Bootstrapping an NMHV amplitude through three loops

Lance J. Dixon[a] **and Matt von Hippel**[b]

[a]*SLAC National Accelerator Laboratory, Stanford University,*
Stanford, CA 94309, U.S.A.

[b]*Simons Center for Geometry and Physics, Stony Brook University,*
Stony Brook, NY 11794, U.S.A.

E-mail: lance@slac.stanford.edu, matthew.vonhippel@stonybrook.edu

ABSTRACT: We extend the hexagon function bootstrap to the next-to-maximally-helicity-violating (NMHV) configuration for six-point scattering in planar $\mathcal{N} = 4$ super-Yang-Mills theory at three loops. Constraints from the $\bar{Q}$ differential equation, from the operator product expansion (OPE) for Wilson loops with operator insertions, and from multi-Regge factorization, lead to a unique answer for the three-loop ratio function. The three-loop result also predicts additional terms in the OPE expansion, as well as the behavior of NMHV amplitudes in the multi-Regge limit at one higher logarithmic accuracy (NNLL) than was used as input. Both predictions are in agreement with recent results from the flux-tube approach. We also study the multi-particle factorization of multi-loop amplitudes for the first time. We find that the function controlling this factorization is purely logarithmic through three loops. We show that a function U, which is closely related to the parity-even part of the ratio function V, is remarkably simple; only five of the nine possible final entries in its symbol are non-vanishing. We study the analytic and numerical behavior of both the parity-even and parity-odd parts of the ratio function on simple lines traversing the space of cross ratios (u, v, w), as well as on a few two-dimensional planes. Finally, we present an empirical formula for V in terms of elements of the coproduct of the six-gluon MHV remainder function R_6 at one higher loop, which works through three loops for V (four loops for R_6).

Contents

1 Introduction

The maximally supersymmetric gauge theory in four dimensions, $\mathcal{N} = 4$ super Yang-Mills theory (SYM), has been a valuable proving ground for scattering amplitudes research, especially in the planar limit of a large number of colors. Over the past two decades, calculations in planar $\mathcal{N} = 4$ SYM have pushed further, in terms of loops, legs, and general understanding, than they have in other gauge theories [1–7]. In doing so, they have also offered insight into efficient methods for handling other gauge theories, as well as into the general properties of scattering amplitudes. In addition, empirical results have led to the discovery of many hidden properties of planar $\mathcal{N} = 4$ SYM, such as dual (super)conformal invariance [8–12], and the amplitude-Wilson-loop duality [11, 13–16].

Many of the more powerful approaches to $\mathcal{N} = 4$ supersymmetric scattering amplitudes compute the loop integrand of the theory [1–3, 5, 9, 10, 17–20]. These approaches can

produce the integrand at very high loop order [21–23], but the evaluation of the loop integrals can be quite challenging, in part due to severe infrared divergences. Some of the methods for producing the integrands are only valid exactly in four dimensions in the massless theory, where the integrals are infinite. Even when the integrands can be computed with a regulator in place, it is difficult to isolate the infrared divergences of high-loop order integrals directly at the integrand level. Although there are exceptions, such as the energy-energy correlation [24, 25], finite observables typically require the explicit cancellation of infrared divergences across different loop orders.

In this paper we will follow an alternative approach, the *hexagon function bootstrap* [26–30]. The philosophy of this program is to bypass integrands altogether and focus on infrared-finite quantities from the very beginning. One such finite quantity is the remainder function [31, 32], R_n, defined by dividing the maximally-helicity-violating (MHV) scattering amplitude for n gluons by the BDS ansatz [3]. Another useful observable, starting with the next-to-MHV (NMHV) helicity configuration, is the ratio function $\mathcal{P}$ [12], in which super-amplitudes for other helicity configurations are divided by the MHV super-amplitude. An on-shell superspace [12, 33–35] is used to organize the external states into $\mathcal{N} = 4$ supermultiplets, and the amplitudes into super-amplitudes.

Such finite observables can be constrained directly from their analytic properties, particularly their behavior in kinematical limits where amplitudes factorize and can be computed by other methods. In the case of planar $\mathcal{N} = 4$ super-Yang-Mills theory, we are fortunate to have a rich abundance of such boundary data. Perhaps the most powerful information comes in the near-collinear limit where two of the external states are almost parallel. Thanks to the equivalence between amplitudes and polygonal Wilson loops, this limit corresponds to an operator product expansion (OPE) [36–39]. The relevant operators, whose anomalous dimensions are known exactly [40], generate excitations of a one-dimensional flux tube. These states have integrable $1 + 1$ dimensional scattering matrices. In the past year or so, Basso, Sever and Vieira (BSV) have shown that the OPE is governed by "pentagon transitions", which they argue can be expressed in terms of the integrable S matrices [41] to all orders in the 't Hooft coupling. BSV have worked out the consequences of this picture in increasingly great detail [42–44]. The perturbative expansions of their results provides valuable boundary data for the hexagon function bootstrap. Recently, aspects of the flux-tube approach have been reformulated in terms of Baxter equations [45, 46].

Another important limit is the multi-Regge limit, when the outgoing gluons are well separated in rapidity. In this limit, Lipatov and collaborators have described the factorization of the $\mathcal{N} = 4$ amplitudes in a Fourier-Mellin transformed space [47–54]. Further perspectives on multi-Regge factorization have been provided by Caron-Huot [55]. The factorization limit has a logarithmic ordering, which allows for the efficient recycling of lower-loop information to higher loops [28, 29, 56, 57]. The recycling is aided by the recognition [56] that in the six-point case the functions relevant for the multi-Regge limit are single-valued harmonic polylogarithms (SVHPLs) [58].

Very recently, a proposal for the multi-Regge limit has been made [59] that predicts *all* subleading logarithmic orders. This proposal is based on an analytic continuation from the near-collinear limit, which is similar in spirit to earlier work [52, 60], but now provides much more detailed information.

The near-collinear, multi-Regge, and other physical constraints are most effective in determining an amplitude when they are combined with a suitable ansatz for the space of functions in which the solution lies. For the case of six-point amplitudes, dual conformal invariance implies that the amplitudes depend essentially on only three variables, the dual conformal cross ratios (u, v, w). The analytic solution for the two-loop remainder function $R_6^{(2)}(u, v, w)$ [61, 62], after it was simplified dramatically using the symbol [4], provided the inspiration for an ansatz for the symbol of the remainder function at higher loops [26]. The same ansatz could also be applied to the symbols of a pair of functions $V(u, v, w)$ and $\tilde{V}(u, v, w)$ entering the NMHV ratio function [27]. Those symbols define a class of functions of three variables, iterated integrals called hexagon functions [28]. The number of iterated integrations defines the *weight* of the hexagon function, which should be $2L$ for the L-loop contributions to R_6, V and $\tilde{V}$. Given the hexagon-function ansatz, the near-collinear limit, multi-Regge behavior, and a few other physical constraints uniquely determine the full six-point remainder function at both three [28] and four loops [29]. The uniquenes of the solution, despite the existence of around 6000 unknown parameters in the inital four-loop ansatz, is a testament to the power of the boundary data.

The aim of this paper is to apply the hexagon function bootstrap to the six-gluon NMHV amplitude. In particular, we will compute V and $\tilde{V}$ through three loops, entirely from physical constraints. A similar exercise was performed previously at two loops [27]. However, at that time fewer constraints were available, and so an explicit evaluation of two-loop integrals for special kinematics had to be performed as well, in order to fix all the unknown parameters. Now the bootstrap works unassisted at both two and three loops. The increasing amount of powerful, higher-twist OPE data [43, 44] suggests that it can be carried out to much higher loop order, with the main limiting factor likely to be computing power.

At three loops, the weight (number of iterated integrations) of V and $\tilde{V}$ is six. We characterize the functions in terms of their weight-five $\{5, 1\}$ coproduct components [63, 64], which are essentially their first derivatives. This characterization makes use of a previous classification of hexagon functions through weight five [28]. There are several hundred free parameters (unknown rational numbers) in our initial ansatz. We then apply a series of constraints to reduce the number of parameters. These constraints include fairly simple and obvious ones, such as symmetry, spurious pole cancellations and vanishing collinear limits. Other constraints incorporate more sophisticated information, such as:

- a final-entry condition (a characterization of the first derivative) which comes from [65] the $\bar{Q}$ differential equation in the super-Wilson loop approach [65, 66];

- the near-collinear limits, which are required to match the OPE results of refs. [39, 41–44] in particular;

- the multi-Regge limits, where we match to a formula that is a natural generalization of one proposed for the MHV amplitude [53, 55], and for the leading-logarithmic terms in the NMHV amplitude [54].

Together, these constraints are more than enough to fully determine V and $\tilde{V}$. Indeed, we have powerful cross checks of the consistency of our assumptions, as well as those made by other groups providing these constraints. With the parity-even and parity-odd functions fully determined, we discuss some of their limiting behaviors, plot them, and showcase their interesting features.

This article is organized as follows. In section 2 we explain our setup further, and then apply the first constraints: (anti)symmetry in $u \leftrightarrow w$; vanishing of $\tilde{V}$ under cyclic permutations of u, v, w; the final-entry condition; and the vanishing of spurious poles. These constraints reduce the number of parameters in the ansatz down to 142. In section 3 we apply constraints in the collinear limit, at leading order and at the first near-collinear order, which together determine all but two parameters. In section 4 we inspect the multi-Regge limits, which fix the remaining two parameters in $V(u, v, w)$ and $\tilde{V}(u, v, w)$. The next term in the near-collinear limit is then determined uniquely and agrees precisely with the OPE predictions of ref. [43]. We also extract the NMHV impact factor for the multi-Regge limit through next-to-next-to-leading-logarithm (NNLL), and compare it to the recent predictions of ref. [59]. In section 5, we inspect the multi-particle factorization limit of the NMHV amplitude. We introduce a function U, closely related to V, that plays an important role in this limit. We show that U collapses to a simple polynomial in $\ln(uw/v)$ in the factorization limit. In section 6, we find that U has additional simplicity across the entire space of cross ratios: it has a restricted set of only five final symbol entries, which leads to a simple form for one of its three derivatives. In section 7 we derive formulae for U and $\tilde{V}$ on various lines through the space of cross ratios where they simplify. We also investigate the numerical behavior of V and $\tilde{V}$ on these lines and on some two-dimensional planes. In section 8, we explore an intriguing empirical relation between V and coproduct components of the remainder function R_6 at one higher loop order. In section 9 we discuss our conclusions and directions for future work. In appendix A, we give the $\{2L - 1, 1\}$ coproduct elements that characterize the weight $2L$ functions U (from which V can be derived) and $\tilde{V}$ through three loops.

We also provide ancillary files containing machine-readable expressions for the near-collinear and multi-Regge limits of the ratio function.

2 Setup and first constraints

As in ref. [27], we introduce an on-shell superspace (see e.g. refs. [12, 33–35]). We arrange the different on-shell states of the theory into an on-shell superfield Φ which depends on Grassmann variables η^A transforming in the fundamental representation of $su(4)$,

$$\Phi = G^+ + \eta^A \Gamma_A + \frac{1}{2!}\eta^A \eta^B S_{AB} + \frac{1}{3!}\eta^A \eta^B \eta^C \epsilon_{ABCD}\overline{\Gamma}^D + \frac{1}{4!}\eta^A \eta^B \eta^C \eta^D \epsilon_{ABCD}G^-. \quad (2.1)$$

Here G^+, Γ_A, $S_{AB} = \frac{1}{2}\epsilon_{ABCD}\overline{S}^{CD}$, $\overline{\Gamma}^A$, and G^- are the positive-helicity gluon, gluino, scalar, anti-gluino, and negative-helicity gluon states, respectively.

We then consider superamplitudes, $\mathcal{A}(\Phi_1, \Phi_2, \ldots, \Phi_n)$, which are functions of the superfields Φ_i. The ratio function is the ratio of the full superamplitude to the MHV super-

amplitude, defined as follows [12],

$$\mathcal{A} = \mathcal{A}_{\text{MHV}} \times \mathcal{P}. \tag{2.2}$$

By expanding in the Grassmann degree, i.e. powers of η, we can select out different values of k in the N^kMHV expansion:

$$\mathcal{P} = 1 + \mathcal{P}_{\text{NMHV}} + \mathcal{P}_{\text{N}^2\text{MHV}} + \ldots + \mathcal{P}_{\overline{\text{MHV}}}, \tag{2.3}$$

where successive terms in the expansion carry four more powers of η. For the six-point superamplitude, the only nontrivial term in this expansion is the NMHV one, because N^2MHV is $\overline{\text{MHV}}$, which is related to MHV by parity (reversal of all helicities).

At tree level, the six-point NMHV ratio function is best described in terms of R-invariants, which in turn are defined in terms of dual coordinates (x_i, θ_i):

$$p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha} = x_i^{\alpha\dot\alpha} - x_{i+1}^{\alpha\dot\alpha}, \qquad q_i^{\alpha A} = \lambda_i^\alpha \eta_i^A = \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A}. \tag{2.4}$$

The usual dual conformal cross ratios are denoted by

$$u = u_1 = \frac{x_{13}^2\, x_{46}^2}{x_{14}^2\, x_{36}^2}, \qquad v = u_2 = \frac{x_{24}^2\, x_{51}^2}{x_{25}^2\, x_{41}^2}, \qquad w = u_3 = \frac{x_{35}^2\, x_{62}^2}{x_{36}^2\, x_{52}^2}, \tag{2.5}$$

where $x_{ij}^2 \equiv (x_i^\mu - x_j^\mu)^2$.

Using the coordinates (x_i, θ_i) we may define momentum (super)twistors [67, 68]

$$\mathcal{Z}_i = (Z_i \,|\, \chi_i), \qquad Z_i^{R=\alpha,\dot\alpha} = (\lambda_i^\alpha, x_i^{\beta\dot\alpha}\lambda_{i\beta}), \qquad \chi_i^A = \theta_i^{\alpha A}\lambda_{i\alpha}. \tag{2.6}$$

The momentum (super)twistors $\mathcal{Z}_i$ transform linearly under dual (super)conformal symmetry, so that $\langle abcd \rangle = \epsilon_{RSTU} Z_a^R Z_b^S Z_c^T Z_d^U$ is a dual conformal invariant. If we label our six external lines as a, b, c, d, e, f, then the R-invariants can be written as

$$(f) \equiv [abcde] = \frac{\delta^4\big(\chi_a \langle bcde \rangle + \text{cyclic}\big)}{\langle abcd \rangle \langle bcde \rangle \langle cdea \rangle \langle deab \rangle \langle eabc \rangle}. \tag{2.7}$$

In general, R-invariants obey many identities; see for example refs. [12, 69]. At six points, the only identity we need is [12]

$$(1) - (2) + (3) - (4) + (5) - (6) = 0. \tag{2.8}$$

Using this identity, the NMHV tree amplitude may be written as

$$\mathcal{P}_{\text{NMHV}}^{(0)} = [12345] + [12356] + [13456] = (6) + (4) + (2) = (1) + (3) + (5). \tag{2.9}$$

Beyond tree level, the R-invariants will be dressed with transcendental functions of the dual conformal cross ratios (u, v, w), which we will assume are hexagon functions.

Hexagon functions are a particular class of iterated integrals [70] or multiple polylogarithms [71, 72], which we will also refer to as pure (transcendental) functions. When a weight-n pure function f is differentiated, the result can be written as

$$df = \sum_{s_k \in \mathcal{S}} f^{s_k} d\ln s_k, \tag{2.10}$$

where $\mathcal{S}$ is a finite set of rational expressions, called the letters of the symbol of f, and f^{s_k} are weight-$(n-1)$ pure functions. The functions f^{s_k} describe the $\{n-1,1\}$ component of a coproduct Δ associated with a Hopf algebra for iterated integrals [73–76]. Similarly, each f^{s_k} can be differentiated,

$$df^{s_k} = \sum_{s_j \in \mathcal{S}} f^{s_j s_k} d \ln s_j\,, \tag{2.11}$$

thereby defining the weight-$(n-2)$ functions $f^{s_j s_k}$, which describe the $\{n-2,1,1\}$ components of Δ. The maximal iteration of this procedure defines the symbol of f, an n-fold tensor product of elements of $\mathcal{S}$ (each standing for a $d\ln$).

Hexagon functions are functions whose symbols have letters drawn from a particular nine-letter set:

$$\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}\,, \tag{2.12}$$

where

$$y_u = \frac{u - z_+}{u - z_-}\,, \qquad y_v = \frac{v - z_+}{v - z_-}\,, \qquad y_w = \frac{w - z_+}{w - z_-}\,, \tag{2.13}$$

and

$$z_{\pm} = \frac{1}{2}\Big[-1 + u + v + w \pm \sqrt{\Delta}\Big]\,, \qquad \Delta = (1 - u - v - w)^2 - 4uvw\,. \tag{2.14}$$

These nine letters are related to the 15 projectively invariant ratios of momentum-twistor four-brackets $\langle abcd \rangle$, which can be factored into nine independent combinations.

Hexagon functions are defined by one additional property: their branch cuts should only be at 0 or ∞ in the variables u, v, w, which means that the first entry of their symbol is restricted to just these three letters [38].

We note that a cyclic permutation of the six external legs sends $u \to v \to w \to u$, while the y_i variables transform as $y_u \to 1/y_v \to y_w \to 1/y_u$. A three-fold cyclic rotation amounts to a space-time parity transformation, under which the cross ratios are invariant while the y_i variables invert. It is useful to classify hexagon functions by their transformation properties under parity. Many additional properties of hexagon functions, and methods for constructing them, are detailed in refs. [28, 30].

The six-point NMHV ratio function can be written in terms of two functions, a parity-even function $V(u, v, w)$ and a parity-odd function $\tilde{V}(y_u, y_v, y_w)$ as follows [12, 27]:

$$\mathcal{P}_{\mathrm{NMHV}} = \frac{1}{2}\Big[[(1)+(4)]V(u,v,w) + [(2)+(5)]V(v,w,u) + [(3)+(6)]V(w,u,v) \tag{2.15}$$
$$+ [(1)-(4)]\tilde{V}(y_u, y_v, y_w) - [(2)-(5)]\tilde{V}(y_v, y_w, y_u) + [(3)-(6)]\tilde{V}(y_w, y_u, y_v)\Big]\,.$$

It is better to think of the parity-odd function $\tilde{V}$ as a function of the y_i variables, because its properties under cyclic permutations are then captured correctly. The loop expansions of V and $\tilde{V}$ are given by

$$V = 1 + \sum_{L=1}^{\infty} a^L V^{(L)}\,, \tag{2.16}$$

$$\tilde{V} = \sum_{L=1}^{\infty} a^L \tilde{V}^{(L)} \,, \tag{2.17}$$

where $a = g_{\mathrm{YM}}^2 N_c/(8\pi^2)$ is our loop expansion parameter, in terms of the Yang-Mills coupling constant g_{YM} and the number of colors N_c. We remark that the expansion parameter conventionally used for the Wilson loop, g^2, is related to our parameter by $g^2 = a/2$.

The fundamental assumption in this paper, which was also used at two loops [27], is that $V^{(L)}$ and $\tilde{V}^{(L)}$ are weight $2L$ hexagon functions, with even and odd parity respectively. The same basic assumption for the (parity-even) remainder function $R_6^{(L)}$ [26, 28] results in a consistent solution through four loops [28, 29].

In this paper, we will work directly with hexagon functions, rather than their symbols. Through three loops, we only need hexagon functions through weight six. According to eq. (2.10), the $\{5, 1\}$ coproduct elements of a weight-six function f completely specify the function in terms of the weight-five functions f^{s_k} up to a single constant of integration, which we can take to be the value of f at the point $(u, v, w) = (1, 1, 1)$. In ref. [28], all the hexagon functions were classified through weight five. We use this information to construct the space of weight-six hexagon functions, by writing the most general $\{5, 1\}$ coproduct elements leading to consistent mixed partial derivatives, i.e. $d^2 f = 0$. Including lower-weight functions multiplied by Riemann ζ values, there are a total of 639 parity-even weight-six hexagon functions, and 122 parity-odd ones. Our initial ansatz for $V^{(3)}$ is the most general linear combination of the parity-even functions with 639 unknown rational-number coefficients. Similarly, the ansatz for $\tilde{V}^{(3)}$ is constructed from the 122 parity-odd functions. We then impose constraints on $V^{(3)}$ and $\tilde{V}^{(3)}$, as described in the remainder of this section and in the following two sections, until all 761 parameters are fixed.

Before carrying out this procedure at three loops, we recall what is known about the functions $V^{(L)}$ and $\tilde{V}^{(L)}$ at lower loop orders. At one loop, the parity-odd function vanishes, while the parity-even one is nontrivial [12]:

$$V^{(1)}(u, v, w) = \frac{1}{2}\Big[H_2^u + H_2^v + H_2^w + (\ln u + \ln w)\ln v - \ln u \ln w - 2\zeta_2\Big], \tag{2.18}$$

$$\tilde{V}^{(1)}(u, v, w) = 0. \tag{2.19}$$

The vanishing of the weight-two parity-odd function $\tilde{V}^{(1)}$ can be understood simply from the fact that there are no such hexagon functions. The first parity-odd hexagon function, $\tilde{\Phi}_6$, is related to the one-loop massless hexagon integral in six dimensions [77, 78], and it has weight three.

In ref. [27], the two-loop ratio function was determined up to ten symbol-level parameters and one beyond-the-symbol parameter, using general constraints, including the leading-discontinuity part of the NMHV OPE [39]. These eleven parameters were then fixed via an explicit evaluation of the relevant loop integrals on the line in which all three cross ratios are equal, (u, u, u). This procedure led to the following expressions for $V^{(2)}(u, v, w)$

and $\tilde{V}^{(2)}(u,v,w)$:

$$V^{(2)} = -\frac{1}{4}\Big\{ \Omega^{(2)}(u,v,w) + \Omega^{(2)}(v,w,u) + 2\,\Omega^{(2)}(w,u,v) + 5\,(H_4^u + H_4^w) + H_{3,1}^u + H_{3,1}^w$$

$$- 3\,(H_{2,1,1}^u + H_{2,1,1}^w) - 2\left[(H_2^u)^2 + (H_2^w)^2\right] - 4\,(\ln u\,H_3^u + \ln w\,H_3^w)$$

$$+ \frac{1}{2}(\ln^2 u\,H_2^u + \ln^2 w\,H_2^w) + 4H_4^v - 2H_{3,1}^v - \frac{3}{2}(H_2^v)^2 - 2\ln v(2H_3^v - H_{2,1}^v) + \ln^2 v\,H_2^v$$

$$- 2\left[(H_2^u + H_2^w)\,H_2^v + H_2^u\,H_2^w\right] + \ln(u/v)\,(H_3^w + H_{2,1}^w) + \ln(w/v)\,(H_3^u + H_{2,1}^u)$$

$$- \Big[\ln u\,\ln(v/w) + 2\ln v\,\ln w\Big]H_2^u - \Big[\ln w\,\ln(v/u) + 2\ln v\,\ln u\Big]H_2^w$$

$$- \left[\frac{1}{2}\ln^2(u/w) + \ln(uw)\,\ln v\right]H_2^v - \frac{1}{2}\ln(uw)\,\ln v\Big[\ln(uw)\,\ln v - \ln u\,\ln w\Big]$$

$$- \frac{1}{4}\ln^2 u\,\ln^2 w + \zeta_2\Big[4\,(H_2^u + H_2^w) + 2\,H_2^v - \ln^2 u - \ln^2 w - 2\ln^2 v$$

$$+ 6\,(\ln(uw)\,\ln v - \ln u\,\ln w)\Big] - 12\,\zeta_4\Big\}, \tag{2.20}$$

$$\tilde{V}^{(2)} = \frac{1}{8}\Big[-F_1(u,v,w) + F_1(w,u,v) + \ln(u/w)\tilde{\Phi}_6(u,v,w)\Big]. \tag{2.21}$$

Here we have rewritten the results in terms of harmonic polylogarithms (HPLs) [79], as well as the other functions constituting the basis of hexagon functions through weight four, namely $\Omega^{(2)}$, $\tilde{\Phi}_6$ and F_1 [28].

The HPLs we need have weight vectors containing only 0 and 1. They can be defined recursively by

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t), \tag{2.22}$$

except for $H_{0_n}(u)$ which is defined by $H_{0_n}(u) = \frac{1}{n!}\log^n u$. We choose a basis for the HPLs in which the point $u = 1$ is regular, by letting the argument be $1 - u$, and restricting to weight vectors whose last entry is 1. We also use a compressed notation where $(k-1)$ 0's followed by a 1 is replaced by k in the weight vector, and the argument $(1 - u)$ is replaced by the superscript u [28]. So, for example, $H_{3,1}^u = H_{0,0,1,1}^u = H_{0,0,1,1}(1-u)$, and similarly for when the argument is v or w.

In ref. [27], only the leading-discontinuity terms in the OPE were available [39]. Now, thanks to the work of BSV [41–44], who have used integrability to determine the OPE expansion exactly in the coupling, we have access to enough data to fix not only the two-loop, but also the three-loop six-point NMHV ratio function, without resorting to evaluating any loop integrals. The starting ansatz at two loops involves 50 parity-even weight-four hexagon functions for $V^{(2)}$, and 2 parity-odd ones for $\tilde{V}^{(2)}$. (At one loop, there are 7 parity-even weight-two hexagon functions for $V^{(1)}$, and no parity-odd ones for $\tilde{V}^{(1)}$.)

We now begin to determine the various unknown rational numbers by applying many of the same constraints as in ref. [27]. Specifically, the constraints we inherit from that paper are as follows:

- **Symmetry:** under the exchange of u and w, the function V is symmetric while $\tilde{V}$ is antisymmetric:

$$V(w,v,u) = V(u,v,w), \qquad \tilde{V}(y_w, y_v, y_u) = -\tilde{V}(y_u, y_v, y_w). \tag{2.23}$$

At three loops, this constraint reduces the $639 + 122 = 761$ parameters to $363 + 49 = 412$.

- **Spurious Pole Constraints:** Scattering amplitudes have poles corresponding to sums of color-adjacent momenta, of the form $(p_i + p_{i+1} + \ldots + p_{j-1})^2 \equiv x_{ij}^2$. These are produced by four-brackets of the form $\langle i-1, i, j-1, j \rangle$. Poles in other four-brackets do not correspond to sums of color-adjacent momenta, and should not be present in the full amplitude. While such poles never appear in hexagon functions, they are present in the R-invariants. In order for such spurious poles to vanish in the full function, the coefficients of the R-invariants must be such that these poles cancel. The R-invariants (1) and (3) contain poles as $\langle 2456 \rangle \to 0$, with equal and opposite residues. In order for them to cancel, we see from eq. (2.15) that

$$[V(u,v,w) - V(w,u,v) + \tilde{V}(y_u, y_v, y_w) - \tilde{V}(y_w, y_u, y_v)]_{\langle 2456 \rangle \to 0} = 0. \qquad (2.24)$$

The $\langle 2456 \rangle \to 0$ limit can be implemented by taking

$$w \to 1, \quad y_u \to (1-w)\frac{u(1-v)}{(u-v)^2}, \quad y_v \to \frac{1}{(1-w)}\frac{(u-v)^2}{v(1-u)}, \quad y_w \to \frac{1-u}{1-v}. \qquad (2.25)$$

- **Collinear Limit:** As two external particles become collinear, the six-point NMHV amplitude should reduce to either the five-point MHV or $\overline{\text{MHV}}$ amplitude times a splitting function. The five-point ratio function is equal to its tree-level value due to parity (NMHV/MHV is $\overline{\text{MHV}}$/MHV at the five-point level). Therefore, at any nonzero loop order the collinear limit of the six-point ratio function must vanish. In particular, taking $w \to 0$ and $v \to 1-u$ gives a collinear limit in which all R-invariants vanish except for (6) and (1), which become equal. Inserting this condition into eq. (2.15), we find the collinear constraint,

$$[V(u,v,w) + V(w,u,v) + \tilde{V}(y_u, y_v, y_w) - \tilde{V}(y_w, y_u, y_v)]_{w \to 0, \, v \to 1-u} = 0. \qquad (2.26)$$

Parity-odd functions always vanish in the collinear limit [27], so the constraint is really just that $V(u,v,w) + V(w,u,v)$ vanishes in the limit.

In addition to these constraints, we impose several new constraints, here in rough order of simplicity:

- **Cyclic Vanishing:** It turns out that not all of the apparent freedom in $\tilde{V}$ is physically meaningful. It is possible to add a cyclically symmetric function to $\tilde{V}$ that is consistent with its other symmetries, but such a contribution $\tilde{f}(u,v,w)$ vanishes in the full ratio function (2.15):

$$\begin{aligned}
\frac{1}{2}\Big[&[(1)-(4)]\tilde{f}(u,v,w) - [(2)-(5)]\tilde{f}(u,v,w) + [(3)-(6)]\tilde{f}(u,v,w)\Big] \\
&= \frac{1}{2}\Big[[(1)+(3)+(5)] - [(2)+(4)+(6)]\Big]\,\tilde{f}(u,v,w) \qquad (2.27) \\
&= 0,
\end{aligned}$$

using eq. (2.9). A function $\tilde{f}$ of this sort cannot contribute to the ratio function, and so it will never be constrained by any physical limits. Therefore, we might as well set any such contribution to zero. This constraint did not appear at two loops, because there are no cyclicly invariant parity-odd hexagon functions at weight four. However, at weight six there are 10 such functions. We remove them using this constraint, right after imposing the $u \leftrightarrow w$ symmetry constraints.

- **Final-Entry Condition:** Caron-Huot and He have observed that supersymmetry constrains the possible final entries of the symbols of finite quantities in planar $\mathcal{N} = 4$ SYM [65, 80]. Specifically, they express the action of certain dual superconformal generators on the N^kMHV amplitude in terms of lower-loop N^{k+1}MHV quantities. For the MHV remainder function, these constraints imply a set of six possible final entries. For the NMHV ratio function, expressed in our variables, the constraints are that $V(u, v, w)$ and $\tilde{V}(u, v, w)$, the functions multiplying the R-invariant (1), can only have final entries from the following seven-element set:

$$\left\{ \frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v, y_w, \frac{uw}{v} \right\} . \tag{2.28}$$

The other R-invariants multiply functions with final entries from sets related by the appropriate cyclic permutations of the variables. (Technically, these constraints apply to the NMHV amplitude from which infrared divergences have been subtracted using the BDS ansatz, rather than to the ratio function itself. However, these quantities differ by the MHV remainder function, which has final entries in a subset of the NMHV set (2.28) (uw/v is not present). Therefore, this final-entry condition can be applied to the ratio function without modification.) We impose this constraint right after the cyclic-vanishing constraint. It reduces the $363 + 39$ free parameters down to $166 + 16 = 182$. Then we impose the vanishing of the spurious poles, which fixes another 40 parameters, and mixes the parity-even and parity-odd sectors so that we can no longer count their parameters separately.

- **Near-Collinear Limits:** BSV use integrability to evaluate the OPE for Wilson loops nonperturbatively in the coupling. They proceed order by order in the number of flux-tube excitations, which corresponds to powers of an expansion parameter T. This parameter is proportional to the square root of a vanishing cross ratio (see section 3). By inserting states on the boundaries of the Wilson loop they are able to replicate particular components of the NMHV amplitude. Constraining our results to agree with their expansions at first order in T [42] constrains many parameters. Two parameters that remain can be constrained using BSV's more recent results at order T^2 [43, 81].

- **Multi-Regge Kinematics:** The multi-Regge limit is a generalization of the Regge limit in which the outgoing particles of a $2 \to n$ scattering process are strongly ordered in rapidity. Lipatov, Prygarin, and Schnitzer [54] have investigated the multi-Regge limit of NMHV amplitudes in $\mathcal{N} = 4$ SYM, creating an ansatz for their behavior at leading-logarithmic order that mirrors previous results for the MHV amplitude.

Constraint	$L=1$	$L=2$	$L=3$
1. (Anti)symmetry in u and w	7	52	412
2. Cyclic vanishing of $\tilde{V}$	7	52	402
3. Final-entry condition	4	25	182
4. Spurious-pole vanishing	3	15	142
5. Collinear vanishing	1	8	92
6. $\mathcal{O}(T^1)$ OPE	0	0	2
7. $\mathcal{O}(T^2)$ OPE *or* multi-Regge kinematics	0	0	0

Table 1. Remaining parameters in the function-level ansätze for $V^{(L)}$ and $\tilde{V}^{(L)}$ after each constraint is applied, at each loop order.

In this paper we generalize their results beyond leading-log order, along the lines of refs. [53, 55]. These generalizations are fully consistent with the near-collinear boundary conditions, and thereby serve as an independent check of them. Also, we can derive the NMHV impact factor in the factorization we propose, through NNLL. The NMHV and MHV impact factors are strikingly similar. Our results are completely consistent with the recent all-orders multi-Regge proposal [59].

In practice it can be useful to constrain the symbol of the ratio function first, and then constrain the full function, making use of the coproduct to characterize the beyond-the-symbol terms. Indeed, this was our first approach to obtaining $V^{(3)}$ and $\tilde{V}^{(3)}$. However, as mentioned earlier in this section, it is straightforward to dispense with the symbol altogether, and begin with a function-level ansatz characterized by various coproduct components. We then apply all constraints directly at function level, using the coproduct information to compute the necessary limiting behavior. Because such an approach may well scale better computationally to higher loops than a symbol-level approach, we describe the results of using that approach here. After applying each set of constraints the number of parameters in the ansatz is reduced, as shown in table 1. This table also includes the corresponding numbers for lower loop orders, so that one can appreciate the growth in the number of parameters with loop order.

As shown, after applying the constraints of $u \leftrightarrow w$ (anti)symmetry, cyclic vanishing of $\tilde{V}$, the final entry condition, and the vanishing of spurious poles, we have 142 parameters remaining in our ansatz. In the following sections, we use the collinear constraints, OPE, and multi-Regge limits to fix these final parameters.

3 Collinear and near-collinear limits

In this section, we consider the $w \to 0$ collinear limit. In general, this limit may be expressed via a permutation of a map between the cross ratios (u, v, w) and the variables

$(F, S, T) \equiv (e^{i\phi}, e^{\sigma}, e^{-\tau})$ defined in ref. [41]:

$$
\begin{aligned}
u &= \frac{F}{F + FS^2 + ST + F^2ST + FT^2}, \\
v &= \frac{FS^2}{(1 + T^2)(F + FS^2 + ST + F^2ST + FT^2)}, \\
w &= \frac{T^2}{1 + T^2}, \\
y_u &= \frac{F + ST + FT^2}{F(1 + FST + T^2)}, \\
y_v &= \frac{FS + T}{F(S + FT)}, \\
y_w &= \frac{(S + FT)(1 + FST + T^2)}{(FS + T)(F + ST + FT^2)}.
\end{aligned}
\tag{3.1}
$$

As mentioned in section 2, the combination $V(u, v, w) + V(w, u, v)$ should vanish in this limit. This is a fairly powerful constraint, fixing 50 of the remaining 142 parameters, and leaving 92. To determine the remaining parameters we will match to the OPE results of Basso, Sever and Vieira.

Many features of BSV's approach to the OPE of polygonal Wilson loops carry over to the NMHV helicity configuration with only minor modifications [42]. In general, NMHV scattering amplitudes are dual to Wilson loops dressed with insertions of states that depend on the particular NMHV component being investigated [82, 83]. Two cases are explored by BSV, that of two scalar insertions, one on the bottom cusp and one on the top, and that of a gluonic insertion on the bottom cusp. We will consider each in turn.

BSV found that by inserting a scalar on the top and bottom cusps of the Wilson loop they were able to probe the $\eta_6\eta_1\eta_3\eta_4$ (or "6134") component of the NMHV amplitude. In this configuration, the leading excitations are scalar ones. Inspecting eq. (2.7), we see that all the R-invariants vanish for the $\eta_6\eta_1\eta_3\eta_4$ component except for (2) and (5). Furthermore, the identity (2.8) collapses for this component to

$$
(2) = (5) = \frac{1}{\langle 6134 \rangle} = \frac{e^{-\tau}}{2 \cosh \sigma},
\tag{3.2}
$$

so that only the term multiplying $V(v, w, u)$ survives. Thus this component of $\mathcal{P}$ has a particularly simple representation in terms of a single pure function. Additionally, up to the first order in T the Wilson loop ratio investigated by BSV is equal to the ratio function. As such, we may simply write

$$
\begin{aligned}
\mathcal{W}^{(6134)} &= \frac{e^{-\tau}}{2 \cosh \sigma} \sum_{L=0}^{\infty} \left(\frac{a}{2}\right)^L \sum_{n=0}^{L} \tau^n F_n^{(L)}(\sigma) + \mathcal{O}(e^{-2\tau}) \\
&= \frac{T}{2 \cosh \sigma} \times V(v, w, u)|_{\mathcal{O}(T^0)} + \mathcal{O}(T^2),
\end{aligned}
\tag{3.3}
$$

where the $F_n^{(L)}$ are given explicitly in appendix F of ref. [42]. Note that we only need the T^0 term in $V(v, w, u)$ as $w \to 0$, because the dual superconformal invariant prefactor

carries a power of T in this limit. Applying the constraint (3.3) at three loops, to the 92-parameter ansatz with vanishing collinear limits, leaves 14 parameters unfixed. In an ancillary file, we give the near-collinear limit of $\mathcal{P}^{(6134)}$ through one higher order, T^2 (after all free parameters have been fixed).

Alternatively, one may insert a gluonic excitation at the bottom cusp of the Wilson loop, probing the $(\eta_1)^4$ (or "1111") component. Up to first order in T, the R-invariants in this component become

$$
\begin{aligned}
&(1) \to 0, && (2) \to \frac{FT}{S(1+S^2)} + \mathcal{O}\left(T^2\right), && (3) \to 1 - FST + \mathcal{O}\left(T^2\right), \\
&(4) \to 1 - \frac{FT}{S} + \mathcal{O}\left(T^2\right), && (5) \to \frac{FS^3T}{1+S^2} + \mathcal{O}\left(T^2\right), && (6) \to 0 + \mathcal{O}(T^4).
\end{aligned}
\tag{3.4}
$$

The odd function $\tilde{V}$ vanishes in the collinear limit; it is $\mathcal{O}(T^1)$ for any permutation. Also, we can use eq. (2.26) to eliminate $V(w,u,v)$ in favor of $-V(u,v,w)$, up to terms suppressed by a power of T. Using such relations, we find that the $(\eta_1)^4$ component of the ratio function becomes,

$$
\begin{aligned}
\mathcal{P}^{(1111)} = \frac{1}{2}\Bigg\{ & V(u,v,w) + V(w,u,v) - \tilde{V}(u,v,w) + \tilde{V}(w,u,v) \\
& + FT\left[-\frac{1-S^2}{S}V(u,v,w) + \frac{1+S^4}{S(1+S^2)}V(v,w,u) \right] \Bigg\} + \mathcal{O}(T^2).
\end{aligned}
\tag{3.5}
$$

We note that the terms without an explicit T are also $\mathcal{O}(T)$ due to the collinear-vanishing relations, except for the tree-level term, which is $1 + \mathcal{O}(T)$.

We match the near-collinear limit of eq. (3.5) to BSV's computation [42] of the OPE, in terms of a single gluonic excitation propagating across the Wilson loop. The result is given as an integral over the excitation's rapidity u, involving its anomalous dimension (or energy) $\gamma(u)$, its momentum $p(u)$, a measure factor $\mu(u)$, and the NMHV dressing functions h and $\bar{h}$. The expansions of these quantities through $\mathcal{O}(a^3)$ are given by,

$$
\begin{aligned}
\gamma(u) = a&\left[\psi\left(\tfrac{1}{2} - iu\right) + \psi\left(\tfrac{1}{2} + iu\right) - 2\psi(1) \right] \\
& - \frac{a^2}{4}\left[\psi''\left(\tfrac{3}{2} - iu\right) + \psi''\left(\tfrac{3}{2} + iu\right) + 4\zeta_2\left[\psi\left(\tfrac{1}{2} - iu\right) + \psi\left(\tfrac{1}{2} + iu\right) - 2\psi(1) \right] + 12\zeta_3 \right] \\
& + \frac{a^3}{8}\Bigg[\frac{1}{6}\left[\psi''''\left(\tfrac{3}{2} - iu\right) + \psi''''\left(\tfrac{3}{2} + iu\right) \right] + 2\zeta_2\left[\psi''\left(\tfrac{3}{2} - iu\right) + \psi''\left(\tfrac{3}{2} + iu\right) \right] \\
& \qquad + 44\zeta_4\left[\psi\left(\tfrac{1}{2} - iu\right) + \psi\left(\tfrac{1}{2} + iu\right) - 2\psi(1) \right] - 24\zeta_2\zeta_3\tanh^2 \pi u + 40(2\zeta_5 + \zeta_2\zeta_3) \Bigg] \\
& + \mathcal{O}(a^4),
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
p(u) = 2u &- a\pi\tanh \pi u + \frac{a^2}{4}\pi^3\left[\frac{8}{3}\tanh \pi u - 2\tanh^3 \pi u \right] \\
& + \frac{a^3}{8}\left[\pi^5\left(-\frac{172}{45}\tanh \pi u + \frac{22}{3}\tanh^3 \pi u - 4\tanh^5 \pi u \right) + 4i\zeta_3\left[\psi'\left(\tfrac{3}{2} - iu\right) - \psi'\left(\tfrac{3}{2} + iu\right) \right] \right] \\
& + \mathcal{O}(a^4),
\end{aligned}
\tag{3.7}
$$

and

$$h(u) = \frac{2x^+(u)x^-(u)}{a}, \qquad \bar{h}(u) = \frac{1}{h(u)}, \tag{3.8}$$

where

$$x^\pm(u) = x\left(u \pm \frac{i}{2}\right) \tag{3.9}$$

is given in terms of the Zhukovsky variable

$$x(u) = \frac{1}{2}\Big[u + \sqrt{u^2 - 2a}\Big]. \tag{3.10}$$

The perturbative expansion of the measure $\mu(u)$ can be found in ref. [41]. It is a bit more complicated, but is still expressible in terms of the function $\psi(x) = d\ln\Gamma(x)/dx$ and its derivatives, as well as $\tanh\pi u$. The rapidity u should not be confused with the cross ratio u.

In terms of these functions, the formula for the gluonic flux-excitation contribution to the OPE is,

$$\begin{aligned}
\mathcal{P}^{(1111)} = 1 &+ TF \int_{-\infty}^{\infty} \frac{du}{2\pi} \mu(u)(h(u) - 1)e^{ip(u)\sigma - \gamma(u)\tau} \\
&+ \frac{T}{F} \int_{-\infty}^{\infty} \frac{du}{2\pi} \mu(u)(\bar{h}(u) - 1)e^{ip(u)\sigma - \gamma(u)\tau} + \mathcal{O}(T^2).
\end{aligned} \tag{3.11}$$

We can carry out the integrals over u by deforming the integral into the lower half-plane, which converts it into a sum over residues at $u = -im/2$ for positive integers m. There are methods for performing such sums exactly, see for example refs. [84, 85]. We take a more mundane approach: we truncate the series in m at a suitably large finite value (of order 100). The truncation yields a high-order Taylor expansion in S. Then we write an ansatz for the exact result in terms of HPLs depending on S^2, and match the Taylor expansion of the ansatz against the actual Taylor expansion, in order to determine all of the rational-number coefficients in the ansatz.

After we have expressed the order T term in eq. (3.11) in terms of HPLs, in order to match it against our ansatz we have to expand eq. (3.5), with the ansatz for V and $\tilde{V}$ inserted into it. The ansatz has either 14 or 92 parameters in it (depending on whether or not we have already imposed the order T constraint on $\mathcal{P}^{(6134)}$). We use the differential equations method described in section 5 of ref. [28] to expand all the hexagon functions in this ansatz. The resulting expressions for the expansion of eq. (3.11) are too lengthy to display here, but we provide them in a computer-readable ancillary file attached to this article. The file also includes the next order in the near-collinear expansion of $\mathcal{P}^{(1111)}$, namely order T^2, after all free parameters have been fixed.

After applying the constraints from the T^1 term in the OPE for the 1111 component, eq. (3.5), just two undetermined parameters remain. These parameters multiply the functions $[\tilde{\Phi}_6]^2$ and $V^{(1)} R_6^{(2)}$, where $\tilde{\Phi}_6$ is the pure function associated with the $D = 6$ one-loop hexagon integral [77, 78], $V^{(1)}$ is the one-loop ratio function given in eq. (2.18), and $R_6^{(2)}$ is the two-loop remainder function. It is easy to see that the two parameters cannot be fixed by any OPE information at $\mathcal{O}(T^1)$: because $\tilde{\Phi}_6$ is parity odd, it vanishes proportional

to T, so its square vanishes like T^2. Similarly, $V^{(1)}$ obeys the collinear vanishing condition (2.26), giving one power of T; and $R_6^{(2)}$ is totally symmetric and its vanishing provides an additional power of T in all channels.

Sever, Vieira and Wang [39] have described the leading-discontinuity OPE behavior of the ratio function. This behavior captures the leading $\ln^L T$ behavior at L loops, irrespective of the number of powers of T multiplying it as $T \to 0$. Hence the leading-discontinuity OPE might contain complementary information to the full T^1 OPE. However, in the present case the leading-discontinuity information cannot be used to fix the coefficients of either $[\tilde{\Phi}_6]^2$ or $V^{(1)} R_6^{(2)}$. That is because the functions $\tilde{\Phi}_6$, $V^{(1)}$ and $R_6^{(2)}$ each have only a single discontinuity, so the two weight-6 functions in question have only double discontinuities, not the triple discontinuity which is the leading one at three loops.

We also remark that at $\mathcal{O}(T^1)$, the 1111 component of the OPE is more powerful than the 6134 component: we imposed the 1111 constraint directly on the 92-parameter ansatz with vanishing collinear limits, and found that it still fixed all but two of the parameters, even without any assistance from the 6134 component. Recall that the 6134 component imposed on the same 92-parameter ansatz still left 14 parameters unfixed.

Basso, Sever and Vieira have evaluated the two flux-excitation contributions to the OPE for the ratio function [43] and they have provided us with the small S expansion of the resulting $\mathcal{O}(T^2)$ terms in the OPE [81]. We can use these terms to fix the two remaining parameters in our ansatz. Alternatively, we can use factorization in the multi-Regge limit, as described in the next section. Either approach leads to the same values for the two parameters, providing a very nice consistency check.

4 Multi-Regge limits

In this section we propose a factorization of the NMHV amplitude in the limit of multi-Regge kinematics (MRK), which is a natural extension of previous work by Fadin and Lipatov [53] in the MHV case, and by Lipatov, Prygarin and Schnitzer [54] for the leading-logarithmic behavior of the NMHV amplitude. We use this factorization as one method for fixing the remaining two parameters in our ansatz. We are then able to extract from the fully-fixed ansatz the NMHV impact factor, which we compare to the previously-known MHV impact factor, through next-to-next-to-leading-logarithmic accuracy.

We remind the reader that the multi-Regge limit of a $2 \to (n-2)$ process is the limit in which the $(n-2)$ outgoing particles are strongly ordered in rapidity. For $2 \to 4$ gluon scattering, this means that two of the gluons are emitted at high energy almost parallel to the incoming gluons, while the other two, while still emitted at small angles to the path of the incoming gluons, have smaller energy. Due to helicity conservation on the highest energy lines, the MHV 6-gluon amplitude in the MRK limit can be viewed as having two positive incoming helicities scattering into four positive outgoing ones. The appropriate color-ordering for the $2 \to 4$ process is to take two diagonally opposite legs to be the incoming legs. So we may consider the MHV helicity configuration to be

$$3^+6^+ \; \to \; 2^+4^+5^+1^+ \, , \tag{4.1}$$

where legs 1 and 2 are the highest-energy outgoing gluons. For an NMHV amplitude, one of the two lower-energy outgoing gluons has its helicity reversed, say

$$3^+ 6^+ \to 2^+ 4^- 5^+ 1^+ \,. \tag{4.2}$$

In eqs. (4.1) and (4.2) we are *not* using the all-outgoing helicity convention, in order to emphasize helicity conservation on the high-energy lines.

In this MRK limit, the cross ratios u_1, u_2 and u_3 approach the values

$$u_1 \to 1 \,, \qquad u_2, u_3 \to 0 \,, \tag{4.3}$$

with the ratios

$$\frac{u_2}{1 - u_1} \equiv \frac{1}{(1 + w)(1 + w^*)} \quad \text{and} \quad \frac{u_3}{1 - u_1} \equiv \frac{ww^*}{(1 + w)(1 + w^*)} \tag{4.4}$$

held fixed. In this section, we use (u_1, u_2, u_3) to denote the three cross ratios (2.5), instead of (u, v, w), in order to minimize confusion between the cross-ratio w and the variable w used to parametrize the multi-Regge kinematics.

Fadin and Lipatov [53] proposed a precise factorization relation for the MRK limit of the six-point MHV remainder function, through at least next-to-leading-logarithmic (NLL) accuracy. Caron-Huot [55] suggested that, subject to some reasonable assumptions, the same formula should hold in the planar limit to all subleading logarithms. Some additional evidence for factorization beyond NLL was provided in ref. [29], where the four-loop remainder function was computed and found to be consistent with the proposed MRK limit through at least next-to-next-to-leading-logarithmic (NNLL) accuracy.

The proposal of Fadin and Lipatov is that the remainder function R_6 obeys [53]:

$$
\begin{aligned}
e^{R_6 + i\pi\delta}\big|_{\mathrm{MRK}} \;=\; \cos\pi\omega_{ab} + i\frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{w}{w^*}\right)^{\frac{n}{2}} \int_{-\infty}^{+\infty} \frac{d\nu}{\nu^2 + \frac{n^2}{4}} |w|^{2i\nu}\, \Phi_{\mathrm{Reg}}^{\mathrm{MHV}}(\nu, n) \\
\times \left(-\frac{1}{1 - u_1} \frac{|1 + w|^2}{|w|}\right)^{\omega(\nu, n)} ,
\end{aligned}
\tag{4.5}
$$

where

$$
\begin{aligned}
\omega_{ab} &= \frac{1}{8}\, \gamma_K(a)\, \log |w|^2 \,, \\
\delta &= \frac{1}{8}\, \gamma_K(a)\, \log \frac{|w|^2}{|1 + w|^4} \,,
\end{aligned}
\tag{4.6}
$$

and $\gamma_K(a)$ is the cusp anomalous dimension.

The BFKL eigenvalue $\omega(\nu, n)$ and the MHV impact factor $\Phi_{\mathrm{Reg}}^{\mathrm{MHV}}(\nu, n) = \Phi_{\mathrm{Reg}}(\nu, n)$ may both be expanded perturbatively in a:

$$
\begin{aligned}
\omega(\nu, n) &= -a \left(E_{\nu, n} + a\, E_{\nu, n}^{(1)} + a^2\, E_{\nu, n}^{(2)} + \mathcal{O}(a^3) \right) , \\
\Phi_{\mathrm{Reg}}(\nu, n) &= 1 + a\, \Phi_{\mathrm{Reg}}^{(1)}(\nu, n) + a^2\, \Phi_{\mathrm{Reg}}^{(2)}(\nu, n) + a^3\, \Phi_{\mathrm{Reg}}^{(3)}(\nu, n) + \mathcal{O}(a^4) .
\end{aligned}
\tag{4.7}
$$

Because $\omega(\nu, n)$ starts at order a, while the impact factor $\Phi_{\mathrm{Reg}}(\nu, n)$ is unity at leading order, the highest power of $\ln(1 - u_1)$ that appears at loop order L is $\ln^{L-1}(1 - u_1)$. This property allows the MRK limit to be organized in successive orders of $\ln(1 - u_1)$, beginning with the leading-log approximation, or LLA. At this order, only the leading BFKL eigenvalue $E_{\nu,n}$ contributes nontrivially to the remainder function. The next order in the logarithmic expansion, the term of order $\ln^{L-2}(1 - u_1)$, is called the next-to-leading-log approximation, or NLLA. It is determined by $E_{\nu,n}^{(1)}$ and $\Phi_{\mathrm{Reg}}^{(1)}$, which were computed in ref. [53]. Computations of the remainder function at three and four loops have provided the BFKL eigenvalue through NNLLA ($E_{\nu,n}^{(2)}$), and the MHV impact factor through N^{3}LLA ($\Phi_{\mathrm{Reg}}^{(2)}$ and $\Phi_{\mathrm{Reg}}^{(3)}$) [28, 29, 56].

The BFKL eigenvalue $\omega(\nu, n)$ is a property of the Reggeized gluon ladder being exchanged in the t-channel. It does not depend on the external states attached to the end of the ladder. For the six-point amplitude, no states should be emitted from the middle of the ladder [47]. At seven and higher points, there can be such emission vertices [86].

Using the independence of the BFKL eigenvalue from the external states, Lipatov, Prygarin, and Schnitzer proposed modifying the LLA version of eq. (4.5) for the NMHV case [54], obtaining:

$$R_{\mathrm{NMHV}}^{\mathrm{LLA}} \;=\; -\frac{ia}{2} \sum_{n=-\infty}^{\infty} (-1)^n \int_{-\infty}^{+\infty} \frac{d\nu \, w^{i\nu + n/2} \, w^{*\,i\nu - n/2}}{(i\nu + \frac{n}{2})^2} \left[(1 - u_1)^{a \, E_{\nu,n}} - 1 \right]. \qquad (4.8)$$

Here R_{NMHV} is the NMHV remainder function, a quantity which is particularly convenient to work with in the MRK limit. It can be defined as the product of the NMHV ratio function and the (exponentiated) MHV remainder function:

$$R_{\mathrm{NMHV}} \;=\; \frac{A_{\mathrm{NMHV}}}{A_{\mathrm{BDS}}} \;=\; \frac{A_{\mathrm{NMHV}}}{A_{\mathrm{MHV}}} \times \frac{A_{\mathrm{MHV}}}{A_{\mathrm{BDS}}} \;=\; \mathcal{P}_{\mathrm{NMHV}} \times \exp(R_6). \qquad (4.9)$$

Clearly, the LLA NMHV formula (4.8) is the same as the LLA version of eq. (4.5) for the MHV case, but with the substitution,

$$\frac{1}{-i\nu + \frac{n}{2}} \to -\frac{1}{i\nu + \frac{n}{2}}. \qquad (4.10)$$

We wish to extend this relation beyond the LLA. The same BFKL eigenvalue will enter the NMHV formula, but in general the NMHV impact factor will receive different loop corrections than in the MHV case. We therefore propose the following ansatz:

$$\mathcal{P}_{\mathrm{NMHV}} \times e^{R_6 + i\pi\delta}|_{\mathrm{MRK}} = \cos \pi\omega_{ab} - i\frac{a}{2} \sum_{n=-\infty}^{\infty} (-1)^n \left(\frac{w}{w^*} \right)^{\frac{n}{2}} \int_{-\infty}^{+\infty} \frac{d\nu}{(i\nu + \frac{n}{2})^2} |w|^{2i\nu} \, \Phi_{\mathrm{Reg}}^{\mathrm{NMHV}}(\nu, n)$$
$$\times \left(-\frac{1}{1 - u_1} \frac{|1 + w|^2}{|w|} \right)^{\omega(\nu, n)}. \qquad (4.11)$$

To investigate the validity of this ansatz, we expand $\mathcal{P}_{\mathrm{NMHV}}$ perturbatively in a, and then decompose the L-loop coefficient in successive orders of $\ln(1 - u_1)$, starting with the leading (LLA) behavior proportional to $\ln^{L-1}(1 - u_1)$.

First we recall the analogous decomposition of the MHV remainder function used in ref. [56]:

$$R_6^{(L)}(1-u_1, w, w^*) = 2\pi i \sum_{r=0}^{L-1} \ln^r(1-u_1)\Big[g_r^{(L)}(w, w^*) + 2\pi i h_r^{(L)}(w, w^*)\Big] + \mathcal{O}(1-u_1). \quad (4.12)$$

Here $g_r^{(L)}(w, w^*)$ corresponds to the leading-log approximation (LLA) for $r = L-1$, next-to-LLA (NLLA) for $r = L-2$, and so on. Both $g_r^{(L)}$ and $h_r^{(L)}$ are pure functions, with weight $2L-r-1$ and $2L-r-2$ respectively. In fact, they are single-valued harmonic polylogarithms (SVHPLs) [56, 58], particular linear combinations of harmonic polylogarithms [79] in w and in w^* that are single-valued, or real-analytic, in the (w, w^*) plane.

We take the multi-Regge limit of the $(\eta_4)^4$ component of the ratio function. This corresponds to flipping the helicity of outgoing gluon 4 from plus to minus, as we go from MHV to NMHV in the processes $3^+6^+ \to 2^+4^\pm5^+1^+$ displayed in eqs. (4.1) and (4.2). In this limit, the R-invariants become rational functions of w^*. In particular, we have

$$(1) \to \frac{1}{1+w^*}, \qquad (5) \to \frac{w^*}{1+w^*}, \qquad (6) \to 1, \quad (4.13)$$

and all of the other R-invariants vanish.

Due to parity symmetry, the ratio function in the MRK limit, $\mathcal{P}_{\mathrm{MRK}}$, should be invariant under $(w, w^*) \to (1/w, 1/w^*)$. This leads us to divide up $\mathcal{P}_{\mathrm{MRK}}$ as follows:

$$\begin{aligned}
\mathcal{P}_{\mathrm{MRK}}^{(L)} = 2\pi i \sum_{r=0}^{L-1} \ln^r(1-u_1)\bigg\{ &\frac{1}{1+w^*}\Big[p_r^{(L)}(w, w^*) + 2\pi i\, q_r^{(L)}(w, w^*)\Big] \\
&+ \frac{w^*}{1+w^*}\Big[p_r^{(L)}(w, w^*) + 2\pi i\, q_r^{(L)}(w, w^*)\Big]\Big|_{(w,w^*)\to(1/w,1/w^*)}\bigg\} \\
+ \mathcal{O}(1-u_1).&
\end{aligned} \quad (4.14)$$

The functions $p_r^{(L)}$ and $q_r^{(L)}$ turn out to be pure functions, in fact they are SVHPLs, just like $g_r^{(L)}$ and $h_r^{(L)}$.

In order to extract $p_r^{(L)}$ and $q_r^{(L)}$ from the ratio function (2.15), we use eq. (4.13) to take the MRK limit of the R-invariants, and then we compare with eq. (4.14). We find that,

$$2\pi i \left[p_r^{(L)}(w, w^*) + 2\pi i\, q_r^{(L)}(w, w^*)\right] \quad (4.15)$$

$$= \frac{1}{2}\Big[V^{(L)}(u_1, u_2, u_3) + V^{(L)}(u_3, u_1, u_2) + \tilde{V}^{(L)}(u_1, u_2, u_3) - \tilde{V}^{(L)}(u_3, u_1, u_2)\Big]_{\mathrm{MRK},\ \ln^r(1-u_1)\ \mathrm{term}}.$$

These equations relate the pure functions $p_r^{(L)}$ and $q_r^{(L)}$ to the MRK limits of $V^{(L)}$ and $\tilde{V}^{(L)}$. We can take the MRK limits of these functions (or ansätze for them) using their $\{2L-1, 1\}$ coproduct components as input to the differential equation method established in ref. [28].

On the other hand, $p_r^{(L)}$ and $q_r^{(L)}$, together with the MHV coefficients $g_r^{(L)}$ and $h_r^{(L)}$, can also be related to the BFKL eigenvalue $\omega(\nu, n)$ and the NMHV impact factor $\Phi_{\mathrm{Reg}}^{\mathrm{NMHV}}(\nu, n)$ through the NMHV master formula (4.11). In general, to determine $p_r^{(L)}$ and $q_r^{(L)}$, we have to evaluate the sum over n and the integral over ν in eq. (4.11), for a given loop

order, a given power of $\ln(1 - u_1)$, and either the real or imaginary part. We will not give the details of how we perform the sum and integral, because the general method was described in ref. [56]: we deform the ν integral into a sum over an integer m, and truncate the sum over n and m at some large value. Then we match the truncated sum against the truncated Taylor expansion for a generic linear combination of SVHPLs with the correct transcendental weight for the relevant $p_r^{(L)}$ or $q_r^{(L)}$ coefficient in eq. (4.14), and the appropriate rational prefactors of $1/(1+w^*)$ and $w^*/(1+w^*)$. The matching determines the rational number coefficients in the linear combination. Once these coefficients are all fixed, we can check them using higher-order terms in the truncated sum and Taylor expansion.

At LLA, for which $\omega(\nu, n) = -aE_{\nu,n}$ and $\Phi_{\rm Reg}^{\rm NMHV}(\nu, n) = 1$, the functions $p_{L-1}^{(L)}$ were predicted in ref. [54] through three loops. We find complete agreement with those predictions. In our notation, which follows that of ref. [56], we have for the LLA coefficients at one loop,

$$p_0^{(1)} = \frac{1}{2}\left[\frac{1}{2}L_0^- - L_1^+\right],$$
$$q_0^{(1)} = 0,$$
(4.16)

at two loops,

$$p_1^{(2)} = \frac{1}{4}\left[L_2^- + \frac{1}{2}L_0^- L_1^+ - (L_1^+)^2\right],$$
$$q_1^{(2)} = 0,$$
(4.17)

and at three loops,

$$p_2^{(3)} = \frac{1}{4}\left[L_3^+ + L_{2,1}^- + \frac{1}{2}L_1^+ L_2^- - \frac{1}{16}(L_0^-)^3 - \frac{1}{8}(L_0^-)^2 L_1^+ - \frac{1}{3}(L_1^+)^3 + \zeta_3\right],$$
$$q_2^{(3)} = 0.$$
(4.18)

In principle, these LLA results for $p_2^{(3)}$ and $q_2^{(3)}$ could be used to fix parameters in our three-loop ansatz. However, once we have imposed all the previously-mentioned constraints, through the $\mathcal{O}(T^1)$ terms in the OPE, we find that the two remaining parameters cannot be fixed by the LLA information. To see this, let's consider the MRK behavior of the two functions multiplying these parameters. These functions were $[\tilde{\Phi}_6]^2$ and $V^{(1)}R_6^{(2)}$. From ref. [28], we know that the function $\tilde{\Phi}_6$ is totally symmetric, vanishes in the MRK limit before the analytic continuation, and has a single discontinuity, with no logarithmic ($\ln(1-u_1)$) enhancement [28]: $\Phi_6|_{\rm MRK} = -4\pi i L_2^-$. Hence the square of this function has a double discontinuity, and no logarithmic enhancement:

$$[\tilde{\Phi}_6]^2|_{\rm MRK} = (2\pi i)^2 \times 4(L_2^-)^2.$$
(4.19)

Comparing to eq. (4.15), we see that this function only contributes to $q_0^{(3)}$, that is, to the NNLLA real part.

The other function with an undetermined coefficient is $V^{(1)} R_6^{(2)}$. Recalling that $\tilde{V}^{(1)}$ vanishes, and inspecting eqs. (4.12) and (4.15), we see that its behavior in the MRK limit is

$$V^{(1)} \times R_6^{(2)}|_{\mathrm{MRK}} \;\propto\; 2\pi i\, p_0^{(1)} \times 2\pi i \left[\ln(1 - u_1)\, g_1^{(2)} + g_0^{(2)} \right]. \tag{4.20}$$

Hence this function contributes to both $q_1^{(3)}$ and $q_0^{(3)}$, which means that we can fix its coefficient using the NLLA real part.

In general, if one knows the $\mathrm{N}^{k-1}\mathrm{LLA}$ imaginary part, one can also predict the $\mathrm{N}^k\mathrm{LLA}$ real part from the master formula (4.11). That is because $i\pi$ and $\ln(1 - u_1)$ enter the formula in the same way, through $-(1 - u_1)$. Thus the LLA information also predicts the NLLA real part, as pointed out also in ref. [54]. The NLLA real part vanishes at one loop, but it is given at two loops by,

$$q_0^{(2)} \;=\; \frac{1}{8}\left[L_2^- - \frac{1}{2} L_0^- L_1^+ + (L_1^+)^2 \right], \tag{4.21}$$

and at three loops by,

$$q_1^{(3)} \;=\; \frac{1}{4}\left[L_3^+ + L_{2,1}^- - \frac{1}{4}(L_0^-)^2 L_1^+ - \frac{1}{2} L_0^- (L_1^+)^2 + \frac{2}{3}(L_1^+)^3 + \zeta_3 \right]. \tag{4.22}$$

Matching the MRK behavior of our ansatz to the NLLA real part $q_1^{(3)}$ fixes the coefficient of $V^{(1)} R_6^{(2)}$. It only remains to fix the coefficient of $[\tilde{\Phi}_6]^2$, using the NNLLA real part $q_0^{(3)}$.

In order to predict both the NLLA imaginary part and the NNLLA real part, we first need to determine the NLL NMHV impact factor $\Phi_{\mathrm{Reg}}^{\mathrm{NMHV},(1)}(\nu, n)$ entering the master formula (4.11). This impact factor first contributes to the MRK behavior of the ratio function at two loops, where it determines $p_0^{(2)}$. We take the MRK limits of $V^{(2)}$ and $\tilde{V}^{(2)}$ from ref. [27], and use eq. (4.15) to find

$$p_0^{(2)} \;=\; \frac{1}{8}\bigg[11\, L_3^+ - 2\, L_{2,1}^- + \left(\frac{3}{2} L_0^- + L_1^+ \right) L_2^- - \frac{1}{12}(L_0^-)^3 - \frac{3}{2}(L_0^-)^2 L_1^+ + L_0^- (L_1^+)^2 - \frac{4}{3}(L_1^+)^3$$
$$- 2\,\zeta_2 (L_0^- - 2\, L_1^+) - 2\,\zeta_3 \bigg]. \tag{4.23}$$

Then we ask what NLL NMHV impact factor $\Phi_{\mathrm{Reg}}^{\mathrm{NMHV},(1)}(\nu, n)$ generates this expression for $p_0^{(2)}$, via the NMHV master formula (4.11) evaluated at two loops.

The answer can be expressed quite simply in terms of the corresponding MHV impact factor, plus a simple rational function[1] of ν and n:

$$\Phi_{\mathrm{Reg}}^{\mathrm{NMHV},(1)}(\nu, n) \;=\; \Phi_{\mathrm{Reg}}^{\mathrm{MHV},(1)}(\nu, n) + \frac{i n \nu}{2 \left(-\frac{n}{2} + i\nu \right)^2 \left(\frac{n}{2} + i\nu \right)^2}, \tag{4.24}$$

where $\Phi_{\mathrm{Reg}}^{\mathrm{MHV},(1)}(\nu, n)$ is the MHV NLL impact factor, and is equal to [53]

$$\Phi_{\mathrm{Reg}}^{\mathrm{MHV},(1)}(\nu, n) = -\frac{1}{2} E_{\nu,n}^2 - \frac{3}{8} \frac{n^2}{(\nu^2 + \frac{n^2}{4})^2} - \zeta_2, \tag{4.25}$$

---

[1] We thank Benjamin Basso for suggesting that we try an ansatz of this form.

where

$$E_{\nu,n} = -\frac{1}{2}\frac{|n|}{\nu^2 + \frac{n^2}{4}} + \psi\left(1 + i\nu + \frac{|n|}{2}\right) + \psi\left(1 - i\nu + \frac{|n|}{2}\right) - 2\psi(1) \qquad (4.26)$$

is the leading-order BFKL eigenvalue.

In order to work out the NLL approximation at three loops, we also need the NLL BFKL eigenvalue [53],

$$E_{\nu,n}^{(1)} = -\frac{1}{4}D_\nu^2 E_{\nu,n} + \frac{1}{2}V D_\nu E_{\nu,n} - \zeta_2 E_{\nu,n} - 3\zeta_3, \qquad (4.27)$$

where

$$V \equiv -\frac{1}{2}\left[\frac{1}{i\nu + \frac{|n|}{2}} - \frac{1}{-i\nu + \frac{|n|}{2}}\right] = \frac{i\nu}{\nu^2 + \frac{|n|^2}{4}}, \qquad (4.28)$$

and $D_\nu \equiv -i\partial/\partial\nu$.

Using these functions in the master formula (4.11) at three loops, we obtain both $p_1^{(3)}$ and $q_0^{(3)}$:

$$
\begin{aligned}
p_1^{(3)} = \frac{1}{16}\Bigg[&-4\,L_4^- + 2\,L_{3,1}^+ - 12\,L_{2,1,1}^- + \left(-\frac{1}{2}L_0^- + 15\,L_1^+\right)L_3^+ + \left(L_0^- + 2\,L_1^+\right)L_{2,1}^- \\
&+ \left((L_0^-)^2 + L_0^- L_1^+ + 4\,(L_1^+)^2\right)L_2^- - \frac{1}{24}(L_0^-)^4 - \frac{5}{24}(L_0^-)^3 L_1^+ - \frac{9}{4}(L_0^-)^2(L_1^+)^2 \\
&+ \frac{1}{2}L_0^-(L_1^+)^3 - \frac{5}{3}(L_1^+)^4 - 8\zeta_2\left(L_2^- + \frac{1}{2}L_0^- L_1^+ - (L_1^+)^2\right) \\
&+ \zeta_3\left(L_0^- + 2\,L_1^+\right)\Bigg], \qquad (4.29)
\end{aligned}
$$

$$
\begin{aligned}
q_0^{(3)} = \frac{1}{16}\Bigg[&-2\,L_4^- + L_{3,1}^+ - 6\,L_{2,1,1}^- + \left(\frac{15}{4}L_0^- - \frac{23}{2}L_1^+\right)L_3^+ + \left(\frac{1}{2}L_0^- + 3\,L_1^+\right)L_{2,1}^- \\
&+ \left(\frac{1}{2}(L_0^-)^2 - L_0^- L_1^+ + (L_1^+)^2\right)L_2^- - \frac{1}{48}(L_0^-)^4 - \frac{25}{48}(L_0^-)^3 L_1^+ + \frac{11}{8}(L_0^-)^2(L_1^+)^2 \\
&- \frac{17}{12}L_0^-(L_1^+)^3 + \frac{11}{6}(L_1^+)^4 - 4\zeta_2\left(L_2^- - \frac{1}{2}L_0^- L_1^+ + (L_1^+)^2\right) \\
&+ \zeta_3\left(\frac{1}{2}L_0^- + 3\,L_1^+\right)\Bigg]. \qquad (4.30)
\end{aligned}
$$

The MRK limit of our ansatz for $V^{(3)}$ and $\tilde{V}^{(3)}$, inserted into eq. (4.15), yields complete agreement with these expressions. The agreement with $q_0^{(3)}$ fixes the one remaining parameter in the ansatz, namely the coefficient of $[\tilde{\Phi}_6]^2$.

Finally, having fixed the ansatz, we turn to NNLLA. Three loops is the first order in which a truly NNLLA quantity appears, namely $p_0^{(3)}$. Thus $p_0^{(3)}$ cannot be predicted using lower-loop information. Extracting it from our function gives novel data. We find that

$$
\begin{aligned}
p_0^{(3)} = \frac{1}{16}\Bigg[&-87\,L_5^+ + 4\,L_{4,1}^- - 14\,L_{3,1,1}^+ + 12\,L_{2,1,1,1}^- - \left(7\,L_0^- + 2\,L_1^+\right)L_4^- + \left(\frac{1}{2}L_0^- + L_1^+\right)L_{3,1}^+ \\
&- 3\left(L_0^- + 2\,L_1^+\right)L_{2,1,1}^- + \left(\frac{45}{4}(L_0^-)^2 - \frac{1}{2}L_0^- L_1^+ + 11\,(L_1^+)^2\right)L_3^+
\end{aligned}
$$

$$+ \left(-(L_0^-)^2 + 4\, L_0^-\, L_1^+ - 2\, (L_1^+)^2 \right) L_{2,1}^-$$

$$+ \left(\frac{17}{16}\, (L_0^-)^3 + \frac{3}{8}\, (L_0^-)^2\, L_1^+ + \frac{5}{4}\, L_0^-\, (L_1^+)^2 + \frac{3}{2}\, (L_1^+)^3 \right) L_2^- + \frac{3}{80}\, (L_0^-)^5 - \frac{5}{4}\, (L_0^-)^4\, L_1^+$$

$$+ \frac{1}{24}\, (L_0^-)^3\, (L_1^+)^2 - \frac{13}{6}\, (L_0^-)^2\, (L_1^+)^3 + \frac{1}{2}\, L_0^-\, (L_1^+)^4 - \frac{8}{15}\, (L_1^+)^5$$

$$+ \zeta_2 \left(-32\, L_3^+ - 16\, L_{2,1}^- - 4\, (L_0^- - 2\, L_1^+)\, L_2^- + \frac{5}{6}\, (L_0^-)^3 + 5\, (L_0^-)^2\, L_1^+ - 4\, L_0^-\, (L_1^+)^2 \right.$$

$$\left. + \frac{40}{3}\, (L_1^+)^3 \right) + \zeta_3 \left(\frac{5}{2}\, (L_0^-)^2 + 3\, L_0^-\, L_1^+ - 6\, (L_1^+)^2 \right) + 22\, \zeta_4\, (L_0^- - 2\, L_1^+)$$

$$+ 30\, \zeta_5 - 16\, \zeta_2\, \zeta_3 \Bigg]. \tag{4.31}$$

In an ancillary file, we provide computer-readable expressions for all the $p_r^{(L)}$ and $q_r^{(L)}$ functions for $L = 1, 2, 3$.

Knowledge of $p_0^{(3)}$ allows us to fix the NNLL impact factor $\Phi_{\text{Reg}}^{\text{NMHV},(2)}(\nu, n)$, in the same way that we used $p_0^{(2)}$ to determine the NMHV impact factor at NLL. Again we find that the NMHV impact factor can be expressed simply in terms of the MHV impact factor and rational functions of ν and n:

$$\Phi_{\text{Reg}}^{\text{NMHV},(2)}(\nu, n) = \Phi_{\text{Reg}}^{\text{MHV},(2)}(\nu, n) \; + \; \left(\Phi_{\text{Reg}}^{\text{MHV},(1)}(\nu, n) + \zeta_2 \right) \frac{i n \nu}{2 \left(-\frac{n}{2} + i \nu \right)^2 \left(\frac{n}{2} + i \nu \right)^2}$$

$$- \frac{i n \nu \left(n^2 - i n \nu - 2 \nu^2 \right)}{8 \left(-\frac{n}{2} + i \nu \right)^4 \left(\frac{n}{2} + i \nu \right)^4}. \tag{4.32}$$

This formula has recently been reproduced, and extended to all orders, using a kind of analytic continuation from the near-collinear limit [59].

The all-orders formula is expressed in terms of the Zhukovsky variable $x(u)$ defined in eq. (3.10). It reads,[2] in our definition of Φ,

$$\Phi_{\text{Reg}}^{\text{NMHV}}(\nu, n) = \Phi_{\text{Reg}}^{\text{MHV}}(\nu, n) \times \frac{\nu - \frac{in}{2}}{\nu + \frac{in}{2}} \frac{x(u + \frac{in}{2})}{x(u - \frac{in}{2})}. \tag{4.33}$$

The rapidity u in this expression is related to the variable ν by an integral expression [59]. (Our ν is defined to be precisely $1/2$ of the ν defined in ref. [59], while our n is just their m.) The integrals can be performed in the weak coupling expansion, and the equation for $\nu(u, n)$ can be inverted to solve for $u(\nu, n)$, order-by-order in the coupling. The first three orders are enough for us here,

$$u = \nu - \frac{i}{2}\, a\, V + \frac{i}{8}\, a^2\, V\, (N^2 + 4\, \zeta_2) + \mathcal{O}(a^3), \tag{4.34}$$

where $V = i\nu/(\nu^2 + n^2/4)$ is defined in eq. (4.28) and $N = n/(\nu^2 + n^2/4)$. Through this order, the relation between u and ν only involves rational functions of ν and n. Inserting the expansion (4.34) into eq. (4.33) yields both eq. (4.24) for $\Phi_{\text{Reg}}^{\text{NMHV},(1)}(\nu, n)$ and eq. (4.32) for $\Phi_{\text{Reg}}^{\text{NMHV},(2)}(\nu, n)$. At the next loop order, the relation (4.34) begins to contain ψ functions,

---

[2]We thank Benjamin Basso for discussions on these points prior to the appearance of ref. [59].
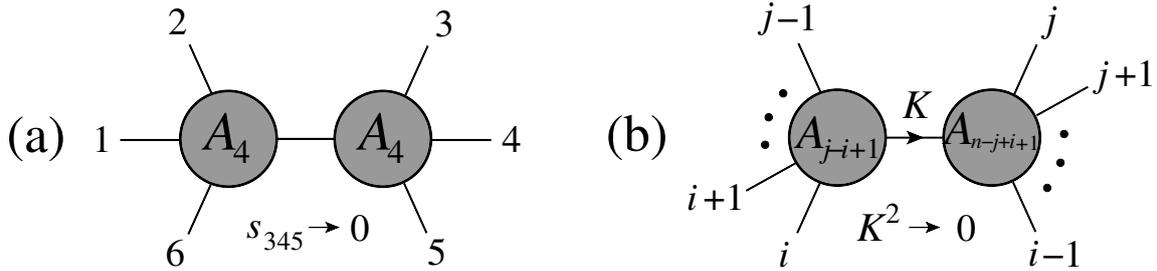
Figure 1. (a) Multi-particle factorization of a six-point amplitude into two four-point amplitudes, in the limit $s_{345} \to 0$. (b) The most general multi-particle factorization of an n-point amplitude into a $(j - i + 1)$-point amplitude and an $(n - j + i + 1)$-point amplitude, in the limit that $K^2 = s_{i,i+1,...,j-1} \to 0$.

which should then enter the formula for $\Phi_{\rm Reg}^{{\rm NMHV},(3)}(\nu, n)$ in terms of $\Phi_{\rm Reg}^{\rm MHV}(\nu, n)$. It would be interesting to check this statement once the four-loop ratio function is determined.

The ratio in eq. (4.33) might appear to be upside-down with respect to ref. [59]. However, we defined $\Phi^{\rm NMHV}$ for the $(\eta_4)^4$ Grassmann component of the NMHV super-amplitude, while it was defined for the $(\eta_1)^4$ component in ref. [59]. The two components are related by the cyclic permutation that inverts all the y_i variables, which exchanges $w \leftrightarrow w^*$ and therefore takes $n \leftrightarrow -n$.

Using eq. (4.32) and the known BFKL NNLL eigenvalue, we have the information necessary to find the NNLLA imaginary part and N³LLA real part to all loop orders. (Of course, the very recent all-orders formulae [59] could be used to go well beyond this.) While we do not pursue this exercise here, such fixed-order data in the (w, w^*) space will prove quite useful during the construction of the ratio function at four loops and beyond.

5 Multi-particle factorization

A six-point amplitude can factorize onto a product of four-point amplitudes in the limit that a three-particle momentum invariant goes on shell, $s_{i,i+1,i+2} \equiv (k_i + k_{i+1} + k_{i+2})^2 \to 0$. This limit is called a multi-particle factorization limit, in order to distinguish it from the two-particle factorization limits, or collinear limits. The multi-particle factorization limit of the six-gluon amplitude, in which the invariant $s_{345} \to 0$, is shown in figure 1(a). We will discuss this limit first, and later consider the most general multi-particle factorization of an n-point amplitude, shown in figure 1(b).

In supersymmetric theories, all multi-particle poles of MHV amplitudes have zero residue, because of the same helicity counting rules that apply at tree level [87]: each four-point amplitude needs to have two negative and two positive external helicities. One negative and one positive helicity must be assigned to the virtual gluon crossing the pole, leaving three negative- and three positive-helicity external gluons, i.e. the NMHV helicity configuration. Using the three-loop ratio function, we can extract the multi-particle factorization behavior of six-point amplitudes in planar $\mathcal{N} = 4$ SYM through three loops. We will find that

it is remarkably simple, containing no function more complicated than the logarithm. The simplicity of the six-point factorization leads to a natural conjecture for the n-point case.

First we review the general factorization behavior and what is known at one loop from the work of Bern and Chalmers [88]. For definiteness, we will factorize the amplitude in the limit that $s_{345} = K^2 \to 0$, where $K = k_3 + k_4 + k_5$. The only two dual superconformal invariants "(i)" that contain a pole in the s_{345} channel are (1) and (4). They become equal in this limit. Furthermore the dual conformal cross ratios u and w contain s_{345} in the denominator, while v does not contain it. Therefore the factorization limit of the ratio function $\mathcal{P}$ in the s_{345} channel will be obtained by letting $u, w \to \infty$ in $V(u, v, w)$, with u/w and v held fixed. The odd part $\tilde{V}(u, v, w)$ will not contribute at any loop order in this limit, because it multiplies $(1) - (4)$, which is power-suppressed in the limit.

We also assume that the component of the NMHV amplitude has been chosen so that there are two negative helicities on one side of the pole, and one negative-helicity on the other side, so that the multi-particle factorization is non-trivial at tree-level. Then we can define an all loop order factorization function F_6 by,

$$A_6^{\text{NMHV}}(k_i) \overset{s_{345} \to 0}{\longrightarrow} A_4(k_6, k_1, k_2, K) \frac{F_6(K^2, s_{i,i+1})}{K^2} A_4(-K, k_3, k_4, k_5), \qquad (5.1)$$

where A_6 and A_4 are all-orders amplitudes. For the given choice of external helicities, there is only one nontrivial assignment of the intermediate gluon helicity.

When we expand eq. (5.1) out to one loop, we obtain [88]

$$\begin{aligned}
A_6^{\text{NMHV (1)}}(k_i) \overset{s_{345} \to 0}{\longrightarrow} \ & A_4^{(1)}(k_6, k_1, k_2, K) \frac{1}{K^2} A_4^{(0)}(-K, k_3, k_4, k_5) \\
& + A_4^{(0)}(k_6, k_1, k_2, K) \frac{1}{K^2} A_4^{(1)}(-K, k_3, k_4, k_5) \\
& + A_4^{(0)}(k_6, k_1, k_2, K) \frac{F_6^{(1)}}{K^2} A_4^{(0)}(-K, k_3, k_4, k_5), \qquad (5.2)
\end{aligned}$$

which defines the one-loop factorization function, $F_6^{(1)}$. This function was computed in $\mathcal{N} = 4$ SYM in ref. [88]. Setting $\mu = 1$, multiplying by $(4\pi)^2/2$ to account for a difference in expansion parameters, and permuting the indices appropriately for the s_{345} channel, the function is,

$$\begin{aligned}
F_6^{(1)} = \ & -\frac{1}{\epsilon^2} \Big[(-s_{23})^{-\epsilon} - (-s_{61})^{-\epsilon} - (-s_{45})^{-\epsilon} \Big] - \frac{1}{2\epsilon^2} \frac{(-s_{61})^{-\epsilon}(-s_{45})^{-\epsilon}}{(-s_{23})^{-\epsilon}} \\
& + \frac{1}{2} \ln^2 \Big(\frac{-s_{23}}{-s_{345}} \Big) - \frac{1}{2} \ln^2 \Big(\frac{-s_{45}}{-s_{345}} \Big) - \frac{1}{2} \ln^2 \Big(\frac{-s_{61}}{-s_{345}} \Big) - 2\zeta_2 \\
& + \{ k_3 \leftrightarrow k_6, \, k_4 \leftrightarrow k_1, \, k_5 \leftrightarrow k_2 \} \qquad (5.3) \\
= \ & \frac{1}{2} \left\{ \frac{1}{\epsilon^2} \left[\left(\frac{(-s_{12})(-s_{34})}{(-s_{56})} \right)^{-\epsilon} + \left(\frac{(-s_{45})(-s_{61})}{(-s_{23})} \right)^{-\epsilon} \right] \right. \\
& \left. - \frac{1}{2} \left[\ln\left(\frac{(-s_{12})(-s_{34})}{(-s_{56})} \right) - \ln\left(\frac{(-s_{45})(-s_{61})}{(-s_{23})} \right) \right]^2 - \frac{1}{2} \ln^2(uw/v) - 8\zeta_2 \right\}. \quad (5.4)
\end{aligned}$$

Using this function, together with the one-loop MHV four-point and six-point amplitudes (which also enter the BDS ansatz), we will be able to predict the behavior of the ratio

function in the factorization limit at one loop. Later we will turn the argument around, and use the factorization behavior of the two- and three-loop NMHV amplitudes to determine the higher-loop factorization functions $F_6^{(L)}$.

First we record the required one-loop amplitudes, after dividing by their respective tree amplitudes, which can be factored out of eq. (5.2). Ref. [1] gives the sum of the two required four-point amplitudes, in terms of functions called V_4 there,

$$\frac{1}{2}\Big[V_4 + V_4'\Big] = \frac{1}{2}\Big\{ -\frac{2}{\epsilon^2}\Big[(-s_{34})^{-\epsilon} + (-s_{45})^{-\epsilon} + (-s_{61})^{-\epsilon} + (-s_{12})^{-\epsilon}\Big]$$
$$+ \ln^2\Big(\frac{-s_{34}}{-s_{45}}\Big) + \ln^2\Big(\frac{-s_{61}}{-s_{12}}\Big) + 12\,\zeta_2 \Big\}. \tag{5.5}$$

The six-point amplitude is also given in terms of the function V_6,

$$\frac{1}{2}V_6 = \frac{1}{2}\Big\{ \sum_{i=1}^{6}\Big[-\frac{1}{\epsilon^2}(-s_{i,i+1})^{-\epsilon} - \ln\Big(\frac{-s_{i,i+1}}{-s_{i,i+1,i+2}}\Big)\ln\Big(\frac{-s_{i+1,i+2}}{-s_{i,i+1,i+2}}\Big) + \frac{1}{4}\ln^2\Big(\frac{-s_{i,i+1,i+2}}{-s_{i+1,i+2,i+3}}\Big)\Big]$$
$$- \mathrm{Li}_2(1-u) - \mathrm{Li}_2(1-v) - \mathrm{Li}_2(1-w) + 6\,\zeta_2 \Big\} \tag{5.6}$$

$$= \frac{1}{2}\Big\{ \sum_{i=1}^{6}\Big[-\frac{1}{\epsilon^2}\Big(1 - \epsilon\ln(-s_{i,i+1})\Big) - \ln(-s_{i,i+1})\ln(-s_{i+1,i+2}) + \frac{1}{2}\ln(-s_{i,i+1})\ln(-s_{i+3,i+4})\Big]$$
$$- Y(u,v,w) + 6\,\zeta_2 \Big\}, \tag{5.7}$$

where

$$Y(u,v,w) \equiv H_2^u + H_2^v + H_2^w + \frac{1}{2}\Big(\ln^2 u + \ln^2 v + \ln^2 w\Big). \tag{5.8}$$

We will be interested in the combination,

$$\frac{1}{2}\Big[V_4 + V_4' - V_6\Big] = \frac{1}{2}\Big\{ -\frac{1}{\epsilon^2}\Big[\Big(\frac{(-s_{12})(-s_{34})}{(-s_{56})}\Big)^{-\epsilon} + \Big(\frac{(-s_{45})(-s_{61})}{(-s_{23})}\Big)^{-\epsilon}\Big]$$
$$+ \frac{1}{2}\Big[\ln\Big(\frac{(-s_{12})(-s_{34})}{(-s_{56})}\Big) - \ln\Big(\frac{(-s_{45})(-s_{61})}{(-s_{23})}\Big)\Big]^2$$
$$+ Y(u,v,w) + 6\,\zeta_2 \Big\}. \tag{5.9}$$

We note that the sum of this quantity with $F_6^{(1)}$ is dual conformal invariant, even before we enter the factorization limit:

$$F_6^{(1)} + \frac{1}{2}\Big[V_4 + V_4' - V_6\Big] = \frac{1}{2}Y(u,v,w) - \frac{1}{4}\ln^2(uw/v) - \zeta_2\,. \tag{5.10}$$

At one loop, when we divide the left-hand side of eq. (5.1) by A_6^{MHV}, and expand out to first order, the tree factors correspond to the superconformal invariant (1). The limiting behavior of $V^{(1)}(u,v,w)$ is then given by, using also eq. (5.10),

$$V^{(1)}(u,v,w)\big|_{u,w\to\infty} = F_6^{(1)} + \frac{1}{2}\Big[V_4 + V_4' - V_6\big|_{u,w\to\infty}\Big]$$
$$= -\frac{1}{4}\ln^2(uw/v) - 2\,\zeta_2 + \frac{1}{2}\Big(\mathrm{Li}_2(1-v) + \frac{1}{2}\ln^2 v\Big). \tag{5.11}$$

The result is manifestly finite and dual conformally invariant. It also matches perfectly against the limit $u, w \to \infty$ of the known one-loop expression, in the form (2.18) given in ref. [27]. In fact, we note from eqs. (2.18) and (5.10) that

$$V^{(1)}(u, v, w) = F_6^{(1)} + \frac{1}{2}\Big[V_4 + V_4' - V_6\Big] \tag{5.12}$$

even outside of the factorization limit.

Now we proceed to higher loops. At this point we should be careful to consider the actual NMHV amplitude, not the ratio function. The ratio function does not have a simple factorization limit because it treats the MHV amplitude on the same footing as the NMHV amplitude. However, there is no tree-level pole for the MHV amplitude, so there is no reason for the transcendental function multiplying the tree amplitude to have a simple form in the factorization limit. In order to do this, and still deal with a finite, dual conformally invariant quantity for a while longer, we multiply the ratio function by the (exponentiated) remainder function. It is also convenient to take the logarithm. Whereas the remainder function $R = R_6$ is defined by

$$\frac{A^{\text{MHV}}}{A^{\text{BDS}}} = \exp(R), \tag{5.13}$$

here we define

$$\frac{A^{\text{NMHV}}}{A^{\text{BDS}}} = \frac{A^{\text{NMHV}}}{A^{\text{MHV}}} \times \frac{A^{\text{MHV}}}{A^{\text{BDS}}} = \mathcal{P} \times \exp(R) \equiv (1) \times \exp(\hat{U}). \tag{5.14}$$

We call the function $\hat{U}$ because it will be useful to adjust it slightly later. In the factorization limit, the tree-(super)amplitude prefactor in $\mathcal{P}$ collapses to (1) and we can identify $Ve^R = e^{\hat{U}}$, or

$$\hat{U}(u, v, w) = \ln V(u, v, w) + R_6(u, v, w), \tag{5.15}$$

so that the perturbative expansion of $\hat{U}$ is,

$$\hat{U}^{(1)} = V^{(1)}, \tag{5.16}$$

$$\hat{U}^{(2)} = V^{(2)} - \frac{1}{2}[V^{(1)}]^2 + R_6^{(2)}, \tag{5.17}$$

$$\hat{U}^{(3)} = V^{(3)} + \frac{1}{3}[V^{(1)}]^3 - V^{(1)}V^{(2)} + R_6^{(3)}, \tag{5.18}$$

where we used the fact that the remainder function only becomes nonvanishing starting at two loops.

We also need to evaluate the BDS ansatz [3],

$$\ln A_n^{\text{BDS}} = \sum_{L=1}^{\infty} a^L \bigg(f^{(L)}(\epsilon)\frac{1}{2}V_n(L\epsilon) + C^{(L)} \bigg), \tag{5.19}$$

where

$$f^{(L)}(\epsilon) \equiv f_0^{(L)} + \epsilon f_1^{(L)} + \epsilon^2 f_2^{(L)}. \tag{5.20}$$

Two of the constants,

$$f_0^{(L)} \;=\; \frac{1}{4}\,\gamma_K^{(L)}\,, \qquad f_1^{(L)} \;=\; \frac{L}{2}\,\mathcal{G}_0^{(L)}\,, \tag{5.21}$$

are given in terms of the planar cusp anomalous dimension γ_K (see eq. (8.4) below) and the "collinear" anomalous dimension $\mathcal{G}_0$, while $f_2^{(L)}$ and $C^{(L)}$ are other (zeta-valued) constants. The L-loop coefficient of the combination we need, $\ln(A_4^{\rm BDS} \times A_4^{\rm BDS'}/A_6^{\rm BDS})$, where $A_4^{\rm BDS^{(\prime)}}$ are the ansätze for the two four-point subprocesses, is closely related to eq. (5.9):

$$\ln\!\left(\frac{A_4^{\rm BDS}\,A_4^{\rm BDS'}}{A_6^{\rm BDS}}\right)^{\!(L)} = -\frac{\gamma_K^{(L)}}{8\epsilon^2 L^2}\left(1+2\,\epsilon\,L\,\frac{\mathcal{G}_0^{(L)}}{\gamma_K^{(L)}}\right)\left[\left(\frac{(-s_{12})(-s_{34})}{(-s_{56})}\right)^{\!-L\epsilon}+\left(\frac{(-s_{45})(-s_{61})}{(-s_{23})}\right)^{\!-L\epsilon}\right]$$

$$+\frac{\gamma_K^{(L)}}{8}\left[\frac{1}{2}\ln^2\!\left(\frac{(-s_{12})(-s_{34})}{(-s_{56})}\Big/\frac{(-s_{45})(-s_{61})}{(-s_{23})}\right)+Y(u,v,w)+6\,\zeta_2\right]$$

$$-\frac{f_2^{(L)}}{L^2}-C^{(L)}\,. \tag{5.22}$$

Because of the appearance of the function $Y(u,v,w)$ in $A_6^{\rm BDS}$ and in eq. (5.22), it is useful to define a function $U(u,v,w)$ that absorbs this function:

$$U(u,v,w)=\hat{U}(u,v,w)-\frac{\gamma_K}{8}Y(u,v,w)\,. \tag{5.23}$$

We will see that U has simpler analytic properties than $\hat{U}$, even outside of the factorization limit. At one loop, we have

$$U^{(1)}(u,v,w)=-\frac{1}{4}\ln^2(uw/v)-\zeta_2\,, \tag{5.24}$$

so the polylogarithms have cancelled from $U^{(1)}$.

In ref. [27], $V^{(2)}(u,v,w)$ was given in terms of one-dimensional HPLs, plus the three independent permutations of the function $\Omega^{(2)}(u,v,w)$. The two-loop remainder function $R_6^{(2)}$ was also given, in a similar form. In the sum of $V^{(2)}$ and $R_6^{(2)}$ entering $U^{(2)}$, two of the permutations cancel, and only the permutation $\Omega^{(2)}(w,u,v)$ survives. In total, before taking the factorization limit, $U^{(2)}$ as defined by eqs. (5.17) and (5.23) is given by,

$$U^{(2)}(u,v,w) = \frac{1}{4}\Big\{-\Omega^{(2)}(w,u,v)-H_4^u-H_4^w-3\left(H_{3,1}^u+H_{3,1}^w-H_{2,1,1}^u-H_{2,1,1}^w\right)$$

$$+\frac{1}{2}\left[(H_2^u)^2+(H_2^w)^2\right]+2\left(\ln u\,H_{2,1}^u+\ln w\,H_{2,1}^w\right)-\ln(w/v)\left(H_3^u+H_{2,1}^u\right)$$

$$-\ln(u/v)\left(H_3^w+H_{2,1}^w\right)+\frac{1}{2}\ln(uw/v)\left(\ln(uv/w)\,H_2^u+\ln(wv/u)\,H_2^w\right)$$

$$-\frac{1}{2}\left(\ln u\,\ln w-8\,\zeta_2\right)\left(\ln u\,\ln w-\ln v\,\ln(uw)\right)-\frac{1}{4}\left(\ln^2 u+\ln^2 w\right)\ln^2 v$$

$$-\zeta_2\left[2\left(H_2^u+H_2^w\right)-\ln^2 u-\ln^2 w-2\ln^2 v\right]+15\,\zeta_4\Big\}\,. \tag{5.25}$$

Note that the dependence on v is particularly simple; aside from $\Omega^{(2)}(w,u,v)$, the only function of v that appears is $\ln v$.

We wish to use the coproduct formalism to extract the behavior of $\Omega^{(2)}(w,u,v)$ in the factorization limit. This exercise will be a useful warmup for obtaining the limit of the

NMHV amplitude at three loops. First we recall [28] the formula for the u-derivative of a generic hexagon function F, holding v and w fixed:

$$\frac{\partial F}{\partial u}\bigg|_{v,w} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1-u-v-w}{u\sqrt{\Delta}}F^{y_u} + \frac{1-u-v+w}{(1-u)\sqrt{\Delta}}F^{y_v} + \frac{1-u+v-w}{(1-u)\sqrt{\Delta}}F^{y_w}.$$

(5.26)

We can permute this relation cyclicly in order to obtain the v- and w-derivatives. Now we take the limit of eq. (5.26) as $u, w \to \infty$, finding

$$\partial_u F = \frac{1}{u}\left[F^u + F^{1-u} - \frac{1}{r}(F^{y_u} - F^{y_w}) + \frac{u-w}{(u+w)r}F^{y_v}\right],$$ (5.27)

$$\partial_v F = \frac{1}{v}\left[F^v - \frac{1}{r}F^{y_v}\right] - \frac{1}{1-v}\left[F^{1-v} + \frac{u-w}{(u+w)r}(F^{y_u} - F^{y_w})\right],$$ (5.28)

where

$$r = \sqrt{1 - \frac{4uvw}{(u+w)^2}}.$$ (5.29)

The w-derivative is obtained from the u-derivative simply by exchanging u and w labels everywhere.

We see from eqs. (5.27) and (5.28) that the factorization limit of a hexagon function is likely to be simple, with all the occurrences of r dropping out, if two conditions on the $\{n-1, 1\}$ coproduct elements are met:

$$F^{y_u} = F^{y_w} \quad \text{and} \quad F^{y_v} = 0.$$ (5.30)

We will see that this condition is satisfied by the specific combinations of nontrivial hexagon functions that we need for taking the limits of U, through three loops.

First consider the function $\Omega^{(2)}(w, u, v)$. By permuting eqs. (B.10) and (B.11) of ref. [28], we see that

$$[\Omega^{(2)}(w, u, v)]^{y_u} = [\Omega^{(2)}(w, u, v)]^{y_w} \quad \text{and} \quad [\Omega^{(2)}(w, u, v)]^{y_v} = 0,$$ (5.31)

so $\Omega^{(2)}(w, u, v)$ should have a simple limit. However, we see further that

$$[\Omega^{(2)}(w, u, v)]^v = [\Omega^{(2)}(w, u, v)]^{1-v} = 0, \quad (5.32)$$
$$[\Omega^{(2)}(w, u, v)]^u + [\Omega^{(2)}(w, u, v)]^{1-u} = [\Omega^{(2)}(w, u, v)]^w + [\Omega^{(2)}(w, u, v)]^{1-w} = 0, \quad (5.33)$$

which means that all the derivatives of $\Omega^{(2)}(w, u, v)$ vanish in the factorization limit. Therefore $\Omega^{(2)}(w, u, v)$ can at most be a constant in this limit.

To fix the constant, we consider the line $(u, 1, u)$ as $u \to \infty$. On this line, all hexagon functions collapse to one-dimensional HPLs, so it is easy to take the large u limit. Here we need,

$$\Omega^{(2)}(u, u, 1) = -2\,H_4^u - 2\,H_{3,1}^u + 6\,H_{2,1,1}^u + 2\,(H_2^u)^2 + 2\ln u\,(H_3^u + H_{2,1}^u)$$
$$+ \ln^2 u\,H_2^u + \frac{1}{4}\ln^4 u - 6\,\zeta_4.$$ (5.34)

Using standard identities for inverting the arguments of the HPLs, we find that this function vanishes as $u \to \infty$. Therefore

$$\Omega^{(2)}(w, u, v)\big|_{u,w\to\infty} = 0. \tag{5.35}$$

Aside from $\Omega^{(2)}(w, u, v)$, the remaining terms in eq. (5.25) for $U^{(2)}$ are one-dimensional HPLs with arguments u, v and w. The same HPL argument-inversion identities allow us to extract the limiting behavior of the HPLs in u and w terms. The final result has the simple form,

$$U^{(2)}(u, v, w)\big|_{u,w\to\infty} = \frac{3}{4}\,\zeta_2\,\ln^2(uw/v) - \frac{1}{2}\,\zeta_3\,\ln(uw/v) + \frac{71}{8}\,\zeta_4\,. \tag{5.36}$$

Remarkably, the limit of $U^{(2)}$ is simply a polynomial in $\ln(uw/v)$ with zeta-valued coefficients.

Turning now to three loops, we find that the $\{5, 1\}$ coproducts of $U^{(3)}$ obey the relations (5.30) required for a simple factorization limit. For example,

$$
\begin{aligned}
[U^{(3)}]^{y_u} &= \frac{1}{32}\Big[3\,H_1(u, v, w) + H_1(v, w, u) + H_1(w, u, v)\Big] \\
&\quad - \frac{1}{128}\Big[11\,J_1(u, v, w) + J_1(v, w, u) + J_1(w, u, v)\Big] \\
&\quad + \frac{1}{32}\tilde{\Phi}_6(u, v, w)\Big[\ln^2 u + \ln^2 w + \ln^2 v + 2\Big(\ln u \ln w - \ln(uw) \ln v\Big) - 22\zeta_2\Big].
\end{aligned} \tag{5.37}
$$

Because the functions H_1 and J_1 are symmetric under exchange of their first and third argument [28], and $\tilde{\Phi}_6$ is totally symmetric, we see that eq. (5.37) is symmetric under $u \leftrightarrow w$. But $U^{(3)}(u, v, w)$ is symmetric under the exchange of $u \leftrightarrow w$. Together, these two properties imply that

$$[U^{(3)}]^{y_u} = [U^{(3)}]^{y_w}\,, \tag{5.38}$$

as desired by eq. (5.30). Note that this "bonus" relation holds even outside of the factorization limit, a property to which we will return in the following section.

We also find that the y_v coproduct element of $U^{(3)}$ is proportional to the weight-5 parity odd function $H_1(u, v, w)$:

$$[U^{(3)}]^{y_v} = \frac{1}{8}H_1(u, v, w)\,. \tag{5.39}$$

One can check that H_1 obeys all the same coproduct relations that $\Omega^{(2)}(w, u, v)$ does [28], so that all of its derivatives vanish in the factorization limit. (Its y_v coproduct element is in fact proportional to $\Omega^{(2)}(w, u, v)$.) In the case of H_1, the vanishing of the constant of integration, i.e. the fact that

$$H_1(u, v, w)\big|_{u,w\to\infty} = 0, \tag{5.40}$$

follows simply because all parity-odd functions vanish on the surface $\Delta(u, v, w) = 0$, which contains the line $(u, 1, u)$.

– 29 –

The $F^u + F^{1-u}$, F^v and F^{1-v} coproduct elements contributing to eqs. (5.27) and (5.28) also simplify dramatically for $F = U^{(3)}$ as $u, w \to \infty$; the only functions they contain are logarithms of u, v and w. We find that in the factorization limit,

$$[U^{(3)}]^u + [U^{(3)}]^{1-u} = \zeta_3 \ln^2(uw/v) - \frac{75}{4} \zeta_4 \ln(uw/v) + 7\zeta_5 + 8\zeta_2\zeta_3 = -[U^{(3)}]^v, \quad (5.41)$$

$$[U^{(3)}]^{1-v} = 0. \quad (5.42)$$

It is quite fascinating that eq. (5.42) actually holds for any (u, v, w). We will explore the consequences of this second bonus relation in the next section.

Unlike eq. (5.42), the first relation does not hold for arbitrary (u, v, w), but it does hold in the factorization limit. Using eqs. (5.38), (5.41), and (5.42), as well as the vanishing of $[U^{(3)}]^{y_v}$ in the factorization limit, it is trivial to solve the differential equations (5.27) and (5.28) for $F = U^{(3)}$ in this limit. The result is,

$$U^{(3)}(u, v, w)\big|_{u, w \to \infty} = \frac{1}{3} \zeta_3 \ln^3(uw/v) - \frac{75}{8} \zeta_4 \ln^2(uw/v) + (7\zeta_5 + 8\zeta_2\zeta_3)\ln(uw/v)$$

$$- \frac{161}{2} \zeta_6 - 3(\zeta_3)^2. \quad (5.43)$$

Again we fixed the constant of integration using the limiting behavior on the line $(u, 1, u)$ as $u \to \infty$. Remarkably, at all three loop orders studied so far, the quantity U approaches a simple polynomial in $\ln(uw/v)$ in the factorization limit.

Now we go back to construct the factorization function F_6 for the NMHV six-point amplitude in terms of U. To do this, we observe from eq. (5.1) that (apart from the trivial tree-level term), the log of the factorization function is, using eq. (5.14),

$$\ln F_6 = \ln\left(\frac{A_6^{\mathrm{NMHV}}}{A_4^{\mathrm{BDS}} A_4^{\mathrm{BDS}'}}\right) = \ln\left(\frac{A_6^{\mathrm{NMHV}}}{A_6^{\mathrm{BDS}}} \frac{A_6^{\mathrm{BDS}}}{A_4^{\mathrm{BDS}} A_4^{\mathrm{BDS}'}}\right) = \hat{U}(u, v, w) - \ln\left(\frac{A_4^{\mathrm{BDS}} A_4^{\mathrm{BDS}'}}{A_6^{\mathrm{BDS}}}\right), \quad (5.44)$$

or, using eqs. (5.22) and (5.23),

$$[\ln F_6]^{(L)} = \frac{\gamma_K^{(L)}}{8\epsilon^2 L^2}\left(1 + 2\epsilon L \frac{\mathcal{G}_0^{(L)}}{\gamma_K^{(L)}}\right)\left[\left(\frac{(-s_{12})(-s_{34})}{(-s_{56})}\right)^{-L\epsilon} + \left(\frac{(-s_{45})(-s_{61})}{(-s_{23})}\right)^{-L\epsilon}\right]$$

$$- \frac{\gamma_K^{(L)}}{8}\left[\frac{1}{2} \ln^2\left(\frac{(-s_{12})(-s_{34})}{(-s_{56})} \Big/ \frac{(-s_{45})(-s_{61})}{(-s_{23})}\right) + 6\zeta_2\right]$$

$$+ U^{(L)}(u, v, w)\big|_{u, w \to \infty} + \frac{f_2^{(L)}}{L^2} + C^{(L)}. \quad (5.45)$$

From the explicit formulae for $U^{(1)}$ (eq. (5.24)), $U^{(2)}$ (eq. (5.36)) and $U^{(3)}$ (eq. (5.43)) in the factorization limit, we see that the dependence of the factorization function F_6 on the vanishing three-particle invariant s_{345} only appears through the logarithm of the ratio,

$$\frac{uw}{v} = \frac{s_{12}s_{34}}{s_{56}} \cdot \frac{s_{45}s_{61}}{s_{23}} \cdot \frac{1}{s_{345}^2}. \quad (5.46)$$

We note that the same ratios that we have assembled into the divergent factors in eq. (5.45) also appear in eq. (5.46).

Consider now the more general multi-particle factorization of an n-point amplitude in planar $\mathcal{N} = 4$ SYM, in which $s_{i,i+1,\ldots,j-1} \to 0$ as shown in figure 1(b). The corresponding factorization formula is,

$$A_n^{\mathrm{NMHV}}(k_i) \longrightarrow A_{j-i+1}(k_i, k_{i+1}, \ldots, k_{j-1}, K) \frac{F_n(K^2, s_{l,l+1})}{K^2} A_{n-(j-i)+1}(-K, k_j, k_{j+1}, \ldots, k_{i-1}),$$
(5.47)

where $K = k_j + k_{j+1} + \cdots + k_{i-1}$ and all indices are mod n. We conjecture that $\ln F_n$ can be extracted from formula (5.45) for $\ln F_6$ by the simple replacements,

$$\frac{s_{12} s_{34}}{s_{56}} \to \frac{s_{i,K} \, s_{i-1,-K}}{s_{i-1,i}},$$
(5.48)

$$\frac{s_{45} s_{61}}{s_{23}} \to \frac{s_{j,-K} \, s_{j-1,K}}{s_{j-1,j}},$$
(5.49)

$$\frac{uw}{v} \to \frac{s_{i,K} \, s_{i-1,-K} \cdot s_{j,-K} \, s_{j-1,K}}{s_{i-1,i} \cdot s_{j-1,j} \cdot (K^2)^2} = \frac{x_{i+1,j}^2 \, x_{j,i-1}^2 \, x_{j+1,i}^2 \, x_{i,j-1}^2}{x_{i-1,i+1}^2 \, x_{j-1,j+1}^2 \, (x_{i,j}^2)^2}.$$
(5.50)

Note that uw/v gets replaced by a dual conformal cross ratio for the n-point amplitude. The other two ratios involve the invariants near the factorization channel, and their appearance in the singular terms in ϵ is dictated by the general structure of the infrared divergences.

6 Coproduct relations for U and $\tilde{V}$

In the course of inspecting the coproducts of $U^{(3)}$, even *before* taking the factorization limit, we found the following three relations,

$$U^u + U^{1-u} = U^w + U^{1-w} = -(U^v + U^{1-v}),$$
(6.1)

$$U^{1-v} = 0,$$
(6.2)

$$U^{y_u} = U^{y_w},$$
(6.3)

which hold for *any* (u, v, w), at least through three loops.

The first relation is not unexpected. It also holds for the parity-even part of the ratio function V, and it corresponds to the existence of the seventh final entry uw/v in eq. (2.28). Suppose such an entry were not present. Then the final entry $u/(1-u)$ would correspond to the coproduct condition $V^u + V^{1-u} = 0$, and similarly $V^v + V^{1-v} = 0$ and $V^w + V^{1-w} = 0$. With the seventh final entry, these conditions are violated, but they have to be violated in the particular form shown in eq. (6.1), namely

$$V^u + V^{1-u} = V^w + V^{1-w} = -(V^v + V^{1-v}).$$
(6.4)

Taking the logarithm of V does not spoil eq. (6.4), and neither does adding R_6, since it obeys $R_6^{u_i} + R_6^{1-u_i} = 0$ for all three u_i. Finally, the function Y also obeys $Y^{u_i} + Y^{1-u_i} = 0$. Because eq. (6.4) follows from the analysis of Caron-Huot and He [65], so does eq. (6.1).

However, the other two relations, (6.2) and (6.3), are rather unexpected. One virtue of these relations is that they simplify the derivative of U with respect to y_u/y_w.

Recall [28] the formula for this derivative,

$$\sqrt{\Delta}\,\frac{\partial U}{\partial \ln(y_u/y_w)} = (1-u)(1-v)U^u - (u-w)(1-v)U^v - (1-v)(1-w)U^w - u(1-v)U^{1-u}$$

$$+ (u-w)v\,U^{1-v} + w(1-v)\,U^{1-w} + \sqrt{\Delta}\,U^{y_u} - \sqrt{\Delta}\,U^{y_w}\,. \qquad (6.5)$$

Using eqs. (6.3), (6.1) and (6.2), the differential equation (6.5) can be simplified dramatically to,

$$\sqrt{\Delta}\,\frac{\partial U}{\partial \ln(y_u/y_w)} \;=\; (1-v)(U^u - U^w)\,. \qquad (6.6)$$

Only a single nontrivial coproduct combination, $U^u - U^w$, enters this equation, at any loop order!

We can find these coproduct combinations using the results in appendix A. The combination $U^u - U^w$ is generally simpler than either U^u or its $u \leftrightarrow w$ image U^w. At one loop, we have trivially,

$$[U^{(1)}]^u - [U^{(1)}]^w = 0. \qquad (6.7)$$

At two loops, the combination is,

$$[U^{(2)}]^u - [U^{(2)}]^w = H_{2,1}^u + \frac{1}{2}\ln u\,H_2^u - H_{2,1}^w - \frac{1}{2}\ln w\,H_2^w - \frac{1}{4}\Big(\ln(v/w)H_2^u - \ln(v/u)H_2^w\Big)$$

$$+ \frac{1}{8}\ln(u/w)\Big(2\,H_2^v - \ln(uvw)\ln v + 3\ln u\ln w - 8\,\zeta_2\Big)\,. \qquad (6.8)$$

At three loops, it is,

$$[U^{(3)}]^u - [U^{(3)}]^w = A^{u-w}(u,v,w) - A^{u-w}(w,v,u), \qquad (6.9)$$

where

$$A^{u-w}(u,v,w) = \frac{1}{16}\Bigg\{ -M_1(w,u,v) + \frac{128}{3}\,Q_{\mathrm{ep}}(v,u,w)$$

$$- \frac{1}{2}\ln(u/w)\Big(2\,\Omega^{(2)}(u,v,w) - \Omega^{(2)}(w,u,v)\Big) + 12\,H_{4,1}^u + 10\,H_{3,2}^u - 72\,H_{3,1,1}^u - 26\,H_{2,2,1}^u$$

$$- 40\,H_{2,1,1,1}^u - 2\,H_2^u\,(3\,H_3^u - 7\,H_{2,1}^u) - 2\,H_2^u\,(2\,H_{2,1}^v + \ln v\,H_2^v) - 2\,(\ln u + 2\ln v - 3\ln w)\,H_{3,1}^u$$

$$- \frac{1}{2}\ln u\,\Big(4\,H_4^v + 40\,H_{3,1}^v + 4\,H_{2,1,1}^v - 11\,(H_2^v)^2 - 4\ln v\,(H_3^v - H_{2,1}^v)\Big) - 12\,\ln(uw/v)\,H_4^u$$

$$- 2\,(13\ln u - 6\ln(v/w))\,H_{2,1,1}^u + \Big(8\ln u\,\ln(uw/v) - 2\ln^2 v + 4\ln v\ln w\Big)\,H_3^u$$

$$+ \frac{1}{2}\,(11\ln u - \ln w)\,(H_2^u)^2 - \Big(8\,(\ln^2 u + \ln^2 v) - 2\ln u\,(\ln v - \ln w) - 14\ln v\ln w\Big)\,H_{2,1}^u$$

$$- \frac{11}{3}\ln^2 w\,(H_3^u + H_{2,1}^u) - \frac{2}{3}\,H_2^w\,\Big(5\,(H_3^u - \ln u\,H_2^u) - 7\,H_{2,1}^u\Big)$$

$$- \frac{1}{6}\Big(8\ln^3 u + 4\ln^3 v - 3\ln^3 w - 7\ln u\ln^2 w + 3\ln(u/w)\ln v\,(3\ln v - 4\ln w)\Big)\,H_2^u$$

$$- \frac{1}{2}\ln(u/w)\,\Big(\ln^2 u + 4\ln u\ln v - 5\ln w\ln u\Big)\,H_2^v$$

$$+ \frac{1}{12}\Big(\ln^3 u\,(11\ln^2 v - 28\ln v\ln w + 35\ln^2 w) + \ln^2 u\,\ln^2 v\,(8\ln v - 27\ln w)\Big)$$

$$- \zeta_2\Big[H_3^u + 36\,H_{2,1}^u + 13\ln u\,H_2^u - \frac{5}{3}\ln^3 u + 2\ln u\,(9\,H_2^v - 4\ln^2 v) + 2\ln w\,(H_2^u + 12\ln^2 u)$$

$$- 4\ln v\,(H_2^u + 2\ln^2 u)\Big] - 2\,\zeta_3\,(3\ln^2 u + 4\,H_2^u) + 122\,\zeta_4\ln u \Bigg\}\,. \qquad (6.10)$$

The y_u/y_w differential equation (6.6) is relatively simple analytically. In ref. [28] it was discussed how this differential equation has natural boundary conditions at $(u, v, w) = (1, 0, 0)$ and $(0, 0, 1)$. They are natural from the point of view that they correspond to surfaces in the coordinates (y_u, y_v, y_w); therefore only a boundary condition at one point is needed to integrate up to any (u, v, w). However, it was also mentioned in ref. [28] that there could be issues of regularization for an even function like U near the endpoints. Indeed, for $U^{(2)}$ or $U^{(3)}$ there are such issues, which would have to be cured by subtracting a suitable function in order for the endpoints $(1, 0, 0)$ and $(0, 0, 1)$ to be usable.

Another strategy is to use the y_u/y_w differential equation to integrate not off a single point, but off a surface. For example, since it is odd in $u \leftrightarrow w$, one could use this differential equation to move off the surface $u = w$, once one has determined the function on the surface using a different strategy.

Independently of the best numerical approach, the coproduct relations for U indicate a simplified analytic structure for this function. In terms of a final-entry condition, the coproduct relations $U^{1-v} = 0$ and $U^{y_u} = U^{y_w}$ reduce the seven member set (2.28) to only five entries:

$$\left\{ \frac{u}{1-u}, \frac{w}{1-w}, y_u y_w, y_v, \frac{uw}{v} \right\} . \tag{6.11}$$

It would be very interesting to try to find an explanation for this property, which at the moment has only been observed empirically through three loops. In the next section we will see that the function U is in many ways simpler than V, and even simpler than the remainder function R_6.

Before doing that, we close this section by remarking that the potential seventh final entry uw/v, which we also allowed for the parity-odd function $\tilde{V}(u, v, w)$, does not actually appear. In other words, at least through three loops, $\tilde{V}$ obeys the coproduct relations,

$$\tilde{V}^u + \tilde{V}^{1-u} = \tilde{V}^v + \tilde{V}^{1-v} = \tilde{V}^w + \tilde{V}^{1-w} = 0 . \tag{6.12}$$

The corresponding set of final entries for $\tilde{V}$ is

$$\left\{ \frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v, y_w \right\} , \tag{6.13}$$

which is the same set as for the remainder function. It would be interesting to understand this property better as well.

7 Quantitative behavior

In this section we examine the analytical behavior of the components of the ratio function on special lines through the three-dimensional space of cross ratios. On some of these lines, V and $\tilde{V}$ collapse to simpler functions, such as HPLs of a single argument. Some of the analytic formulas, for the function U in particular, exhibit intriguing simplicity. We also plot V and $\tilde{V}$, or various ratios, on these special lines and on some two-dimensional surfaces, such as the plane $u + v + w = 1$, and as a function of u and w for particular values of v.

After the imposition of the MRK constraints, the coproducts of $V^{(3)}$ and $\tilde{V}^{(3)}$ are fully fixed. Of course, given the remainder function R_6 and the function Y defined in eq. (5.8), we can go back and forth between V and U, using the relations

$$U(u, v, w) = \ln V(u, v, w) + R_6(u, v, w) - \frac{\gamma_K}{8} Y(u, v, w) \,, \tag{7.1}$$

$$V(u, v, w) = \exp\left[U(u, v, w) - R_6(u, v, w) + \frac{\gamma_K}{8} Y(u, v, w) \right] \,. \tag{7.2}$$

The $\{n - 1, 1\}$ coproduct elements for U through three loops are given in appendix A.1. This information completely specifies the first derivatives of $U^{(L)}$ and $\tilde{V}^{(L)}$.

We should also fix the functions by giving their values at one point, say $(u, v, w) = (1, 1, 1)$. This point is on the surface $\Delta = 0$, on which all parity-odd hexagon functions vanish. Hence

$$\tilde{V}^{(L)}(1, 1, 1) = 0 \qquad \text{for all } L. \tag{7.3}$$

On the other hand, parity-even functions such as V have nontrivial values at this point. The constant term in $V^{(3)}(1, 1, 1)$ is fixed when the collinear vanishing constraints are applied. It is actually fixed in the vicinity of the collinear limit lines, such as $v = 0$, $u + w = 1$. We can use the simple analytic behavior of hexagon functions on the lines $(u, u, 1)$ and $(u, 1, u)$ (see the next subsection) to carry the information about the constant out to the point $(1, 1, 1)$. We find that

$$V^{(3)}(1, 1, 1) = -\frac{243}{4} \zeta_6 \,. \tag{7.4}$$

This value can be compared to the corresponding values for the one- and two-loop ratio functions,

$$\begin{aligned}
V^{(1)}(1, 1, 1) &= -\zeta_2 \,, \\
V^{(2)}(1, 1, 1) &= 9\, \zeta_4 \,.
\end{aligned} \tag{7.5}$$

We also quote the values of U at this point:

$$\begin{aligned}
U^{(1)}(1, 1, 1) &= -\zeta_2 \,, \\
U^{(2)}(1, 1, 1) &= \frac{21}{4} \zeta_4 \,, \\
U^{(3)}(1, 1, 1) &= -\frac{117}{4} \zeta_6 + (\zeta_3)^2 \,,
\end{aligned} \tag{7.6}$$

where the $(\zeta_3)^2$ term in $U^{(3)}(1, 1, 1)$ comes entirely from $R_6^{(3)}(1, 1, 1)$ [28].

With V and $\tilde{V}$ now completely fixed at three loops, we can investigate their analytic and numerical behavior. In the remainder of this section, we describe lines on which the analytic behavior simplifies, and then we plot the functions V and $\tilde{V}$ on these lines and on various planes in the space of cross ratios (u, v, w).

7.1 The lines $(u, u, 1)$ and $(u, 1, u)$

When one of the cross ratios is equal to unity and the other two are equal to each other, the hexagon functions collapse to pure HPLs. Because $\Delta(u, u, 1) = 0$, all parity-odd functions

vanish on this line, including $\tilde{V}$. On the other hand, V is nontrivial but relatively simple. The simplest way to present V is to give U, and then V can be obtained using eq. (7.2). We also use the "linearized" representation for the HPLs discussed in ref. [29], in which we expand all products of HPLs in terms of a linear combination of HPLs of maximum weight using the shuffle algebra. In that reference a compressed notation for the HPLs was also introduced. Here we will not need that notation, because the formulas are not too lengthy through three loops, and because it obscures some of the patterns in which HPL weight vectors occur.

In the linearized representation, we have,

$$U^{(1)}(u, u, 1) = -\zeta_2 , \tag{7.7}$$

$$U^{(2)}(u, u, 1) = \frac{1}{2}\left(H^u_{0,1,0,1} + H^u_{1,1,0,1}\right) - \frac{3}{2}\left(H^u_{0,1,1,1} + H^u_{1,1,1,1}\right) - \zeta_2\left(H^u_{0,1} + H^u_{1,1}\right) + \frac{21}{4}\zeta_4 , \tag{7.8}$$

$$\begin{aligned}
U^{(3)}(u, u, 1) = {} & H^u_{0,1,0,1,0,1} + H^u_{1,1,0,1,0,1} - 4\left(H^u_{0,1,0,1,1,1} + H^u_{1,1,0,1,1,1}\right) - 5\left(H^u_{0,1,1,0,1,1} + H^u_{1,1,1,0,1,1}\right) \\
& - 4\left(H^u_{0,1,1,1,0,1} + H^u_{1,1,1,1,0,1}\right) + 10\left(H^u_{0,1,1,1,1,1} + H^u_{1,1,1,1,1,1}\right) \\
& - 2\zeta_2\left[H^u_{0,1,0,1} + H^u_{1,1,0,1} - 4\left(H^u_{0,1,1,1} + H^u_{1,1,1,1}\right)\right] + 8\zeta_4\left(H^u_{0,1} + H^u_{1,1}\right) \\
& - \frac{117}{4}\zeta_6 + (\zeta_3)^2 . \tag{7.9}
\end{aligned}$$

For reference, we also give

$$Y(u, u, 1) = 2\left(H^u_{0,1} + H^u_{1,1}\right) , \tag{7.10}$$

and the remainder function in the same notation is,

$$R_6^{(2)}(u, u, 1) = H^u_{0,0,0,1} + H^u_{1,0,0,1} - H^u_{0,0,1,1} - H^u_{1,0,1,1} - \frac{5}{2}\zeta_4 , \tag{7.11}$$

$$\begin{aligned}
R_6^{(3)}(u, u, 1) = {} & -\frac{1}{2}\Big[H^u_{0,0,1,0,0,1} + H^u_{1,0,1,0,0,1} + H^u_{0,1,0,1,0,1} + H^u_{1,1,0,1,0,1} + 2\left(H^u_{0,0,1,1,0,1} + H^u_{1,0,1,1,0,1}\right) \\
& + 3\left(H^u_{0,0,1,0,1,1} + H^u_{1,0,1,0,1,1} + H^u_{0,1,0,0,0,1} + H^u_{1,1,0,0,0,1} + H^u_{0,1,0,1,1,1} + H^u_{1,1,0,1,1,1} \right. \\
& - H^u_{0,0,0,1,0,1} - H^u_{1,0,0,1,0,1} - H^u_{0,1,0,0,1,1} - H^u_{1,1,0,0,1,1}\left.\right) + 6\left(H^u_{0,0,0,0,0,1} + H^u_{1,0,0,0,0,1}\right) \\
& + 9\left(H^u_{0,0,0,1,1,1} + H^u_{0,0,1,1,1,1}\right) - 10\left(H^u_{0,0,0,0,1,1} + H^u_{1,0,0,0,1,1}\right)\Big] \\
& - \zeta_2\left[H^u_{0,0,0,1} + H^u_{1,0,0,1} - 3\left(H^u_{0,0,1,1} + H^u_{1,0,1,1}\right) - 2\left(H^u_{0,1,0,1} + H^u_{1,1,0,1}\right)\right] \\
& - 2\zeta_4\left(H^u_{0,1} + H^u_{1,1}\right) + \frac{413}{24}\zeta_6 + (\zeta_3)^2 . \tag{7.12}
\end{aligned}$$

Although $U^{(2)}$ is slightly lengthier than $R_6^{(2)}$ in this representation, $U^{(3)}$ is considerably shorter than $R_6^{(3)}$.

Note that in the formulas for both U and R_6, the HPL weight vectors always end in 1. This restriction simply guarantees that there are no branch cuts developing at $u = 1$. Also, for both U and R_6, there is a pairing of terms of the form $H_{0,\vec{m}} + H_{1,\vec{m}}$, where $\vec{m}$ is a sequence of 0's and 1's. This pairing is a consequence of the final-entry condition, as discussed in ref. [29] for R_6. It also holds for U on the line $(u, u, 1)$, basically because the extra entry uw/v reduces to 1 on this line.

On the other hand, the function U exhibits two patterns not found in R_6. First of all, the second weight-vector entry m_2 in $H_{m_1 m_2 \ldots m_n}$ is always 1 for U. Second of all, U has a symmetry on reversing the order of $m_3 \ldots m_{n-1}$; the coefficients of the two HPLs with the

weight-vectors swapped in this way are always equal. It will be interesting to see if these patterns are accidents of the first three orders, or hold up in further orders in perturbation theory; and if the latter, what they signify.

Next we turn to the line $(u, 1, u)$. On this line, R_6, which is cyclically symmetric, is still given by eqs. (7.11) and (7.12), but the formulas for U are different:

$$U^{(1)}(u, 1, u) = -2\, H^u_{1,1} - \zeta_2\,, \tag{7.13}$$

$$U^{(2)}(u, 1, u) = \frac{1}{2}\, H^u_{0,1,0,1} - \frac{3}{2}\, H^u_{0,1,1,1} + H^u_{1,0,0,1} - H^u_{1,0,1,1} - \frac{1}{2}\, H^u_{1,1,0,1} - \frac{9}{2}\, H^u_{1,1,1,1}$$
$$- \zeta_2\,(H^u_{0,1} - 3\, H^u_{1,1}) + \frac{21}{4}\, \zeta_4\,, \tag{7.14}$$

$$U^{(3)}(u, 1, u) = H^u_{0,1,0,1,0,1} - H^u_{0,1,0,1,1,1} - H^u_{0,1,1,1,0,1} - H^u_{0,1,1,0,1,1} - 20\, H^u_{0,1,1,1,1,1} - 2\,(H^u_{1,0,0,0,1,1}$$
$$+ H^u_{1,0,0,1,0,1} + H^u_{1,0,1,0,0,1} + H^u_{1,0,1,0,1,1} + H^u_{1,1,0,0,0,1} + H^u_{1,1,0,1,0,1} + H^u_{1,1,1,1,0,1})$$
$$- 3\,(H^u_{1,0,0,0,0,1} - H^u_{1,0,0,1,1,1} - H^u_{1,1,1,0,0,1}) - 4\,(H^u_{1,0,1,1,0,1} + H^u_{1,1,1,0,1,1})$$
$$+ 5\, H^u_{1,1,0,0,1,1} - 9\, H^u_{1,1,0,1,1,1} - 16\, H^u_{1,0,1,1,1,1} - 55\, H^u_{1,1,1,1,1,1}$$
$$- 2\, \zeta_2\,(H^u_{0,1,0,1} - H^u_{0,1,1,1} - 3\, H^u_{1,0,1,1} - 3\, H^u_{1,1,0,1} - 8\, H^u_{1,1,1,1})$$
$$+ \zeta_4\,(8\, H^u_{0,1} - 21\, H^u_{1,1}) - \frac{117}{4}\, \zeta_6 + (\zeta_3)^2\,. \tag{7.15}$$

On this line, U is not as simple as it is on $(u, u, 1)$.

In figure 2 and figure 3 we plot $V^{(1)}$, $V^{(2)}$ and $V^{(3)}$ on the lines $(u, u, 1)$ and $(u, 1, u)$, respectively. We normalize the functions by dividing by their values at $(1, 1, 1)$. Note that for small u, the functions' values on $(u, 1, u)$ are approximately the negative of those on $(u, u, 1)$. This approximate relation becomes exact if we drop power-suppressed terms as $u \to 0$. Then we find,

$$V^{(1)}(u, u, 1) \sim \frac{1}{2}\, \ln^2 u\,, \tag{7.16}$$

$$V^{(2)}(u, u, 1) \sim \frac{1}{16}\, \ln^4 u - \frac{3}{4}\, \zeta_2\, \ln^2 u + \zeta_3\, \ln u + \frac{5}{8}\, \zeta_4\,, \tag{7.17}$$

$$V^{(3)}(u, u, 1) \sim \frac{1}{288}\, \ln^6 u - \frac{5}{24}\, \zeta_2\, \ln^4 u + \frac{71}{16}\, \zeta_4\, \ln^2 u - 2\,(2\, \zeta_5 + \zeta_2\, \zeta_3)\, \ln u$$
$$- \frac{77}{16}\, \zeta_6 + \frac{1}{2}\, (\zeta_3)^2\,, \tag{7.18}$$

$$V^{(1)}(u, 1, u) \sim -V^{(1)}(u, u, 1)\,, \tag{7.19}$$

$$V^{(2)}(u, 1, u) \sim -V^{(2)}(u, u, 1)\,, \tag{7.20}$$

$$V^{(3)}(u, 1, u) \sim -V^{(3)}(u, u, 1)\,. \tag{7.21}$$

The small value of the coefficient of $\ln^6 u$ in $V^{(3)}(u, u, 1)$, relative to that of $\ln^4 u$, causes the blue curve in figure 2 to oscillate as $u \to 0$: it reaches a maximum around $u = 0.0005$ and then goes negative for even smaller u.

7.2 The lines $(u, 1, 1)$ and $(1, v, 1)$

With two of the cross ratios equal to unity, the hexagon functions also collapse to pure HPLs. On these lines, Δ is nonzero, and so the function $\tilde{V}(u, 1, 1)$ is nonvanishing. The
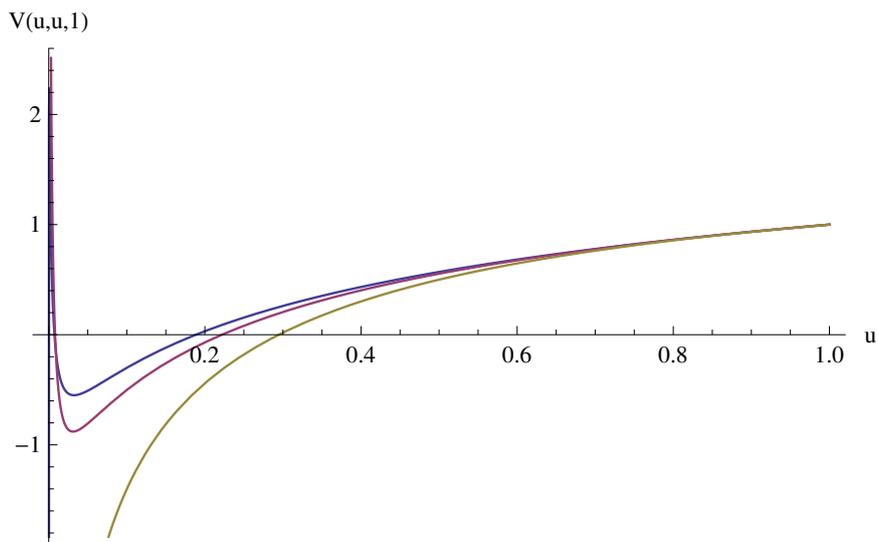
Figure 2. $V^{(1)}(u,u,1)$, $V^{(2)}(u,u,1)$ and $V^{(3)}(u,u,1)$, normalized to unity at $(1,1,1)$. One loop is in green, two loops is in purple, and three loops is in blue.
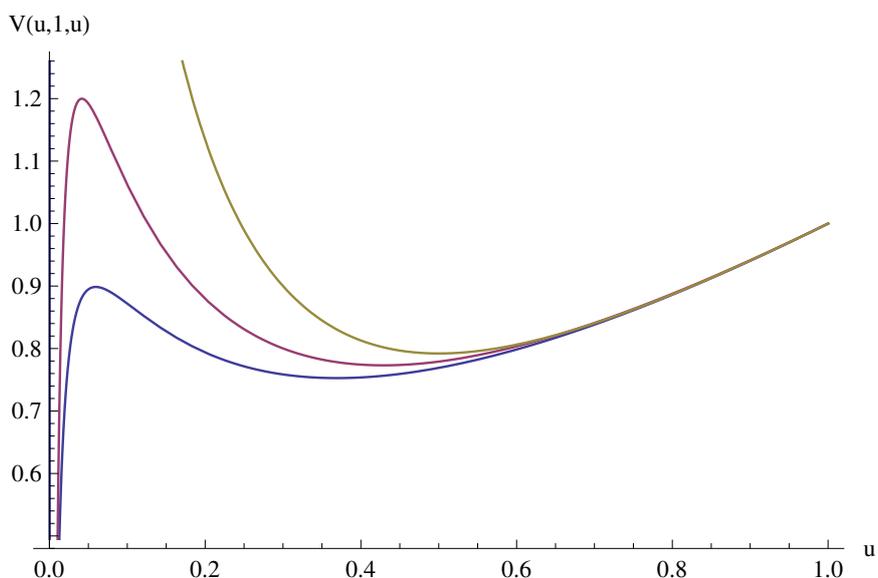
Figure 3. $V^{(1)}(u,1,u)$, $V^{(2)}(u,1,u)$ and $V^{(3)}(u,1,u)$, normalized to unity at $(1,1,1)$. One loop is in green, two loops is in purple, and three loops is in blue.

function $\tilde{V}(1,v,1)$ vanishes, though, due to $u \leftrightarrow w$ antisymmetry. Again we give U rather than V, and use the linearized HPL representation.

On $(u,1,1)$, the function $\tilde{V}$ becomes,

$$\tilde{V}^{(2)}(u,1,1) = \frac{1}{4}\left(H^u_{1,0,0,1} + H^u_{1,0,1,1} + H^u_{1,1,0,1}\right) - \frac{1}{2}\zeta_2 H^u_{1,1}, \tag{7.22}$$

$$\tilde{V}^{(3)}(u,1,1) = \frac{1}{8}\Big[H^u_{0,1,0,1,0,1} - H^u_{1,0,1,1,0,1} - H^u_{1,1,0,1,0,1} + 2\left(H^u_{0,1,0,0,1,1} + H^u_{0,1,1,0,0,1} + H^u_{1,0,1,0,1,1}\right.$$
$$\left. + H^u_{1,1,0,0,1,1} - H^u_{1,0,0,1,0,1}\right) + 3\left(H^u_{0,1,0,1,1,1} + H^u_{0,1,1,1,0,1} + H^u_{1,0,0,1,1,1} + H^u_{1,1,0,1,1,1}\right.$$

$$+ H^u_{1,1,1,0,1,1} - H^u_{1,1,1,0,0,1}) + 4\,(H^u_{0,1,1,0,1,1} - H^u_{1,0,1,0,0,1})$$
$$-\ 6\,(H^u_{1,0,0,0,0,1} + H^u_{1,1,0,0,0,1})\Big]$$
$$-\ \frac{1}{4}\,\zeta_2\left[3\,(H^u_{0,1,1,1} + H^u_{1,0,1,1}) + H^u_{1,0,0,1} + 2\,H^u_{1,1,0,1}\right] + \frac{21}{4}\,\zeta_4\,H^u_{1,1}\,. \tag{7.23}$$

The function U becomes,

$$U^{(1)}(u,1,1) = -\frac{1}{2}\,H^u_{1,1} - \zeta_2\,, \tag{7.24}$$

$$U^{(2)}(u,1,1) = \frac{1}{4}\,(H^u_{0,1,0,1} + H^u_{1,0,0,1} + H^u_{1,0,1,1} + H^u_{1,1,0,1}) - \frac{1}{2}\,\zeta_2\,(H^u_{0,1} - H^u_{1,1}) + \frac{21}{4}\,\zeta_4\,, \tag{7.25}$$

$$U^{(3)}(u,1,1) = -\frac{1}{8}\Big[H^u_{1,0,1,0,1,1} - 2\,(H^u_{0,1,0,1,1,1} + H^u_{0,1,1,1,0,1} - H^u_{1,0,1,1,0,1} - H^u_{1,1,0,0,1,1})$$
$$+\ 3\,(H^u_{1,0,0,1,1,1} + H^u_{1,1,0,1,0,1} + H^u_{1,1,1,0,0,1} - H^u_{1,1,0,1,1,1} - H^u_{1,1,1,0,1,1})$$
$$-\ 4\,(H^u_{0,1,0,1,0,1} + H^u_{0,1,1,0,1,1}) + 5\,(H^u_{1,0,0,1,0,1} + H^u_{1,0,1,0,0,1})$$
$$+\ 6\,(H^u_{1,0,0,0,0,1} + H^u_{1,0,0,0,1,1} + H^u_{1,1,0,0,0,1})\Big]$$
$$-\ \frac{1}{2}\,\zeta_2\,(2\,H^u_{0,1,0,1} + H^u_{0,1,1,1} + H^u_{1,0,1,1}) + 4\,\zeta_4\,(H^u_{0,1} - H^u_{1,1})$$
$$-\ \frac{117}{4}\,\zeta_6 + (\zeta_3)^2\,, \tag{7.26}$$

and

$$U^{(1)}(1,u,1) = -\frac{1}{2}\,H^u_{1,1} - \zeta_2\,, \tag{7.27}$$

$$U^{(2)}(1,u,1) = \frac{1}{4}\,H^u_{0,1,0,1} - \frac{1}{2}\,\zeta_2\,(H^u_{0,1} - 2\,H^u_{1,1}) + \frac{21}{4}\,\zeta_4\,, \tag{7.28}$$

$$U^{(3)}(1,u,1) = \frac{1}{2}\,(H^u_{0,1,0,1,0,1} + H^u_{0,1,1,0,1,1}) + \frac{1}{4}\,(H^u_{0,1,0,1,1,1} + H^u_{0,1,1,1,0,1} + H^u_{1,0,1,0,1,1} + H^u_{1,0,1,1,0,1})$$
$$-\ \frac{1}{2}\,\zeta_2\,(2\,H^u_{0,1,0,1} + H^u_{0,1,1,1} + H^u_{1,0,1,1}) + 4\,\zeta_4\,(H^u_{0,1} - 2\,H^u_{1,1})$$
$$-\ \frac{117}{4}\,\zeta_6 + (\zeta_3)^2\,. \tag{7.29}$$

We see that U is simpler on the line $(1,u,1)$ than on the line $(u,1,1)$.

A combination that seems exceptionally simple, at least through three loops, is the difference between U on the line $(u,1,1)$ and on the line $(1,u,1)$. Defining

$$\Delta U \equiv U(u,1,1) - U(1,u,1)\,, \tag{7.30}$$

we find

$$\Delta U^{(1)} = 0\,, \tag{7.31}$$

$$\Delta U^{(2)} = \frac{1}{4}\,(H^u_{1,0,0,1} + H^u_{1,0,1,1} + H^u_{1,1,0,1}) - \frac{1}{2}\,\zeta_2\,H^u_{1,1}\,, \tag{7.32}$$

$$\Delta U^{(3)} = -\frac{1}{8}\Big[2\,H^u_{1,1,0,0,1,1} + 3\,(H^u_{1,0,0,1,1,1} + H^u_{1,1,1,0,0,1} + H^u_{1,0,1,0,1,1} + H^u_{1,1,0,1,0,1}$$
$$-\ H^u_{1,1,0,1,1,1} - H^u_{1,1,1,0,1,1}) + 4\,H^u_{1,0,1,1,0,1} + 5\,(H^u_{1,0,0,1,0,1} + H^u_{1,0,1,0,0,1})$$
$$+\ 6\,(H^u_{1,0,0,0,0,1} + H^u_{1,0,0,0,1,1} + H^u_{1,1,0,0,0,1})\Big] + 4\,\zeta_4\,H^u_{1,1}\,. \tag{7.33}$$

We observe a similar pattern to that for $U(u,u,1)$, with the role of the second weight vector entry in $U(u,u,1)$ played by the first weight vector entry in ΔU. In other words, the first

entry as well as the last entry in ΔU is always 1. Also, ΔU is a *palindrome*: reversing the ordering of the letters (weight vector entries) leaves it invariant.

In figures 4, 5, and 6 we plot the functions V and $\tilde{V}$ through three loops. The even functions are normalized so that they are all equal to one at $(1,1,1)$. The parity-odd functions vanish at $(1,1,1)$, so we can't normalize them there. However, $\tilde{V}^{(2)}(u,1,1)$ and $\tilde{V}^{(3)}(u,1,1)$ are both proportional to $\ln^2 u$ as u goes to zero. Hence we instead normalize by the coefficient of that divergence: $-\frac{1}{8}\zeta_2$ for $\tilde{V}^{(2)}$ and $\frac{47}{32}\zeta_4$ for $\tilde{V}^{(3)}$. Remarkably, with this choice of normalization, the odd functions are almost indistinguishable on the line $(u,1,1)$.

The small u behavior of the parity-even V functions on the lines $(u,1,1)$ and $(1,u,1)$ is milder than that on $(u,u,1)$ and $(u,1,u)$, having at most $\ln^2 u$ behavior:

$$V^{(1)}(u,1,1) \sim -\frac{1}{2}\,\zeta_2\,, \tag{7.34}$$

$$V^{(2)}(u,1,1) \sim \frac{1}{8}\,\zeta_2\,\ln^2 u - \frac{1}{2}\,\zeta_3\,\ln u + \frac{31}{8}\,\zeta_4\,, \tag{7.35}$$

$$V^{(3)}(u,1,1) \sim -\frac{27}{16}\,\zeta_4\,\ln^2 u + \frac{1}{4}\,(11\,\zeta_5 + 5\,\zeta_2\,\zeta_3)\,\ln u - \frac{97}{4}\,\zeta_6\,, \tag{7.36}$$

$$V^{(1)}(1,u,1) \sim -\frac{1}{2}\,\zeta_2\,, \tag{7.37}$$

$$V^{(2)}(1,u,1) \sim \frac{1}{4}\,\zeta_2\,\ln^2 u - \frac{1}{2}\,\zeta_3\,\ln u + \frac{67}{16}\,\zeta_4\,, \tag{7.38}$$

$$V^{(3)}(1,u,1) \sim -\frac{101}{32}\,\zeta_4\,\ln^2 u + \frac{1}{4}\,(11\,\zeta_5 + 5\,\zeta_2\,\zeta_3)\,\ln u - \frac{3447}{128}\,\zeta_6 + \frac{1}{4}\,(\zeta_3)^2\,. \tag{7.39}$$

As mentioned above, the small u behavior of the parity-odd V functions on the line $(u,1,1)$ is of the same order,

$$\tilde{V}^{(2)}(u,1,1) \sim -\frac{1}{8}\,\zeta_2\,\ln^2 u - \frac{5}{16}\,\zeta_4\,, \tag{7.40}$$

$$\tilde{V}^{(3)}(u,1,1) \sim \frac{47}{32}\,\zeta_4\,\ln^2 u + \frac{343}{128}\,\zeta_6 - \frac{1}{4}\,(\zeta_3)^2\,. \tag{7.41}$$

The ratio of the $\ln^2 u$ coefficient for $\tilde{V}^{(3)}$ to that for $\tilde{V}^{(2)}$ is about -7.7, while for the constant term it is about -7. This numerical similarity accounts for some of the indistinguishability of the two curves in figure 6, but only at small u.

7.3 The line (u,u,u)

When all three cross ratios are equal, the parity-odd function vanishes by $u \leftrightarrow w$ symmetry, $\tilde{V}(u,u,u) = 0$. In contrast to the behavior on the previous lines, on the line (u,u,u) the ratio function V does not collapse to ordinary HPLs. We can still study its asymptotic behavior analytically, and we can evaluate it numerically in order to inspect how zero crossings of the ratio function change with loop order. (For the remainder function, these zero crossings, at each loop order and at strong coupling, are all very close to $u = \frac{1}{3}$ [29].)

Like all hexagon functions, the ratio function on the line (u,u,u) can be expressed [29] in terms of the *cyclotomic* HPLs defined in ref. [89]. At the level of the symbol, this correspondence is easy to see because on the line (u,u,u) we have $u = y/(1+y)^2$ and $1 - u = (1 + y + y^2)/(1+y)^2$, where $y \equiv y_u$. Therefore the symbol entries are all drawn
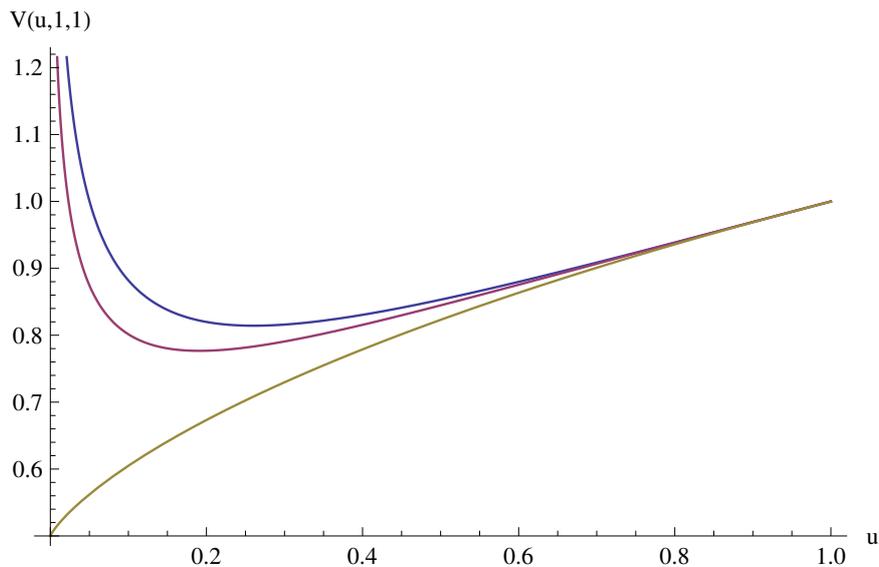
Figure 4. $V^{(1)}(u,1,1)$, $V^{(2)}(u,1,1)$ and $V^{(3)}(u,1,1)$, normalized to unity at $(1,1,1)$. One loop is in green, two loops is in purple, and three loops is in blue.
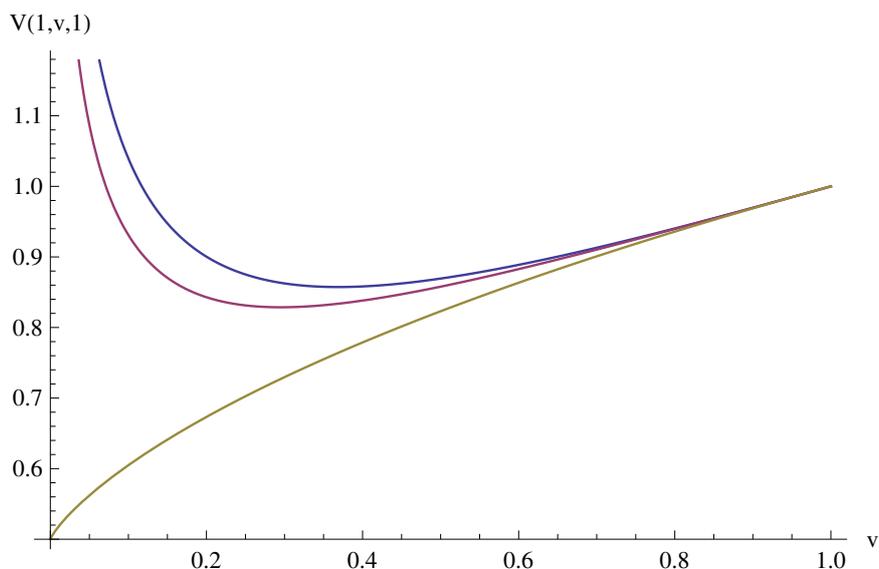
Figure 5. $V^{(1)}(1,v,1)$, $V^{(2)}(1,v,1)$ and $V^{(3)}(1,v,1)$, normalized to unity at $(1,1,1)$. One loop is in green, two loops is in purple, and three loops is in blue.

from the set $\{y,\ 1+y,\ 1+y+y^2\}$. The latter two elements of this set are the second and third cyclotomic polynomials in y, with roots $e^{i\pi}$ and $e^{\pm 2\pi i/3}$.

Here we do not make explicit use of the cyclotomic polylogarithm correspondence. In order to obtain a numerical representation, we simply series expand to high orders (of order 100 terms) about $u = 0, 1, \infty$. Such series representations have overlapping domains of convergence. In figure 7 we plot $V(u,u,u)$ for each loop order, normalized to unity at $(1,1,1)$. The point at which $V(u,u,u)$ crosses the zero line, in the neighborhood of $u = \frac{1}{3}$,
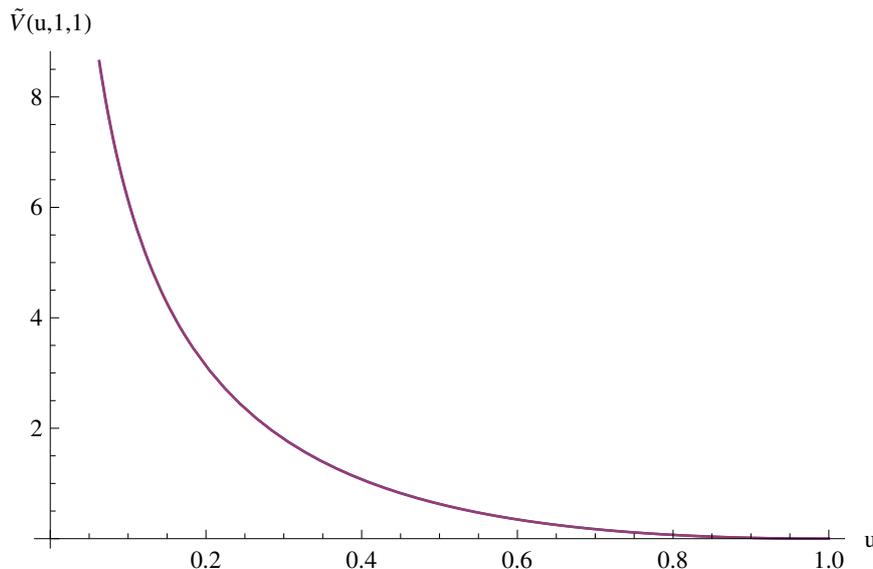
Figure 6. $\tilde{V}^{(2)}(u,1,1)$ and $\tilde{V}^{(3)}(u,1,1)$, normalized so their $u \to 0$ limit is $\ln^2 u$ with unit coefficient. Two loops is in purple and three loops is in blue. At this scale, the lines are indistinguishable.

decreases gradually with increasing loop order. We define the crossing values $u_0^{(L)}$ by the condition $V^{(L)}(u_0^{(L)}, u_0^{(L)}, u_0^{(L)}) = 0$. They are given by

$$u_0^{(1)} = 0.372098\ldots, \qquad u_0^{(2)} = 0.352838\ldots, \qquad u_0^{(3)} = 0.347814\ldots. \tag{7.42}$$

As in the case of the line $(u,u,1)$, there are oscillations and additional zero crossings at higher loop order. The two-loop result has a zero crossing near 0.0015. The three-loop function crosses near 0.007 and again near 1.3×10^{-6}.

These zero crossings are again dictated by the small u asymptotic behavior,

$$V^{(1)}(u,u,u) \sim \frac{1}{2}\ln^2 u + \frac{1}{2}\zeta_2\,, \tag{7.43}$$

$$V^{(2)}(u,u,u) \sim \frac{1}{16}\ln^4 u - \frac{3}{2}\zeta_2\ln^2 u + \frac{1}{2}\zeta_3\ln u - \frac{53}{16}\zeta_4\,, \tag{7.44}$$

$$V^{(3)}(u,u,u) \sim \frac{1}{288}\ln^6 u - \frac{41}{96}\zeta_2\ln^4 u + \frac{1}{8}\zeta_3\ln^3 u + \frac{419}{32}\zeta_4\ln^2 u - \left(2\zeta_5 + \frac{3}{4}\zeta_2\zeta_3\right)\ln u$$
$$+ \frac{2589}{128}\zeta_6 - \frac{1}{4}(\zeta_3)^2\,. \tag{7.45}$$

The leading-log behavior at each order has exactly the same coefficients as did the small-u expansion on the line $(u,u,1)$. The subleading-log coefficients are different, however.

7.4 The plane $u + v + w = 1$

The plane $u+v+w = 1$ intersects the positive octant in an equilateral triangle. This triangle is bounded by the lines corresponding to the collinear limits: $v = 0$, $u + w = 1$, and cyclic permutations of this line. The remainder function $R_6^{(3)}$ comes very close to vanishing on
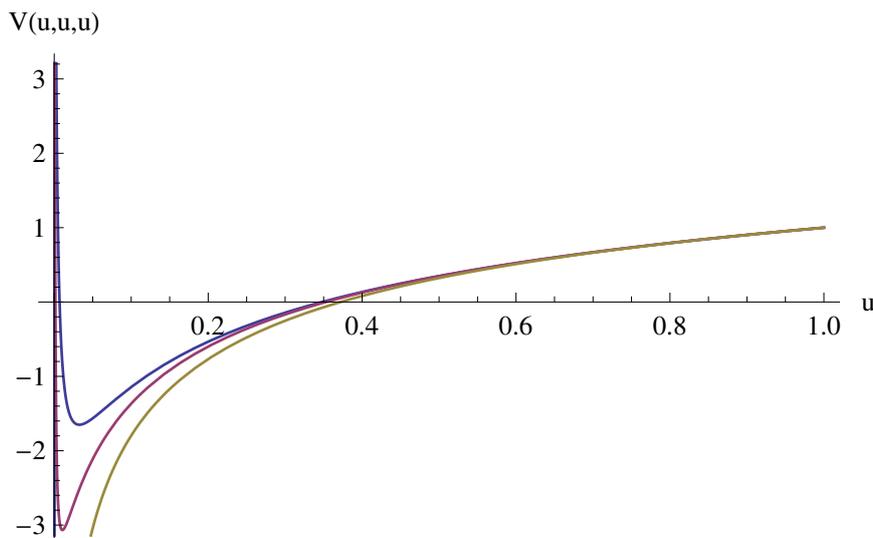
Figure 7. $V^{(1)}(u, u, u)$, $V^{(2)}(u, u, u)$ and $V^{(3)}(u, u, u)$, normalized to unity at $(1, 1, 1)$. One loop is in green, two loops is in purple, and three loops is in blue.

this equilateral triangle [28], which may not be so surprising, given that the remainder function is identically zero on all three edges of the triangle. In contrast, the collinear limits of the ratio function involve two different permutations of V. For this reason, the behavior of the ratio function on the plane $u + v + w = 1$ can be much less uniform than is the remainder function. Both V and $\tilde{V}$ show an interesting range of behavior, and their zero-crossing surfaces slice through this plane.

Figure 8 plots $V^{(3)}(u, v, w)$ on the equilateral triangle. In this region, the function reaches its highest values near the triangle's vertices at $u = 1$ and $w = 1$, and its lowest values near the vertex at $v = 1$. The function crosses zero on a curve in between; the vanishing curve is not far from the circle of radius $\frac{1}{2}$ centered at $(0, 1, 0)$. From eq. (3.3), we can see that $V^{(L)}(v, w, u)$ diverges like $\ln^L T$ in the collinear limit with $w \approx T^2 \to 0$. In fact, all permutations of $V^{(L)}$ diverge like $\ln^L T$ in the near-collinear limit. Therefore $V^{(3)}$ actually becomes infinite on each edge of the equilateral triangle in figure 8. However, it diverges extremely slowly, so that the divergence is not apparent at all in the plot.

What is visible in the plot is the symmetry of $V^{(3)}$ under $u \leftrightarrow w$, which exchanges the lower-left and top corners of the triangle. It is also clear from the plot that on the lower edge of the triangle $V^{(3)}$ is odd under reflection about the edge's midpoint. By symmetry, the same reflection-odd property holds along the upper-right edge of the triangle. Using the $u \leftrightarrow w$ symmetry of $V(u, v, w)$, this property is just a consequence of the collinear vanishing constraint (2.26),

$$V(u, v, w) + V(v, u, w) \ \to \ 0, \qquad \text{as } w \to 0, \ v \to 1 - u. \tag{7.46}$$

So the fact that the vanishing surface intersects two of the edges of the equilateral triangle at their midpoints is no surprise.
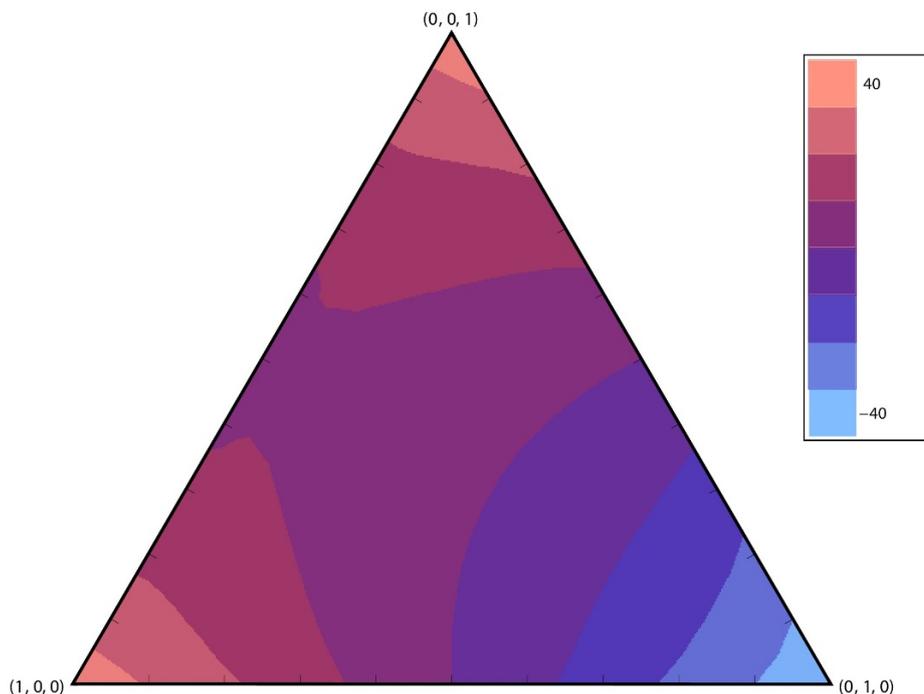
Figure 8. $V^{(3)}(u, v, w)$ evaluated on the plane where $u + v + w = 1$. The corners are labeled with their (u, v, w) values.

The parity-odd function $\tilde{V}^{(3)}(u, v, w)$ is plotted on the same equilateral triangle in figure 9. Parity-odd pure functions are pure imaginary when $\Delta < 0$, as in this region, so we divide $\tilde{V}^{(3)}$ by i before plotting it. This function is antisymmetric under the exchange $u \leftrightarrow w$ and therefore it vanishes on the line $u = w$. It is positive for large w and negative for large u. Like any parity-odd function, $\tilde{V}^{(3)}(u, v, w)$ vanishes in the collinear limits, on the edges of the triangle. However, this vanishing happens so slowly that it is not evident in the plot.

The sign of $\tilde{V}^{(3)}/i$ depends on the value of (y_u, y_v, y_w), not just (u, v, w). For each point (u, v, w), there are two points (y_u, y_v, y_w), related by flipping the sign of $\sqrt{\Delta}$ in eq. (2.13). This sign flip inverts the three y_i. On the plane $u + v + w = 1$, $\sqrt{\Delta}$ is imaginary, and the y_i are pure phases, satisfying $y_u y_v y_w = -1$. The sign flip conjugates the three phases. The branch of $\tilde{V}^{(3)}/i$ plotted in figure 9 is for positive $\sqrt{\Delta}/i$, corresponding to negative imaginary parts for all three y_i.

7.5 Planes in v

In order to get a complete view of the ratio function's behavior, it is useful to plot it as a function of u and w for successive values of v.

In figure 10, we plot $V^{(3)}(u, v, w)$ on the planes $v = \frac{3}{4}$, $v = \frac{1}{2}$, and $v = \frac{1}{4}$. A similar plot has been made for the remainder function $R_6^{(3)}$, as figure 8 of ref. [28]. (The roles of v and
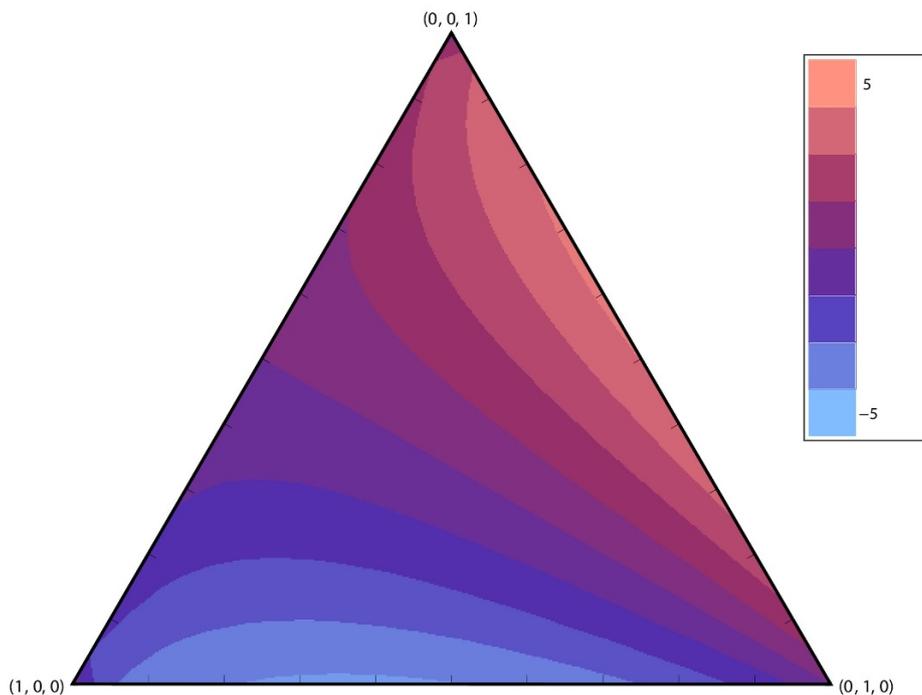
Figure 9. $\tilde{V}^{(3)}(u, v, w)/i$ evaluated on the plane where $u + v + w = 1$. The corners are labeled with their (u, v, w) values.

w are reversed in that plot, but of course that is irrelevant for the remainder function, since it is totally symmetric.) Much as in the case of the remainder function $R_6^{(3)}$, the function $V^{(3)}(u, v, w)$ looks monotonic in v, but actually the $v = \frac{1}{2}$ and $v = \frac{1}{4}$ planes intersect near $u = w = 1$.

The functions $V^{(3)}(u, v, w)$ and $V^{(2)}(u, v, w)$ cross zero on different surfaces. The difference in zero-crossing locations means that plotting the ratio of $V^{(3)}(u, v, w)$ to $V^{(2)}(u, v, w)$ is relatively uninformative. Instead, in figure 11 we plot $V^{(2)}(u, v, w)$ on the same planes in v for comparison.

In contrast, $\tilde{V}^{(3)}(u, v, w)$ and $\tilde{V}^{(2)}(u, v, w)$ are constrained to cross zero at the exact same place, on the plane $u = w$. Parity-odd functions are either real or imaginary based on the sign of $\Delta(u, v, w)$, and they vanish on the surface $\Delta = 0$. For these reasons, it is simpler to plot the ratio between two odd functions than to plot one odd function alone. Omitting points for which $u = w$, and for which Δ vanishes, we plot the ratio $\tilde{V}^{(3)}(u, v, w)/\tilde{V}^{(2)}(u, v, w)$ in figure 12. Given the vanishings of both numerator and denominator within the region of the plot, it is remarkable that the ratio $\tilde{V}^{(3)}/\tilde{V}^{(2)}$ stays within a fairly limited range and has no dramatic behavior. On the other hand, it is clear that it is not totally constant in u, v, or w.
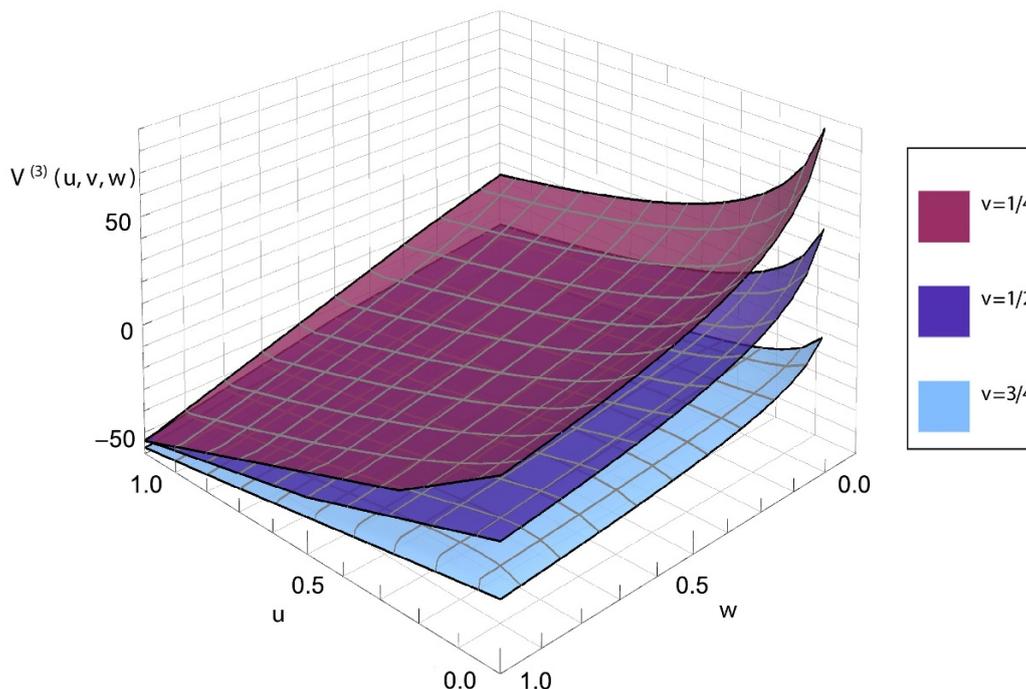
Figure 10. $V^{(3)}(u, v, w)$ evaluated on successive planes in v.

8 Relation between V and coproduct elements of R_6

In section 6, we observed that the function U had surprising simplicity, possessing only five independent $\{n-1, 1\}$ coproduct components. In this section, we will describe another interesting empirical observation. This one relates the even-parity NMHV function at L loops, $V^{(L)}$, to the elements of the $\{2L, 1, 1\}$ coproduct component of the remainder function at one higher loop, $R_6^{(L+1)}$. The relation was originally noticed while inspecting the behavior of both functions on the diagonal line (u, u, u), for the purpose of making the plot in section 7.3. However, it can be extended to a relation that holds throughout the (u, v, w) space.

We denote the $\{2L, 1, 1\}$ component of the coproduct of $R_6^{(L+1)}$ by $\Delta_{2L,1,1}(R_6^{(L+1)})$. Its elements can be represented as,

$$\Delta_{2L,1,1}(R_6^{(L+1)}) = \sum_{s_i, s_j \in \mathcal{S}_u} [R_6^{(L+1)}]^{s_i, s_j} \otimes \ln s_i \otimes \ln s_j \,, \tag{8.1}$$

and they each have weight $2L$. We found the following relation:

$$V^{(L)}(u, v, w) = [R_6^{(L+1)}]^{Z,Z} + E^{(L+1)} + \frac{1}{8} \gamma_K^{(L+1)} \,, \tag{8.2}$$
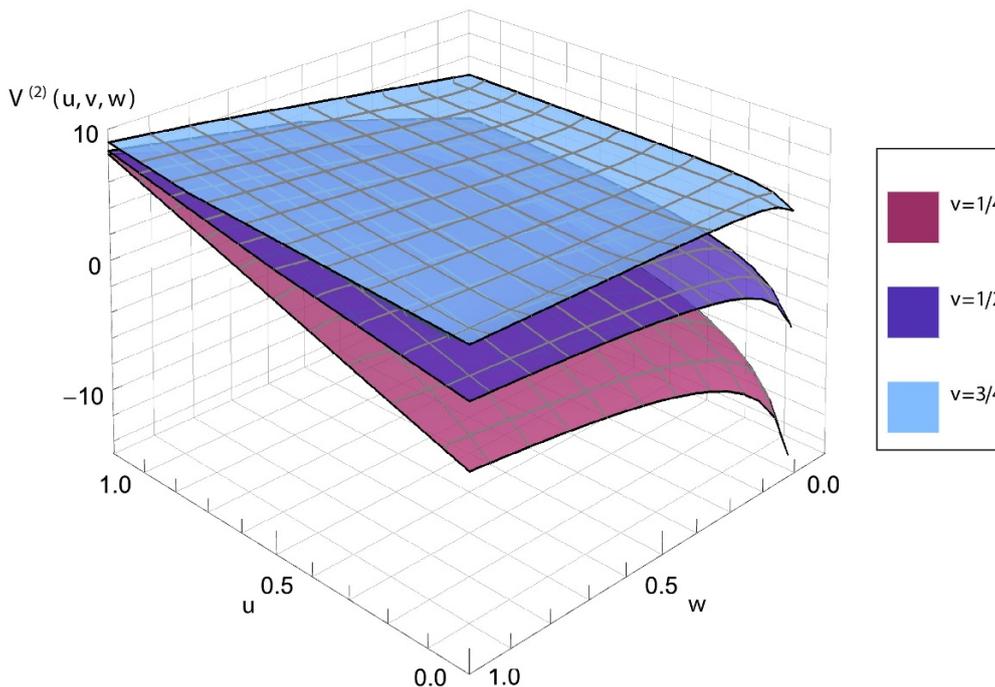
Figure 11. $V^{(2)}(u, v, w)$ evaluated on successive planes in v.

where we define the "Z, Z" linear combination of coproduct elements of a hexagon function X to be,

$$X^{Z,Z} \equiv -X^{v,v} - X^{1-v,v} + X^{y_u,y_u} + X^{y_w,y_w} - 3\, X^{y_v,y_v} + 2\left(X^{y_w,y_v} + X^{y_u,y_v}\right) - X^{y_u,y_w} - X^{y_w,y_u}\,.$$
(8.3)

Recall that the cusp anomalous dimension is given through four loops by,

$$\gamma_K(a) = \sum_{L=1}^{\infty} a^L \gamma_K^{(L)} = 4a - 4\zeta_2\, a^2 + 22\zeta_4\, a^3 - 4\left(\frac{219}{8}\zeta_6 + (\zeta_3)^2\right)a^4 + \mathcal{O}(a^5)\,. \qquad (8.4)$$

The extra term E is only needed so far at zero loops (which is definitely a special case), and at four loops, where it is proportional to the square of the P-odd $D = 6$ hexagon integral:

$$E^{(1)} = \frac{1}{2}\,, \qquad E^{(2)} = E^{(3)} = 0\,, \qquad E^{(4)} = \frac{1}{16}\,[\tilde{\Phi}_6]^2\,. \qquad (8.5)$$

For $L = 0$, the relation (8.2) is simply $1 = 0 + \frac{1}{2} + \frac{1}{2}$.

For $L = 1$, it is straightforward to compute the $\{2, 1, 1\}$ coproduct component of $R_6^{(2)}$ from the form given in ref. [27] in terms of classical polylogarithms and the function $\Omega^{(2)}$, whose $\{3, 1\}$ coproduct component is given in ref. [28]. We find,

$$-[R_6^{(2)}]^{v,v} - [R_6^{(2)}]^{1-v,v} = [R_6^{(2)}]^{y_v,y_v} + \frac{\zeta_2}{2} = [R_6^{(2)}]^{y_v,y_u} + \frac{\zeta_2}{2} = \frac{1}{4}\Big[H_2^u + H_2^v + H_2^w + \ln u\, \ln w\Big]\,.$$
(8.6)
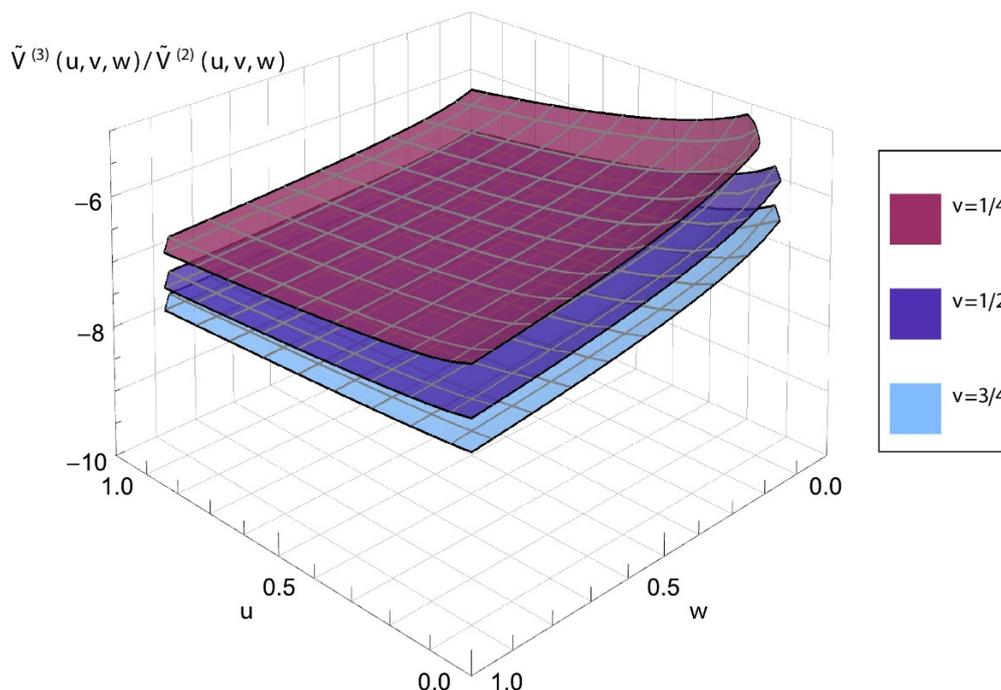
Figure 12. The ratio $\tilde{V}^{(3)}(u,v,w)/\tilde{V}^{(2)}(u,v,w)$ evaluated on successive planes in v.

Because $R_6^{(2)}(u,v,w)$ is totally symmetric in u,v,w, we can obtain all the other coproduct elements entering $[R_6^{(2)}]^{Z,Z}$ by permuting the ones given in eq. (8.6). It is then easy to check that the right-hand side of eq. (8.2) for $L = 1$ adds up to yield $V^{(1)}$ as given in eq. (2.18).

To check the relation (8.2) for $L = 2$, we use the formulae for the elements $R_6^{(3),u}$ and $R_6^{(3),y_u}$ of the $\{5,1\}$ coproduct component of $R_6^{(3)}$ given in ref. [28], as well as the final-entry relation $R_6^{(3),1-u} = -R_6^{(3),u}$. These formulae are given in terms of the basis of weight-five hexagon functions, whose $\{4,1\}$ coproducts are also tabulated in ref. [28]. This information makes it straightforward to extract the $\{4,1,1\}$ coproduct component of $R_6^{(3)}$ from the $\{5,1\}$ coproduct component:

$$[R_6^{(3)}]^{v,v} = \frac{1}{32}\Big[3\left(\Omega^{(2)}(u,v,w) + \Omega^{(2)}(v,w,u)\right) + 2\,\Omega^{(2)}(w,u,v) + \text{HPLs} + 16\,\zeta_4\Big], \quad (8.7)$$

$$[R_6^{(3)}]^{1-v,v} = \frac{1}{32}\Big[\Omega^{(2)}(u,v,w) + \Omega^{(2)}(v,w,u) + 2\,\Omega^{(2)}(w,u,v) + \text{HPLs} + 24\,\zeta_4\Big], \quad (8.8)$$

$$[R_6^{(3)}]^{y_v,y_v} = -\frac{1}{32}\Big[9\left(\Omega^{(2)}(u,v,w) + \Omega^{(2)}(v,w,u)\right) + 6\,\Omega^{(2)}(w,u,v) + \text{HPLs} - 24\,\zeta_4\Big], \quad (8.9)$$

$$[R_6^{(3)}]^{y_v,y_u} = -\frac{1}{32}\Big[8\,\Omega^{(2)}(u,v,w) + 7\left(\Omega^{(2)}(v,w,u) + \Omega^{(2)}(w,u,v)\right) + \text{HPLs} - 36\,\zeta_4\Big]. \quad (8.10)$$

The portions of the expressions containing harmonic polylogarithms are fairly lengthy, so we do not present them here. Using these results and their various permutations, we can assemble the right-hand side of eq. (8.2) and verify that it agrees with $V^{(2)}$ as given in eq. (2.20).

For $L = 3$, we first check the first derivative of eq. (8.2) with respect to u, v and w. We do this using the fact that the derivatives of each of the elements of the $\{6, 1, 1\}$ coproduct component of $R_6^{(4)}$ can be expressed in terms of the $\{5, 1, 1, 1\}$ coproduct elements, using the general eq. (5.26). The $\{5, 1, 1, 1\}$ coproduct elements of $R_6^{(4)}$ can in turn be written in terms of the weight-five basis of hexagon functions [29]. On the left-hand side of eq. (8.2), we compute the derivative of $V^{(3)}(u, v, w)$ using the $\{5, 1\}$ coproduct component for $U^{(3)}$ presented in appendix A.1, together with the relation between $V^{(3)}$ and $U^{(3)}$ given in eq. (7.2) or eq. (5.18). Expanding both sides of the derivative of eq. (8.2) in terms of the basis of weight-five functions, they agree perfectly. Having checked the first derivative, we should check the relation at one point in order to establish that the constant of integration is also correct. It is convenient to choose the point $(u, v, w) = (1, 1, 1)$. At this point, we have

$$[R_6^{(4)}]^{v,v}(1, 1, 1) = \frac{73}{8}\,\zeta_6 - \frac{1}{2}(\zeta_3)^2\,, \tag{8.11}$$

$$[R_6^{(4)}]^{1-v,v}(1, 1, 1) = 0\,, \tag{8.12}$$

$$[R_6^{(4)}]^{y_v,y_v}(1, 1, 1) = -\frac{607}{16}\,\zeta_6\,, \tag{8.13}$$

$$[R_6^{(4)}]^{y_v,y_u}(1, 1, 1) = -\frac{607}{16}\,\zeta_6\,. \tag{8.14}$$

Using these relations, plus eq. (7.4) for $V^{(3)}(1, 1, 1)$ and eq. (8.4) for the four-loop cusp anomalous dimension, as well as the fact that $\tilde{\Phi}_6(1, 1, 1) = 0$, it is easy to verify that eq. (8.2) holds for $L = 3$ at $(u, v, w) = (1, 1, 1)$.

There are a number of linear relations among the $\{2L, 1, 1\}$ coproduct elements, which follow from integrability, i.e. from the consistency of mixed partial derivatives of the original weight-$2(L + 1)$ function $R_6^{(L+1)}$. These integrability relations make it possible to rewrite eq. (8.2) in various ways. In eq. (8.2), we used these relations to eliminate the "off-diagonal" even-even coproduct elements, $[R_6^{(L+1)}]^{v,u}$, $[R_6^{(L+1)}]^{1-v,u}$, and permutations thereof. It is possible that using the integrability relations in a different way might lead to a version of eq. (8.2) that is more revealing of its origin. No matter how it is rewritten, though, the appearance of the cusp anomalous dimension in an equation that holds throughout the full (u, v, w) space of cross ratios is very interesting.

The appearance of the extra $[\tilde{\Phi}_6]^2$ term at four loops is presumably related to the fact that we are using a logarithmic definition of the remainder function, eq. (5.13). The ratio function is not defined by taking any logarithms. Let's define a modified remainder function $\bar{R}_6$ by

$$\frac{A^{\text{MHV}}}{A^{\text{BDS}}} = \bar{R}_6 = \exp(R_6)\,. \tag{8.15}$$

Then $\bar{R}_6$ and V are on the same footing. The zero-loop value of $\bar{R}_6$ differs from that of R_6: $\bar{R}_6^{(0)} = 1$, while $R_6^{(0)} = 0$. (This shift does not affect eq. (8.2), of course.) Otherwise, R_6 and $\bar{R}_6$ are identical until four loops, at which point they are related by,

$$\bar{R}_6^{(4)} = [\exp(R_6)]^{(4)} = R_6^{(4)} + \frac{1}{2}\Big[R_6^{(2)}\Big]^2\,. \tag{8.16}$$

We can rewrite eq. (8.2) in terms of coproducts of $\bar{R}_6$ instead of R_6. In order to do that, according to eq. (8.16) we need to compute the Z, Z coproduct of $[R_6^{(2)}]^2$. We find that

$$\left\{\left[R_6^{(2)}\right]^2\right\}^{Z,Z} = \frac{1}{8}[\tilde{\Phi}_6]^2 + 2\,[R_6^{(2)}]^{Z,Z}\,R_6^{(2)} \tag{8.17}$$

$$= \frac{1}{8}[\tilde{\Phi}_6]^2 + 2\left(V^{(1)} - \frac{1}{8}\gamma_K^{(2)}\right)R_6^{(2)}. \tag{8.18}$$

This relation implies that we can rewrite eq. (8.2) in terms of $\bar{R}_6$ as,

$$V^{(L)}(u,v,w) = [\bar{R}_6^{(L+1)}]^{Z,Z} + \bar{E}^{(L+1)} + \frac{1}{8}\gamma_K^{(L+1)}, \tag{8.19}$$

where

$$\bar{E}^{(1)} = \frac{1}{2}, \qquad \bar{E}^{(2)} = \bar{E}^{(3)} = 0, \qquad \bar{E}^{(4)} = -[\bar{R}_6^{(2)}]^{Z,Z}\,\bar{R}_6^{(2)}. \tag{8.20}$$

Next, we further "improve" eq. (8.19) by removing the $\bar{E}$ term. We do this by considering not V but $V\bar{R}_6 = V\exp(R_6)$, much as we did in section 5 when studying the multi-particle factorization properties. As discussed in section 5, $V\bar{R}_6$ is a pure NMHV quantity, as the finite part of the MHV amplitude has been cleared out of the denominator. First, let's multiply eq. (8.19) by a^{L+1} and sum over L to obtain,

$$a\left(V - \frac{1}{2}\right) - \frac{1}{8}\gamma_K = \bar{R}_6^{Z,Z} + \left(\bar{E} - \frac{a}{2}\right), \tag{8.21}$$

where $\bar{E} - a/2$ vanishes until four loops. Now multiply by $\bar{R}_6$:

$$\left[a\left(V - \frac{1}{2}\right) - \frac{1}{8}\gamma_K\right]\bar{R}_6 = \bar{R}_6^{Z,Z}\,\bar{R}_6 + \left(\bar{E} - \frac{a}{2}\right)\bar{R}_6. \tag{8.22}$$

Through four loops, using eq. (8.20), we have,

$$\bar{R}_6^{Z,Z}\,\bar{R}_6 = \bar{R}_6^{Z,Z} + a^4[\bar{R}_6^{(2)}]^{Z,Z}\,\bar{R}_6^{(2)} + \mathcal{O}(a^5), \tag{8.23}$$

$$\left(\bar{E} - \frac{a}{2}\right)\bar{R}_6 = -a^4[\bar{R}_6^{(2)}]^{Z,Z}\,\bar{R}_6^{(2)} + \mathcal{O}(a^5), \tag{8.24}$$

so the explicit a^4 terms cancel in the sum.

We are left with,

$$\left[a\left(V - \frac{1}{2}\right) - \frac{1}{8}\gamma_K\right]\bar{R}_6 = \bar{R}_6^{Z,Z}, \tag{8.25}$$

which is valid at least through order a^4. Except for the factor of $1/2$, this equation is a relation for the difference between the NMHV and MHV amplitudes, $V\bar{R}_6 - \bar{R}_6$. The right-hand side looks naively like a second-order differential operator, but of course the coproduct operation is not the same as taking a derivative. Nevertheless, it might be useful to try to prove eq. (8.25) using the $\bar{Q}$ differential equation found in the super-Wilson loop approach [65, 66].

Given this interesting relation for the P-even function V, we investigated whether it was possible to write the P-odd part of the ratio function, $\tilde{V}$, as a linear combination of the

P-odd $\{6, 1, 1\}$ coproduct elements of R_6 at one higher loop. At one loop, or weight two, both sides of such a relation vanish trivially, because there are no P-odd weight-2 hexagon functions. For $\tilde{V}^{(2)}$, we found multiple solutions; however, the space of P-odd weight-4 hexagon functions is quite small, so most of the solutions are presumably accidental. Because $\tilde{V}$ is not itself totally physical, it is better to consider the difference of two cyclic permutations, $\tilde{V}(u, v, w) - \tilde{V}(w, u, v)$. Note that this combination is symmetric under the exchange $u \leftrightarrow v$.

At three loops, we tried to write $\tilde{V}^{(3)}$ as a generic linear combination of the odd elements of the $\{6, 1, 1\}$ coproduct component of $R_6^{(4)}$, imposing $u \leftrightarrow v$ symmetry and taking into account the integrability relations among the elements. We could not find a solution for $\tilde{V}^{(3)}$ without also introducing the odd coproducts of $R_6^{(3)}$, multiplied by a single logarithm. Allowing for this, we found:

$$\tilde{V}^{(3)}(u, v, w) - \tilde{V}^{(3)}(w, u, v) = 4\Big[2\, R_6^{(4)\, y_w, w} + R_6^{(4)\, 1-w, y_w} - R_6^{(4)\, y_u, w} - R_6^{(4)\, y_v, w} \Big]$$
$$- 2\, H_1^w \Big[R_6^{(3)\, y_u} + R_6^{(3)\, y_v} - R_6^{(3)\, y_w} \Big]$$
$$+ \tilde{\Phi}_6 \left[\frac{1}{4} H_1^w\, \Omega^{(1)} - \frac{1}{2}\Big(H_3^w - H_{2,1}^w + H_1^w\, H_2^w \Big) \right], \quad (8.26)$$

where $\tilde{\Phi}_6 = -4\, R_6^{(2)\, y_w}$. Again integrability relations allow one to rewrite this linear combination in different ways.

This solution is nice in that it descends smoothly to one loop lower, in which the final term is absent:

$$\tilde{V}^{(2)}(u, v, w) - \tilde{V}^{(2)}(w, u, v) = 4\Big[2\, R_6^{(3)\, y_w, w} + R_6^{(3)\, 1-w, y_w} - R_6^{(3)\, y_u, w} - R_6^{(3)\, y_v, w} \Big]$$
$$- 2\, H_1^w \Big[R_6^{(2)\, y_u} + R_6^{(2)\, y_v} - R_6^{(2)\, y_w} \Big]. \quad (8.27)$$

However, the structure of this relation does not yet seem as simple as the one for the parity-even part V.

At the moment, the ultimate significance of these relations is still quite unclear. It would be interesting to investigate their meaning in the near-collinear and multi-Regge limits, where the OPE approach of Basso, Sever and Vieira, and the recent work of Basso, Caron-Huot and Sever, respectively, provide information at much higher loop order,

Recently, BSV have investigated a double-scaling limit in which $T \to 0$ but TF is held fixed. In this limit, only gluonic flux-tube excitations contribute [44]. This limit corresponds to taking $v \to 0$ with u and w held fixed. In this limit, the letters of the symbols for hexagon functions (after extracting powers of $\ln v$) can be shown to collapse to a simple five parameter set, $\{u, w, 1-u, 1-w, 1-u-w\}$. This means that hexagon functions approach a subset of the 2dHPL function space introduced by Gehrmann and Remiddi [90] in order to solve for the master integrals for the process $\gamma^* \to qg\bar{q}$ at two loops. (The 2dHPLs also allow for the letter $(u + w)$, which does not appear here.) BSV have a simple rule for an insertion factor $h_a(u)$ (where u is the rapidity) that relates NMHV to MHV Wilson loops. At leading order, the insertion factor leads to a relation for the 1111 component of the NMHV Wilson loop in terms of a second-order Laplacian operator

acting on the MHV Wilson loop. This relation looks superficially similar to eq. (8.25), although the NMHV side of the relation involves two permutations each of V and $\tilde{V}$, and it is clear that the relation will have to be modified at higher loop orders.

In general, there might be other, cleaner ways to rewrite the parity-even and parity-odd coproduct relations found in this section, which might better reveal their origin. We shall leave such investigations for future work.

9 Conclusions and outlook

In this paper we successfully extended the bootstrap program, initiated in ref. [26], to calculate the three-loop six-point NMHV ratio function in planar $\mathcal{N} = 4$ super Yang-Mills theory. We began with an ansatz for the coproduct of the desired functions, built out of the hexagon functions introduced in ref. [28]. By constraining this ansatz with the known behavior of the ratio function in various kinematic limits, we were able to uniquely determine the NMHV coefficient functions V and $\tilde{V}$ through three loops.

At three loops, we began with a 412-parameter ansatz. After applying several constraints, including the vanishing of the cyclicly symmetric part of $\tilde{V}^{(3)}$, a final-entry condition drawn from the $\bar{Q}$ differential equation from ref. [65], the vanishing of spurious poles, and vanishing in the collinear limit, we had 92 parameters remaining.

We were then able to fix those parameters with near-collinear data obtained from the work of Basso, Sever and Vieira [41]. The first-order T^1 correction in the near-collinear limit, from single flux excitations [42], was sufficient to fix all but two parameters in our ansatz. Those two parameters can be fixed if we also incorporate BSV's recently published results for the contributions of two flux excitations, at order T^2 [43]. The rest of the order T^2 results then serves as an extensive check on BSV's results.

Alternatively, the remaining two parameters can be fixed by examining the multi-Regge limits of the amplitude. By generalizing the predictions of ref. [54] beyond the leading-logarithmic limit, we were able to fix the form of $V^{(3)}$ and $\tilde{V}^{(3)}$ independently of BSV's T^2 data, letting the T^2 comparison serve as an entirely independent check. Using the NLLA and NNLLA functions derived from the two- and three-loop NMHV ratio functions we are able to find all contributions to the MRK limit at any loop order, up to NNLLA in the imaginary part and N^{3}LLA in the real part. These results will serve as important input for the calculation of the ratio function at higher loops.

With access to an NMHV amplitude at this loop order, we are uniquely positioned to investigate multi-particle factorization behavior at three loops. In constructing the multi-particle factorization function we find remarkable simplicity. We conjecture that our results should be straightforwardly generalizable beyond six points.

In investigating multi-particle factorization, we found remarkable relations in the co-product entries of the ratio function, relations that go beyond those predicted by Caron-Huot and He [65, 80]. While we do not yet understand the source of these relations, if they continue to higher loops they might serve as useful constraints on further bootstraps. Similarly, the simplicity of the function U along certain kinematic lines (and in particular the status of ΔU as a palindrome) suggest deeper properties.

By plotting V and $\tilde{V}$ on a variety of lines and planes, we have observed how its quantitative behavior changes with loop order. While the overall behavior is not nearly as consistent between loop orders as it was for the remainder function in ref. [28], we do find that the ratio between three loops and two loops at least stays in a confined range over much of the space, being particularly tightly constrained for $\tilde{V}$. Time will tell if this behavior becomes more regular at higher loop orders.

In general, the success of the hexagon function program for the four-loop remainder function [29] indicates that the same program should be viable for the four-loop NMHV ratio function. Deriving the NMHV ratio function at four loops would allow us to confirm the trends observed at three loops, with an eye towards understanding their origins.

More generally, we have conjectured that the relative constant ratios of successive loop orders for the remainder function and the ratio function (in suitable regimes) are a byproduct of the convergence of perturbation theory in the planar $\mathcal{N} = 4$ theory. This possibility, discussed in ref. [29], could be investigated in more detail using BSV's approach to the OPE. Since the quantities they calculate are fully non-perturbative, it may be possible to look at their behavior at higher orders and thereby gain an understanding of why quantities like the remainder function and $\tilde{V}$ have such clean inter-loop ratios even at comparatively low loop order. Such an understanding could lead to a merging of the two approaches, with the goal of understanding amplitudes in planar $\mathcal{N} = 4$ super Yang-Mills for any value of the coupling and any kinematics.

A Coproduct elements of U and $\tilde{V}$

Because of the coproduct relations for U and $\tilde{V}$, and their (anti)symmetry under $u \leftrightarrow w$, only four independent $\{n-1, 1\}$ coproduct elements need to be specified in each case. We take these four components to be u, v, y_u and y_v. (In the case of the even function U, we should also specify the constant of integration by giving the value of the function at a particular point, say $(u, v, w) = (1, 1, 1)$, which we do elsewhere in this article.)

A.1 U

For the function U, the other $\{n-1, 1\}$ coproduct elements are given in terms of U^u, U^v, U^{y_u} and U^{y_v} as follows:

$$U^w(u,v,w) = U^u(w,v,u)\,, \tag{A.1}$$

$$U^{1-u}(u,v,w) = -U^u(u,v,w) - U^v(u,v,w)\,, \tag{A.2}$$

$$U^{1-v}(u,v,w) = 0\,, \tag{A.3}$$

$$U^{1-w}(u,v,w) = U^{1-u}(w,v,u)\,, \tag{A.4}$$

$$U^{y_w}(u,v,w) = U^{y_u}(u,v,w)\,. \tag{A.5}$$

In the rest of this subsection, we give the four independent coproduct elements for U through three loops.

The one-loop independent coproduct elements are trivial, given eq. (5.24) for $U^{(1)}$:

$$[U^{(1)}]^u = -\frac{1}{2}\ln(uw/v)\,, \tag{A.6}$$

$$[U^{(1)}]^v = \frac{1}{2}\ln(uw/v)\,, \tag{A.7}$$

$$[U^{(1)}]^{y_u} = 0\,, \tag{A.8}$$

$$[U^{(1)}]^{y_v} = 0\,. \tag{A.9}$$

The two-loop independent coproduct elements can be computed from eq. (5.25) for $U^{(2)}$, but we list them here for convenience:

$$
\begin{aligned}
[U^{(2)}]^u = \frac{1}{8}\Big[&-2\,H_3^u + 4\,H_{2,1}^u + 2\,H_{2,1}^v - 2\,H_3^w - 4\,H_{2,1}^w + (3\ln u + \ln(v/w))\,H_2^u \\
&+ \ln(uv/w)\,H_2^v - (3\ln(u/v) + \ln w)\,H_2^w - \ln^2 v\,\ln u \\
&- \ln^2 w\,(3\ln u - \ln v) + 3\ln u\,\ln v\,\ln w + 2\,\zeta_2\,(\ln u - 5\ln(v/w))\Big]\,,
\end{aligned} \tag{A.10}
$$

$$
\begin{aligned}
[U^{(2)}]^v = \frac{1}{4}\Big[&H_3^u + H_{2,1}^u + H_3^w + H_{2,1}^w - \ln(v/w)\left(H_2^u + \frac{1}{2}\ln^2 u\right) \\
&- \ln(v/u)\left(H_2^w + \frac{1}{2}\ln^2 w\right) - 4\,\zeta_2\,\ln(uw/v)\Big]\,,
\end{aligned} \tag{A.11}
$$

$$[U^{(2)}]^{y_u} = \frac{1}{8}\,\tilde{\Phi}_6(u,v,w)\,, \tag{A.12}$$

$$[U^{(2)}]^{y_v} = 0\,. \tag{A.13}$$

The independent parity-even $\{5, 1\}$ coproduct elements of $U^{(3)}$ are

$$
\begin{aligned}
[U^{(3)}]^u = \frac{1}{32}\Big\{ &-M_1(w,u,v) + M_1(u,w,v) - \frac{128}{3}\,(Q_{\mathrm{ep}}(v,w,u) - Q_{\mathrm{ep}}(v,u,w)) \\
&- \ln(u/w)\,(\Omega^{(2)}(u,v,w) + \Omega^{(2)}(v,w,u)) - (3\ln u - 4\ln v + 5\ln w)\,\Omega^{(2)}(w,u,v) \\
&+ 24\,H_5^u - 4\,H_{4,1}^u + 10\,H_{3,2}^u + 96\,H_{3,1,1}^u + 22\,H_{2,2,1}^u - 72\,H_{2,1,1,1}^u - 2\,H_2^u\,(3\,H_3^u + 5\,H_{2,1}^u) \\
&- \frac{3}{2}\ln u\left(24\,H_4^u - 20\,H_{3,1}^u + 28\,H_{2,1,1}^u - (H_2^u)^2\right) + 4\ln^2 u\,(4\,H_3^u - 3\,H_{2,1}^u) - 2\ln^3 u\,H_2^u \\
&- 96\,H_{3,1,1}^v - 32\,H_{2,2,1}^v - 16\,H_{2,1,1,1}^v + 16\,H_2^v\,H_{2,1}^v - 4\ln v\left(4\,H_{3,1}^v + 2\,H_{2,1,1}^v - (H_2^v)^2\right)
\end{aligned}
$$

$$+ \frac{2}{3} \ln^3 v \, H_2^v + 24 \, H_5^w - 28 \, H_{4,1}^w - 10 \, H_{3,2}^w + 240 \, H_{3,1,1}^w + 74 \, H_{2,2,1}^w + 8 \, H_{2,1,1,1}^w$$

$$+ 2 \, H_2^w \left(3 \, H_3^w - 19 \, H_{2,1}^w \right) - \frac{1}{2} \ln w \left(24 \, H_4^w - 68 \, H_{3,1}^w - 20 \, H_{2,1,1}^w + 19 \, (H_2^w)^2 \right) + 4 \ln^2 w \, H_{2,1}^w$$

$$+ \frac{2}{3} \ln^3 w \, H_2^w - 4 \left(H_2^u - H_2^w + 2 \left(\ln^2 u + \ln^2 w \right) \right) H_{2,1}^v - \frac{1}{2} \ln(u/w) \Big(4 \, H_4^v + 40 \, H_{3,1}^v + 4 \, H_{2,1,1}^v$$

$$- 11 \, (H_2^v)^2 \Big) - \frac{1}{2} \left(\ln^3 u - \ln^3 w \right) H_2^v + 2 \ln v \Big(6 \left(H_4^u - H_4^w \right) - 18 \, H_{3,1}^u - 14 \, H_{3,1}^w - 2 \, H_{2,1,1}^u$$

$$- 14 \, H_{2,1,1}^w + 4 \left((H_2^u)^2 + (H_2^w)^2 \right) - \left(H_2^u - H_2^w \right) H_2^v - \ln u \left(H_{2,1}^u - H_3^v + 3 \, H_{2,1}^v \right)$$

$$- \ln^2 u \left(H_2^u + 3 \, H_2^v \right) + \ln w \left(8 \, H_3^w - 3 \, H_{2,1}^w - H_3^v - H_{2,1}^v \right) - \ln^2 w \left(H_2^u + H_2^v \right) \Big)$$

$$- \frac{1}{4} \ln^2 v \Big(24 \, H_3^u + 2 \ln u \left(3 \, H_2^u + 4 \, H_2^v \right) - \ln^3 u + 8 \, H_3^w - 64 \, H_{2,1}^w - 2 \ln w \left(3 \, H_2^w - 4 \, H_2^v \right)$$

$$+ \frac{19}{3} \ln^3 w \Big) - \frac{2}{3} \ln^3 v (H_2^u - H_2^w - 2 \ln^2 u) - \frac{10}{3} (H_2^w H_3^u - H_2^u H_3^w) + \frac{14}{3} (H_2^w H_{2,1}^u - H_2^u H_{2,1}^w)$$

$$+ \frac{1}{6} \ln u \Big(72 \, H_4^w + 156 \, H_{3,1}^w + 168 \, H_{2,1,1}^w - 45 \, (H_2^w)^2 + 20 \, H_2^u H_2^w - 12 \ln w \left(8 \, H_3^w - 3 \, H_{2,1}^w \right.$$

$$- H_{2,1}^u \Big) + \ln^2 w \left(12 \, H_2^w + 7 \, H_2^u \right) \Big) - \frac{1}{6} \ln w \Big(72 \, H_4^u - 228 \, H_{3,1}^u - 24 \, H_{2,1,1}^u + 51 \, (H_2^u)^2$$

$$+ 20 \, H_2^u H_2^w + \ln^2 u \left(7 \, H_2^w - 12 \, H_2^u \right) \Big) - \frac{1}{4} \ln^3 u \left(2 \, H_2^w - \ln^2 w \right) + \frac{1}{12} \ln^3 w \left(6 \, H_2^u - 67 \ln^2 u \right)$$

$$- \frac{1}{3} \ln^2 w \left(23 H_3^u - 13 H_{2,1}^u \right) - \frac{1}{3} \ln^2 u \left(H_3^w - 35 H_{2,1}^w \right) + \frac{1}{6} \ln u \ln w \left(144 H_{2,1}^v + 33 \ln(u/w) \, H_2^v \right)$$

$$+ \ln v \Big(2 \ln u \left(2 \, H_3^w - 15 \, H_{2,1}^w \right) + 2 \ln w \left(6 \, H_3^u - H_{2,1}^u \right) + 2 \ln u \ln w \left(H_2^u + 6 \, H_2^v - H_2^w \right)$$

$$+ 2 \left(\ln^2 u \, H_2^w - \ln^2 w \, H_2^u \right) - \ln^3 u \ln w + 10 \ln^2 u \ln^2 w + \frac{11}{3} \ln u \ln^3 w \Big)$$

$$- \frac{1}{4} \ln^2 v \Big(6 \left(\ln u \, H_2^w - \ln w \, H_2^u \right) + 25 \ln^2 u \ln w + 7 \ln u \ln^2 w \Big) + \frac{2}{3} \ln^3 v \ln u \ln w$$

$$- \zeta_2 \Big[H_3^u - 12 \, H_{2,1}^u - 3 \ln u \, H_2^u - 3 \ln^3 u + 32 \, H_{2,1}^v + 16 \ln v \, H_2^v + \frac{4}{3} \ln^3 v - H_3^w - 84 \, H_{2,1}^w$$

$$- 29 \ln w \, H_2^w + \frac{1}{3} \ln^3 w + 2 \left(14 \ln v - 15 \ln w \right) H_2^u - 4 \ln^2 u \left(\ln v - \ln w \right)$$

$$- \ln u \left(34 \, H_2^w + 44 \ln^2 w - 18 \, H_2^v + 20 \ln^2 v - 56 \ln v \ln w \right) + 12 \ln v \left(3 \, H_2^w + \ln^2 w \right)$$

$$- 2 \ln w \left(9 \, H_2^v + 2 \ln^2 v \right) \Big] - 2 \zeta_3 \Big[4 \left(H_2^u - H_2^w \right) + 3 \left(\ln^2 u - \ln^2 w \right) \Big]$$

$$- 2 \zeta_4 \Big[35 \ln u - 160 \ln v + 157 \ln w \Big] \bigg\} , \tag{A.14}$$

and

$$[U^{(3)}]^v = A^v(u,v,w) + A^v(w,v,u), \tag{A.15}$$

where

$$A^v(u,v,w) = -\frac{1}{8} \bigg\{ \left(\frac{1}{2} \ln v - \ln u \right) \Omega^{(2)}(w,u,v) + 6 \, H_5^u - 4 \, H_{4,1}^u + 66 \, H_{3,1,1}^u + 20 \, H_{2,2,1}^u$$

$$- 4 \, H_{2,1,1,1}^u - 10 \, H_2^u H_{2,1}^u - \ln u \left(6 \, H_4^u - 12 \, H_{3,1}^u + 2 \, H_{2,1,1}^u + 2 \, (H_2^u)^2 \right)$$

$$+ \ln^2 u \left(2 \, H_3^u - H_{2,1}^u \right) - \frac{1}{3} \ln^3 u \, H_2^u$$

$$- \ln(v/w) \left(8 \, H_{3,1}^u + 4 \, H_{2,1,1}^u - 2 \ln u \, H_3^u - 2 \, (H_2^u)^2 \right)$$

$$+\ln^2(v/w)\left(-H_3^u + 4\,H_{2,1}^u + \ln u\,H_2^u - \frac{1}{3}\,\ln^3 u\right)$$

$$+\zeta_2\left[20\,H_{2,1}^u + 8\,\ln u\,H_2^u + \frac{2}{3}\,\ln^3 u - 8\,\ln(v/w)\left(H_2^u + \frac{1}{2}\,\ln^2 u\right)\right]$$

$$-32\,\zeta_4\,(2\,\ln u - \ln v)\Bigg\}\,. \tag{A.16}$$

The parity-odd coproducts of $U^{(3)}$ are given by,

$$[U^{(3)}]^{y_u} = \frac{1}{32}\Bigg\{3\,H_1(u,v,w) + H_1(v,w,u) + H_1(w,u,v) - \frac{11}{4}\,J_1(u,v,w)$$

$$-\frac{1}{4}\big(J_1(v,w,u) + J_1(w,u,v)\big) + \tilde{\Phi}_6(u,v,w)\Big[\ln^2 u + \ln^2 w + \ln^2 v$$

$$+2\Big(\ln u\,\ln w - \ln(uw)\,\ln v\Big) - 22\zeta_2\Big]\Bigg\}\,, \tag{A.17}$$

$$[U^{(3)}]^{y_v} = \frac{1}{8}\,H_1(u,v,w)\,. \tag{A.18}$$

A.2 $\tilde{V}$

For the function $\tilde{V}$, the other $\{n-1,1\}$ coproduct elements are given in terms of $\tilde{V}^u$, $\tilde{V}^v$, $\tilde{V}^{y_u}$ and $\tilde{V}^{y_v}$ as follows:

$$\tilde{V}^w(u,v,w) = -\tilde{V}^u(w,v,u)\,, \tag{A.19}$$

$$\tilde{V}^{1-u}(u,v,w) = -\tilde{V}^u(u,v,w)\,, \tag{A.20}$$

$$\tilde{V}^{1-v}(u,v,w) = -\tilde{V}^v(u,v,w)\,, \tag{A.21}$$

$$\tilde{V}^{1-w}(u,v,w) = -\tilde{V}^{1-u}(w,v,u)\,, \tag{A.22}$$

$$\tilde{V}^{y_w}(u,v,w) = -\tilde{V}^{y_u}(w,v,u)\,. \tag{A.23}$$

In the rest of this subsection, we give the four independent coproduct elements for $\tilde{V}$ through three loops.

The one-loop function $\tilde{V}^{(1)}$ vanishes. The two-loop function $\tilde{V}^{(2)}$ is given in eq. (2.21). Its $\{3,1\}$ coproduct elements are,

$$[\tilde{V}^{(2)}]^u = \frac{1}{8}\,\tilde{\Phi}_6(u,v,w)\,, \tag{A.24}$$

$$[\tilde{V}^{(2)}]^v = 0\,, \tag{A.25}$$

$$[\tilde{V}^{(2)}]^{y_u} = \frac{1}{4}\left[H_3^u - H_{2,1}^v - H_3^w - \frac{1}{2}\,\ln(u/w)\left(H_2^u + H_2^v + H_2^w + \ln v\,\ln w\right)\right.$$

$$\left.-\frac{1}{2}\,\ln v\,(H_2^u + H_2^v - H_2^w) + \zeta_2\,\ln(uv/w)\right]\,, \tag{A.26}$$

$$[\tilde{V}^{(2)}]^{y_v} = \frac{1}{4}\left[H_3^u - H_{2,1}^u - \ln u\,H_2^u - H_3^w + H_{2,1}^w + \ln w\,H_2^w\right.$$

$$\left.-\ln(u/w)\left(H_2^v + \frac{1}{2}\,\ln u\,\ln w - 2\,\zeta_2\right)\right]\,. \tag{A.27}$$

The independent parity-odd $\{5,1\}$ coproduct elements of $\tilde{V}^{(3)}$ are given by,

$$[\tilde{V}^{(3)}]^u = \frac{1}{96}\left\{-H_1(u,v,w) + H_1(v,w,u) + 3\,H_1(w,u,v) - \frac{23}{4}\,J_1(u,v,w)\right.$$
$$-\frac{13}{4}\,J_1(v,w,u) - \frac{3}{4}\,J_1(w,u,v) - 6\ln u\,\Big(F_1(u,v,w) - F_1(w,u,v)\Big)$$
$$+ 3\,\tilde{\Phi}_6(u,v,w)\Big[3\ln^2 u + \ln^2 v + \ln^2 w$$
$$\left.+ 2\,(H_2^u + H_2^v + H_2^w - \ln u\,\ln w) - 26\,\zeta_2\Big]\right\}, \tag{A.28}$$

$$[\tilde{V}^{(3)}]^v = \frac{1}{96}\left\{2\,\Big(H_1(v,w,u) - H_1(w,u,v)\Big) + \frac{5}{2}\,\Big(J_1(v,w,u) - J_1(w,u,v)\Big)\right.$$
$$\left.- 6\ln v\Big(F_1(u,v,w) - F_1(w,u,v) - \ln(u/w)\,\tilde{\Phi}_6(u,v,w)\Big)\right\}. \tag{A.29}$$

The independent parity-even coproducts of $\tilde{V}^{(3)}$ are given by,

$$[\tilde{V}^{(3)}]^{y_u} = \frac{1}{96}\left\{M_1(w,v,u) - M_1(v,w,u) + 3\,(M_1(w,u,v) - M_1(u,w,v))\right.$$
$$-\frac{64}{3}\,\Big(2\,(Q_{\text{ep}}(w,v,u) - Q_{\text{ep}}(w,u,v)) + 7\,(Q_{\text{ep}}(v,w,u) - Q_{\text{ep}}(v,u,w))\Big)$$
$$+ (3\ln u + \ln v - 4\ln w)\,\Omega^{(2)}(u,v,w) - (3\ln u - \ln v - 2\ln w)\,(\Omega^{(2)}(w,u,v) - \Omega^{(2)}(v,w,u))$$
$$- 72\,H_5^u + 72\,H_{4,1}^u + 15\,H_{3,2}^u + 36\,H_{3,1,1}^u + 27\,H_{2,2,1}^u - 9\,H_2^u\,H_{2,1}^u + 3\ln u\,\Big(12\,H_4^u - 4\,H_{3,1}^u$$
$$+ 3\,H_{2,1,1}^u - (H_2^u)^2\Big) - 3\ln^2 u\,H_3^u - \frac{3}{2}\ln^3 u\,H_2^u - 12\,H_{4,1}^v - 10\,H_{3,2}^v + 168\,H_{3,1,1}^v + 58\,H_{2,2,1}^v$$
$$+ 56\,H_{2,1,1,1}^v + 6\,H_2^v\,(H_3^v - 7\,H_{2,1}^v) + \frac{1}{2}\ln v\,\Big(24\,H_4^v + 36\,H_{3,1}^v + 68\,H_{2,1,1}^v - 31\,(H_2^v)^2\Big)$$
$$- 8\ln^2 v\,(H_3^v - H_{2,1}^v) + \frac{2}{3}\ln^3 v\,H_2^v + 72\,H_5^w - 60\,H_{4,1}^w - 5\,H_{3,2}^w + 84\,H_{3,1,1}^w + 11\,H_{2,2,1}^w$$
$$- 8\,H_{2,1,1,1}^w - 3\,H_2^w\,(2\,H_3^w + 3\,H_{2,1}^w) - \frac{1}{2}\ln w\,\Big(96\,H_4^w - 84\,H_{3,1}^w + 38\,H_{2,1,1}^w - (H_2^w)^2\Big)$$
$$+ \ln^2 w\,(11\,H_3^w - 8\,H_{2,1}^w) - \frac{7}{6}\ln^3 w\,H_2^w - \frac{1}{4}\ln^3 u\,(5\ln^2 v - 2\,H_2^v)$$
$$+ \frac{1}{12}\ln^2 u\,\Big(3\ln^3 v + 2\ln v\,(35\,H_2^v + 24\,H_2^u) + 44\,H_3^v + 164\,H_{2,1}^v\Big)$$
$$+ \ln u\,\Big(\ln^2 v\,\Big(4\,H_2^v + \frac{1}{6}\,H_2^u\Big) + \ln v\,\Big(-18\,(H_3^u + H_3^v) + 16\,H_{2,1}^u + 12\,H_{2,1}^v\Big) + 18\,H_4^v$$
$$+ 32\,H_{3,1}^v + 10\,H_{2,1,1}^v - \frac{37}{2}\,(H_2^v)^2 - \frac{20}{3}\,H_2^v\,H_2^u\Big) + \frac{1}{6}\ln^3 v\,H_2^u + \frac{1}{3}\ln^2 v\,(13\,H_3^u + 7\,H_{2,1}^u)$$
$$+ \ln v\,\Big(14\,H_4^u + 16\,H_{3,1}^u + 22\,H_{2,1,1}^u - \frac{31}{2}\,(H_2^u)^2 - \frac{28}{3}\,H_2^v\,H_2^u\Big) + \frac{2}{3}\,H_2^u\,(5\,H_3^v - 7\,H_{2,1}^v)$$
$$+ \frac{26}{3}\,H_2^v\,(H_3^u + H_{2,1}^u) - \ln^3 u\,(\ln^2 w - H_2^w) + \frac{1}{12}\ln^2 u\,\Big(15\ln^3 w + 2\ln w\,(19\,H_2^w - 24\,H_2^u)$$
$$- 26\,H_3^w + 70\,H_{2,1}^w\Big) + \frac{1}{3}\ln u\,\Big(4\ln^2 w\,(H_2^u + 6\,H_2^w) + 6\ln w\,\Big(9\,H_3^w - 8\,H_{2,1}^w - 12\,(H_3^w - H_{2,1}^w)\Big)$$
$$+ 72\,H_4^w - 6\,H_{3,1}^w + 96\,H_{2,1,1}^w - \frac{69}{2}\,(H_2^w)^2 - 7\,H_2^u\,H_2^w\Big) - \frac{13}{6}\ln^3 w\,H_2^u + \frac{1}{6}\ln^2 w\,(H_3^u + 13\,H_{2,1}^u)$$
$$- \frac{1}{3}\ln w\,\Big(42\,H_4^u + 66\,(H_{3,1}^u + H_{2,1,1}^u) - 51\,(H_2^u)^2 - H_2^u\,H_2^w\Big) - \frac{1}{3}\,H_2^u\,(H_3^w + 49\,H_{2,1}^w)$$

$$+ \frac{1}{3} H_2^w \left(H_3^u + 37 H_{2,1}^u \right) + \frac{1}{12} \ln^3 v \left(7 \ln^2 w + 10 H_2^w \right) + \frac{1}{6} \ln^2 v \left(13 \ln^3 w + 6 \ln w \left(7 H_2^w - 4 H_2^v \right) \right.$$

$$\left. - 48 H_3^w + 60 H_{2,1}^w \right) - \frac{1}{2} \ln v \left(\ln^2 w \left(4 H_2^w - 25 H_2^v \right) + 12 \ln w \left(2 H_{2,1}^v - 3 H_3^v - 4 \left(H_3^w - H_{2,1}^w \right) \right) \right.$$

$$\left. + 48 H_4^w - 20 H_{3,1}^w + 64 H_{2,1,1}^w - 19 \left(H_2^w \right)^2 + 12 H_2^v H_2^w \right) - \frac{1}{3} \ln^3 w H_2^v + 2 \ln^2 w \left(H_3^v + 7 H_{2,1}^v \right)$$

$$+ \ln w \left(-18 H_4^v - 34 H_{3,1}^v - 10 H_{2,1,1}^v + 19 \left(H_2^v \right)^2 + 12 H_2^v H_2^w \right) - 24 H_2^w H_{2,1}^v - 12 H_2^v H_3^w$$

$$+ 3 \ln^3 u \ln v \ln w - \frac{1}{4} \ln^2 u \left(4 \ln^2 v \ln w + \ln v \left(17 \ln^2 w + 18 H_2^w \right) + 24 \ln w H_2^v \right)$$

$$+ \ln u \left(-\frac{1}{3} \ln^3 v \ln w - \frac{1}{4} \ln^2 v \left(3 \ln^2 w - 10 H_2^w \right) + \ln v \left(\frac{4}{3} \ln^3 w - 2 \ln w \left(8 H_2^w + 3 H_2^u \right) \right. \right.$$

$$\left. \left. + 13 H_2^v \right) + 4 H_3^w - 24 H_{2,1}^w \right) + \frac{13}{2} \ln^2 w H_2^v - 2 \ln w \left(2 H_3^v + 21 H_{2,1}^v \right) - 12 H_2^v H_2^w \right)$$

$$- \frac{3}{2} \ln w \left(4 \ln^2 v H_2^u - \ln v \left(9 H_2^u \ln w + 4 H_{2,1}^u \right) - 8 H_2^u H_2^v \right)$$

$$+ \zeta_2 \left[-\frac{57}{2} H_3^u + H_3^v + \frac{55}{2} H_3^w - 18 H_{2,1}^u + 92 H_{2,1}^v + 46 H_{2,1}^w \right.$$

$$+ \frac{1}{2} H_2^u \left(45 \ln u + 92 \ln v - 104 \ln w \right) + H_2^v \left(58 \ln u + 53 \ln v - 60 \ln w \right)$$

$$+ \frac{1}{2} H_2^w \left(76 \ln u - 12 \ln v - 7 \ln w \right) - \frac{3}{2} \ln^3 u - \frac{1}{3} \ln^3 v + \frac{35}{6} \ln^3 w$$

$$\left. - 8 \left(\ln^2 u \ln(v/w) + \ln^2 v \ln(u/w) \right) - 4 \ln^2 w \left(4 \ln u + 11 \ln v \right) + 72 \ln u \ln v \ln w \right]$$

$$- \zeta_3 \left[66 H_2^u - 8 H_2^v - 58 H_2^w + 27 \ln^2 u - 6 \ln^2 v - 21 \ln^2 w \right]$$

$$- 2 \zeta_4 \left[117 \ln u + 123 \ln v - 114 \ln w \right] \Bigg\}, \tag{A.30}$$

and

$$[\tilde{V}^{(3)}]^{y_v} = B^{y_v}(u, v, w) - B^{y_v}(w, v, u), \tag{A.31}$$

where

$$B^{y_v}(u, v, w)$$

$$= \frac{1}{96} \Bigg\{ -3 \left(M_1(u, v, w) + M_1(v, w, u) \right) + 2 M_1(w, u, v) + \frac{320}{3} \left(Q_{\mathrm{ep}}(u, v, w) + Q_{\mathrm{ep}}(w, u, v) \right)$$

$$- \left(\ln u - 6 \ln v + 5 \ln w \right) \Omega^{(2)}(u, v, w) + \ln u \, \Omega^{(2)}(w, u, v) - 72 H_5^u + 48 H_{4,1}^u - 5 H_{3,2}^u$$

$$+ 84 H_{3,1,1}^u + 47 H_{2,2,1}^u + 64 H_{2,1,1,1}^u + 3 H_2^u \left(4 H_3^u - 11 H_{2,1}^u \right) + \ln u \left(60 H_4^u - 24 H_{3,1}^u \right.$$

$$\left. + 53 H_{2,1,1}^u - 16 \left(H_2^u \right)^2 \right) - \ln^2 u \left(19 H_3^u - 16 H_{2,1}^u \right) + \frac{11}{6} \ln^3 u H_2^u - \ln^3 u \ln^2 v$$

$$+ \frac{1}{6} \ln^2 u \left(25 H_3^v + H_{2,1}^v - 7 \ln v H_2^v \right) + \frac{1}{3} H_2^u \left(25 H_3^v - 11 H_{2,1}^v \right)$$

$$+ 2 \ln u \left(14 H_4^v + 19 H_{3,1}^v + 22 H_{2,1,1}^v - \frac{65}{4} \left(H_2^v \right)^2 - \ln v \left(18 H_3^v - 16 H_{2,1}^v \right) + 4 \ln^2 v H_2^v \right)$$

$$- \frac{1}{4} \ln^3 v \left(\ln^2 u + 2 H_2^u \right) + \frac{1}{6} \ln^2 v \left(35 H_3^u + 47 H_{2,1}^u + 16 \ln u H_2^u \right)$$

$$+ 2 \ln v \left(-3 H_4^u + 17 H_{3,1}^u - 11 H_{2,1,1}^u - \frac{7}{2} \left(H_2^u \right)^2 + 3 \ln u \left(H_3^u - 2 H_{2,1}^u \right) - 2 \ln^2 u H_2^u \right)$$

$$+ \frac{1}{3} H_2^v \left(7 \ln^3 u + 11 H_3^u + 35 H_{2,1}^u - \left(29 \ln u + 13 \ln v \right) H_2^u \right)$$

$$
\begin{aligned}
&- \frac{1}{12} \ln^3 u \, (19 \ln^2 w - 14 \, H_2^w) - \frac{11}{2} \ln^2 u \, \ln w \, H_2^w \\
&+ 2 \ln u \left(-\frac{19}{4} \, (H_2^w)^2 + 3 \ln w \, (H_3^w - 2 \, H_{2,1}^w) + \ln^2 w \, H_2^w \right) \\
&+ 2 \ln w \, (3 \, H_4^u - 22 \, H_{3,1}^u + 11 \, H_{2,1,1}^u) + 2 \ln^2 w \, (5 \, H_3^u + 2 \, H_{2,1}^u) \\
&+ 6 \, H_2^w \, (2 \, H_3^u - 4 \, H_{2,1}^u - 3 \ln u \, H_2^u) + \frac{1}{4} \ln^2 v \, \ln u \, (6 \, H_2^w - 13 \ln^2 w) \\
&+ \frac{3}{2} \, H_2^v \, \ln w \, (8 \, H_2^u - 13 \ln^2 u) \\
&+ \frac{1}{3} \ln v \left(-5 \ln^3 u \, \ln w - 12 \ln^2 u \, H_2^w + 6 \ln u \, (4 \, H_3^w + 9 \, H_{2,1}^w + 5 \ln w \, H_2^w) \right) \\
&+ \zeta_2 \left[-\frac{53}{2} \, H_3^u + 46 \, H_{2,1}^u + H_2^u \left(\frac{113}{2} \ln u + 20 \ln v - 54 \ln w \right) + 98 \ln u \, H_2^v - \frac{37}{6} \ln^3 u \right. \\
&\qquad \left. + \ln^2 u \, (8 \ln v + 52 \ln w) - 16 \ln u \, \ln^2 v \right] \\
&- \zeta_3 \, (50 \, H_2^u + 15 \ln^2 u) - 474 \, \zeta_4 \, \ln u \Bigg\} .
\end{aligned}
\tag{A.32}
$$

References

[1] Z. Bern, L.J. Dixon, D.C. Dunbar and D.A. Kosower, *One loop n point gauge theory amplitudes, unitarity and collinear limits*, *Nucl. Phys.* **B 425** (1994) 217 [hep-ph/9403226] [INSPIRE].

[2] Z. Bern, J.S. Rozowsky and B. Yan, *Two loop four gluon amplitudes in $\mathcal{N} = 4$ super Yang-Mills*, *Phys. Lett.* **B 401** (1997) 273 [hep-ph/9702424] [INSPIRE].

[3] Z. Bern, L.J. Dixon and V.A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev.* **D 72** (2005) 085001 [hep-th/0505205] [INSPIRE].

[4] A.B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Classical polylogarithms for amplitudes and Wilson loops*, *Phys. Rev. Lett.* **105** (2010) 151605 [arXiv:1006.5703] [INSPIRE].

[5] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, S. Caron-Huot and J. Trnka, *The all-loop integrand for scattering amplitudes in planar $\mathcal{N} = 4$ SYM*, *JHEP* **01** (2011) 041 [arXiv:1008.2958] [INSPIRE].

[6] N. Arkani-Hamed and J. Trnka, *The Amplituhedron*, arXiv:1312.2007 [INSPIRE].

[7] N. Arkani-Hamed and J. Trnka, *Into the Amplituhedron*, arXiv:1312.7878 [INSPIRE].

[8] J.M. Drummond, J. Henn, V.A. Smirnov and E. Sokatchev, *Magic identities for conformal four-point integrals*, *JHEP* **01** (2007) 064 [hep-th/0607160] [INSPIRE].

[9] Z. Bern, M. Czakon, L.J. Dixon, D.A. Kosower and V.A. Smirnov, *The four-loop planar amplitude and cusp anomalous dimension in maximally supersymmetric Yang-Mills theory*, *Phys. Rev.* **D 75** (2007) 085010 [hep-th/0610248] [INSPIRE].

[10] Z. Bern, J.J.M. Carrasco, H. Johansson and D.A. Kosower, *Maximally supersymmetric planar Yang-Mills amplitudes at five loops*, *Phys. Rev.* **D 76** (2007) 125020 [arXiv:0705.1864] [INSPIRE].

[11] L.F. Alday and J.M. Maldacena, *Gluon scattering amplitudes at strong coupling*, *JHEP* **06** (2007) 064 [arXiv:0705.0303] [INSPIRE].

[12] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *Dual superconformal symmetry of scattering amplitudes in* $\mathcal{N} = 4$ *super-Yang-Mills theory*, *Nucl. Phys.* **B 828** (2010) 317 [arXiv:0807.1095] [INSPIRE].

[13] J.M. Drummond, G.P. Korchemsky and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops*, *Nucl. Phys.* **B 795** (2008) 385 [arXiv:0707.0243] [INSPIRE].

[14] A. Brandhuber, P. Heslop and G. Travaglini, *MHV amplitudes in* $\mathcal{N} = 4$ *super Yang-Mills and Wilson loops*, *Nucl. Phys.* **B 794** (2008) 231 [arXiv:0707.1153] [INSPIRE].

[15] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *On planar gluon amplitudes/Wilson loops duality*, *Nucl. Phys.* **B 795** (2008) 52 [arXiv:0709.2368] [INSPIRE].

[16] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes*, *Nucl. Phys.* **B 826** (2010) 337 [arXiv:0712.1223] [INSPIRE].

[17] L.F. Alday and R. Roiban, *Scattering amplitudes, Wilson loops and the string/gauge theory correspondence*, *Phys. Rept.* **468** (2008) 153 [arXiv:0807.1889] [INSPIRE].

[18] J.J.M. Carrasco and H. Johansson, *Generic multiloop methods and application to* $\mathcal{N} = 4$ *super-Yang-Mills*, *J. Phys.* **A 44** (2011) 454004 [arXiv:1103.3298] [INSPIRE].

[19] T. Adamo, M. Bullimore, L. Mason and D. Skinner, *Scattering amplitudes and Wilson loops in twistor space*, *J. Phys.* **A 44** (2011) 454008 [arXiv:1104.2890] [INSPIRE].

[20] N. Arkani-Hamed et al., *Scattering amplitudes and the positive Grassmannian*, arXiv:1212.5605 [INSPIRE].

[21] J.L. Bourjaily, A. DiRe, A. Shaikh, M. Spradlin and A. Volovich, *The soft-collinear bootstrap:* $\mathcal{N} = 4$ *Yang-Mills amplitudes at six and seven loops*, *JHEP* **03** (2012) 032 [arXiv:1112.6432] [INSPIRE].

[22] B. Eden, P. Heslop, G.P. Korchemsky and E. Sokatchev, *Constructing the correlation function of four stress-tensor multiplets and the four-particle amplitude in* $\mathcal{N} = 4$ *SYM*, *Nucl. Phys.* **B 862** (2012) 450 [arXiv:1201.5329] [INSPIRE].

[23] Z. Bern et al., $D = 5$ *maximally supersymmetric Yang-Mills theory diverges at six loops*, *Phys. Rev.* **D 87** (2013) 025018 [arXiv:1210.7709] [INSPIRE].

[24] A.V. Belitsky, S. Hohenegger, G.P. Korchemsky, E. Sokatchev and A. Zhiboedov, *Event shapes in* $\mathcal{N} = 4$ *super-Yang-Mills theory*, *Nucl. Phys.* **B 884** (2014) 206 [arXiv:1309.1424] [INSPIRE].

[25] A.V. Belitsky, S. Hohenegger, G.P. Korchemsky, E. Sokatchev and A. Zhiboedov, *Energy-energy correlations in* $\mathcal{N} = 4$ *supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **112** (2014) 071601 [arXiv:1311.6800] [INSPIRE].

[26] L.J. Dixon, J.M. Drummond and J.M. Henn, *Bootstrapping the three-loop hexagon*, *JHEP* **11** (2011) 023 [arXiv:1108.4461] [INSPIRE].

[27] L.J. Dixon, J.M. Drummond and J.M. Henn, *Analytic result for the two-loop six-point NMHV amplitude in* $\mathcal{N} = 4$ *super Yang-Mills theory*, *JHEP* **01** (2012) 024 [arXiv:1111.1704] [INSPIRE].

[28] L.J. Dixon, J.M. Drummond, M. von Hippel and J. Pennington, *Hexagon functions and the three-loop remainder function*, *JHEP* **12** (2013) 049 [arXiv:1308.2276] [INSPIRE].

[29] L.J. Dixon, J.M. Drummond, C. Duhr and J. Pennington, *The four-loop remainder function and multi-Regge behavior at NNLLA in planar* $\mathcal{N} = 4$ *super-Yang-Mills theory*, *JHEP* **06** (2014) 116 [arXiv:1402.3300] [INSPIRE].

[30] L.J. Dixon, J.M. Drummond, C. Duhr, M. von Hippel and J. Pennington, *Bootstrapping six-gluon scattering in planar* $\mathcal{N} = 4$ *super-Yang-Mills theory*, *PoS(LL2014)077* [arXiv:1407.4724] [INSPIRE].

[31] Z. Bern et al., *The two-loop six-gluon MHV amplitude in maximally supersymmetric Yang-Mills theory*, *Phys. Rev.* **D 78** (2008) 045007 [arXiv:0803.1465] [INSPIRE].

[32] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *Hexagon Wilson loop = six-gluon MHV amplitude*, *Nucl. Phys.* **B 815** (2009) 142 [arXiv:0803.1466] [INSPIRE].

[33] V.P. Nair, *A current algebra for some gauge theory amplitudes*, *Phys. Lett.* **B 214** (1988) 215 [INSPIRE].

[34] G. Georgiou, E.W.N. Glover and V.V. Khoze, *Non-MHV tree amplitudes in gauge theory*, *JHEP* **07** (2004) 048 [hep-th/0407027] [INSPIRE].

[35] N. Arkani-Hamed, F. Cachazo and J. Kaplan, *What is the simplest quantum field theory?*, *JHEP* **09** (2010) 016 [arXiv:0808.1446] [INSPIRE].

[36] L.F. Alday, D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, *An operator product expansion for polygonal null Wilson loops*, *JHEP* **04** (2011) 088 [arXiv:1006.2788] [INSPIRE].

[37] D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, *Bootstrapping null polygon Wilson loops*, *JHEP* **03** (2011) 092 [arXiv:1010.5009] [INSPIRE].

[38] D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, *Pulling the straps of polygons*, *JHEP* **12** (2011) 011 [arXiv:1102.0062] [INSPIRE].

[39] A. Sever, P. Vieira and T. Wang, *OPE for super loops*, *JHEP* **11** (2011) 051 [arXiv:1108.1575] [INSPIRE].

[40] B. Basso, *Exciting the GKP string at any coupling*, *Nucl. Phys.* **B 857** (2012) 254 [arXiv:1010.5237] [INSPIRE].

[41] B. Basso, A. Sever and P. Vieira, *Spacetime and flux tube S-matrices at finite coupling for* $\mathcal{N} = 4$ *supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **111** (2013) 091602 [arXiv:1303.1396] [INSPIRE].

[42] B. Basso, A. Sever and P. Vieira, *Space-time S-matrix and flux tube S-matrix II. Extracting and matching data*, *JHEP* **01** (2014) 008 [arXiv:1306.2058] [INSPIRE].

[43] B. Basso, A. Sever and P. Vieira, *Space-time S-matrix and flux-tube S-matrix III. The two-particle contributions*, *JHEP* **08** (2014) 085 [arXiv:1402.3307] [INSPIRE].

[44] B. Basso, A. Sever and P. Vieira, *Space-time S-matrix and flux-tube S-matrix IV. Gluons and fusion*, *JHEP* **09** (2014) 149 [arXiv:1407.1736] [INSPIRE].

[45] A.V. Belitsky, S.E. Derkachov and A.N. Manashov, *Quantum mechanics of null polygonal Wilson loops*, *Nucl. Phys.* **B 882** (2014) 303 [arXiv:1401.7307] [INSPIRE].

[46] A.V. Belitsky, *Nonsinglet pentagons and NHMV amplitudes*, arXiv:1407.2853 [INSPIRE].

[47] J. Bartels, L.N. Lipatov and A. Sabio Vera, *BFKL Pomeron, reggeized gluons and Bern-Dixon-Smirnov amplitudes*, *Phys. Rev.* **D 80** (2009) 045002 [arXiv:0802.2065] [INSPIRE].

[48] J. Bartels, L.N. Lipatov and A. Sabio Vera, $\mathcal{N} = 4$ *supersymmetric Yang-Mills scattering amplitudes at high energies: the Regge cut contribution*, *Eur. Phys. J.* **C 65** (2010) 587 [arXiv:0807.0894] [INSPIRE].

[49] L.N. Lipatov and A. Prygarin, *Mandelstam cuts and light-like Wilson loops in $\mathcal{N} = 4$ SUSY*, *Phys. Rev.* **D 83** (2011) 045020 [arXiv:1008.1016] [INSPIRE].

[50] L.N. Lipatov and A. Prygarin, *BFKL approach and six-particle MHV amplitude in $\mathcal{N} = 4$ super Yang-Mills*, *Phys. Rev.* **D 83** (2011) 125001 [arXiv:1011.2673] [INSPIRE].

[51] J. Bartels, L.N. Lipatov and A. Prygarin, *MHV amplitude for $3 \rightarrow 3$ gluon scattering in Regge limit*, *Phys. Lett.* **B 705** (2011) 507 [arXiv:1012.3178] [INSPIRE].

[52] J. Bartels, L.N. Lipatov and A. Prygarin, *Collinear and Regge behavior of $2 \rightarrow 4$ MHV amplitude in $\mathcal{N} = 4$ super Yang-Mills theory*, arXiv:1104.4709 [INSPIRE].

[53] V.S. Fadin and L.N. Lipatov, *BFKL equation for the adjoint representation of the gauge group in the next-to-leading approximation at $\mathcal{N} = 4$ SUSY*, *Phys. Lett.* **B 706** (2012) 470 [arXiv:1111.0782] [INSPIRE].

[54] L. Lipatov, A. Prygarin and H.J. Schnitzer, *The multi-Regge limit of NMHV amplitudes in $\mathcal{N} = 4$ SYM theory*, *JHEP* **01** (2013) 068 [arXiv:1205.0186] [INSPIRE].

[55] S. Caron-Huot, *When does the gluon reggeize?*, arXiv:1309.6521 [INSPIRE].

[56] L.J. Dixon, C. Duhr and J. Pennington, *Single-valued harmonic polylogarithms and the multi-Regge limit*, *JHEP* **10** (2012) 074 [arXiv:1207.0186] [INSPIRE].

[57] J. Pennington, *The six-point remainder function to all loop orders in the multi-Regge limit*, *JHEP* **01** (2013) 059 [arXiv:1209.5357] [INSPIRE].

[58] F.C.S. Brown, *Single-valued multiple polylogarithms in one variable*, *Compt. Rend. Acad. Sci. Paris, Ser. I* **338** (2004) 527.

[59] B. Basso, S. Caron-Huot and A. Sever, *Adjoint BFKL at finite coupling: a short-cut from the collinear limit*, arXiv:1407.3766 [INSPIRE].

[60] Y. Hatsuda, *Wilson loop OPE, analytic continuation and multi-Regge limit*, arXiv:1404.6506 [INSPIRE].

[61] V. Del Duca, C. Duhr and V.A. Smirnov, *An analytic result for the two-loop hexagon Wilson loop in $\mathcal{N} = 4$ SYM*, *JHEP* **03** (2010) 099 [arXiv:0911.5332] [INSPIRE].

[62] V. Del Duca, C. Duhr and V.A. Smirnov, *The two-loop hexagon Wilson loop in $\mathcal{N} = 4$ SYM*, *JHEP* **05** (2010) 084 [arXiv:1003.1702] [INSPIRE].

[63] C. Duhr, H. Gangl and J.R. Rhodes, *From polygons and symbols to polylogarithmic functions*, *JHEP* **10** (2012) 075 [arXiv:1110.0458] [INSPIRE].

[64] C. Duhr, *Hopf algebras, coproducts and symbols: an application to Higgs boson amplitudes*, *JHEP* **08** (2012) 043 [arXiv:1203.0454] [INSPIRE].

[65] S. Caron-Huot and S. He, *Jumpstarting the all-loop S-matrix of planar $\mathcal{N} = 4$ super Yang-Mills*, *JHEP* **07** (2012) 174 [arXiv:1112.1060] [INSPIRE].

[66] M. Bullimore and D. Skinner, *Descent equations for superamplitudes*, arXiv:1112.1056 [INSPIRE].

[67] A. Hodges, *Eliminating spurious poles from gauge-theoretic amplitudes*, *JHEP* **05** (2013) 135 [arXiv:0905.1473] [INSPIRE].

[68] L.J. Mason and D. Skinner, *Dual superconformal invariance, momentum twistors and Grassmannians*, *JHEP* **11** (2009) 045 [arXiv:0909.0250] [INSPIRE].

[69] J.M. Drummond, J. Henn, G.P. Korchemsky and E. Sokatchev, *Generalized unitarity for $\mathcal{N} = 4$ super-amplitudes*, *Nucl. Phys.* **B 869** (2013) 452 [arXiv:0808.0491] [INSPIRE].

[70] K.T. Chen, *Iterated path integrals*, *Bull. Amer. Math. Soc.* **83** (1977) 831 [INSPIRE].

[71] F.C.S. Brown, *Multiple zeta values and periods of moduli spaces $\mathfrak{M}_{0,n}$*, *Annales Sci. Ecole Norm. Sup.* **42** (2009) 371 [math.AG/0606419] [INSPIRE].

[72] A.B. Goncharov, *A simple construction of Grassmannian polylogarithms*, arXiv:0908.2238 [INSPIRE].

[73] A.B. Goncharov, *Multiple polylogarithms and mixed Tate motives*, math.AG/0103059 [INSPIRE].

[74] A.B. Goncharov, *Galois symmetries of fundamental groupoids and noncommutative geometry*, *Duke Math. J.* **128** (2005) 209 [math.AG/0208144] [INSPIRE].

[75] F. Brown, *On the decomposition of motivic multiple zeta values*, arXiv:1102.1310 [INSPIRE].

[76] F. Brown, *Mixed Tate motives over Z*, arXiv:1102.1312.

[77] L.J. Dixon, J.M. Drummond and J.M. Henn, *The one-loop six-dimensional hexagon integral and its relation to MHV amplitudes in $\mathcal{N} = 4$ SYM*, *JHEP* **06** (2011) 100 [arXiv:1104.2787] [INSPIRE].

[78] V. Del Duca, C. Duhr and V.A. Smirnov, *The massless hexagon integral in $D = 6$ dimensions*, *Phys. Lett.* **B 703** (2011) 363 [arXiv:1104.2781] [INSPIRE].

[79] E. Remiddi and J.A.M. Vermaseren, *Harmonic polylogarithms*, *Int. J. Mod. Phys.* **A 15** (2000) 725 [hep-ph/9905237] [INSPIRE].

[80] S. Caron-Huot and S. He, private communication.

[81] B. Basso, A. Sever and P. Vieira, private communication.

[82] L.J. Mason and D. Skinner, *The complete planar S-matrix of $\mathcal{N} = 4$ SYM as a Wilson loop in twistor space*, *JHEP* **12** (2010) 018 [arXiv:1009.2225] [INSPIRE].

[83] S. Caron-Huot, *Notes on the scattering amplitude/Wilson loop duality*, *JHEP* **07** (2011) 058 [arXiv:1010.1167] [INSPIRE].

[84] G. Papathanasiou, *Hexagon Wilson loop OPE and harmonic polylogarithms*, *JHEP* **11** (2013) 150 [arXiv:1310.5735] [INSPIRE].

[85] G. Papathanasiou, *Evaluating the 6-point remainder function near the collinear limit*, arXiv:1406.1123 [INSPIRE].

[86] J. Bartels, A. Kormilitzin, L.N. Lipatov and A. Prygarin, *BFKL approach and $2 \to 5$ maximally helicity violating amplitude in $\mathcal{N} = 4$ super-Yang-Mills theory*, *Phys. Rev.* **D 86** (2012) 065026 [arXiv:1112.6366] [INSPIRE].

[87] S.J. Parke and T.R. Taylor, *An amplitude for n gluon scattering*, *Phys. Rev. Lett.* **56** (1986) 2459 [ɪɴSPIRE].

[88] Z. Bern and G. Chalmers, *Factorization in one loop gauge theory*, *Nucl. Phys.* **B 447** (1995) 465 [hep-ph/9503236] [ɪɴSPIRE].

[89] J. Ablinger, J. Blumlein and C. Schneider, *Harmonic sums and polylogarithms generated by cyclotomic polynomials*, *J. Math. Phys.* **52** (2011) 102301 [arXiv:1105.6063] [ɪɴSPIRE].

[90] T. Gehrmann and E. Remiddi, *Two loop master integrals for $\gamma^* \to 3$ jets: the planar topologies*, *Nucl. Phys.* **B 601** (2001) 248 [hep-ph/0008287] [ɪɴSPIRE].