Constraint algebra of general relativity from a formal continuum limit of canonical tensor model

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ABSTRACT: Canonical tensor model (CTM for short below) is a rank-three tensor model formulated as a totally constrained system in the canonical formalism. In the classical case, the constraints form a first-class constraint Poisson algebra with structures similar to that of the ADM formalism of general relativity, qualifying CTM as a possible discrete formalism for quantum gravity. In this paper, we show that, in a formal continuum limit, the constraint Poisson algebra of CTM with no cosmological constant exactly reproduces that of the ADM formalism. To this end, we obtain the expression of the metric tensor field in general relativity in terms of one of the dynamical rank-three tensors in CTM, and determine the correspondence between the constraints of CTM and those of the ADM formalism. On the other hand, the cosmological constant term of CTM seems to induce non-local dynamics, and is inconsistent with an assumption about locality of the continuum limit.

Contents

1 Introduction

Tensor models [1–3] were originally introduced as models for quantum gravity in $D > 2$ dimensions, extending the matrix models which are considered to successfully describe quantum gravity in $D = 2$. Subsequently, tensor models with group-valued indices [4, 5], called group field theories [6–8], were introduced, which are especially studied in the context of loop quantum gravity. The central idea of tensor models is that Feynman diagrams in tensor models may correspond to dual diagrams of discretized spacetimes. Though the original models were not successful due to some difficulties [2, 9], colored tensor models [10, 11] with promising properties appeared, and have extensively been analyzed with interesting concrete results (see for instance [12–16] for some recent developments). The colored tensor models also stimulated the renormalization group procedures of group field theories (see for instance [17–20] for recent developments). There also appeared a new approach to random volumes in terms of matrix models [21, 22], which are in relation with colored tensor models.

The analysis of the colored tensor models has shown that leading orders of $1/N^1$ expansions of the partition functions are dominated by branched polymers composed of melonic diagrams [11, 23, 24]. Naively, this would be an obstacle for a model of our spacetime, since branched polymers do not seem to represent extending entities like our real space, though there have been some interesting directions of study to change the situation by considering higher orders [13, 25–28]. On the other hand, it might be possible that the existence of a time-like direction is essentially important in quantum gravity, while the tensor models

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[1]N represents the number of discrete labels which an index takes: any index a of tensors is assumed to take, say, $a = 1, 2, \ldots, N$, in this paper.

above basically deal with Euclidean signatures. This possibility arises from the fact that Causal Dynamical Triangulation has succeeded in generating de Sitter-like spacetimes [29], while the Euclidean cousin, Dynamical Triangulation, is not successful in this respect.[2]

Motivated by these considerations with the expectation that a time-like direction plays a model-independent important role in quantum gravity, one of the present authors has introduced a rank-three tensor model in Hamilton formalism [32–34] (There exists another Hamiltonian approach [35] in the framework of group field theories.). Here the minimum choice of rank-three over matrices has been taken, based on a belief that rank-three is enough to describe any dimensional space; this is a conclusion from the past works by one of the present authors on Euclidean rank-three tensor models [36, 37], though these models themselves are not successful due to serious necessities of fine-tuning. A time-like direction has been introduced by constructing Hamiltonian constraints forming a first-class constraint Poisson algebra with kinematical symmetries, which are the analog of spatial diffeomorphism in general relativity. This way of introducing a time-like direction as gauge symmetry would be necessary for such a model aiming for quantum gravity to reproduce general relativity in a (presently unknown) classical limit of continuous spacetime. The requirement of the first-class nature of the constraint algebra is so strong that the constraints and the algebraic structure of such a tensor model are unique under some physically reasonable assumptions [33]. Thus, our tensor model in the Hamilton formalism (canonical tensor model or CTM for short below) has turned out to be formulated as a totally constrained system with first-class constraints, which have a Poisson algebraic structure very similar to the constraint Poisson algebra [38–40] of the Arnowitt-Deser-Misner (ADM) formalism of general relativity [41, 42].

The subsequent analyses have revealed some remarkable properties of CTM. The $N = 1$ case exactly reproduces the mini-superspace approximation of general relativity with a cosmological constant [43]. CTM can consistently be quantized [44], and a number of exact physical wave functions for general N, namely the exact solutions to the CTM analogue of the Wheeler-DeWitt equations, have been found [45]. There is an intimate relation between CTM and statistical systems on random networks: the Hamiltonian constraint of CTM generates the renormalization group flow of randomly connected tensor networks [46–48]. This insight was remarkably useful in constructing the exact physical wave functions mentioned above [45].

In fact, the results on CTM mentioned above partially support the idea that introducing a time-like direction may be essentially important for models of quantum gravity to generate semi-classical space-times. It has been observed that the physical wave functions of CTM have peaks or singularities at configurations where locality is maximized [44] or group symmetries are enhanced [45]. This is in sharp contrast with the situation in our Euclidean rank-three tensor models mentioned above [36, 37], in which the background configurations were initially selected out from plenty of minima of the fine-tuned actions.

The main purpose of the present paper is to consider a formal continuum limit of CTM to find a relation with general relativity more general than the mini-superspace ap-

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proximation mentioned above. We will show that, in the formal continuum limit, the first-class constraint Poisson algebra of CTM exactly agrees with that of the ADM formalism of general relativity by properly taking into account a difference of weights between the Hamiltonian constraints of both the theories. The continuum limit contains the following two main assumptions: the indices of tensors can be replaced by continuous D-dimensional coordinates with an implicit assumption of very large N, and one of the dynamical tensors of CTM must have an almost diagonal form. Here, the off-diagonal components of the tensor are essentially important: the lowest orders of a moment expansion for the off-diagonal components will be identified with the (inverse) metric tensor field in general relativity. It should be noted that we take the continuum limit in a formal manner, and the limit must be justified by the dynamics of CTM in future study. We would also like to mention that a similar derivation of the constraint algebra of the ADM formalism from that of CTM was done in a previous work [32] by one of the present authors. The previous work, however, was obviously insufficient, because a specific Gaussian distribution of off-diagonal components was assumed for computational simplicity, and coordinate dependences of variables were not fully accounted. On the contrary, the treatment of this paper is general and thorough.

This paper is organized as follows. In section 2, the formalism of CTM is briefly recapitulated. In section 3, the first-class constraint Poisson algebra of CTM is computed in the formal continuum limit. Here, we introduce a moment expansion for the off-diagonal components of one of the dynamical tensors of CTM, and use the moments to express the continuum limit of the algebra. In section 4, we interpret the continuum limit in terms of the ADM formalism of general relativity. This is successfully done by taking into account differences of weights of the gauge parameters between CTM and the ADM formalism, and by introducing an assumed relation between the lowest orders of the moment expansion and the (inverse) metric tensor field. In section 5, we consider the Hamiltonian constraint obtained by multiplying that of the ADM formalism by a weight of half-density, and study the constraint Poisson algebra among the newly defined Hamiltonian constraint and the momentum constraint. The algebra certainly reproduces the continuum limit of that of CTM, and hence this proves the equivalence of the two. The final section is devoted to summary and discussions.

2 Canonical tensor model

The set of the dynamical variables of the canonical tensor model (CTM) [32–34] in the minimal setting [44] is given by a canonical conjugate pair of symmetric real rank-three tensors, M_{abc} and P_{abc} ($a, b, c = 1, 2, \ldots, N$), satisfying

$$\{M_{abc}, P_{def}\} = \sum_{\sigma} \delta_{a\sigma_d}\delta_{b\sigma_e}\delta_{c\sigma_f}, \quad \{M_{abc}, M_{def}\} = \{P_{abc}, P_{def}\} = 0, \tag{2.1}$$

where $\{\ ,\ \}$ denotes the Poisson bracket, and the summation is over all the permutations of d, e, f to incorporate the symmetry of the tensors under all the permutations of the indices. The Hamiltonian is given by

$$H = \xi_a \mathcal{H}_a + \eta_{[ab]} \mathcal{J}_{[ab]}, \tag{2.2}$$

where ξ_a and $\eta_{[ab]}$ are Lagrange multipliers, repeated indices are summed over, and

$$\mathcal{H}_a = \frac{1}{2} \left(P_{abc} P_{bde} M_{cde} - \lambda M_{abb} \right), \tag{2.3}$$

$$\mathcal{J}_{[ab]} = \frac{1}{4} \left(P_{acd} M_{bcd} - P_{bcd} M_{acd} \right). \tag{2.4}$$

Here the square brackets in the indices symbolically represent the anti-symmetry, $\mathcal{J}_{[ab]} = -\mathcal{J}_{[ba]}$, $\eta_{[ab]} = -\eta_{[ba]}$, and $\mathcal{J}_{[ab]}$ and $\mathcal{H}_a$ are the generators of the SO(N)-kinematical symmetry and of the symmetry analogous to the temporal diffeomorphism in general relativity, respectively. Following the naming in the ADM formalism [41, 42] of general relativity, we call $\mathcal{H}_a$ and $\mathcal{J}_{[ab]}$ *Hamiltonian constraint* and *momentum constraint* of CTM, respectively. λ is a real undetermined constant, which we call *cosmological constant.* The last naming comes from the fact that the $N = 1$ case exactly agrees with the mini-superspace approximation of general relativity with a cosmological constant proportional to λ [43]. Here we would like to stress that the form of the Hamiltonian constraint in (2.3) is the most general one under some physically reasonable assumptions [33], and therefore we cannot ignore the cosmological constant term from the beggining.[3]

As in the case of the ADM formalism, $\mathcal{H}_a$ and $\mathcal{J}_{[ab]}$ form a first-class constraint Poisson algebra given by

$$\begin{aligned}
\{\mathcal{H}(\xi^1), \mathcal{H}(\xi^2)\} &= \mathcal{J}([\tilde{\xi}^1, \tilde{\xi}^2] + 2\lambda[[\xi^1, \xi^2]]), \\
\{\mathcal{J}(\eta), \mathcal{H}(\xi)\} &= \mathcal{H}(\eta\,\xi), \\
\{\mathcal{J}(\eta^1), \mathcal{J}(\eta^2)\} &= \mathcal{J}\left([\eta^1, \eta^2]\right),
\end{aligned} \tag{2.5}$$

where $\mathcal{H}(\xi) \equiv \xi_a \mathcal{H}_a$, $\mathcal{J}(\eta) \equiv \eta_{[ab]} \mathcal{J}_{[ab]}$, $\tilde{\xi}_{ab} \equiv P_{abc}\xi_c$. Here, on the right-hand sides, $[\,,\,]$ denotes the matrix commutator, and $[[\xi^1, \xi^2]]_{ab} \equiv \xi_a^1 \xi_b^2 - \xi_a^2 \xi_b^1$. It is important to note that the algebra (2.5) has a structure depending on P on the right-hand side in the first line, and therefore it is not a genuine Lie algebra. This is a similar situation as the constraint algebra [38–40] of general relativity in the ADM formalism, and will be essential for (2.5) to reproduce the constraint algebra of the ADM formalism in a formal continuum limit. The constraints (2.3), (2.4) and hence the first-class algebra (2.5) are unique under some physically reasonable assumptions [33].

3 Constraint algebra of CTM in a formal continuum limit

In this section, we take a continuum limit of the constraint algebra (2.5) of CTM by assuming emergence of a continuum space. Note that this assumption is imposed without any justifications. Ideally, such an assumption should be derived as infrared effective dynamics of CTM, but this is out of our reach at present. Namely, the derivation in this section should be considered as a formal continuum limit ignoring the presently unknown

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[3]In connection with randomly connected tensor networks, the first term of (2.3) describes the simplest operation inserting a tensor vertex [48], while it is easy to see that the cosmological constant term is the simplest operation deleting a tensor vertex. It would be natural for the Hamiltonian constraint to contain both the creation/annihilation operations.

real dynamics of CTM. On the other hand, it will suggest a plausible way for emergence of space and general relativity in the framework of CTM, and give directions of future study to finally justify the assumption.

First of all, the assumption is translated to that the indices of the tensors can be replaced by continuous coordinates of *space* as

$$a \to x \in \mathrm{R}^D, \tag{3.1}$$

where D denotes the spatial dimension. Then, we also assume that index contractions are replaced by integrations as

$$\sum_{a=1}^{N} \to \int d^D x. \tag{3.2}$$

In general, there could exist a non-trivial integration measure as $\int d^D x \, \rho(x)$, but this could be canceled by a Jacobian $|\frac{\partial x'}{\partial x}|$ after an appropriate transformation of the coordinate $x'(x)$. Note that there are no contradictions in regarding x to be fixed labels as here, since the transformations below apply to the dynamical variables of CTM but not to the indices represented by the coordinates.

A continuum space has an intrinsic concept of locality, and we will pick up a local part of the constraint algebra (2.5). We will also introduce an assumption about the form of P which is in accord with the locality. We will show that this reduction to a local part of the algebra can consistently be done, except for the cosmological constant term in (2.3). We will comment on the cosmological constant term in the final section.

3.1 $\{\mathcal{J}, \mathcal{J}\}$ part

Let us first discuss the Poisson algebra of $\mathcal{J}$. The local part of the algebra with respect to the continuum space would be picked up by putting the D-dimensional delta-function $\delta^D(x-y)$ into the argument of $\mathcal{J}(\eta)$ in some manner. Since η must be anti-symmetric, the lowest order should be expressed by using the first derivatives of $\delta^D(x-y)$, and we are lead to the form,

$$\eta_{[xy]} = \frac{1}{2} \left(v^\mu(x) + v^\mu(y) \right) \delta^D_\mu(x-y), \tag{3.3}$$

where v is an arbitrary vector field on the space, the Greek index μ denotes the spatial indices, $\mu = 1, 2, \ldots, D$, and

$$\delta^D_\mu(x-y) \equiv \frac{\partial}{\partial x^\mu} \delta^D(x-y). \tag{3.4}$$

The third equation of (2.5) implies that the algebra of $\mathcal{J}$ is equivalent to the commutator algebra of η. Products of distributions can be computed rather easily by using test functions. In the present case, for a test function f, we obtain

$$(\eta f)(x) \equiv \int d^D y \, \eta_{[xy]} f(y)$$

$$= \frac{1}{2} \int d^D y \, \left(v^\mu(x) + v^\mu(y) \right) \delta^D_\mu(x-y) f(y)$$

$$= \frac{1}{2} \int d^D y \, \delta^D(x-y) \frac{\partial}{\partial y^\mu} \big((v^\mu(x) + v^\mu(y)) f(y) \big)$$

$$= \left(\frac{1}{2}(\partial_\mu v^\mu) + v^\mu \partial_\mu \right) f(x). \tag{3.5}$$

So, we obtain

$$[\eta^1, \eta^2] f = \eta^1(\eta^2 f) - \eta^2(\eta^1 f)$$

$$= \left(\frac{1}{2}(\partial_\mu v_1^\mu) + v_1^\mu \partial_\mu \right) \left(\frac{1}{2}(\partial_\mu v_2^\mu) f + v_2^\mu \partial_\mu \right) f - (1 \leftrightarrow 2)$$

$$= \left(\frac{1}{2}(\partial_\mu v_3^\mu) + v_3^\mu \partial_\mu \right) f, \tag{3.6}$$

where $v_{1,2}$ are respectively related to $\eta^{1,2}$ as (3.3), and

$$v_3^\mu = [v_1, v_2]^\mu = v_1^\nu \partial_\nu v_2^\mu - v_2^\nu \partial_\nu v_1^\mu. \tag{3.7}$$

Note that (3.6) has exactly the same form as (3.5). Therefore,

$$[\eta^1, \eta^2] = \eta^3, \tag{3.8}$$

where η^3 is (3.3) with $v = v_3$. Thus, the commutator algebra of η with the form (3.3) closes, and the representative vectors v form a commutation algebra given in (3.7).

3.2 $\{\mathcal{J}, \mathcal{H}\}$ part

Under the assumption of the continuum limit, the vector ξ of $\mathcal{H}(\xi)$ should be replaced by a function $\xi(x)$ on the D-dimensional space. Then, the second line of (2.5) implies that the Poisson bracket between $\mathcal{J}$ and $\mathcal{H}$ is equivalent to the matrix-vector product $\eta \xi$. Thus, by using (3.3) and (3.5), we obtain

$$\eta \xi = \frac{1}{2}(\partial_\mu v^\mu)\xi + v^\mu \partial_\mu \xi, \tag{3.9}$$

where v is the representative vector of η.

3.3 $\{\mathcal{H}, \mathcal{H}\}$ part

This is the most non-trivial part of our discussions. We firstly assume that almost local configurations of P become dominant in the formal continuum limit.[4] Then, we will see that we have to take into account not only the leading order in an expansion of P regarding locality but also the next leading orders to obtain a non-vanishing result. The next leading orders are dimensionally higher than the leading one by (length)^2. This would mean that $\{\mathcal{H}, \mathcal{H}\}$ part contains an effective length scale, and the way it appears suggests that it is intimately related with the smallest length scale in a space or the Planck length. In this paper, its definitive interpretation is beyond our scope, and is left for future study.

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[4]The validity of this assumption may be checked by investigating peaks of the exact wave functions obtained in [45]. This remains as a future study.

Let us first consider the part independent of the cosmological constant λ, namely $[\tilde{\xi}^1, \tilde{\xi}^2]$ on the right-hand side in the first line of (2.5). Let us first try a strictly local form of P, $P^\delta_{xyz} \equiv \delta^D(x-y)\delta^D(x-z)$, and evaluate $[\tilde{\xi}^1, \tilde{\xi}^2]$. The operation $\tilde{\xi}$ on a test function f can be computed as

$$(\tilde{\xi}f)(x) = \int d^D y d^D z P^\delta_{xyz} \xi(y) f(z)$$
$$= (\xi f)(x), \tag{3.10}$$

and hence

$$[\tilde{\xi}^1, \tilde{\xi}^2]f = \tilde{\xi}^1(\tilde{\xi}^2 f) - \tilde{\xi}^2(\tilde{\xi}^1 f)$$
$$= \xi^1 \xi^2 f - \xi^2 \xi^1 f$$
$$= 0. \tag{3.11}$$

Therefore, P needs to be smeared for $[\tilde{\xi}^1, \tilde{\xi}^2]$ to be non-vanishing.

To characterize such smearing of P in the order of (length)^2,[5] let us introduce the following moments,

$$\int d^D y d^D z P_{xyz} = \alpha(x),$$
$$\int d^D y d^D z P_{xyz} \delta y^\mu = \int d^D y d^D z P_{xyz} \delta z^\mu = \beta^\mu(x),$$
$$\int d^D y d^D z P_{xyz} \delta y^\mu \delta y^\nu = \int d^D y d^D z P_{xyz} \delta z^\mu \delta z^\nu = \gamma^{\mu\nu}(x), \tag{3.12}$$
$$\int d^D y d^D z P_{xyz} \delta y^\mu \delta z^\nu = \tilde{\gamma}^{\mu\nu}(x),$$

where $\delta y = y - x$, $\delta z = z - x$, and we have taken into account $P_{xyz} = P_{xzy}$, which is one of the permutation symmetries of P. In fact, the remaining permutation symmetry of P gives more restrictions on the moments in (3.12). To see this, let us consider three independent test functions f_i $(i = 1, 2, 3)$. By using (3.12) and performing partial integrations, we obtain

$$\int d^D x d^D y d^D z \, P_{xyz} f_1(x) f_2(y) f_3(z)$$
$$= \int d^D x d^D y d^D z \, P_{xyz} f_1 \left(f_2 + \delta y^\mu f_{2,\mu} + \frac{1}{2} \delta y^\mu \delta y^\nu f_{2,\mu\nu} \right) \left(f_3 + \delta z^\mu f_{3,\mu} + \frac{1}{2} \delta z^\mu \delta z^\nu f_{3,\mu\nu} \right) + \cdots$$
$$= \int d^D x \left(\alpha f_1 f_2 f_3 + \beta^\mu f_1 (f_2 f_3)_{,\mu} + \tilde{\gamma}^{\mu\nu} f_1 f_{2,\mu} f_{3,\nu} + \frac{1}{2} \gamma^{\mu\nu} f_1 f_2 f_{3,\mu\nu} + \frac{1}{2} \gamma^{\mu\nu} f_1 f_{2,\mu\nu} f_3 \right) + \cdots$$
$$= \int d^D x \left(\left(\alpha - \beta^\mu_{,\mu} + \frac{1}{2} \gamma^{\mu\nu}_{,\mu\nu} \right) f_1 f_2 f_3 + \left(\gamma^{\mu\nu}_{,\nu} - \beta^\mu \right) f_{1,\mu} f_2 f_3 + \left(\gamma^{\mu\nu}_{,\nu} - \tilde{\gamma}^{\mu\nu}_{,\nu} \right) f_1 f_2 f_{3,\mu} \right.$$
$$\left. - f_2 \tilde{\gamma}^{\mu\nu} (f_1 f_{3,\mu\nu} + f_{1,\mu} f_{3,\nu}) + \frac{1}{2} f_2 \gamma^{\mu\nu} (2 f_1 f_{3,\mu\nu} + f_{1,\mu\nu} f_3 + 2 f_{1,\mu} f_{3,\nu}) \right) + \cdots, \tag{3.13}$$

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[5]As shown below, β looks like in the first order in (3.12), but turns out to be in the second order due to the permutation symmetry of P.

where we have expanded the test functions $f_2(y), f_3(z)$ around x, suppressing the obvious argument x, and have used the shorthand notations, $f_{,\mu} = \partial_\mu f$, etc. Here the dots represent terms with higher moments and will be ignored. The third line contains no derivatives of f_1, while the final lines contain no derivatives of f_2. Since the permutation symmetry of P requires that (3.13) must be symmetric under the permutations of the test functions, the third and the final lines should have the same expression after the exchange of f_1 and f_2. This demands

$$\alpha - \beta^\mu_{,\mu} + \frac{1}{2}\gamma^{\mu\nu}_{,\mu\nu} = \alpha,$$
$$\gamma^{\mu\nu}_{,\nu} - \beta^\mu = \gamma^{\mu\nu}_{,\nu} - \tilde{\gamma}^{\mu\nu}_{,\nu} = \beta^\mu,$$
$$\gamma^{\mu\nu} - \tilde{\gamma}^{\mu\nu} = \tilde{\gamma}^{\mu\nu}, \tag{3.14}$$
$$\gamma^{\mu\nu} - \tilde{\gamma}^{\mu\nu} = \frac{1}{2}\gamma^{\mu\nu},$$

and the solution is

$$\beta^\mu = \frac{1}{2}\gamma^{\mu\nu}_{,\nu},$$
$$\tilde{\gamma}^{\mu\nu} = \frac{1}{2}\gamma^{\mu\nu}. \tag{3.15}$$

Thus we obtain that the moment expansion (3.12) is characterized thoroughly by α and γ. Here, it is important to recall that we are *not* performing derivative expansions for dynamical variables, but rather we are performing expansions in moments of "fuzziness" of space: we count the number of derivatives on test functions as orders, but not derivatives on dynamical variables such as α, γ. Therefore, the orders associated to α and γ are zeroth and second, respectively, and β should be considered to be in the same order as γ, irrespective of a derivative on γ in (3.15).

By using (3.12) and (3.15), the operation $\tilde{\xi}$ in (2.5) on a test function f can be computed as

$$(\tilde{\xi}f)(x) = \int d^D y\, d^D z\, P_{xyz}\xi(y)f(z)$$
$$= \int d^D y\, d^D z\, P_{xyz}\left(\xi(x) + \delta y^\mu \xi_{,\mu}(x) + \frac{1}{2}\delta y^\mu \delta y^\nu \xi_{,\mu\nu}(x)\right)$$
$$\times \left(f(x) + \delta z^\mu f_{,\mu}(x) + \frac{1}{2}\delta z^\mu \delta z^\nu f_{,\mu\nu}(x)\right) + \cdots$$
$$= \left(\alpha\xi f + \frac{1}{2}\gamma^{\mu\nu}_{,\mu}(\xi f)_{,\nu} + \frac{1}{2}\gamma^{\mu\nu}\left(\xi_{,\mu}f_{,\nu} + \xi f_{,\mu\nu} + f\xi_{,\mu\nu}\right)\right)(x) + \cdots. \tag{3.16}$$

Then, by using (3.16), we obtain the commutator $[\tilde{\xi}^1, \tilde{\xi}^2]$ as

$$[\tilde{\xi}^1, \tilde{\xi}^2]f = \tilde{\xi}^1(\tilde{\xi}^2 f) - \tilde{\xi}^2(\tilde{\xi}^1 f)$$
$$= \frac{1}{2}\alpha\xi^1 \gamma^{\mu\nu}_{,\mu}\xi^2_{,\nu}f + \frac{1}{2}\gamma^{\mu\nu}\left(\alpha\xi^1\xi^2_{,\mu\nu}f + 2\alpha\xi^1\xi^2_{,\mu}f_{,\nu} + \xi^1\alpha_{,\mu}\xi^2_{,\nu}f\right) - (1 \leftrightarrow 2), \tag{3.17}$$

where we have discarded terms with orders $(\gamma)^2$ and higher. In (2.5), $[\tilde{\xi}^1, \tilde{\xi}^2]$ is the argument of the generator $\mathcal{J}$, and, surprisingly, (3.17) has exactly the form of (3.5) with

$$v^\mu = \alpha\gamma^{\mu\nu}\left(\xi^1\xi^2_{,\nu} - \xi^2\xi^1_{,\nu}\right). \tag{3.18}$$

This shows that the local form of η in (3.3) is consistent with the algebra of the Hamiltonian constraints in the second order of the moment expansion of P, except for the part depending on the cosmological constant λ.

As for the term depending on the cosmological constant λ on the right-hand side in the first equation of (2.5), we obtain

$$\left[\left[\xi^1, \xi^2\right]\right]_{xy} = \xi^1(x)\xi^2(y) - \xi^2(x)\xi^1(y). \tag{3.19}$$

Since we may take any x, y, the expression is generally non-local with respect to the continuum space.

4   Interpretation in terms of geometrodynamics

In section 3, we have analyzed the structure of the constraint algebra of CTM on the assumption of the emergence of a continuous D-dimensional space with its intrinsic locality. In this section, we will compare the result with the constraint algebra [38–40] of the ADM formalism [41, 42] of general relativity, and will construct an exact correspondence except for the part proportional to the cosmological constant λ.

In the Hamilton formalism, general relativity can be formulated as a totally constrained system due to the gauge symmetry of the spacetime diffeomorphism. In the ADM formalism, the Hamiltonian is given by

$$H^{ADM} = \int d^D x \left(n(x)\mathcal{H}^{GR}(x) + w^\mu(x)\mathcal{J}^{GR}_\mu(x)\right), \tag{4.1}$$

where $\mathcal{H}^{GR}$ and $\mathcal{J}^{GR}_\mu$ are respectively the Hamiltonian and momentum constraints, and n and w^μ are the lapse function and the shift vector, respectively. The constraints satisfy the first-class constraint Poisson algebra,

$$\begin{aligned} \{H^{GR}(n_1), H^{GR}(n_2)\} &= J^{GR}(\tilde{n}), \\ \{J^{GR}(w), H^{GR}(n)\} &= H^{GR}(\mathcal{L}_w n), \\ \{J^{GR}(w_1), J^{GR}(w_2)\} &= J^{GR}(\mathcal{L}_{w_1} w_2), \end{aligned} \tag{4.2}$$

where

$$H^{GR}(n) = \int d^D x \, n(x) \, \mathcal{H}^{GR}(x), \tag{4.3}$$

$$J^{GR}(w) = \int d^D x \, w^\mu(x) \, \mathcal{J}^{GR}_\mu(x), \tag{4.4}$$

$$\tilde{n}^\mu = g^{\mu\nu}(n_1\partial_\nu n_2 - n_2\partial_\nu n_1), \tag{4.5}$$

and $\mathcal{L}$ denotes the Lie derivative,

$$\mathcal{L}_w n = w^\mu \partial_\mu n, \tag{4.6}$$

$$(\mathcal{L}_{w_1} w_2)^\mu = [w_1, w_2]^\mu = w_1^\nu \partial_\nu w_2^\mu - w_2^\nu \partial_\nu w_1^\mu. \tag{4.7}$$

An important feature of (4.2) is that the algebraic structure depends on the dynamical field, the spatial inverse metric $g^{\mu\nu}(x)$, as in (4.5), and the algebra is therefore not a genuine Lie algebra. This structure is an essence of geometrodynamics, as thoroughly discussed in [39]. Dependence on the dynamical variable P exists similarly in the constraint algebra (2.5) of CTM, and was important in deriving the constraint algebra in the continuum limit of CTM in section 3.

Let us first compare $\{\mathcal{J}, \mathcal{J}\}$ with $\{J^{GR}, J^{GR}\}$. By comparing (3.7) with (4.7), we can simply identify

$$v = w. \tag{4.8}$$

Thus $\mathcal{J}(\eta)$ with (3.3) of CTM can be identified with the spatial diffeomorphism in the continuum limit.

Here we would like to stress that we are not arguing that $so(N)$ Lie algebra of CTM can be identified with the spatial diffeomorphism in the continuum limit. Rather, we have shown that the spatial diffeomorphism can be embedded into the orthogonal Lie algebra in the continuum limit; this simply gives a unitary (real orthogonal) representation of the spatial diffeomorphism.[6] In fact, the $so(N)$ symmetry of CTM contains or is actually dominated by other transformations which are non-local in the sense of the assumed continuous space.

Next, let us compare $\{\mathcal{J}, \mathcal{H}\}$ with $\{J^{GR}, H^{GR}\}$. With the identification (4.8), it is not possible to identify ξ and n, because there is a difference of weights indicated by (3.9) and (4.6). Geometrically, n is a scalar, but ξ is a scalar half-density. The difference can be balanced by assuming

$$\xi = g^{\frac{1}{4}} n, \tag{4.9}$$

where $g = \text{Det}[g_{\mu\nu}]$. In fact, by assuming (4.6) and using $\mathcal{L}_w g_{\mu\nu} = \nabla_\mu w_\nu + \nabla_\nu w_\mu$, one obtains

$$\begin{aligned} \mathcal{L}_w(g^A n) &= 2A(\nabla_\mu w^\mu) g^A n + g^A w^\mu \partial_\mu n \\ &= 2A(\partial_\mu w^\mu + \Gamma^\nu_{\nu\mu} w^\mu) g^A n + g^A w^\mu \partial_\mu n \\ &= 2A(\partial_\mu w^\mu) g^A n + w^\mu \partial_\mu (g^A n), \end{aligned} \tag{4.10}$$

where A is a number. For $A = \frac{1}{4}$, (4.10) agrees with (3.9) under (4.8), and therefore we should choose as (4.9).

Lastly, we compare $\{\mathcal{H}, \mathcal{H}\}$ with $\{H^{GR}, H^{GR}\}$. By putting (4.9) into (3.18), we obtain

$$v^\mu = \alpha \gamma^{\mu\nu} g^{\frac{1}{4}} \left(n_1 \partial_\nu (g^{\frac{1}{4}} n_2) - n_2 \partial_\nu (g^{\frac{1}{4}} n_1) \right)$$

---

[6]A simple illustrating example is an N-dimensional unitary (real orthogonal) representation of $so(3)$. This gives an embedding of $so(3)$ into $so(N)$, but of course $so(3)$ cannot be identified with $so(N)$ in any sense.

$$= \alpha\gamma^{\mu\nu}g^{\frac{1}{2}}\left(n_1\partial_\nu n_2 - n_2\partial_\nu n_1\right). \tag{4.11}$$

By comparing this with (4.5) under (4.8), we obtain

$$\alpha\gamma^{\mu\nu} = g^{-\frac{1}{2}}g^{\mu\nu}. \tag{4.12}$$

As for the the part proportional to the cosmological constant on the righthand side in the first equation of (2.5), it seems difficult to associate a geometrodynamical interpretation due to the non-local character shown in (3.19).

Finally, we will make a comment on the weights appearing in (4.9) and (4.12). As shown in (3.5), with the identification (4.8), a vector in CTM is translated to a quantity which transforms as a scalar half-density by $\mathcal{J}(\eta)$ with (3.3). This applies to $\xi(x)$ as in (3.9) as well as to each index of P. Therefore, we make assignments,

$$[\xi]_w = \frac{1}{2}, \tag{4.13}$$

$$[P]_w = \frac{3}{2}, \tag{4.14}$$

where we have introduced $[\ ]_w$ to denote an additive weight of a quantity; the weight is defined so that a quantity q transforms as a scalar density, if $[q]_w = 1$. On the other hand, we have

$$\left[\int d^D x\right]_w = -1, \tag{4.15}$$

since an integration over the space cancels the transformation of a quantity $q(x)$ with $[q]_w = 1$ as

$$\int d^D x \left((\partial_\mu w^\mu)\, q + w^\mu \partial_\mu q\right) = \int d^D x\, \partial_\mu (w^\mu q) = 0, \tag{4.16}$$

where we have ignored possibilities of boundary contributions. By applying (4.14) and (4.15) to the quantities in (3.12), we obtain

$$[\alpha]_w = [\gamma^{\mu\nu}]_w = -\frac{1}{2}, \tag{4.17}$$

and therefore $[\alpha\gamma^{\mu\nu}]_w = -1$. This explains the power of g in (4.12).

5   Modified constraint algebra in the ADM formalism

In section 4, it has been shown that, to relate CTM to the ADM formalism, the gauge parameters ξ and n associated respectively to the Hamiltonian constraints of CTM and the ADM formalism should have different weights as in (4.9). In the discussions, a factor $g^{\frac{1}{4}}$ has been introduced to modify the wight of the gauge parameter n, and $g_{\mu\nu}(x)$ in it was treated as a metric tensor field on the space. In the Hamilton formalism, however, $g_{\mu\nu}(x)$ is a dynamical variable, and therefore the wight factor should be considered to be a modification of the Hamiltonian constraint rather than the gauge parameter. Then, the additional weight factor multiplied on the Hamiltonian constraint may potentially ruin the correspondence argued in section 4 on account of the Poisson brackets between the

weight factor and the constraints. Therefore, in this section, we will explicitly write down the constraint algebra after the change of the weight of the Hamiltonian constraint in the ADM formalism, and will show that it actually agrees with the continuum limit of the constraint algebra of CTM, namely (3.9) and (3.18) with (4.12) (Since there is no change in the momentum constraint, (3.7) does not need to be checked.).

Let us define the modified Hamiltonian constraint $\tilde{\mathcal{H}}$ (and $\tilde{H}$) as

$$\tilde{\mathcal{H}} = g^B \mathcal{H}^{GR}, \tag{5.1}$$

$$\tilde{H}(\xi) = \int d^D x \, \xi(x) \tilde{\mathcal{H}}(x), \tag{5.2}$$

where $g = \text{Det}(g_{\mu\nu})$ and B is a number. Here, we have changed the weight of the Hamiltonian constraint, and, from (4.9), we expect that the constraint algebra of CTM can be obtained for $B = -\frac{1}{4}$.

The Hamiltonian constraint of the ADM formalism satisfies

$$\{g_{\mu\nu}(x), \mathcal{H}^{GR}(y)\} = C_{\mu\nu}(x)\delta^D(x - y), \tag{5.3}$$

where $C_{\mu\nu}(x) = -2K_{\mu\nu}(x)$, being proportional to the extrinsic curvature expressed in terms of $g_{\mu\nu}(x)$ and its conjugate momentum $\pi^{\mu\nu}(x)$ [38–40]. Here, what matters in the following discussion is only the fact that the right-hand side of (5.3) is strictly local: it does not contain derivatives of $\delta^D(x - y)$. From (5.3), we obtain

$$\{g(x)^B, \mathcal{H}^{GR}(y)\} = Bg(x)^B C(x)\, \delta^D(x - y), \tag{5.4}$$

where $C = g^{\mu\nu} C_{\mu\nu}$. Then, we find that

$$
\begin{aligned}
\{\tilde{\mathcal{H}}(x), \tilde{\mathcal{H}}(y)\} &= \{g(x)^B \mathcal{H}^{GR}(x), g(y)^B \mathcal{H}^{GR}(y)\} \\
&= g(x)^B \mathcal{H}^{GR}(y)\{\mathcal{H}^{GR}(x), g(y)^B\} + g(y)^B \mathcal{H}^{GR}(x)\{g(x)^B, \mathcal{H}^{GR}(y)\} \\
&\quad + g(x)^B g(y)^B \{\mathcal{H}^{GR}(x), \mathcal{H}^{GR}(y)\} \\
&= g(x)^B g(y)^B \{\mathcal{H}^{GR}(x), \mathcal{H}^{GR}(y)\},
\end{aligned} \tag{5.5}
$$

where the two terms in the second line have canceled with each other due to the strict local form of (5.4). Therefore, by using (4.2), (4.3), (4.4), (4.5), and (5.5), we obtain

$$
\begin{aligned}
\{\tilde{H}(\xi^1), \tilde{H}(\xi^2)\} &= \int d^D x d^D y \, \xi^1(x)\xi^2(y)\{\tilde{\mathcal{H}}(x), \tilde{\mathcal{H}}(y)\} \\
&= \int d^D x d^D y \, g(x)^B \xi^1(x) g(y)^B \xi^2(y)\{\mathcal{H}^{GR}(x), \mathcal{H}^{GR}(y)\} \\
&= \int d^D x \, g^{\mu\nu}(x) \left(g(x)^B \xi^1(x)\partial_\nu(g(x)^B \xi^2(x)) - (1 \leftrightarrow 2)\right) \mathcal{J}_\mu^{GR}(x) \\
&= J^{GR}(v),
\end{aligned} \tag{5.6}
$$

where

$$v^\mu = g^{\mu\nu} g^{2B} (\xi^1 \partial_\nu \xi^2 - \xi^2 \partial_\nu \xi^1). \tag{5.7}$$

Indeed, for $B = -\frac{1}{4}$, this agrees with (3.18) under (4.12).

We will next compute $\{J^{GR}(v), \tilde{H}(\xi)\}$. Let us first remind the explicit expression of the momentum constraint [38–40] in the ADM formalism,

$$\mathcal{J}_\mu^{GR} = -2D_\nu \pi_\mu{}^\nu, \tag{5.8}$$

where D_μ denotes the covariant derivative, and π satisfies

$$\{g_{\mu\nu}(x), \pi^{\rho\sigma}(y)\} = \frac{1}{2}\left(\delta_\mu^\rho \delta_\nu^\sigma + \delta_\mu^\sigma \delta_\nu^\rho\right) \delta^D(x-y). \tag{5.9}$$

Then, for a vector v, we obtain

$$
\begin{aligned}
\{J^{GR}(v), g(x)\} &= \int d^D y \; v^\mu(y)\{J_\mu^{GR}(y), g(x)\} \\
&= \int d^D y \; v^\mu(y)\{-2D_\nu \pi_\mu{}^\nu(y), g(x)\} \\
&= 2\int d^D y \; (D_\nu v^\mu(y))\{\pi_\mu{}^\nu(y), g(x)\} \\
&= -2g(x)D_\mu v^\mu(x).
\end{aligned} \tag{5.10}
$$

Therefore,

$$
\begin{aligned}
\{J^{GR}(v), \tilde{H}(\xi)\} &= \int d^D x \; \{\mathcal{J}^{GR}(v), \xi(x)g(x)^B \mathcal{H}^{GR}(x)\} \\
&= \int d^D x \left(\xi(x)\{J^{GR}(v), g(x)^B\}\mathcal{H}^{GR}(x) + \xi(x)g(x)^B\{J^{GR}(v), \mathcal{H}^{GR}(x)\}\right) \\
&= \int d^D x \left(-2B\xi(x)g(x)^B D_\mu v^\mu(x) + v^\mu(x)\partial_\mu\left(\xi(x)g(x)^B\right)\right) \mathcal{H}^{GR}(x),
\end{aligned} \tag{5.11}
$$

where we have used (5.10), (4.2), and (4.6). Then, by substituting an identity,

$$
\begin{aligned}
D_\mu v^\mu(x) &= \partial_\mu v^\mu(x) + \Gamma_{\nu\mu}^\nu(x)v^\mu(x) \\
&= \partial_\mu v^\mu(x) + \frac{v^\mu(x)\partial_\mu g(x)}{2g(x)},
\end{aligned} \tag{5.12}
$$

into (5.11), we obtain

$$
\begin{aligned}
\{J^{GR}(v), \tilde{H}(\xi)\} &= \int d^D x \; (-2B\xi(x)\partial_\mu v^\mu(x) + v^\mu(x)\partial_\mu\xi(x))g(x)^B\mathcal{H}^{GR}(x) \\
&= \tilde{H}(-2B\xi\partial_\mu v^\mu + v^\mu\partial_\mu\xi).
\end{aligned} \tag{5.13}
$$

Thus, for $B = -\frac{1}{4}$, we certainly obtain (3.9) of CTM.

6 Summary and discussions

The canonical tensor model (CTM) is a rank-three tensor model formulated as a totally constrained system with a number of first-class constraints, which have the Poisson algebraic structure similar to the constraint Poisson algebra of the ADM formalism of general relativity. In this paper, we consider a formal continuum limit of CTM and have shown that,

in the continuum limit, the constraint algebra of CTM coincides with that of the ADM formalism by properly taking into account the weight difference: the Hamiltonian constraints of CTM and the ADM formalism are different with each other by half-density. We have obtained the expression of the (inverse) metric tensor field of general relativity in terms of the dynamical rank-three tensor P of CTM. Here the continuum limit assumes an almost diagonal form of P, and the lowest and the next to the lowest order coefficients of a moment expansion for the off-diagonal components of P give the expression of the (inverse) metric tensor field. This explicit correspondence between P and the metric tensor field would be the most useful achievement of this paper to guide future study of the dynamics of CTM.

A specific form of P, almost diagonal, is assumed in the continuum limit. This is obviously an insufficient treatment, since P is a dynamical variable, and its form must be determined dynamically rather than formally assumed. One possible way to justify it would be to study the exact physical wave functions which have been obtained so far [44, 45], and check whether such configurations can appear as peaks of these wave functions. A nontrivial issue in such a study would be that we have to take a certain large N limit, which would be necessary for indices to be effectively represented by continuous coordinates.

We have seen that the cosmological constant of CTM generates non-local dynamics which is in contradiction with the locality assumption of the continuum limit. Therefore, for the consistency of our discussions, the cosmological constant must vanish. In fact, we have previously shown that the value of the cosmological constant can be changed by shifting P [45]. It would be interesting to study the exact physical wave functions [44, 45] to see whether there is a dynamical mechanism which tunes P to cancel the cosmological constant. Note that the dynamics of the cosmological constant term of CTM is obviously different from that of general relativity due to the non-local property, and this would give a certain chance to circumvent the common difficulties to tune the cosmological constant to the observed value [49]. Another possibility is that the cosmological constant term is prohibited by the consistency of the constraint algebra of CTM. As shown in a previous paper [33], this actually occurs, if we impose the generalized Hermiticity condition on the dynamical variables of CTM, instead of imposing the reality condition as in this paper.

We have chosen a local class of momentum constraints of CTM from all, and have shown that they correspond to the momentum constraints in the ADM formalism. However, we have not discussed consequences of ignoring the non-local momentum constraints of CTM. One way to justify the ignorance would be to gauge-fix the non-local ones, while the local ones are left intact. Then, a consequence in quantum case would be that a constraint, say $\hat{\mathcal{C}}$, would be modified by a similarity transformation $\hat{\mathcal{C}}_{\text{eff}} = \sqrt{V}\hat{\mathcal{C}}\sqrt{V}^{-1}$ [50], where V is the gauge volume of the non-local gauge transformations. In a more general treatment, V could also contain a Jacobian generated from a process of taking the coefficients of the moment expansions as coarse grained collective coordinates for P [50]. Such a similarity transformation does not change the algebraic structure, but there will be some consequences of physical importance. One is that an almost diagonal form will be required for $\hat{P}_{\text{eff}} = \sqrt{V}\hat{P}\sqrt{V}^{-1}$ instead of $\hat{P}$ in the continuum limit. Another will be that Hamiltonian constraint $\hat{\mathcal{H}}_{\text{eff}} = \sqrt{V}\hat{\mathcal{H}}\sqrt{V}^{-1}$ will be changed from the original form $\hat{\mathcal{H}}$. Since the original Hamiltonian constraint of CTM seems lacking a term corresponding to the spatial

curvature term in that of the ADM formalism, it would be highly interesting to see whether the corrections will modify the Hamiltonian constraint in a desired manner or not.

To clarify the meaning of semi-classical limit in CTM in a satisfactory manner even formally, we have to make a complete map between variables of general relativity and of CTM in the formal continuum limit. This is an important future work. A primary question is what variables of CTM correspond to the conjugate momentum of the spatial metric in general relativity. It is supposed to be related to M in some manner, but some complications exist: the spatial metric is related to the off-diagonal components of P^2, rather than P itself as shown in this paper, and also M may contain some non-local components suggested by the non-local character of the cosmological constant term. Another interesting future direction is to discuss the Lagrangian corresponding to our constrained Hamiltonian system. It is straightforward to write down a first-order form of a Lagrangian in terms of M and P, while it seems difficult to obtain a second-order form in terms of M due to a difficulty in solving the equation of motion of P with respect to M. We hope we can report some progress on these aspects in future publications.

Acknowledgments

References

[1] J. Ambjørn, B. Durhuus and T. Jonsson, *Three-dimensional simplicial quantum gravity and generalized matrix models*, [*Mod. Phys. Lett.* **A 6** (1991) 1133](https://doi.org/) [iɴSPIRE].

[2] N. Sasakura, *Tensor model for gravity and orientability of manifold*, [*Mod. Phys. Lett.* **A 6** (1991) 2613](https://doi.org/) [iɴSPIRE].

[3] N. Godfrey and M. Gross, *Simplicial quantum gravity in more than two-dimensions*, [*Phys. Rev.* **D 43** (1991) 1749](https://doi.org/) [iɴSPIRE].

[4] D.V. Boulatov, *A model of three-dimensional lattice gravity*, [*Mod. Phys. Lett.* **A 7** (1992) 1629](https://doi.org/) [[hep-th/9202074](https://arxiv.org/abs/hep-th/9202074)] [iɴSPIRE].

[5] H. Ooguri, *Topological lattice models in four-dimensions*, [*Mod. Phys. Lett.* **A 7** (1992) 2799](https://doi.org/) [[hep-th/9205090](https://arxiv.org/abs/hep-th/9205090)] [iɴSPIRE].

[6] R. De Pietri, L. Freidel, K. Krasnov and C. Rovelli, *Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space*, [*Nucl. Phys.* **B 574** (2000) 785](https://doi.org/) [[hep-th/9907154](https://arxiv.org/abs/hep-th/9907154)] [iɴSPIRE].

[7] L. Freidel, *Group field theory: an overview*, *Int. J. Theor. Phys.* **44** (2005) 1769 [hep-th/0505016] [INSPIRE].

[8] D. Oriti, *The microscopic dynamics of quantum space as a group field theory*, arXiv:1110.5606 [INSPIRE].

[9] R. De Pietri and C. Petronio, *Feynman diagrams of generalized matrix models and the associated manifolds in dimension* 4, *J. Math. Phys.* **41** (2000) 6671 [gr-qc/0004045] [INSPIRE].

[10] R. Gurau, *Colored group field theory*, *Commun. Math. Phys.* **304** (2011) 69 [arXiv:0907.2582] [INSPIRE].

[11] R. Gurau and J.P. Ryan, *Colored tensor models — a review*, *SIGMA* **8** (2012) 020 [arXiv:1109.4812] [INSPIRE].

[12] T. Delepouve and R. Gurau, *Phase transition in tensor models*, *JHEP* **06** (2015) 178 [arXiv:1504.05745] [INSPIRE].

[13] V. Bonzom, T. Delepouve and V. Rivasseau, *Enhancing non-melonic triangulations: a tensor model mixing melonic and planar maps*, *Nucl. Phys.* **B 895** (2015) 161 [arXiv:1502.01365] [INSPIRE].

[14] T. Delepouve and V. Rivasseau, *Constructive tensor field theory: the T_3^4 model*, arXiv:1412.5091 [INSPIRE].

[15] V.A. Nguyen, S. Dartois and B. Eynard, *An analysis of the intermediate field theory of T^4 tensor model*, *JHEP* **01** (2015) 013 [arXiv:1409.5751] [INSPIRE].

[16] S. Dartois, *A Givental-like formula and bilinear identities for tensor models*, *JHEP* **08** (2015) 129 [arXiv:1409.5621] [INSPIRE].

[17] V. Lahoche, D. Oriti and V. Rivasseau, *Renormalization of an Abelian tensor group field theory: solution at leading order*, *JHEP* **04** (2015) 095 [arXiv:1501.02086] [INSPIRE].

[18] D. Benedetti, J. Ben Geloun and D. Oriti, *Functional renormalisation group approach for tensorial group field theory: a rank-3 model*, *JHEP* **03** (2015) 084 [arXiv:1411.3180] [INSPIRE].

[19] J. Ben Geloun and R. Toriumi, *Parametric representation of rank d tensorial group field theory: Abelian models with kinetic term $\sum_s |p_s| + \mu$*, *J. Math. Phys.* **56** (2015) 093503 [arXiv:1409.0398] [INSPIRE].

[20] S. Carrozza, *Discrete renormalization group for* SU(2) *tensorial group field theory*, *Ann. Inst. Henri Poincaré Comb. Phys. Interact.* **2** (2015) 49 [arXiv:1407.4615] [INSPIRE].

[21] M. Fukuma, S. Sugishita and N. Umeda, *Putting matters on the triangle-hinge models*, arXiv:1504.03532 [INSPIRE].

[22] M. Fukuma, S. Sugishita and N. Umeda, *Random volumes from matrices*, *JHEP* **07** (2015) 088 [arXiv:1503.08812] [INSPIRE].

[23] V. Bonzom, R. Gurau, A. Riello and V. Rivasseau, *Critical behavior of colored tensor models in the large-N limit*, *Nucl. Phys.* **B 853** (2011) 174 [arXiv:1105.3122] [INSPIRE].

[24] R. Gurau and J.P. Ryan, *Melons are branched polymers*, *Annales Henri Poincaré* **15** (2014) 2085 [arXiv:1302.4386] [INSPIRE].

[25] M. Raasakka and A. Tanasa, *Next-to-leading order in the large-N expansion of the multi-orientable random tensor model*, *Annales Henri Poincaré* **16** (2015) 1267 [`arXiv:1310.3132`] [ɪɴSPIRE].

[26] S. Dartois, R. Gurau and V. Rivasseau, *Double scaling in tensor models with a quartic interaction*, *JHEP* **09** (2013) 088 [`arXiv:1307.5281`] [ɪɴSPIRE].

[27] W. Kamiński, D. Oriti and J.P. Ryan, *Towards a double-scaling limit for tensor models: probing sub-dominant orders*, *New J. Phys.* **16** (2014) 063048 [`arXiv:1304.6934`] [ɪɴSPIRE].

[28] R. Gurau, *The 1/N expansion of tensor models beyond perturbation theory*, *Commun. Math. Phys.* **330** (2014) 973 [`arXiv:1304.2666`] [ɪɴSPIRE].

[29] J. Ambjørn, J. Jurkiewicz and R. Loll, *Emergence of a 4D world from causal quantum gravity*, *Phys. Rev. Lett.* **93** (2004) 131301 [`hep-th/0404156`] [ɪɴSPIRE].

[30] S. Horata, H.S. Egawa, N. Tsuda and T. Yukawa, *Phase structure of four-dimensional simplicial quantum gravity with a* U(1) *gauge field*, *Prog. Theor. Phys.* **106** (2001) 1037 [`hep-lat/0004021`] [ɪɴSPIRE].

[31] J. Ambjørn, K.N. Anagnostopoulos and J. Jurkiewicz, *Abelian gauge fields coupled to simplicial quantum gravity*, *JHEP* **08** (1999) 016 [`hep-lat/9907027`] [ɪɴSPIRE].

[32] N. Sasakura, *Canonical tensor models with local time*, *Int. J. Mod. Phys.* **A 27** (2012) 1250020 [`arXiv:1111.2790`] [ɪɴSPIRE].

[33] N. Sasakura, *Uniqueness of canonical tensor model with local time*, *Int. J. Mod. Phys.* **A 27** (2012) 1250096 [`arXiv:1203.0421`] [ɪɴSPIRE].

[34] N. Sasakura, *A canonical rank-three tensor model with a scaling constraint*, *Int. J. Mod. Phys.* **A 28** (2013) 1350030 [`arXiv:1302.1656`] [ɪɴSPIRE].

[35] D. Oriti, *Group field theory as the* 2nd *quantization of loop quantum gravity*, `arXiv:1310.7786` [ɪɴSPIRE].

[36] N. Sasakura, *Emergent general relativity on fuzzy spaces from tensor models*, *Prog. Theor. Phys.* **119** (2008) 1029 [`arXiv:0803.1717`] [ɪɴSPIRE].

[37] N. Sasakura, *Gauge fixing in the tensor model and emergence of local gauge symmetries*, *Prog. Theor. Phys.* **122** (2009) 309 [`arXiv:0904.0046`] [ɪɴSPIRE].

[38] B.S. DeWitt, *Quantum theory of gravity. 1. The canonical theory*, *Phys. Rev.* **160** (1967) 1113 [ɪɴSPIRE].

[39] S.A. Hojman, K. Kuchar and C. Teitelboim, *Geometrodynamics regained*, *Annals Phys.* **96** (1976) 88 [ɪɴSPIRE].

[40] C. Teitelboim and J. Zanelli, *Dimensionally continued topological gravitation theory in Hamiltonian form*, *Class. Quant. Grav.* **4** (1987) L125 [ɪɴSPIRE].

[41] R.L. Arnowitt, S. Deser and C.W. Misner, *Canonical variables for general relativity*, *Phys. Rev.* **117** (1960) 1595 [ɪɴSPIRE].

[42] R.L. Arnowitt, S. Deser and C.W. Misner, *The dynamics of general relativity*, *Gen. Rel. Grav.* **40** (2008) 1997 [`gr-qc/0405109`] [ɪɴSPIRE].

[43] N. Sasakura and Y. Sato, *Interpreting canonical tensor model in minisuperspace*, *Phys. Lett.* **B 732** (2014) 32 [`arXiv:1401.2062`] [ɪɴSPIRE].

[44] N. Sasakura, *Quantum canonical tensor model and an exact wave function*, *Int. J. Mod. Phys.* **A 28** (2013) 1350111 [arXiv:1305.6389] [inSPIRE].

[45] G. Narain, N. Sasakura and Y. Sato, *Physical states in the canonical tensor model from the perspective of random tensor networks*, *JHEP* **01** (2015) 010 [arXiv:1410.2683] [inSPIRE].

[46] N. Sasakura and Y. Sato, *Ising model on random networks and the canonical tensor model*, *Prog. Theor. Exp. Phys.* **2014** (2014) 053B03 [arXiv:1401.7806] [inSPIRE].

[47] N. Sasakura and Y. Sato, *Exact free energies of statistical systems on random networks*, *SIGMA* **10** (2014) 087 [arXiv:1402.0740] [inSPIRE].

[48] N. Sasakura and Y. Sato, *Renormalization procedure for random tensor networks and the canonical tensor model*, *Prog. Theor. Exp. Phys.* **2015** (2015) 043B09 [arXiv:1501.05078] [inSPIRE].

[49] S. Weinberg, *The cosmological constant problem*, *Rev. Mod. Phys.* **61** (1989) 1 [inSPIRE].

[50] A. Jevicki and B. Sakita, *The quantum collective field method and its application to the planar limit*, *Nucl. Phys.* **B 165** (1980) 511 [inSPIRE].