2}}} \frac{z_2^n}{(1 - 432z_1)^{3n}} \right], \quad (\text{D.66})$$

and determine $f_n(q_1)$ recursively comparing with the solution (D.51) and using the deformation (D.53). From the same reasonings above we see $f_n(q_1)$ contains no negative power

of $E_4(q_1)$ and no fractional powers, so it is a quasi-modular form. We only need to show it is E_2 independent. We compute the $E_2(\tilde{q})$ derivative in the X_0 solution in (D.51)

$$\begin{aligned} \partial_{E_2(\tilde{q})} X_0 &= E_4(\tilde{q})^{\frac{1}{4}} \left[\sum_{n=1}^{\infty} \frac{a_{3n}}{2^{3n}(n!)^3 E_4(\tilde{q})^{\frac{9n}{2}}} \frac{z_2^n}{(1-432z_1)^{3n}} \right] \\ &= \frac{t_1 - \tilde{t}}{12} X_0, \end{aligned} \tag{D.67}$$

where in the last step we use (D.52). On the other hand, we can also compute the $E_2(\tilde{q})$ derivative using the ansatz (D.66) and the formula (D.63) with the modular weight $k = 1$, and we get

$$\partial_{E_2(\tilde{q})} X_0 = \frac{t_1 - \tilde{t}}{12} X_0 + \sum_{n=1}^{\infty} \frac{\partial_{E_2(q_1)} f_n(q_1)}{E_4(q_1)^{\frac{18n-1}{4}}} \frac{z_2^n}{(1-432z_1)^{3n}}. \tag{D.68}$$

Comparing the two results (D.67), (D.68) we find that $f_n(q_1)$ has no E_2 dependence. Thus we have proven Conjecture 2. We see the case of X_0 is quite interesting. It is quasi-modular when we use the \tilde{q} parameter, but the E_2 dependence vanishes and it becomes purely modular when we use the physical Kähler parameter q_1 of the compact model.

We can also show that the modular forms $f_n(q_1)$ in (D.66) contain a factor of $\eta^{24} \sim E_4^3 - E_6^2$, as can be seen from the explicit formulae in the expansion (D.27). Since $f_n(q_1)$ is a modular form, it is sufficient to show that $f_n(q_1) = 0$ at $q_1 = 0$ in order to be divisible by $\eta^{24}(q_1)$. We note that the derivative of any rational function of modular forms and their fractional powers vanish at q_1 , as long as it is not singular at $q_1 = 0$, due to its q -series expansion. So the contributions to $f_n(0)$ from lower order terms in (D.66) vanish. We also need to evaluate the solution (D.51) for X_0 at $\tilde{q} = 0$. We denote the value of $a_n(\tilde{q}), b_n(\tilde{q})$ in the solution at $\tilde{q} = 0$ simply as $a_n(0), b_n(0)$. The recursion relation (D.49) at $\tilde{q} = 0$ is

$$\begin{aligned} a_{n+1}(0) &= -\left(2n - \frac{1}{6}\right) a_n(0) + \frac{1}{6} b_n(0), \\ b_{n+1}(0) &= -\left(2n + \frac{1}{6}\right) b_n(0) - \frac{1}{6} a_n(0). \end{aligned} \tag{D.69}$$

Adding together we find

$$a_{n+1}(0) + b_{n+1}(0) = -2n[a_n(0) + b_n(0)], \tag{D.70}$$

so the sum $a_n(0) + b_n(0)$ vanishes for $n \geq 1$. The solution is simply $X_0 = 1$ for $\tilde{q} = 0$, and does not contribute to higher order terms either. Thus $f_n(q_1)$ vanish at $q_1 = 0$, i.e. is divisible by $\eta^{24}(q_1)$.

Next we would like to find the solution to Picard-Fuchs equation with leading $\log(z_2)$ behavior. We make the ansatz

$$X_2 = X_0 \left[\log(z_2) - \frac{3}{2} t_1 - \frac{3}{2} \log\left(\frac{1-432z_1}{z_1}\right) \right] + \sum_{n=0}^{\infty} \alpha_n(z_1) z_2^n, \tag{D.71}$$

where X_0 and $X_0 t_1 = X_1$ are the two solutions found previously in (D.51). The terms $\frac{3}{2} t_1 + \frac{3}{2} \log\left(\frac{1-432z_1}{z_1}\right)$ comes from (D.11) which is the solution to the z_2 zero order equation

with the correct leading asymptotic behavior, so that we can choose the initial function $\alpha_0(z_1) = 0$. For convenience, we denote

$$d_n = \frac{E_4(\tilde{q})^{\frac{1}{4}}}{(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^n} (a_n E_2(\tilde{q}) + b_n), \quad (\text{D.72})$$

where a_n, b_n are modular forms defined in (D.48), and satisfy the recursion relation (D.49), so that the power series solution

$$X_0 = \sum_{n=0}^{\infty} \frac{d_{3n}}{(n!)^3} z_2^n. \quad (\text{D.73})$$

Again after some calculations, we find recursion relation for $\alpha_n(z_1)$ in (D.71) from the second Picard-Fuchs equation

$$\begin{aligned} \alpha_{n+1}(z_1) = & \frac{1}{(n+1)^3} \prod_{i=0}^2 (\theta_1 - 3n - i) \alpha_n(z_1) - \frac{3d_{3n+3}}{(n+1)(n+1)!^3} \\ & - \frac{9E_6(\tilde{q})d_{3n+2}}{(n+1)!^3(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))} + \frac{9(E_4(\tilde{q})^3 + E_6(\tilde{q})^2)d_{3n+1}}{(n+1)!^3(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^2} \\ & - \frac{6E_6(\tilde{q})(3E_4(\tilde{q})^3 + E_6(\tilde{q})^2)d_{3n}}{(n+1)!^3(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^3}. \end{aligned} \quad (\text{D.74})$$

From the recursion relation and initial condition $\alpha_0 = 0$, it is clear that the coefficients $\alpha_n(z_1)$ have the following structure

$$\alpha_n = \frac{E_4(\tilde{q})^{\frac{1}{4}}}{(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^{3n}} (A_n(\tilde{q})E_2(\tilde{q}) + B_n(\tilde{q})), \quad (\text{D.75})$$

where $A_n(\tilde{q}), B_n(\tilde{q})$ are modular forms of \tilde{q} of weight $18n - 2, 18n$. Some low order formulae are

$$\begin{aligned} A_1 &= \frac{E_4(31E_4^3 + 113E_6^2)}{72}, & B_1 &= -\frac{E_6(1297E_4^3 + 575E_6^2)}{72}, \\ A_2 &= -\frac{E_4E_6}{6912} (233753E_4^6 + 812750E_4^3E_6^2 + 252953E_6^4), \\ B_2 &= \frac{1}{82944} (3717955E_4^9 + 59441331E_4^6E_6^2 + 66006225E_4^3E_6^4 + 5867321E_6^6). \end{aligned} \quad (\text{D.76})$$

The second flat coordinate is then

$$t_2 = \frac{X_2}{X_0} = \log(z_2) - \frac{3}{2}t_1 - \frac{3}{2} \log\left(\frac{1 - 432z_1}{z_1}\right) + \frac{1}{X_0} \sum_{n=1}^{\infty} \alpha_n(z_1) z_2^n, \quad (\text{D.77})$$

where α_n are expressed in (D.75) in terms of the modular forms A_n and B_n . It is clear we can write a series expansion

$$\log(Q) = \log(Z) + \sum_{n=1}^{\infty} f_n(q_1) Z^n, \quad (\text{D.78})$$

where $Q = \frac{\exp(t_2 + \frac{3t_2}{2})}{\eta(q_1)^{36}}$, $Z = \frac{z_2}{(1-432z_1)^3 E_4(q_1)^{\frac{9}{2}}}$ are the variables defined in (D.32), (D.23).

Using the relation between t_1 and \tilde{t} as in equation (D.53) we see that $f_n(q_1)$ here must be quasi-modular forms of weight $18n$. Because the coefficients of z_2^n have zero modular weight, the derivative of $E_2(\tilde{q})$ and $E_2(q_1)$ are the the same for $\log(Q)$ according to the formula (D.63). In order to prove conjecture (3), we should confirm the equation (D.40). We compute the E_2 derivative

$$\begin{aligned} \partial_{E_2(q_1)} \log(Q) &= \partial_{E_2(\tilde{q})} \log(Q) \\ &= -\frac{t_1 - \tilde{t}}{12X_0} \sum_{n=1}^{\infty} \alpha_n(z_1) z_2^n + \frac{1}{X_0} \sum_{n=1}^{\infty} \frac{E_4(\tilde{q})^{\frac{1}{4}} A_n(\tilde{q})}{(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^{3n}} z_2^n, \end{aligned} \quad (\text{D.79})$$

where we have used (D.67). We shall prove the above equation is equal to the $\frac{1}{12} \partial_{t_2} \mathcal{F}_{\text{inst}}^{(0)}(t_1, t_2)$, where $\mathcal{F}_{\text{inst}}^{(0)}$ is the instanton part of the the prepotential.

The classical contribution in the prepotential for our model is

$$\mathcal{F}_{\text{classical}}^{(0)} = \frac{3t_1^3}{2} + \frac{3}{2} t_1^2 t_2 + \frac{1}{2} t_1 t_2^2 \quad (\text{D.80})$$

The prepotential is determined by the fact that $X_0 \partial_{t_i} \mathcal{F}^{(0)}$ are the double logarithmic solutions of the Picard-Fuchs (PF) equations (2.10). In order to confirm the equation (D.40), we shall show the following is a solution to the PF equations

$$\begin{aligned} w_3 &= X_0 \left(\frac{3}{2} t_1^2 + t_1 t_2 \right) - (t_1 - \tilde{t}) \sum_{n=1}^{\infty} \alpha_n(z_1) z_2^n + 12 \sum_{n=1}^{\infty} \frac{E_4(\tilde{q})^{\frac{1}{4}} A_n(\tilde{q})}{(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^{3n}} z_2^n \\ &= X_1 \left[\log(z_2) - \frac{3}{2} \log \left(\frac{1 - 432z_1}{z_1} \right) \right] + \tilde{t} \sum_{n=1}^{\infty} \alpha_n(z_1) z_2^n + 12 \sum_{n=1}^{\infty} \frac{E_4(\tilde{q})^{\frac{1}{4}} A_n(\tilde{q})}{(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^{3n}} z_2^n, \end{aligned} \quad (\text{D.81})$$

where we have use the solution for X_2 in (D.71).

After some tedious calculations, we check that (D.81) is indeed a solution. We can provide some details in the following. At zero order z_2 , we need to check the equation from the first PF operator

$$[\theta_1^2 - 12z_1(6\theta_1 + 1)(6\theta_1 + 5)] \left[-\frac{3}{2} \tilde{t} E_4(\tilde{q})^{\frac{1}{4}} \log \left(\frac{1 - 432z_1}{z_1} \right) \right] = 3\theta_1 [\tilde{t} E_4(\tilde{q})^{\frac{1}{4}}]. \quad (\text{D.82})$$

Assuming the two Picard-Fuchs equations are consistent, we only need to further check the second PF equation which provide recursion relation for the coefficients of z_2^n . We denote the coefficient of z_2^n in the solution (D.81) as

$$\beta_n = \tilde{t} \alpha_n + \frac{12 E_4(\tilde{q})^{\frac{1}{4}} A_n(\tilde{q})}{(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^{3n}}. \quad (\text{D.83})$$

We see the similar structure for the ansatz of X_2 in (D.71). From the same calculations we derive the recursion relation for $\beta_n(z_1)$ from the second Picard-Fuchs equation, similar

to (D.74),

$$\begin{aligned} \beta_{n+1}(z_1) = & \frac{1}{(n+1)^3} \prod_{i=0}^2 (\theta_1 - 3n - i) \beta_n(z_1) - \frac{3e_{3n+3}}{(n+1)(n+1)!^3} \\ & - \frac{9E_6(\tilde{q})e_{3n+2}}{(n+1)!^3(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))} + \frac{9(E_4(\tilde{q})^3 + E_6(\tilde{q})^2)e_{3n+1}}{(n+1)!^3(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^2} \\ & - \frac{6E_6(\tilde{q})(3E_4(\tilde{q})^3 + E_6(\tilde{q})^2)e_{3n}}{(n+1)!^3(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^3}, \end{aligned} \quad (\text{D.84})$$

where we denote

$$e_n = \frac{E_4(\tilde{q})^{\frac{1}{4}}}{(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^n} [\tilde{t}(a_n E_2(\tilde{q}) + b_n) + 12a_n], \quad (\text{D.85})$$

so that the solution $X_1 = \sum_{n=0}^{\infty} \frac{e_{3n}}{n!^3} z_2^n$. We further denote

$$\frac{E_4(\tilde{q})^{\frac{1}{4}} [A_n^{(k)} E_2(\tilde{q}) + B_n^{(k)}]}{(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^{3n+k+1}} = \prod_{i=0}^k (\theta_1 - 3n - i) \alpha_n, \quad (\text{D.86})$$

where $A_n^{(k)}$ and $B_n^{(k)}$ are modular forms of \tilde{q} of weight $6(3n+k+1) - 2$ and $6(3n+k+1)$. Then similar to the case of solution X_1 , one can show inductively

$$\frac{E_4(\tilde{q})^{\frac{1}{4}} [\tilde{t}(A_n^{(k)} E_2(\tilde{q}) + B_n^{(k)}) + 12A_n^{(k)}]}{(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^{3n+k+1}} = \prod_{i=0}^k (\theta_1 - 3n - i) \left(\tilde{t} \alpha_n + \frac{12E_4(\tilde{q})^{\frac{1}{4}} A_n(\tilde{q})}{(E_4(\tilde{q})^{\frac{3}{2}} + E_6(\tilde{q}))^{3n}} \right).$$

From the recursion relation for α_n in (D.74), we can write the recursion separately for A_n and B_n as

$$\begin{aligned} A_{n+1} = & \frac{A_n^{(2)}}{(n+1)^3} - \frac{3a_{3n+3}}{(n+1)(n+1)!^3} - \frac{9E_6(\tilde{q})a_{3n+2}}{(n+1)!^3} + \frac{9(E_4(\tilde{q})^3 + E_6(\tilde{q})^2)a_{3n+1}}{(n+1)!^3} \\ & - \frac{6E_6(\tilde{q})(3E_4(\tilde{q})^3 + E_6(\tilde{q})^2)a_{3n}}{(n+1)!^3}, \\ B_{n+1} = & \frac{B_n^{(2)}}{(n+1)^3} - \frac{3b_{3n+3}}{(n+1)(n+1)!^3} - \frac{9E_6(\tilde{q})b_{3n+2}}{(n+1)!^3} + \frac{9(E_4(\tilde{q})^3 + E_6(\tilde{q})^2)b_{3n+1}}{(n+1)!^3} \\ & - \frac{6E_6(\tilde{q})(3E_4(\tilde{q})^3 + E_6(\tilde{q})^2)b_{3n}}{(n+1)!^3}. \end{aligned} \quad (\text{D.87})$$

Now it is straightforward to check that β_n in (D.83) satisfy the recursion relation (D.84). This completes the proof of Conjecture 3.

E Expansion around a point on the conifold divisor

We now study the expansion of the higher genus topological string amplitudes around a point on the conifold divisor in terms of the local flat coordinates, and use the gap condition to fix the holomorphic ambiguity. Since the involution symmetry exchanges the

two conifold divisors, it is sufficient to consider the simpler one $\Delta_2 = 1 + 27z_2$. The simplest point to consider is the intersection point of the divisor $z_1 = 0$ and $\Delta_2 = 1 + 27z_2$. This is a normal intersection, and we can choose the local coordinates as

$$(z_{c1}, z_{c2}) = \left(z_1, z_2 + \frac{1}{27} \right) \tag{E.1}$$

We solve the PF equations (2.10) perturbatively around this point. There should be 6 linearly independent solutions. We only need 3 of them with at most one logarithmic leading term in order to compute the flat coordinates. The three solutions are two power series solutions X_0, X_1 and one logarithmic solution X_2 as the following

$$\begin{aligned} X_0 = & \left[1 + 60z_{c1} + 13860z_{c1}^2 + \frac{9529520z_{c1}^3}{3} + 148728580z_{c1}^4 - 569333004240z_{c1}^5 \right. \\ & \left. - \frac{5035405372197200z_{c1}^6}{9} + \mathcal{O}(z_{c1}^7) \right] + [24504480z_{c1}^3 + 32125373280z_{c1}^4 \\ & + 27949074753600z_{c1}^5 + 19093358660376000z_{c1}^6 + \mathcal{O}(z_{c1}^7)]z_{c2} \\ & + [15162373053828000z_{c1}^6 + \mathcal{O}(z_{c1}^7)]z_{c2}^2 + \mathcal{O}(z_{c2}^3) \end{aligned} \tag{E.2}$$

$$X_1 = z_{c2} + \left(\frac{33}{2} + 270z_{c1} \right) z_{c2}^2 + (327 + 7020z_{c1} + 374220z_{c1}^2) z_{c2}^3 + \mathcal{O}(z_{c2}^4), \tag{E.3}$$

$$\begin{aligned} X_2 = & X_0 \log(z_{c1}) + \left[372z_{c1} + 98442z_{c1}^2 + \frac{224976472z_{c1}^3}{9} + \frac{7098994823z_{c1}^4}{3} + \mathcal{O}(z_{c1}^5) \right] \\ & + [-540z_{c1} - 124740z_{c1}^2 + 132742656z_{c1}^3 + 220660710516z_{c1}^4 + \mathcal{O}(z_{c1}^5)]z_{c2} \\ & + [27 - 4860z_{c1} - 1683990z_{c1}^2 - 606485880z_{c1}^3 - 307198881990z_{c1}^4 + \mathcal{O}(z_{c1}^5)]z_{c2}^2 \\ & + \mathcal{O}(z_{c2}^3) \end{aligned} \tag{E.4}$$

The local flat coordinates are computed as the ratio of the solutions $t_{c1} = \frac{X_1}{X_0}$ and $t_{c2} = \frac{X_2}{X_0}$. For the power series solution, the flat coordinate goes like $t_{c1} \sim z_{c2} = z_2 + \frac{1}{27}$, which is the expected asymptotic behavior of near a conifold point. For the logarithmic solution, the flat coordinate goes like $t_{c2} \sim \log(z_{c1})$, which is similar to the behavior near large volume point, and we should use the exponential of the flat coordinate as the expansion parameter. So we define the expansion parameters in terms of flat coordinates

$$t_c := t_{c1} = \frac{X_1}{X_0}, \quad q_c := \exp(t_{c2}). \tag{E.5}$$

We shall compute the asymptotic expansion of the propagators S^{ij}, S^i, S , where we have written the propagators in z_i coordinate and for simplicity we denote $S^{ij} = S^{z_i z_j}$ and $S^i = S^{z_i}$. Here the propagators should simply transform as tensors. In the case of involution symmetry, there is a minus sign and shifts for S^i and S because the involution map the point to a different branch of moduli space. But this should not happen locally. The three point function also transforms as tensor without minus sign. The equations (B.2)

are coordinate invariant if we take the holomorphic ambiguities $h_i^{jk}, h_i^j, h_i, h_{ij}$ to transform as tensors, while the Christoffel symbol Γ_{ij}^k and the holomorphic ambiguity s_{ij}^k transforms with a shift according to the well known transformation rule of Christoffel symbol as in (B.4).

We can compute the asymptotic expansion of the Christoffel symbol $\Gamma_{z_{ci}z_{cj}}^{z_{ck}}$ in the local coordinates (E.1) by the well known formula from special geometry $\Gamma_{z_{ci}z_{cj}}^{z_{ck}} = \frac{\partial z_{ck}}{\partial t_{cl}} \frac{\partial^2 t_{cl}}{\partial z_{ci} \partial z_{cj}}$ in the holomorphic limit, and compute the propagators $S^{z_{ci}z_{cj}}, S^{z_{ci}}, S$ according to (B.4). Here the transformation between the coordinates z_{ci} and z_i are particularly simple because the Jacobian transformation matrix $\frac{\partial z_{ci}}{\partial z_j}$ is an identity matrix. So the propagators S^{ij}, S^i, S and holomorphic ambiguities $h_i^{jk}, h_i^j, h_i, h_{ij}, s_{ij}^k$ are actually invariant under the coordinate transformation (E.1).

We find that after including the well known scaling factor X_0^{2g-2} , the asymptotic expansion of the higher genus topological string amplitudes with the correct holomorphic ambiguity satisfy the gap condition near the point $(z_1, z_2) = (0, -\frac{1}{27})$. Specifically, we find

$$X_0^{2g-2} \mathcal{F}^{(g)} = \frac{B_{2g}}{4g(g-1)3^{5(g-1)}} \frac{1}{t_c^{2(g-1)}} + \sum_{n=0}^{\infty} f_n(q_c) t_c^n, \tag{E.6}$$

where $f_n(q_c)$ are some power series of q_c .

Now we can study how the gap condition (E.6) fixes holomorphic ambiguity at higher genus. First as in previous situations, suppose we have found a particular holomorphic ambiguity at genus g such that the gap condition (E.6) is satisfied. Then we may add an additional piece of rational function of z_1, z_2 to $\mathcal{F}^{(g)}$ such that the gap condition (E.6) is not affected. Since near the point $(z_1, z_2) = (0, -\frac{1}{27})$, we have the asymptotic behavior $X_0 \sim 1$ and $t_c \sim z_{c2} \sim \Delta_2$, this additional rational function should have no pole at the conifold divisor Δ_2 in order to be regular at $t_c \sim 0$. Together with the involution symmetry as defined by the spaces in (C.2), we consider the following linear space of holomorphic ambiguity

$$X^{(g,m)} := \left\{ f \mid f \in \sum_{n=0}^{\infty} \oplus V_2^{(g,m,n)} \text{ and has no pole at } \Delta_2 \right\}. \tag{E.7}$$

In particular, in the case of $m = \lceil \frac{19}{3}(g-1) \rceil$, according to the proposition (3), the dimension of $X^{(g, \lceil \frac{19}{3}(g-1) \rceil)}$ is exactly the remaining number of unknown constants in the genus g holomorphic ambiguity after we impose both the fiber modularity and the gap conditions.

Clearly if $f \in X^{(g,m)}$, then its involution transformation $\tilde{f} = (-1)^{g-1} f \in X^{(g,m)}$ as well. Since the involution transformation symmetry exchanges the two conifold divisors, we infer that f must have no pole at Δ_1 either, so it is actually a polynomial of z_1, z_2 . Canceling the pole Δ_1^{2g-2} reduces the degree of z_1 in the numerator by $6(g-1)$. So the space can be determined as

$$X^{(g,m)} = \begin{cases} \sum_{n=0}^{\infty} \oplus (V_-^{(1,m-6(g-1),2n)} \oplus V_+^{(1,m-6(g-1),2n+1)}), & \text{if } g \text{ is even;} \\ \sum_{n=0}^{\infty} \oplus (V_+^{(1,m-6(g-1),2n)} \oplus V_-^{(1,m-6(g-1),2n+1)}), & \text{if } g \text{ is odd.} \end{cases} \tag{E.8}$$

It is now easy to compute the dimension of $X^{(g,m)}$ with the formulae (C.13). We can consider in particular the case of $m = \lfloor \frac{19}{3}(g-1) \rfloor$, and find the number of remaining unknown constants at low genus $\text{Dim}(X^{(g, \lfloor \frac{19}{3}(g-1) \rfloor)}) = 0, 1, 1, 1, 1, 2, 1, 2, 3, 2, 3$ for $g = 2, 3, \dots, 12$ respectively.

We can also estimate the asymptotic behavior of $\text{Dim}(X^{(g, \lfloor \frac{19}{3}(g-1) \rfloor)})$ at large genus. Ignoring sub-leading contributions, we find

$$\text{Dim}(X^{(g, \lfloor \frac{19}{3}(g-1) \rfloor)}) = \sum_{n=0}^{\frac{g-1}{9}} \left(\frac{g-1}{3} - 3n \right) \sim \frac{g^2}{54}. \tag{E.9}$$

Since the elements of the space $X^{(g,m)}$ are regular at both conifold divisors Δ_1, Δ_2 , they would not affect the gap conditions at any other points on the conifold divisors. For example, studying the expansion of topological string amplitudes around the intersection of Δ_1 and Δ_2 would not give more boundary conditions to fix the holomorphic ambiguity. It is thus sufficient to consider the gap condition around only one point $(z_1, z_2) = (0, -\frac{1}{27})$ here.

At low genus we can fix the remaining holomorphic ambiguity (E.8) by the boundary conditions at large volume point $(z_1, z_2) = (0, 0)$, which include the known higher genus constant map contributions and some vanishing property of GV invariants. For example, the GV invariants vanish $n_{d_E, d_B}^g = 0$ for base degree $d_B = 0$ and $g \geq 2$ [3]. We see that only the holomorphic ambiguities with zero z_2 degree in the space (E.8) contribute to the zero base degree GV invariants. So this property alone can eliminate the subspace $V_{\pm}^{(1, \lfloor \frac{g-1}{3}, 0)}$ in the remaining holomorphic ambiguities, which are now actually completely fixed up to genus $g \leq 9$.

We consider also the local limit. The topological string amplitudes on local \mathbb{P}^2 model which has only the base class are solvable to arbitrary high genus. So we know the zero fiber degree GV invariants n_{d_E, d_B}^g for $d_E = 0$ from local model calculations, which may fix some more holomorphic ambiguity for the compact model. However, at a closer look we find that the local limit actually does not help. It is easy to check that except one element which is the constant in $V_{+}^{(1, m-6(g-1), 0)}$ or $1 - 864z_1$ in $V_{-}^{(1, m-6(g-1), 0)}$, all other elements in $X^{(g,m)}$ are polynomially divisible by z_1 , so they do not affect the zero fiber degree GV invariants. So the local limit only fix the constant or $1 - 864z_1$ in $X^{(g,m)}$, which is already fixed by the known constant map contribution in Gromov-Witten theory.

We can discuss a little more detail on fixing the holomorphic ambiguity with GV invariants. From the definition of the spaces V_{\pm} in (C.10), (C.11), we find that the elements in $V_{\pm}^{(1, m-6(g-1), n)}$ are polynomially divisible by z_1^{3n} , so they only affect the GV invariants n_{d_E, d_B}^g with fiber degree $d_E \geq 3n$. So in order to fix the elements in $V_{\pm}^{(1, m-6(g-1), n)}$ we need to use the GV invariants n_{d_E, d_B}^g with fiber degree $d_E \geq 3n$ and $d_B \geq n$.

These B-model considerations are combined with the use of weak Jacobi forms to fix the topological string amplitudes. Suppose we have the exact formulas up to a base degree d_B which are valid for all genus and fiber class, then in the following we shall show that we have sufficient boundary conditions to fix the B-model formula up to any genus no bigger

than $9(d_B + 1)$. To see this, we further define a vector space

$$Y^{(g,d_B)} = \sum_{n=d_B+1}^{\infty} \oplus V_{(-1)^{g+n+1}}^{(1, [\frac{g-1}{3}], n)}, \tag{E.10}$$

where we denote the \pm subscript universally as $(-1)^{g+n+1}$ depending on the sign. Here we have also set $m = [\frac{19(g-1)}{3}]$ in (E.8). The space $Y^{(g,d_B)}$ is exactly the space of remaining holomorphic ambiguities at genus g , after we impose the boundary conditions at the conifold and orbifold loci, and use the exact formulas for base degrees up to d_B . Now we consider $g = 9(d_B + 1)$, then according to (C.10), (C.11), the dimension of the first linear space in the direct sum (E.10) is

$$\text{Dim}(V_{-}^{(1, 3d_B+2, d_B+1)}) = 0. \tag{E.11}$$

The dimensions of the other spaces in (E.10) are smaller and also vanish, so we have $\text{Dim}(Y^{(9(d_B+1), d_B)}) = 0$, i.e. all holomorphic ambiguities can be fixed at genus $9(d_B + 1)$. On the other hand, consider one genus higher at $g = 9(d_B + 1) + 1$, then the dimension of the first linear space is computed similarly

$$\text{Dim}(V_{+}^{(1, 3d_B+3, d_B+1)}) = 1 > 0. \tag{E.12}$$

so we have $\text{Dim}(Y^{(9(d_B+1)+1, d_B)}) \geq \text{Dim}(V_{+}^{(1, 3d_B+3, d_B+1)}) > 0$, i.e. there are some unfixed holomorphic ambiguities.

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