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Spinning strings and minimal surfaces in AdS_3 with mixed 3-form fluxes

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ABSTRACT: Motivated by the recent proposal for the S-matrix in $AdS_3 \times S^3$ with mixed three form fluxes, we study classical folded string spinning in AdS_3 with both Ramond and Neveu-Schwarz three form fluxes. We solve the equations of motion of these strings and obtain their dispersion relation to the leading order in the Neveu-Schwarz flux b . We show that dispersion relation for the spinning strings with large spin $\mathcal{S}$ acquires a term given by $-\frac{\sqrt{\lambda}}{2\pi}b^2\log^2\mathcal{S}$ in addition to the usual $\frac{\sqrt{\lambda}}{\pi}\log\mathcal{S}$ term where $\sqrt{\lambda}$ is proportional to the square of the radius of AdS_3 . Using $SO(2,2)$ transformations and re-parametrizations we show that these spinning strings can be related to light like Wilson loops in AdS_3 with Neveu-Schwarz flux b . We observe that the logarithmic divergence in the area of the light like Wilson loop is also deformed by precisely the same coefficient of the $b^2\log^2\mathcal{S}$ term in the dispersion relation of the spinning string. This result indicates that the coefficient of $b^2\log^2\mathcal{S}$ has a property similar to the coefficient of the $\log\mathcal{S}$ term, known as cusp-anomalous dimension, and can possibly be determined to all orders in the coupling λ using the recent proposal for the S-matrix.

KEYWORDS: AdS-CFT Correspondence, Bosonic Strings, Long strings

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1 Introduction

Classical string solutions propagating in various $AdS \times M$ background have played an important role in the study of the AdS/CFT correspondence. The anomalous dimensions of various operators of the field theory with large charges can be obtained by examining the dispersion of classical strings. Non-local operators like Wilson loops are also represented as minimal surfaces in the dual geometry. One particular classical solution which has been crucial in the detailed study of $\mathcal{N} = 4$ Yang-Mills is the folded spinning string solution in AdS_5 . This solution was originally found by [1] and was studied in more detail in [2]. The dispersion relation of the string which effectively moves in a AdS_3 subspace of AdS_5 with large spin $\mathcal{S}$ is given by

$$\Delta = \mathcal{S} + \frac{\sqrt{\lambda}}{\pi} \log \frac{\mathcal{S}}{\sqrt{\lambda}}, \quad \lambda \rightarrow \infty. \quad (1.1)$$

Here Δ is the energy of the string. The spinning folded string is dual to twist two operators of the form $\text{Tr}(\Phi \partial^{\mathcal{S}} \Phi)$, where the Φ is one of the adjoint scalars in $\mathcal{N} = 4$ Yang-Mills and ∂ indicated spatial derivatives. From a perturbative analysis in the field theory it can be shown that in the planar limit the anomalous dimensions of these operators with large $\mathcal{S}$ obey the relation

$$\Delta - (\mathcal{S} + 2) = f(\lambda) \log \mathcal{S} + O(1/\mathcal{S}). \quad (1.2)$$

The function $f(\lambda)$ at weak t'Hooft coupling is given by $f(\lambda) = \lambda$ while the behaviour of $f(\lambda)$ at strong coupling can be read out from the dispersion relation of the spinning string in (1.1). $f(\lambda)$ is related to a variety of different physical observables [3]. One particular relation which is of interest in this paper is that $f(\lambda)$ determines the expectation value of

the Wilson loop operator which has a cusp in its contour. It can be shown entirely from a field theory analysis [4, 5] that $f(\lambda)$ determines the logarithmic divergence of a Wilson loop which makes a turn of angle γ from the straight line. In the limit of large cusp angle the Wilson loop is given by

$$W = \left(\frac{L}{\epsilon}\right)^{-f(\lambda)\gamma}, \quad (1.3)$$

where L and ϵ are the ultra-violet and Infra-red cut off respectively. The function $f(\lambda)$ is called the cusp anomalous dimensions in literature. The area of the minimal surface corresponding to the cusp Wilson loop in the dual geometry exhibits a similar logarithmic divergence and $f(\lambda) = \frac{\sqrt{\lambda}}{\pi}$ [6, 7]. The minimal surface also lies only in a AdS_3 subspace of AdS_5 . Indeed the classical spinning string solution after a series of conformal transformations and re-parametrizations can be related to the minimal surface solution [8]. The function $f(\lambda)$ is that it has been determined to all orders in λ by using integrability [9].

Another example of holographic dual pair is that the case of strings on $AdS_3 \times S^3 \times T^4$ and the $\mathcal{N} = (4, 4)$ super conformal field theory associated with the D1-D5 system. Motivated by studying integrability of the string theory in this background semi-classical solutions like magnons and the folded spinning strings have also been studied in this background [10–19]. Recently it has been shown that the background $AdS_3 \times S^3$ is supported by both Ramond and Neveu-Schwarz three form fluxes, the string theory is integrable [20, 21]. There is a proposal for the S-matrix in this background [22–24]. The giant magnon solution and the finite gap equations has been studied in this background by [25] and [26] respectively. Short string solutions in presence of Neveu Schwarz B field in $AdS_5 \times S^5$ geometry were studied in [27]. Our goal in this paper is to study the behaviour of the spinning folded string solution in the background of $AdS_3 \times S^3$ with both Ramond as well as Neveu-Schwarz three form fluxes. This study is motivated from the recent proposal of the S-matrix for this background.

Once the Neveu-Schwarz three form flux is turned on, the equations of motion of the string are affected and therefore the folded spinning string solution cannot be simply embedded in AdS_3 . Earlier studies of classical strings in AdS_3 with Neveu-Schwarz fluxes are [28–30]. In this paper we solve the equations of motion of the classical folded spinning string solution in AdS_3 in terms of a perturbative expansion in the Neveu-Schwarz flux b . We show that the dispersion relation of the spinning strings in the large $\mathcal{S}$ limit is given by

$$\Delta = \mathcal{S} + \frac{\sqrt{\lambda}}{\pi} \log \mathcal{S} - \frac{\sqrt{\lambda}}{2\pi} b^2 \log^2 \mathcal{S}, \quad (1.4)$$

where λ is related to the radius of AdS_3 and we have retained the leading terms in the large $\mathcal{S}$ limit. The order of perturbative expansion which results in the dispersion relation (1.4) is the following. We first perform a perturbative expansion in b . At each order in b we retain the leading term in the large $\mathcal{S}$ limit. We also require $b \log \mathcal{S} \ll 1$ so that the expansion in (1.4) makes sense. $\log^2 \mathcal{S}$ terms are known to occur in the anomalous dimensions of twist two operators in 4 dimensional theories with lesser super symmetries compared to $\mathcal{N} = 4$ Yang-Mills [31]. But these terms are always suppressed by $1/\mathcal{S}$ and therefore are not relevant in the large $\mathcal{S}$ limit.

From (1.4) we note that at order b^2 the leading term is proportional to $\log^2 \mathcal{S}$. As we have discussed earlier, the coefficient of the $\log \mathcal{S}$ term in the dispersion relation of the spinning string is also the coefficient of the logarithmic divergence seen in the area of the minimal surface corresponding to the cusp Wilson line. When these classical solutions are embedded in AdS_5 , the reason that these coefficients must agree has a purely perturbative field theoretic explanation [4, 5]. It was argued in [8] from the dual gravity side the reason that the coefficient of the $\log \mathcal{S}$ term agreed with the cusp anomalous dimension is that the classical solutions can be related to each other by $SO(2, 2)$ transformations. In fact in [8], the one loop corrections around both the spinning string and the cusp Wilson line were evaluated and shown to agree. In light of this fact it is interesting to study the area of the minimal surface in presence of the Neveu-Schwarz field. We show that the equations of motion for the minimal surface and its action in presence of the Neveu-Schwarz field can be solved exactly. We show that the logarithmic divergence of the modulus of the expectation value of Wilson loop is determined by the function

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} - \frac{\sqrt{\lambda}}{2\pi} b^2 + O(b^4). \quad (1.5)$$

We observe that the coefficient of b^2 is identical to the coefficient of the $b^2 \log^2 \mathcal{S}$ in the dispersion relation of the spinning string. To argue that there is a relation of the coefficient of b^2 in the logarithmic divergence of the area of the minimal surface corresponding to the Wilson loop to the coefficient of $b^2 \log^2 \mathcal{S}$ in the dispersion relation of the spinning string, we note the following.

1. We first relate the scaling limit of the spinning string solution found to $O(b^2)$ to the minimal surface corresponding to the cusp Wilson line by performing a series of $SO(2, 2)$ transformations¹ and re-parametrizations as done by [8] in the absence of the Neveu-Schwarz field.
2. We also show that it is only the logarithmic divergence in the area of the minimal surface that is universal. There is a class of minimal surfaces which ends on the cusp Wilson line and the logarithmic divergence in the area remains the same but admit other divergences which depends on the parameters of the minimal surface.

Using these facts, if we extrapolate the observation of [8] to $b \neq 0$ it is natural to compare the coefficient of the $b^2 \log^2 \mathcal{S}$ to the coefficient in the logarithmic divergence of the area of the minimal surface. The observation that these coefficients are same suggests that the coefficient of $b^2 \log^2 \mathcal{S}$ has a property similar to that of the $\log \mathcal{S}$ term in the dispersion relation of the spinning string. It will be interesting if the S-matrix proposal of [22, 23] can be used to derive this dispersion relation and whether one can determine the b^2 term to all orders in λ .

This paper is organized as follows. In section 2 after a brief review of the supergravity background with both Neveu-Schwarz and Ramond 3-form fluxes in AdS_3 we solve the equations of motion for the spinning string in AdS_3 with angular momentum in the S^3 to

¹Note that the world sheet sigma model in the presence of both Ramond and Neveu-Schwarz fields has global $SO(2, 2)$ symmetry.

the leading order in the Neveu-Schwarz flux b . We show that in the large spin limit the dispersion relation is given by (1.4). We will again derive this dispersion relation in the scaling limit of the long string. As a check on our perturbation theory, we also obtain the dispersion relation for small strings and show that we obtain the BMN dispersion relation. In section 3 we first derive the minimal surface corresponding to the cusp Wilson line in the background with mixed 3-form flux and evaluate its area exactly. We show that the coefficient of b^2 in the area is precisely the same as the coefficient of $b^2 \log^2 \mathcal{S}$ term in the dispersion relation (1.4). We then relate the scaling limit of the spinning string solution to that of cusp Wilson line using the $SO(2, 2)$ symmetry of the sigma model and re-parametrization invariance. Section 4 contains our conclusions.

2 Strings in $AdS_3 \times S_3$ with mixed form fluxes

To set up our notations and conventions we first write down the background solution which we will work in. It is a solution of the type IIB action with $AdS_3 \times S^3 \times M^4$ geometry. M^4 can be T^4 or K^3 . This will not be relevant for the discussions in this paper. The solution has RR and NS-NS fluxes along the AdS_3 and S^3 directions. The type IIB background fields which are turned on in this solution are

$$\begin{aligned} ds^2 &= ds_{AdS_3}^2 + ds_{S^3}^2 + ds_{T^4}^2, \\ ds_{AdS_3}^2 &= -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2, \\ ds_{S^3}^2 &= d\beta_1^2 + \cos^2 \beta_1 (d\beta_2^2 + \cos^2 \beta_2 d\beta_3^2). \end{aligned} \quad (2.1)$$

Here we have assumed that the compact manifold to be the torus T^4 for definiteness. The NS-NS and RR 3-form fluxes are given by

$$\begin{aligned} H_{t\rho\phi}^{(3)} &= -2b \cosh \rho \sinh \rho, & F_{t\rho\phi}^{(3)} &= -2\sqrt{1-b^2} \cosh \rho \sinh \rho, \\ H_{\beta_1\beta_2\beta_3}^{(3)} &= -2b \cos^2 \beta_1 \cos \beta_2, & F_{\beta_1\beta_2\beta_3}^{(3)} &= -2\sqrt{1-b^2} \cos^2 \beta_1 \cos \beta_2, \end{aligned} \quad (2.2)$$

where $0 \leq b \leq 1$. As the parameter b is tuned from 0 to 1, the solution interpolates from purely RR 3-form flux to purely NS-NS flux. All the remaining fluxes are set to zero, the dilaton, Φ is constant and can be set to zero. We have taken the radius of AdS_3 and S^3 to be unity. We will incorporate the radius of AdS_3 in the sigma model coupling. This background is the solution of type IIB equations of motion in the Einstein frame. For completeness and as a check of our normalizations we write down the equations of motion for the metric

$$\begin{aligned} R_{MN} - \frac{1}{2} G_{MN} R &= \frac{1}{2} \left(\partial_M \Phi \partial_N \Phi - \frac{1}{2} G_{MN} \partial_P \Phi \partial^P \Phi \right) \\ &+ \frac{1}{2} e^{2\Phi} \left(F_M^{(1)} F_N^{(1)} - \frac{1}{2} G_{MN} F_1^2 \right) + \frac{1}{2.3!} e^\Phi \left(3 F_{MPQ}^{(3)} F_N^{(3)PQ} - \frac{1}{2} G_{MN} F_3^2 \right) \\ &+ \frac{5}{4.5!} F_{MPQRS}^{(5)} F_N^{(5)PQRS} + \frac{1}{2.3!} e^{-\Phi} \left(3 H_{MPQ}^{(3)} H_N^{(3)PQ} - \frac{1}{2} G_{MN} H_3^2 \right). \end{aligned} \quad (2.3)$$

The background metric in (2.1) and the fluxes (2.2) can be easily shown to satisfy the equation (2.3) with the dilaton set to zero.

2.1 Classical solution for spinning strings in $AdS_3 \times S^3$

To begin deriving the classical solution of strings in $AdS_3 \times S^3$ with mixed 3-form fluxes we first write down the sigma model in the background given by (2.1) and (2.2). The Polyakov action is given by

$$S_{pol} = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[\sqrt{-h} h^{ab} G_{mn} \partial_a X^m \partial_b X^n - \epsilon^{ab} B_{mn} \partial_a X^m \partial_b X^n \right]. \quad (2.4)$$

where h^{ab} in conformal gauge is given by $h^{ab} = \text{diag}(-1, 1)$. The antisymmetric tensor $\epsilon^{01} = 1$ and the indices m, n run from 0 to 5. The directions 0, 1, 2 label AdS_3 while 3, 4, 5 label the S^3 . The classical solutions of interest in this paper have no dynamics along the T^4 , therefore from now on we will ignore these directions. Note that we have re-instated the radius of AdS_3 and S^3 in the sigma model coupling $\sqrt{\lambda}$ which is given by

$$\sqrt{\lambda} = \frac{R^2}{\alpha'}, \quad (2.5)$$

where R is the radius of AdS_3 and α' is the string length squared. The interesting dynamics of the spinning string solutions we consider will take place in AdS_3 , for which we find it convenient to work with the following global metric

$$ds^2 = -(1 + r^2) d\tilde{t}^2 + \frac{dr^2}{1 + r^2} + r^2 d\tilde{\phi}^2. \quad (2.6)$$

This metric is related to the AdS_3 metric given in (2.1) by the coordinate transformation $r = \sinh \rho$. The NS-NS flux along AdS_3 in this co-ordinate system is given by

$$B_{t\phi} = br^2. \quad (2.7)$$

The metric on S^3 and the NS-NS flux on S^3 is taken to be as given in (2.1) and (2.2) respectively. We choose the following ansatz for the classical solutions we consider.

$$\tilde{t} = c_1 \tau + t(\sigma), \quad \tilde{\phi} = c_2 \tau + \phi(\sigma), \quad r = r(\sigma), \quad \beta_1 = \beta_2 = 0, \quad \beta_3 = \omega \tau. \quad (2.8)$$

Here $\tilde{\phi}$ is the angle in AdS_3 and $\tilde{t}$ is the global time while r is the radial direction. We look for solutions which satisfy the condition

$$t(\sigma + 2\pi) = t(\sigma), \quad r(\sigma + 2\pi) = r(\sigma). \quad (2.9)$$

Just as in the giant magnon solution of [32] we do not impose periodic boundary conditions on the co-ordinate ϕ . We will however show that a closed string solution can be constructed by considering several periods in the world sheet σ direction. This ansatz in (2.8) is a generalization of the folded spinning string solution in the absence of NS-NS flux studied in [2]. The functions $t(\sigma)$ and $\phi(\sigma)$ vanish when $b = 0$. The angle co-ordinate $\tilde{\phi}$ is function of σ when $b \neq 0$. The solution we construct at a given instance of time in the $r, \tilde{\phi}$ plane will not be a simple line, which is the signature of the folded string. Since $\tilde{\phi}$ is a function of σ it will look like a smoothed or a blown up version of the folded string, turning around

smoothly in the $r, \tilde{\phi}$ plane. We will continue to call this the folded spinning string as it reduces to the folded solution when $b = 0$.

Substituting the ansatz into the world sheet action (2.4), we find that it reduces to

$$S = \frac{\sqrt{\lambda}}{4\pi} \int d\tau d\sigma \left[-(1+r^2)(-c_1^2 + t'^2) + \frac{1}{1+r^2} r'^2 + r^2(-c_2^2 + \phi'^2) - 2br^2(c_1\phi' - c_2t') \right]. \quad (2.10)$$

The NS-NS flux in the S^3 direction does not contribute to the action since the ansatz in (2.8) does not have world sheet σ direction in the S^3 directions. Note that the action in (2.10) does not explicitly depend on t and ϕ , therefore the corresponding conjugate momenta are conserved. This leads to the following equations

$$t' = \frac{bc_2r^2 - k_1}{1+r^2}, \quad \phi' = \frac{bc_1r^2 - k_2}{r^2}, \quad (2.11)$$

where k_1, k_2 are constants of motion. The Virasoro constraint corresponding to the vanishing of world sheet momentum leads to the constraint

$$c_1k_1 = c_2k_2. \quad (2.12)$$

The Virasoro constraint corresponding to the vanishing of world sheet energy leads to a first order differential equation for the function r .

$$r'^2r^2 = Ar^6 + Br^4 + Cr^2 + D, \quad (2.13)$$

where

$$\begin{aligned} A &= c_1^2 + c_2^2b^2 - c_2^2 - c_1^2b^2, \\ B &= 2c_1^2 - c_2^2 - c_1^2b^2 - 2bc_2k_1 + 2bc_1k_2 - \omega^2, \\ C &= c_1^2 + k_1^2 + 2bc_1k_2 - k_2^2 - \omega^2, \\ D &= -k_2^2. \end{aligned} \quad (2.14)$$

It can be verified that the solutions for t', ϕ' and r' given in (2.11) and (2.13) solves the second order equations of motion derived from the sigma model.

Note that the equations simplify when $b = 1$, that is the situation when there is a pure NS-NS flux. The coefficient A vanishes and the polynomial on the right hand side of (2.13) reduces to a quartic polynomial. This is the limit studied in [29] for which exact solutions were obtained. We will first write down the solutions for a general b and then set up a perturbative expansion about $b = 0$. As we have mentioned earlier, $t(\sigma)$ and $\phi(\sigma)$ must vanish when $b = 0$ for the solution to reduce to the folded spinning string. From (2.11), this implies that k_1, k_2 must vanish when $b = 0$. Assuming there exists a well defined perturbation theory about $b = 0$ we look for solutions with k_1, k_2 vanish linearly with b . Therefore k_1, k_2 admits an expansion given by

$$k_1 = k_1^{(0)}b + k_1^{(1)}b^2 + \dots, \quad k_2 = \frac{c_1k_1}{c_2}. \quad (2.15)$$

We can now integrate the equation for r' . For this we need to find the turning points of the equation in (2.13). We see that from the expressions for A, B, C, D in (2.14) and the expansion in (2.15), when $b = 0$ the three roots of cubic polynomial in r^2 given by $Ar^6 + Br^4 + Cr^2 + D$ which determines r' are $-1, 0, \frac{c_1^2 - \omega^2}{c_2^2 - c_1^2}$, we label these roots as $R_1^{(0)}, R_3^{(0)}$ and $R_2^{(0)}$ respectively. The turning points for r' for the folded spinning string solutions are at 0 and $\frac{c_1^2 - \omega^2}{c_2^2 - c_1^2}$, that is $R_3^{(0)}$ and $R_2^{(0)}$. Let R_3 and R_2 be the roots continuously connected to the roots $R_3^{(0)}$ and $R_2^{(0)}$ respectively when b is set to zero. Let us rewrite the equation (2.13) as

$$r' = \frac{\sqrt{A(r^2 - R_1)(r^2 - R_2)(r^2 - R_3)}}{r}, \quad (2.16)$$

R_1, R_2, R_3 are the roots continuously connected to $R_1^{(0)}, R_2^{(0)}$ and $R_3^{(0)}$ respectively. The folded string satisfies the periodicity property $r(\sigma + 2\pi) = r(\sigma)$. This is ensured as follows. The interval $0 \leq \sigma < 2\pi$ is split into 4 segments. For $0 \leq \sigma < \pi/2$, $r(\sigma)$ increases from $\sqrt{R_3}$ to $\sqrt{R_2}$. Then $r(\sigma)$ decreases back to $\sqrt{R_2}$ as σ goes from $\pi/2$ to π . Integrating the equation in (2.16) between $\sqrt{R_3}$ to $\sqrt{R_2}$, $r(\sigma)$ we obtain

$$2\pi = \int_0^{2\pi} d\sigma = 4 \int_{\sqrt{R_3}}^{\sqrt{R_2}} \frac{r dr}{\sqrt{A(r^2 - R_1)(r^2 - R_2)(r^2 - R_3)}}. \quad (2.17)$$

We can reduce the integral in (2.17) to a known function by substitution $r^2 = x + R_3$. The integral then becomes

$$2\pi = \frac{4}{2\sqrt{A}} \int_0^{z_1} \frac{dx}{\sqrt{x(x - z_1)(x - z_2)}}, \quad (2.18)$$

where $z_1 = R_2 - R_3$ and $z_2 = R_1 - R_3$. It is now easy to recognize that after an appropriate scaling the integral reduces to the hypergeometric function. Thus we obtain the condition

$$\sqrt{Az_2} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, 1, \frac{z_1}{z_2}\right). \quad (2.19)$$

Let us now examine the difference $\phi(\sigma + 2\pi) - \phi(\sigma)$. After substituting the expression for ϕ' from (2.11) in the above equation we obtain

$$\phi(\sigma + 2\pi) - \phi(\sigma) = bc_1 2\pi - \frac{c_1 k_1}{c_2} \int_0^{2\pi} \frac{d\sigma}{r^2}. \quad (2.20)$$

The integral in (2.20), can be rewritten as

$$\int_0^{2\pi} \frac{d\sigma}{r^2} = \frac{4}{\sqrt{A}} \int_{\sqrt{R_3}}^{\sqrt{R_2}} \frac{dr}{r \sqrt{(r^2 - R_1)(r^2 - R_2)(r^2 - R_3)}}. \quad (2.21)$$

After changing variables to $x = r^2 - R_3$, the integral reduces to

$$\begin{aligned} \int_0^{2\pi} \frac{d\sigma}{r^2} &= \frac{2}{\sqrt{A}} \int_0^{z_1} \frac{dx}{(x + R_3) \sqrt{x(x - z_1)(x - z_2)}}, \\ &= \frac{4}{R_3 \sqrt{Az_2}} \Pi(u_1 | v_1), \end{aligned} \quad (2.22)$$

where $u_1 = -\frac{z_1}{R_3}$, $v_1 = \frac{z_1}{z_2}$. $\Pi(a|b)$ is the complete elliptic integral of the third kind defined by

$$\Pi(u|v) = \frac{1}{2} \int_0^1 \frac{dy}{(1-uy)\sqrt{y(1-y)(1-vy)}}. \quad (2.23)$$

Therefore we obtain

$$\phi(2\pi) - \phi(0) = 2\pi bc_1 - \frac{4k_2}{R_3\sqrt{z_2}} \Pi(u_1|v_1). \quad (2.24)$$

In general this difference will not vanish, but after sufficiently large number of periods in the world sheet σ direction we can ensure that the string closes. This will be shown explicitly in the scaling limit 2.4. Finally we examine the closed string condition $t(\sigma + 2\pi) = t(\sigma)$. This can be written as

$$\int_0^{2\pi} d\sigma t' = 0. \quad (2.25)$$

Substituting the value of t' from (2.11) and using similar change of variables, the above condition can be written again in terms of elliptic function of the third kind. This results in the following condition

$$2\pi bc_2 - \frac{4(bc_2 + k_1)}{(1 + R_3)\sqrt{Az_2}} \Pi(u_2|v_2) = 0. \quad (2.26)$$

where $u_2 = -\frac{z_1}{1+R_3}$ and $v_2 = \frac{z_1}{z_2}$. Periodicity in the co-ordinate t enforces constraints on the parameters of the solution. Using (2.26) we can eliminate say the parameter k_1 in terms of c_1, c_2, ω . We will see that it is crucial to impose periodicity in time to obtain the dispersion relation for these strings.

The energy Δ and spin $\mathcal{S}$ of the string, which are the conserved charges corresponding to time translations and shifts in ϕ are given by the following formulae respectively

$$\Delta = \sqrt{\lambda} E = \sqrt{\lambda} \int_0^{2\pi} [(1+r^2)c_1 - br^2\phi'] \frac{d\sigma}{2\pi}, \quad (2.27)$$

$$\mathcal{S} = \sqrt{\lambda} S = \sqrt{\lambda} \int_0^{2\pi} [r^2 c_2 - br^2 t'] \frac{d\sigma}{2\pi}. \quad (2.28)$$

An important point to note is that the integrands in the above expressions for the conserved charges are independent of the world sheet co-ordinate τ . Therefore one can perform the integral for an arbitrary length in the σ direction and still expect a conserved quantity. The angular momentum corresponding to rotations in S^3 is given by

$$J = \sqrt{\lambda} \omega. \quad (2.29)$$

Using similar change of variables and manipulations as done to obtain (2.19) we can write the integral in the expression for the energy given in (2.27) as

$$E = (c_1 + bk_2) + c_1(1-b^2) \left(R_3 + \frac{z_1}{2\sqrt{Az_2}} {}_2F_1\left(\frac{1}{2}, \frac{3}{2}, 2, \frac{z_1}{z_2}\right) \right). \quad (2.30)$$

To further simplify the expression for the spin, it is convenient to relate the S and the energy E . From the equations in (2.27) we obtain

$$E - \frac{c_1}{c_2} S = c_1 - b \int_0^{2\pi} \left(r^2 \phi' - r^2 t' \frac{c_1}{c_2} \right) \frac{d\sigma}{2\pi}. \quad (2.31)$$

Using 2.11 and 2.12 we can simplify the expression in the integrand. This results in the following

$$r^2 \left(\phi' - t' \frac{c_1}{c_2} \right) = \frac{c_1}{c_2} \frac{bc_2 r^2 - k_1}{1 + r^2} = \frac{c_1}{c_2} t'. \quad (2.32)$$

Since the string is closed we have $t(\sigma + 2\pi) = t(\sigma)$. Therefore the integral of the above expression from 0 to 2π vanishes. Thus the dispersion relation between energy and spin takes the following form

$$E = \frac{c_1}{c_2} S + c_1. \quad (2.33)$$

We emphasize the fact that the above dispersion relation is true only when one imposes the fact the string is closed in the t direction. Closure in ϕ direction is not crucial for deriving the dispersion relation. Using the above relation and (2.31) we can write the following equation for the spin

$$S = bk_1 + c_2(1 - b^2) \left(R_3 + \frac{z_1}{2\sqrt{Az_2}} {}_2F_1 \left(\frac{1}{2}, \frac{3}{2}, 2, \frac{z_1}{z_2} \right) \right). \quad (2.34)$$

Naively the dispersion relation in (2.33) does not involve the Neveu-Schwarz field b . But as we will see subsequently we can use (2.19), (2.26) and (2.34) to eliminate the independent parameters c_1, c_2, k_1 in favour of the spin S . We will derive this dispersion relation perturbatively to order b^2 .

2.2 Perturbation theory in b

We have formally written the conditions for the general solution of the equations of motion of the spinning string in presence of the NS-NS field. In this section we show that these conditions can indeed be satisfied by setting up a perturbative expansion in b . We will show that the crucial condition (2.26) can be satisfied at the linear order in b . The condition (2.24) which states that the closed string must be wound integer times will also be shown to be satisfied explicitly in the scaling limit in section 2.4 at the linear order in b ,

To proceed further we derive the corrections to the roots R_1, R_2, R_3 to order b^2 assuming the expansion (2.15). These are given by

$$R_1 = -1 + a_2 b^2 + O(b^3), \quad R_2 = \frac{c_1^2 - \omega^2}{c_2^2 - c_1^2} + a_1 b^2 + O(b^3), \quad R_3 = ab^2 + O(b^3), \quad (2.35)$$

where

$$\begin{aligned} a_2 &= -\frac{(c_2 + k_1^{(0)})^2}{c_2^2 - \omega^2}, \\ a_1 &= -\frac{\omega^2(c_1^2(k_1^{(0)} + c_2) - c_2(c_2 k_1^{(0)} + \omega^2))^2}{c_2^2(c_2^2 - c_1^2)(c_1^2 - \omega^2)(c_2^2 - \omega^2)}, \\ a &= \frac{c_1^2(k_1^{(0)})^2}{c_2^2(c_1^2 - \omega^2)}. \end{aligned} \quad (2.36)$$

Recall that the turning points are at $r = \sqrt{R_3}$ and $r = \sqrt{R_2}$. From the expression in (2.31) and the identity (2.32) we see that to obtain the dispersion relation to order b^2 it is sufficient

to satisfy the closed string boundary conditions to linear order in b in the t direction. This is because t' occurs with a factor of b in (2.31). Therefore satisfying the closed string boundary conditions to order b will ensure that these terms start at order b^3 and therefore are of higher order in the dispersion relation.

Let us now examine the condition (2.26) to order $O(b)$. From (2.11) we have

$$\begin{aligned} t' &= bc_2 - \frac{bc_2 + k_1}{1 + r^2}, \\ &= bc_2 - b \left(\frac{c_2 + k_1^{(0)}}{1 + r^2} \right), \end{aligned} \quad (2.37)$$

where in the second line of the above equation we have kept terms to the linear order in b . Integrating the world sheet co-ordinate σ from 0 to 2π the condition (2.26) can be written

$$2\pi bc_2 - b(c_2 + k_1^{(0)}) \int_0^{2\pi} \frac{d\sigma}{1 + r^2} = 0. \quad (2.38)$$

We can convert the integration over σ to over r using (2.16). Performing similar change of variables as discussed earlier in the paper and working to the leading order in b we obtain the condition

$$c_2^{(0)} - \frac{(c_2^{(0)} + k_1^{(0)})}{\sqrt{(c_2^{(0)})^2 - (c_1^{(0)})^2}} {}_2F_1 \left(\frac{3}{2}, \frac{1}{2}, 1, -\frac{(c_1^{(0)})^2 - \omega^2}{(c_2^{(0)})^2 - (c_1^{(0)})^2} \right) = 0. \quad (2.39)$$

where the superscript (0) to indicate the zero order contributions of c_1 and c_2 . We have integrated over r between the turning points $\sqrt{R_3}$ and $\sqrt{R_2}$. Since there is an overall factor of b in the condition given in (2.38) these turning points and all other terms are multiplying the equation are evaluated at the zeroth order in b . Finally to write the equation in (2.39) we have factored out the overall b . This equation can be used to solve $k_1^{(0)}$ in terms of $c_1^{(0)}, c_2^{(0)}, \omega$.

Let us now examine the leading behaviour of the difference $\phi(2\pi) - \phi(0)$ given in (2.24). After a re-scaling of the variables by a change of variables the integral in (2.21) reduces to

$$\int_0^{2\pi} \frac{d\sigma}{r^2} = \frac{4}{\sqrt{AR_3}} \int_1^{\sqrt{\frac{R_2}{R_3}}} \frac{dy}{y \sqrt{(R_3 y^2 - R_1)(R_3 y^2 - R_2)(y^2 - 1)}}. \quad (2.40)$$

Note that the upper limit of the integral in the limit $b \rightarrow 0$ tends to infinity since $R_3 \sim O(b^2)$. To obtain the leading contribution of this integral we can perform a Taylor series expansion of the factor $(R_3 y^2 - R_1)(R_3 y^2 - R_2)$ in b . The leading term is given by

$$\begin{aligned} k_2 \int_0^{2\pi} \frac{d\sigma}{r^2} &= \frac{4k_2}{\sqrt{AR_3 R_1 R_2}} \left(\int_1^\infty \frac{dy}{y \sqrt{y^2 - 1}} + O(b^1) \right), \\ &= \left[-\tan^{-1} \left(\frac{1}{\sqrt{y^2 - 1}} \right) \right]_1^\infty + O(b^1), \\ &= 4 \left(m + \frac{1}{2} \right) \frac{\pi}{2} + 4 \frac{k_2}{|k_2|} g(c_1^{(0)}, c_2^{(0)}, k_1^{(0)}) + O(b^2). \end{aligned} \quad (2.41)$$

Here we have called the linear term in b the function g which depends on the zeroth order coefficients $c_1^{(0)}, c_2^{(0)}, k_1^{(0)}$,² m is any integer and in the last line we have used the relation $AR_1R_2R_3 = k_2^2$. Thus the difference in the end points of the string in (2.24) reduces to

$$\phi(2\pi) - \phi(0) = 2\pi bc_1^{(0)} - (2m+1)\pi - 4\frac{k_2}{|k_2|}g(c_1^{(0)}, c_2^{(0)}, k_1^{(0)}) + O(b^2). \quad (2.42)$$

Now we need to solve k_1^0 from equation (2.39) and then substitute in (2.42) and check whether one obtains $2n\pi$. In general the difference

$$\delta\phi = 2\pi bc_1^{(0)} - b\frac{k_2^{(0)}}{|k_2^{(0)}|}g(c_1^{(0)}, c_2^{(0)}, k_1^{(0)}). \quad (2.43)$$

will not vanish and therefore the string will not be closed for a single period. This situation is similar to the giant magnon solution of [32]. We will show that we can construct a closed string solution after sufficiently large number of periods in σ . That is we consider

$$\phi(2N\pi) - \phi(0) = 2m\pi + N\delta\phi, \quad (2.44)$$

and we demand $N\delta\phi = 2m'\pi$, N, m' are integers. This implies that $\delta\phi$ is a rational multiple of π . We will explicitly discuss this method of obtaining a closed string solution in the scaling limit in section 2.4 where we find the function g . We will also see that $\delta\phi$ can be chosen to be a rational multiple of π . So for the purposes of this paper we look for open string solutions in the ϕ direction for a single world sheet period, but closed in the t direction. We will assume that a closed string in ϕ can be constructed.

The strategy to obtain the dispersion relation is first solve $k_1^{(0)}$ in terms of $c_1^{(0)}, c_2^{(0)}, \omega$ using (2.39). Then we can use (2.19) to solve for say c_2 in terms of c_1 and ω . We use the equations (2.34) and (2.29) to eliminate c_1 and ω in favour of the spin S and angular momentum J . Finally we substitute these values of c_1, c_2, ω in the relation for the energy in (2.33) to obtain the dispersion relation in terms of the spin and angular momentum. All these relations involve hypergeometric functions and therefore inverting them is possible in certain limits. We will now restrict ourselves to three limits, the long string, the scaling limit and the small string in which these functions simplify.

2.3 Long string limit

Let us first consider the long string limit. This limit is obtained by pushing the length of the string proportional to the difference in the turning points $z_1 = R_2 - R_3$ to infinity with R_3 held fixed. This is achieved by taking the parameters c_2 and c_1 to be almost equal. From now on we will restrict ourselves to the case in which c_1, c_2, k_1, k_2, b all are positive. We will set $\omega = 0$ to simplify our calculations. It is straight forward to repeat the analysis with $\omega \neq 0$. Under this limit, the hypergeometric functions simplify to

$${}_2F_1\left(\frac{1}{2}, \frac{3}{2}; 2, \frac{z_1}{z_2}\right) \simeq \sqrt{-\frac{z_2}{z_1}} \frac{4}{\pi}, \quad (2.45)$$

² $k_2^{(0)}$ can be determined using the Virasoro constraint.

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, \frac{z_1}{z_2}\right) \simeq \sqrt{-\frac{z_2}{z_1}} \frac{1}{\pi} \log\left(-\frac{z_1}{z_2}\right).$$

We first solve the equation for $k_1^{(0)}$. Using the asymptotic forms for the hypergeometric functions in (2.39) solve for $k_1^{(0)}$. This results in

$$k_1^{(0)} = c_2^{(0)} \left(\frac{\pi c_1^{(0)}}{2} - 1 \right). \quad (2.46)$$

Note that here the superscript (0) to indicate the zero order contributions of c_1, c_2 . Using the asymptotic forms of the hypergeometric functions and working to order b^2 the equation (2.19) reduces to

$$\begin{aligned} b^2 \left(\pi c_1^3 \left((k_1^{(0)})^2 - c_2^2 \right) + 2c_1^2 (c_2 + k_1^{(0)})^2 - \pi c_1 c_2^2 (k_1^{(0)})^2 + 2c_2^2 (k_1^{(0)})^2 \right) \\ = 2(c_1 c_2)^2 \log\left(\frac{c_1^2}{c_2^2 - c_1^2}\right) - 2\pi c_1^3 c_2^2. \end{aligned} \quad (2.47)$$

Note that $k_1^{(0)}$ always occurs with terms suppressed by b^2 , therefore we can substitute for it from equation (2.46). We can now solve c_2 in terms of c_1 to $O(b^2)$. This gives

$$\begin{aligned} c_2 = \sqrt{e^{-\pi c_1} c_1^2 + c_1^2} + \frac{b^2}{16\sqrt{e^{-\pi c_1} + 1}} \times \\ \times \left\{ c_1 e^{-2\pi c_1} (\pi c_1 - 2) \left((\pi c_1 - 2)^2 + e^{\pi c_1} (4 - 4\pi c_1) \right) \right\}, \end{aligned} \quad (2.48)$$

We then parametrize c_1 as

$$c_1 = \frac{\log\left(\frac{1}{\nu}\right)}{\pi}. \quad (2.49)$$

Substituting this parametrization of c_1 into (2.34) and rewriting c_2 in terms of c_1 using (2.48) we obtain

$$\begin{aligned} S = \frac{2\sqrt{\nu+1}}{\pi\nu} - \frac{b^2}{8\pi\nu\sqrt{\nu+1}} \times \\ \times \left\{ \log\left(\frac{1}{\nu}\right) \left(4(\nu+1)(\nu+6) - \log\left(\frac{1}{\nu}\right) \left[\nu^2 \log\left(\frac{1}{\nu}\right) + 10\nu + 8 \right] \right) - 8(\nu+1) \right\}. \end{aligned} \quad (2.50)$$

Here we have kept terms to $O(b^2)$. We can now solve for ν in terms of S to $O(b^2)$, this leads to

$$\begin{aligned} \nu = \frac{2}{\pi S} \sqrt{1 + \frac{1}{\pi^2 S^2}} + \frac{2}{\pi^2 S^2} + \frac{b^2}{2\pi^{5/2} S^{5/2} \sqrt{2 + \pi S}} \times \\ \times \left\{ 2\pi S(\pi S + 2) + \log^3\left(\frac{\pi S}{2}\right) + \pi S(2\pi S + 5) \log^2\left(\frac{\pi S}{2}\right) \right. \\ \left. - 2(\pi S + 2)(3\pi S + 1) \log\left(\frac{\pi S}{2}\right) \right\}. \end{aligned} \quad (2.51)$$

Hence using (2.51), (2.48) and (2.49) we can write down c_1 and c_2 in terms of S . Substituting these in the expression for the energy given in (2.33) and working to the $O(b^2)$ we obtain

$$\begin{aligned} E &= \frac{c_1}{c_2} S + c_1 \\ &= S + \frac{1}{\pi} \log S - \frac{b^2}{2\pi} \log^2 S + O((\log S)^0, b^2 \log S). \end{aligned} \quad (2.52)$$

In the terms of the above equation we have kept the leading order in the spin S at each order in the b^2 expansion. For the coefficient of the $O(b^0)$ term we have neglected terms which are of $O(\log S)^0$. In the coefficient of $O(b^2)$ term we have neglected terms which are of $O(b^2 \log S)$. We can rewrite this dispersion relation in terms of the physical charges as

$$\Delta = \mathcal{S} + \frac{\sqrt{\lambda}}{\pi} \log \mathcal{S} - \frac{b^2 \sqrt{\lambda}}{2\pi} \log^2 \mathcal{S}. \quad (2.53)$$

Therefore the dispersion relation of the large spinning string is corrected at $O(b^2)$. The leading correction at this order is given by $-\frac{b^2 \sqrt{\lambda}}{2\pi} \log^2 \mathcal{S}$. Note that it is clear from our analysis that we have first performed a perturbation in b and then at each order extracted out the leading behaviour in the spin $\mathcal{S}$. We will arrive at the above dispersion relation in the scaling limit of the long string solution in the next section.

2.4 Scaling limit of the long string

There is a further limit of the long string solution in which the solution simplifies [8]. This limit is known as the scaling limit. In this limit it is possible to write down the functional dependence of the radial co-ordinate r on the world sheet σ in terms of a simple function rather than hypergeometric function or elliptic functions. The scaling limit of the long string solution in the absence of the NS-NS B-field has been mapped by $SO(2, 2)$ transformations and world sheet re-parametrizations to the minimal surface corresponding to the cusp Wilson line [8]. One of the goals of this paper is to obtain this mapping in the presence of the NS-NS flux. With this motivation we will study this scaling limit.

$b = 0$. Let us first review the scaling limit of [8] with $b = 0$ in our language which will enable the generalization to the situation with the NS-flux. From the equations (2.48) we see that, the constants c_1 and c_2 are related by

$$c_2 = c_1 + O(c_1 e^{-c_1}). \quad (2.54)$$

Now in the large S limit, (2.49) and (2.51) implies that c_1 is large so we can ignore the exponentially suppressed term. Therefore we can look for a solution with $c_1 = c_2$ to begin with, this solution is the scaling limit of the long string. From (2.11) and $b = 0$ we have

$$t = c_1 \tau, \quad \tilde{\phi} = c_1 \tau. \quad (2.55)$$

The differential equation for r given in (2.13) reduces to³

$$r'^2 = c_1^2 (r^2 + 1). \quad (2.56)$$

³We have set $\omega = 0$.

To ensure that the solution is a closed folded string, the solution is allowed to grow from $r = 0$ to $r_{\max}$ from $\sigma = 0$ to $\sigma = \pi/2$, then r decreases back to zero as σ goes from $\pi/2$ to π . The same motion repeats for the interval $\pi < \sigma \leq 2\pi$. Integrating the equation (2.56) we obtain

$$r = \sinh(c_1\sigma), \quad 0 \leq \sigma \leq \frac{\pi}{2}. \quad (2.57)$$

The energy and spin of the solution is given by

$$E = \frac{4c_1}{2\pi} \int_0^{r_{\max}} \sqrt{1+r^2} dr, \quad S = \frac{4c_1}{2\pi} \int_0^{r_{\max}} \frac{r^2}{\sqrt{1+r^2}} dr. \quad (2.58)$$

Here we have used the fact that S receives contribution from the 4 segments of the folded string and used (2.56) to write the integral in terms of the radial co-ordinate. Integrating the equation for S and using the fact that c_1 is large we obtain

$$S \sim \frac{1}{2\pi} e^{c_1\pi}. \quad (2.59)$$

We can now substitute for c_1 in the dispersion relation (2.33) to obtain

$$E - S = \frac{1}{\pi} \log S \quad (2.60)$$

Re-writing this in terms of physical charges we obtain

$$\Delta - \mathcal{S} = \frac{\sqrt{\lambda}}{\pi} \log \mathcal{S}. \quad (2.61)$$

Note that in the limit $c_1 = c_2$ we have obtained the dispersion relation of the long string. Further more there exists a simple and explicit expression for the solution for the solution $r(\sigma)$.

$b \neq 0$. Let us now generalize the scaling limit in the presence of the NS-flux. As before we choose the following ansatz for the solution

$$t = c_1\tau + t(\sigma), \quad \tilde{\phi} = c_2\tau + \phi(\sigma), \quad r = r(\sigma), \quad (2.62)$$

This ansatz is the same as the one considered in the previous section but with $c_1 = c_2$. The Virasoro constraints (2.12) then reduces to

$$k_1 = k_2 \quad (2.63)$$

From (2.11) we see that the conservation laws for t and ϕ is given by

$$t' = \frac{bc_1r^2 - k_1}{1+r^2}, \quad \phi' = \frac{bc_1r^2 - k_1}{r^2}. \quad (2.64)$$

As we have discussed in the previous section we look for a solution in which k_1 admits a power series in b as given in (2.15). Finally the Virasoro equation (2.13) takes the following form

$$r'^2 r^2 = \tilde{B}r^4 + \tilde{C}r^2 + \tilde{D}, \quad (2.65)$$

where

$$\tilde{B} = c_1^2 - c_1^2 b^2, \quad \tilde{C} = c_1^2 + 2bc_1 k_1, \quad \tilde{D} = -k_1^2. \quad (2.66)$$

The equation for r' simplifies to a quadratic polynomial in r^2 . Let the two roots of the polynomial be $\tilde{R}_1$ and $\tilde{R}_2$, then we have

$$r' = \frac{1}{r} \sqrt{\tilde{B}(r^2 - \tilde{R}_1)(r^2 - \tilde{R}_2)}, \quad (2.67)$$

where to $O(b^2)$ the roots are given by

$$\tilde{R}_1 = \frac{(k_1^{(0)})^2}{(c_1^{(0)})^2} b^2, \quad \tilde{R}_2 = -1 - \frac{(c_1^{(0)} + k_1^{(0)})^2}{(c_1^{(0)})^2} b^2. \quad (2.68)$$

The turning points to integrate the equation for r' is root R_1 and the point $r_{\max}$ which is reached at $\sigma = \pi/2$. The solution for $0 < \sigma < \pi/2$ is given by

$$r^2 = \frac{e^{-2\sqrt{\tilde{B}}\sigma}}{4\alpha\tilde{B}} \left\{ \left(\alpha e^{2\sqrt{\tilde{B}}\sigma} - \tilde{C} \right)^2 - 4\tilde{B}\tilde{D} \right\}, \quad (2.69)$$

$$\alpha = 2\sqrt{\tilde{B}} \sqrt{\tilde{B}\tilde{R}_1^2 + \tilde{C}\tilde{R}_1 + \tilde{D} + 2\tilde{B}\tilde{R}_1 + \tilde{C}}.$$

It can be seen that on setting $b = 0$, this solution reduces to $r = \sinh c_1 \sigma$.

Let us now study the closed string boundary conditions $t(\sigma) = t(\sigma + 2\pi)$. As discussed in the previous section to obtain dispersion relation to $O(b^2)$ it is sufficient to study this constraint to the linear order in b . Integrating the equation for t' in (2.64) and after imposing closed string boundary conditions we obtain

$$\int_0^{2\pi} t' d\sigma = 4 \int_{\sqrt{\tilde{R}_1}}^{r_{\max}} \left(bc_1 - \frac{bc_1 + k_1}{1 + r^2} \right) \frac{dr}{r'} = 0. \quad (2.70)$$

To the linear order in b , the above equation reduces to

$$b \int_0^{r_{\max}} \left(c_1^{(0)} - \frac{c_1^{(0)} + k_1^{(0)}}{1 + r^2} \right) \frac{dr}{\sqrt{1 + r^2}} = 0. \quad (2.71)$$

Note that we have substituted the zeroth order values for the limits and for r' . This equation can be easily integrated by change of variables to the σ coordinate. This leads to the following equation for $k_1^{(0)}$

$$bc_1^{(0)} \frac{\pi}{2} - \frac{bc_1^{(0)} + bk_1^{(0)}}{c_1^{(0)}} \tanh \left(c_1^{(0)} \frac{\pi}{2} \right) = 0. \quad (2.72)$$

Since we are working in the large string limit we can approximate $\tanh(c_1^{(0)} \frac{\pi}{2}) \sim 1$. Therefore we obtain

$$k_1^{(0)} = \frac{(c_1^{(0)})^2 \pi}{2} - c_1^{(0)}. \quad (2.73)$$

Note that this is identical to the value obtained in the previous section in equation (2.46) without taking the scaling limit. Thus we obtain $k_1^{(0)} = k_2^{(0)} > 0$ since $c_2^{(0)} = c_1^{(0)}$ is large. Let us now integrate the equation for ϕ' , the solution is given by

$$\begin{aligned}\tilde{\phi} &= c_1\tau + \int_0^\sigma \phi' d\sigma, \\ &= c_1\tau + \int_{\sqrt{R_1}}^{r(\sigma)} \frac{\phi'}{r'} dr.\end{aligned}\tag{2.74}$$

Substituting the expression for r', ϕ' and integrating we obtain

$$\tilde{\phi} = c_1\tau + c_1b\sigma - \frac{k_2}{|k_2|} \tan^{-1} \left(\sqrt{\left| \frac{R_2}{R_1} \right|} \sqrt{\frac{r^2(\sigma) - R_1}{r^2(\sigma) - R_2}} \right).\tag{2.75}$$

To obtain this solution we have used the relation $k_2^2 = -\tilde{B}R_1R_2$ which follows from the definition of the roots of the quadratic polynomial in (2.65) and the Virasoro constraint (2.63). To the leading order in b , the solution is given by

$$\phi = c_1^{(0)}\tau + c_1^{(0)}b\sigma - \tan^{-1} \left(\frac{c_1^{(0)}}{bk_1^{(0)}} \tanh(c_1^{(0)}\sigma) \right).\tag{2.76}$$

Therefore the difference in the end points in the ϕ direction for a single period is given by

$$\phi(2\pi) - \phi(0) = 2\pi c_1^{(0)}b - (2m+1)\frac{\pi}{2} + 4b\frac{k_1^{(0)}}{c_1^{(0)}}.\tag{2.77}$$

Here we have used the fact that the integral over r can be broken up to 4 integrals from 0 to $r_{\max}$ and $r_{\max} = \sinh(c_1^{(0)}\frac{\pi}{2}) \sim \cosh(c_1^{(0)}\frac{\pi}{2})$ since $c_1^{(0)}$ is large. From the definition of $\delta\phi$ in (2.43) we obtain

$$\begin{aligned}\delta\phi &= 2\pi c_1^{(0)}b + 4b\frac{k_1^{(0)}}{c_1^{(0)}}, \\ &= (4\pi c_1^{(0)} - 4)b.\end{aligned}\tag{2.78}$$

In the second line of the above equation we have used (2.73). Therefore in general the string does not close in the ϕ direction analogous to the giant magnon solution of [32]. But if one considers the difference after N periods with N large enough in the σ direction we can make the difference and integer multiple of 2π . That is we can have

$$N\delta\phi = N(4\pi c_1^{(0)} - 4)b = 2m'\pi,\tag{2.79}$$

where m' is an integer. We see that we can ensure this mild quantization condition on $\delta\phi$ by either choosing b or $c_1^{(0)}$ to satisfy the above condition. Note that it does not put any restriction on $c_1^{(0)}$ or b , that is we can still have $c_1^{(0)} \rightarrow \infty$, $b \ll 1$ and $c_1^{(0)}b \ll 1$. This is seen as follows, since we are working in the large $c_1^{(0)}$ limit the condition in (2.79) reduces to $2Nc_1^{(0)}b = m$. By choosing N sufficiently large we can ensure that the condition $c_1^{(0)}b \ll 1$

necessary for perturbation theory is satisfied.⁴ This implies we can make the string closed after a sufficiently large period in the σ direction.

By using the expression for t' given in (2.64) the equation for the spin S in (2.27) can be written as

$$S = \int_0^{2\pi} (r^2 c_1 (1 - b^2) + b k_1) \frac{d\sigma}{2\pi}. \quad (2.80)$$

Here we have using the closed string boundary condition on the co-ordinate t .⁵ Substituting the explicit solution for $r(\sigma)$ given in (2.69) and performing the integration in σ we obtain the following expression for the spin to $O(b^2)$.

$$2\pi S = (\sinh(\pi c_1) - \pi c_1) + \frac{b^2}{2c_1^2} \left(-\pi c_1^3 \cosh(\pi c_1) + c_1^2 \sinh(\pi c_1) + 4k_{10}^2 \sinh(\pi c_1) + 4c_1 k_1^{(0)} \sinh(\pi c_1) \right). \quad (2.81)$$

We can now solve for c_1 perturbatively in b by assuming an expansion of the form $c_1 = c_1^{(0)} + b^2 c_1^{(1)}$. Substituting this expansion in (2.81) and matching orders to b^2 we obtain

$$c_1^{(0)} = \frac{\log S}{\pi}, \quad c_1^{(1)} = -\pi \frac{(c_1^{(0)})^2}{2} = -\frac{1}{2\pi} \log^2 S. \quad (2.82)$$

We have taken the large $c_1^{(0)}$ limit when we solve for each order in b .⁶ We have also used the value of $k_1^{(0)}$ obtained in (2.73). Substituting this value of c_1 to order b^2 into the dispersion relation, we obtain

$$E - S = c_1^{(0)} + b^2 c_1^{(1)} = \frac{\log S}{\pi} - b^2 \frac{\log^2 S}{2\pi}. \quad (2.83)$$

Note that after expressing this dispersion relation in terms of physical charges agrees with that obtained in (2.61) for the spinning string without the scaling limit.

The scaling limit enables one to find a closed form expression for the solution as a function of σ to $O(b^2)$. We can write down the solutions explicitly to the order of b^2 now. Using (2.69) we write down the solution in the global coordinate r and $\rho = \sinh^{-1} r$ in order of b .

$$\begin{aligned} r &= \sinh(c_1^{(0)} \sigma) + \frac{1}{8} b^2 \{ 4 \sinh(c_1^{(0)} \sigma) - 4 c_1^{(0)} (\pi c_1^{(0)} + 1) \sigma \cosh(c_1^{(0)} \sigma) + \\ &\quad + (\pi c_1^{(0)} - 2) (\pi c_1^{(0)} \cosh(2 c_1^{(0)} \sigma) - 2) \operatorname{csch}(c_1^{(0)} \sigma) \}, \\ \rho &= c_1^{(0)} \sigma + \frac{1}{8} b^2 \left\{ c_1^{(0)} \left(\pi^2 c_1^{(0)} \tanh(c_1^{(0)} \sigma) - 4 (\pi c_1^{(0)} \sigma + \sigma) \right) + (\pi c_1^{(0)} - 2)^2 \coth(c_1^{(0)} \sigma) \right\}. \end{aligned} \quad (2.84)$$

In writing down the expansions in b we have assumed σ is sufficiently away from 0. If σ is close to zero one needs to use the expression in (2.69) directly. Now plugging in the explicit

⁴We need $c_1^{(0)} b \ll 1$ so that $b \log S \ll 1$, this ensures that the perturbative expansion in the dispersion relation makes sense.

⁵We have satisfied this to the linear order in b in (2.70).

⁶Note here again we have first performed the perturbation in b and then at each order in b retained the contribution from the spin.

value of r in (2.64) and using the value of $c_1^{(1)}$ from (2.82) in the expansion of c_1 , we obtain the expression of t and ϕ in terms of τ and σ .

$$\tilde{t} = c_1^{(0)}\tau - \frac{b^2(c_1^{(0)})^2\pi}{2}\tau + \frac{1}{2}bc_1^{(0)}\left\{2\sigma - \pi \tanh(c_1^{(0)}\sigma)\right\}. \quad (2.85)$$

The solution for ϕ can be obtained as an expansion in b from (2.76). This results in

$$\tilde{\phi} = c_1^{(0)}\tau - \frac{b^2(c_1^{(0)})^2\pi}{2}\tau + c_1^{(0)}b\sigma + \frac{b}{2}(\pi c_1^{(0)} - 2)\coth(c_1^{(0)}\sigma) - b(\pi c_1^{(0)} - 1). \quad (2.86)$$

Note that again this expansion is valid for σ sufficiently away from the origin, if σ is close to the origin we can use the solution in (2.76). We have chosen an integration constant in ϕ such that $\phi(\tau = 0, \frac{\pi}{2})$ vanishes for $c_1^{(0)}$ large. This done for convenience in later manipulations.

Let us now re-write the solution is obtained after scaling the worldsheet coordinates by redefining $(\sigma', \tau') = (c_1^{(0)}\sigma, c_1^{(0)}\tau)$. This rescaling of σ with $c_1^{(0)} \rightarrow \infty$ effectively decompactifies σ . After the scaling the solutions look like the following

$$\begin{aligned} \rho &= \sigma + \frac{1}{8}b^2\left\{\pi^2(c_1^{(0)})^2\tanh(\sigma) - 4(\pi c_1^{(0)} + 1)\sigma + (\pi c_1^{(0)} - 2)^2\coth(\sigma)\right\}, \\ \tilde{t} &= \tau - \frac{1}{2}\pi b^2 c_1^{(0)}\tau + b\sigma - \frac{1}{2}\pi b c_1^{(0)}\tanh(\sigma), \\ \tilde{\phi} &= \tau - \frac{1}{2}\pi b^2 c_1^{(0)}\tau + b\sigma + \frac{1}{2}b(\pi c_1^{(0)} - 2)\coth(\sigma) - b(\pi c_1^{(0)} - 1). \end{aligned} \quad (2.87)$$

where we have dropped the primes from τ and σ . One can check that this satisfies all the equations of motions and the Virasoro constraints of the sigma model. We will comment on the choice of integration constants in the solution given in (2.87) later.

2.5 Small string limit

We now consider the opposite limit, that is the limit in which the extent of the string in the r direction is small. To obtain the limit we first set $k_1 = k_2 = 0$. The second Virasoro constraint given in (2.12) is automatically satisfied. Using the Virasoro constraint corresponding to the vanishing of the world sheet energy for this situation leads to the following equation for the radial co-ordinate r

$$r'^2 = \hat{A}r^4 + \hat{B}r^2 + C = \hat{A}(r^2 - \hat{R}_1)(r^2 - \hat{R}_2), \quad (2.88)$$

where

$$\hat{A} = c_1^2 + c_2^2 b^2 - c_2^2 - c_1^2 b^2, \quad \hat{B} = 2c_1^2 - c_2^2 - c_1^2 b^2 - \omega^2, \quad \hat{C} = c_1^2 - \omega^2. \quad (2.89)$$

The situation now is analogous to the case of $b = 0$ in which there were only 2 roots. The roots of the quadratic equation in (2.88) to order b^2 are given by

$$\hat{R}_1 = -\frac{b^2 c_2^2}{c_2^2 - \omega^2} - 1, \quad \hat{R}_2 = \frac{b^2 \omega^2 (c_1^2 - \omega^2)}{(c_1^2 - c_2^2)(c_2^2 - \omega^2)} + \frac{\omega^2 - c_1^2}{c_1^2 - c_2^2}. \quad (2.90)$$

The solution for $r(\sigma)$ now begins at the origin reaches $\sqrt{R_2}$ at $\sigma = \pi/2$ and then turns back and returns to the origin in the next quarter period. The same behaviour is repeated in the interval $\pi < \sigma \leq 2\pi$. Therefore integrating the equation for r leads to

$$2\pi = \int_0^{2\pi} d\sigma = 4 \int_0^{\sqrt{\hat{R}_2}} \frac{dr}{\sqrt{\hat{A}(r^2 - \hat{R}_1)(r^2 - \hat{R}_2)}}. \quad (2.91)$$

After similar manipulations the integral can be performed in terms of the hypergeometric function and it leads to

$$\sqrt{\hat{R}_1 \hat{A}} = {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}, 1, \frac{\hat{R}_2}{\hat{R}_1} \right). \quad (2.92)$$

The small string approximation is essentially the fact that the maximum extent of the string $\hat{R}_2 \rightarrow 0$. In this limit, the hypergeometric function just reduces to 1. Therefore we obtain the relation

$$\hat{R}_1 \hat{A} = 1. \quad (2.93)$$

Substituting the expressions for $\hat{R}_1, \hat{A}$ from (2.90) leads to the constraint

$$c_2^2 - c_1^2 - 1 + b^2 \omega^2 \left(\frac{c_1^2 - c_2^2}{\omega^2 - c_2^2} \right) = 0. \quad (2.94)$$

Let us now examine if the closure condition for t is satisfied. Integrating the equation for t' in (2.11) with $k_1 = 0$ leads to

$$t(2\pi) - t(0) = 2\pi b c_2 - b c_2 \int_0^{2\pi} \frac{d\sigma}{1 + r^2}. \quad (2.95)$$

Again performing the same manipulations in the integrand and examining the condition to the leading order gives rise to the equation

$$c_2^{(0)} - \frac{c_2^{(0)}}{\sqrt{(c_2^{(0)})^2 - (c_1^{(0)})^2}} {}_2F_1 \left(\frac{3}{2}, \frac{1}{2}, 1, \frac{\hat{R}_2}{\hat{R}_1} \right) = 0. \quad (2.96)$$

Approximating the hypergeometric function by unity since the $R_2 \rightarrow 0$ and using the fact $(c_2^{(0)})^2 - (c_1^{(0)})^2 = 1$ from (2.94), we see that the condition for periodicity in t is satisfied. The equation of motion for ϕ is just $\phi' = b c_1$, here again the string does not close, but as discussed earlier one can consider several periods and ensure the string closes in the ϕ direction.

Let us now obtain the dispersion relation for the string. For this we need the expression for the spin which is given by

$$\begin{aligned} S &= \int_0^{2\pi} [c_2(1 - b^2)r^2] \frac{d\sigma}{2\pi}, \\ &= \frac{\hat{R}_2 c_2(1 - b^2)}{2\sqrt{A\hat{R}_1}} {}_2F_1 \left(\frac{1}{2}, \frac{3}{2}, 2, \frac{\hat{R}_2}{\hat{R}_1} \right), \\ &= \frac{\hat{R}_2}{2} c_2(1 - b^2). \end{aligned} \quad (2.97)$$

To obtain the first line we have used the closed string boundary condition on t . In the last line of the above equation we have approximated the hypergeometric equation by unity and also used (2.93). We now have all the ingredients to obtain the dispersion relation. We first parametrize c_2 by

$$c_2^2 = 1 + \omega^2 + x. \quad (2.98)$$

From (2.94) we obtain c_1 as

$$c_1 = \frac{b^2 \omega^2}{2(x+1)\sqrt{\omega^2+x}} + \sqrt{\omega^2+x} \quad (2.99)$$

We now examine the situation in which $x \ll 1$ studied earlier for the situation with $b = 0$ in [2]. We plug these values of c_1 and c_2 in the equation for spin (2.97) and expand the expression to the linear order of x .

$$S = \frac{1}{2}b^2\sqrt{\omega^2+1}\omega^2 + \frac{x(-2b^2\omega^4 - 3b^2\omega^2 - 2b^2 + 2\omega^2 + 2)}{4\sqrt{\omega^2+1}} \quad (2.100)$$

We then solve for x in terms of S and ω . To the order of b^2 it is

$$x = \frac{2S}{\sqrt{\omega^2+1}} + \frac{b^2(2S\omega^4 + 3S\omega^2 + 2S - \sqrt{\omega^2+1}\omega^2 - \sqrt{\omega^2+1}\omega^4)}{(\omega^2+1)^{3/2}} \quad (2.101)$$

So now we can write down c_1 and c_2 in terms of S and ω and now we can write down the dispersion relation.

$$\begin{aligned} c_1 &= \sqrt{\frac{2S}{\sqrt{\omega^2+1}} + \omega^2} + \\ &+ \frac{b^2 S \sqrt{\frac{2S}{\sqrt{\omega^2+1}} + \omega^2}}{2(\omega^2+1)(2S + \sqrt{\omega^2+1})(4S^2\sqrt{\omega^2+1} + 2S(\omega^2+1)^2 + \omega^2(\omega^2+1)^{3/2})} \times \\ &\times (4S^2\sqrt{\omega^2+1}(2\omega^4+3\omega^2+2) + 4S(\omega^6+3\omega^4+4\omega^2+2) + \sqrt{\omega^2+1}(\omega^4+3\omega^2+2)) \\ c_2 &= \sqrt{\frac{2S}{\sqrt{\omega^2+1}} + \omega^2 + 1} + \frac{b^2(S(2\omega^4+3\omega^2+2) - \omega^2(\omega^2+1)^{3/2})}{2(\omega^2+1)^{3/2}\sqrt{\frac{2S}{\sqrt{\omega^2+1}} + \omega^2 + 1}} \end{aligned} \quad (2.102)$$

We then find the dispersion relation to the order of b^2 in the limit $\omega \gg 1$

$$\begin{aligned} E &= \frac{c_1}{c_2}S + c_1 \\ &= \omega + S + \frac{S}{2\omega^2} + \frac{b^2S}{2} - \frac{b^2S}{4\omega^2} + O(S^2) + O\left(\frac{1}{\omega^3}\right) \end{aligned} \quad (2.103)$$

To obtain this we have neglected higher order terms in the spin S . This is consistent in the small string limit since in this approximation we have to neglect higher powers of x . Let us now recast this dispersion relation in the conventional form by re-defining the spin to $O(b^2)$ as

$$\tilde{S} = \left(1 + \frac{b^2}{2}\right)S \quad (2.104)$$

Then in terms of this rescaled spin,⁷ we obtain the dispersion relation

$$E = \omega + \tilde{S} + \frac{\tilde{S}}{2\omega^2} - b^2 \frac{\tilde{S}}{2\omega^2}. \quad (2.105)$$

We can now write the dispersion relation in terms of macroscopic charges by re-instating $\sqrt{\lambda}$. This results in

$$\Delta - J = \mathcal{S} + \frac{\lambda}{2J^2}(1 - b^2)\mathcal{S}. \quad (2.106)$$

The dispersion relation for the small string can be compared to that of the plane wave spectrum in presence of the mixed 3-form fluxes. This was derived in [33], the dispersion relation is given by⁸

$$\Delta - J = \sum_n N_n \left(1 + 2 \frac{\sqrt{\lambda} n}{J} \sin \beta + \frac{\lambda n^2}{J} \right)^{\frac{1}{2}}. \quad (2.107)$$

We have defined $\beta = \frac{\pi}{2} - \alpha$, where $\sin \beta$ is the coefficient of the Neveu-Schwarz flux. We have also identified $\frac{\sqrt{\lambda}}{J} = \frac{1}{\alpha' p^+}$ and have chosen $\mu = 1$. Let us now expand the square root in (2.107) and also use the level matching condition $\sum_{n=-\infty}^{\infty} n N_n = 0$. We then obtain

$$\Delta - J = \sum_n N_n + \left(\sum_n n^2 N_n \right) \frac{\lambda}{2J^2} (1 - \sin^2 \beta) + O(1/J^3). \quad (2.108)$$

Following [2] we identify the charge $\mathcal{S}$ with the excited state having quantum numbers $n = 1, N_1 = \frac{\mathcal{S}}{2}, n = -1, N_{-1} = \frac{\mathcal{S}}{2}$. We then see that the plane wave dispersion relation (2.108) precisely matches with that obtained from the small string in (2.106) for Neveu-Schwarz flux $\sin \beta = b$.

3 Minimal surfaces with mixed form fields

In 4 dimensional conformal field theories the anomalous dimensions of high spin twist two operators is related to the logarithmic divergence of the expectation value of the Wilson loop which has a cusp in its contour. This relationship can be established entirely in the field theory. In the bulk it was shown in [8] that the classical solution of spinning strings in the scaling limit is related by conformal transformations and re-parametrization to the minimal surface corresponding to the cusped Wilson loop. The 2 dimensional conformal field theories dual to the $AdS_3 \times S^3$ backgrounds are not as well understood as those in 4 dimensions. Therefore it is interesting to ask the question if the anomalous dimensions of high spin operators determines the cusp anomaly for these 2 dimensional theories. If this question is posed in the bulk AdS_3 background supported with purely RR 3-form flux then the same conformal transformations and re-parametrizations found in [8] is sufficient to relate the two classical solutions. This is because the spinning string and minimal surface can be embedded in AdS_3 .

⁷Note that this is just a redefinition of what is called the spin.

⁸See equation (C.3) of [33].

It is less clear how to relate the spinning string in the AdS_3 background supported with NS-NS 3-form flux to minimal surfaces corresponding to cusped Wilson loops. To establish this relation in section 3.1 we first study minimal surfaces which end on a light like cusp. We show that the equation of motion for the minimal surface can be solved exactly to all orders in the NS-field for the special case called the ‘uniform’ minimal surface. In general the equation of motion admits a solution in terms of a perturbative expansion in b . We then evaluate the area of the minimal surface and show that the coefficient of logarithmic divergence of the area proportional to b^2 is precisely the coefficient of the $b^2 \log^2 S$ in the anomalous dimensions of the spinning string solution. In section 3.2 starting from the scaling limit of the spinning string solution given in (2.87) to $O(b^2)$ we perform $SO(2,2)$ transformations and re-parametrizations to relate the solution to the minimal surface corresponding to the cusped Wilson loop. This minimal surface is a one parameter generalization of the ‘uniform’ minimal surface depending of the parameter $c_1^{(0)}$ of the spinning string solution in (2.87). We will see that the logarithmic divergence of the area of the minimal surface is however universal and independent of this parameter and precisely the coefficient of $b^2 \log^2 S$ of the spinning string solution.

3.1 Cusp anomalous dimensions from gravity

We look for minimal surfaces which end on a light like cusp in the Poincaré co-ordinates of AdS_3 given by the following metric

$$ds^2 = \frac{1}{z^2}(-dx_0^2 + dx_1^2 + dz^2) \quad (3.1)$$

The NS-NS 2-form in this co-ordinate system is given by

$$B_{x_0 x_1} = -\frac{b}{z^2} \quad (3.2)$$

It can be easily verified that this background is a solution to the bulk equations of motion. The minimal surface is a solution to the equations of the Nambu-Goto action given by

$$S_{NG} = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \left(\sqrt{\det(G_{\mu\nu} \partial_a X^\mu \partial_b X^\nu)} + \frac{i}{2} \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu \right). \quad (3.3)$$

The equations of motion are affected by the presence of the NS field, therefore the minimal surface corresponding to the cusp Wilson line found by [6] will be modified. Note that we work with the Euclidean world sheet action. The same analysis can be performed in the Minkowski signature on the world sheet with identical results as was done in the absence of the NS-NS 3-form flux in [34]. We look for a minimal surface with the following ansatz

$$\Lambda z = e^{\tau-\sigma} g(\sigma), \quad \Lambda x_0 = e^{\tau-\sigma} \cosh(\sigma + \tau), \quad \Lambda x_1 = e^{\tau-\sigma} \sinh(\sigma + \tau), \quad (3.4)$$

where σ, τ are the world sheet co-ordinates and Λ is an arbitrary scale. We will show that the action of the minimal surface is independent of the scale Λ . The ansatz in (3.4) satisfies the property that

$$\frac{z^2}{x_0^2 - x_1^2} = g^2. \quad (3.5)$$

Therefore if g does not vanish at any point in the co-ordinate σ , the surface reduces to a light cone at the boundary of AdS_3 at $z = 0$. The equations of motion for $g(\sigma)$ following from the Nambu-Goto action is given by

$$8 + 12gg' + 3g^2 [(g')^2 - 4] - g^3 (g'' + 6g') + 4g^4 = i8b (-gg' + g^2 - 1)^{3/2}, \quad (3.6)$$

where the primes refer to derivatives with respect to σ . The above differential equation admits a simple exact solution if we assume $g' = 0$ and therefore $g = c$ where c is a constant. From (3.6) we see that c then must satisfy the algebraic equation

$$c^2 - 2 = i2b\sqrt{(c^2 - 1)}. \quad (3.7)$$

The solutions to this equation are given by

$$\begin{aligned} c &= \sqrt{2}(1 - b^2 \pm ib\sqrt{1 - b^2})^{\frac{1}{2}}, \\ &= \sqrt{2} \pm i\frac{b}{\sqrt{2}} - \frac{3}{4\sqrt{2}}b^2 + \dots \end{aligned} \quad (3.8)$$

Here we have kept the roots which reduce to the solution found by [6] at $b = 0$. Note that this is a complex solution to the equations of the motion when $b \neq 0$. We call this solution the ‘uniform’ solution since $g(\sigma)$ does not depend on the world sheet co-ordinate σ . In general one can solve the equation of motion given in (3.6) perturbatively in b for g which is not uniform in the world sheet co-ordinate. We will see that the spinning string solution in (2.87) can be mapped to a non-uniform solution but in a further scaling limit it reduces to the uniform solution.

Let us now evaluate the action of the ‘uniform’ solution. We follow the regularization procedure adopted by [6]. First we define the following world sheet coordinates.

$$\rho = \tau - \sigma, \quad \xi = \tau + \sigma. \quad (3.9)$$

In these coordinates the induced metric on the world sheet is given by

$$ds^2 = \frac{1}{c^2} \left(\frac{c^2 - 1}{\rho^2} d\rho^2 + d\xi^2 \right). \quad (3.10)$$

Following [6] we take the range for the coordinates ρ, ξ to be (ϵ, L) and $(-\gamma/2, \gamma/2)$ respectively. γ is then the cusp angle, ϵ, L are the UV and IR cutoff’s respectively. Substituting the induced metric into the Nambu-Goto action (3.3) we obtain

$$\begin{aligned} S_{NG} &= \frac{\sqrt{\lambda}}{2\pi} \int_{-\gamma/2}^{\gamma/2} \int_{\epsilon}^L \left(\sqrt{G} + i\frac{b}{z^2} (x'_0 x_1 - x'_1 x_0) \right) d\rho d\xi \\ &= \frac{\sqrt{\lambda}}{2\pi} \left[\frac{\sqrt{c^2 - 1}}{c^2} + i\frac{b}{c^2} \right] \gamma \log \frac{L}{\epsilon}. \end{aligned} \quad (3.11)$$

Substituting the solution for c from (3.8) and expanding in powers of b we obtain

$$S_{NG} = \frac{\sqrt{\lambda}}{2\pi} \left[\frac{(1 - 2b^2 \pm 2ib\sqrt{1 - b^2})^{1/2} + ib}{2(1 - b^2 \pm ib\sqrt{1 - b^2})} \right] \gamma \log \frac{L}{\epsilon}, \quad (3.12)$$

$$= \frac{\sqrt{\lambda}}{2\pi} \left[\frac{1}{2} \pm i \frac{b}{2} - \frac{b^2}{4} + O(\tilde{b}^3) \right] \gamma \log \frac{L}{\epsilon}.$$

Therefore the expectation value of the light like Wilson loop is given by

$$\langle W \rangle = \exp(-S_{NG}) = \left(\frac{L}{\epsilon} \right)^{-\frac{\sqrt{\lambda}}{2\pi} \left[\frac{1}{2} \pm i \frac{b}{2} - \frac{b^2}{4} + O(\tilde{b}^3) \right] \gamma}. \quad (3.13)$$

The imaginary term just contributes to a phase. Note however the modulus of the expectation value or the area is corrected at $O(b^2)$. The coefficient of this correction is precisely 1/2 of the leading term and of the opposite sign. This is precisely the behaviour of the coefficient of leading correction at $O(b^2 \log^2 S)$ of the anomalous dimension seen in (2.33). Though this analysis has been done in Euclidean world sheet signature, it can be repeated with Minkowski world sheet signature to arrive at the same conclusion as done for the $b = 0$ case in [34].

3.2 Relating the Wilson loop and the spinning string

We have observed that the $O(b^2)$ correction of the cusp anomaly precisely agrees with the $O(b^2 \log^2 S)$ term in the anomalous dimension of the spinning string. Just as the $O(b^0)$ of the cusp anomaly agrees with the $O(\log S)$ term in the dispersion relation of the spinning string. To relate these solutions further we follow the method of [8] and perform a set of conformal transformation and re-parametrizations starting from the scaling limit of the spinning string solution in (2.87) and arrive at the light like Wilson loop. We will see that after a further scaling limit, the solution is precisely that of the ‘uniform’ Wilson loop.

To begin let us define the embedding co-ordinates in which AdS_3 is a hyperboloid and its relationship with the AdS_3 global co-ordinate in which the solution (2.87) is written down. The hyperboloid is defined by the constraint

$$-X_0^2 - X_3^2 + X_1^2 + X_2^2 = -1. \quad (3.14)$$

and the metric in the embedding space is given by

$$ds^2 = -dX_0^2 - dX_3^2 + dX_1^2 + dX_2^2. \quad (3.15)$$

The relationship between these co-ordinates and the global co-ordinates of AdS_3 given in (2.1) is given by

$$\begin{aligned} X_0 &= \cosh \rho \cos t, & X_3 &= \cosh \rho \sin t, \\ X_1 &= \sinh \rho \cos \phi, & X_2 &= \sinh \rho \sin \phi. \end{aligned} \quad (3.16)$$

We will now outline the transformations to relate the spinning solution (2.87) to the light-like Wilson loop.

1. We first analytically continue the Minkowski world sheet to Euclidean by replacing

$$\tau = i\tilde{\tau}. \quad (3.17)$$

For convenience we also replace

$$b = i\tilde{b}. \quad (3.18)$$

This is so that we do not have to deal with intermediate factors of i . At the end of all the steps we will reinstate b . On performing this we see that the solution in the embedding co-ordinates is of the form

$$\begin{aligned} X_0 &= \cosh \rho \cosh \tilde{t}, & X_3 &= i \cosh \rho \sinh \tilde{t}, \\ X_1 &= \sinh \rho \cosh \tilde{\phi}, & X_2 &= i \sinh \rho \sinh \tilde{\phi}. \end{aligned} \quad (3.19)$$

where $\rho, \tilde{t}, \tilde{\phi}$ are the solutions given in (2.87) but with τ, b replaced with $\tilde{\tau}, \tilde{b}$ respectively.

2. The next step is to factor out the pre-factor i in X_3 and X_2 in (3.19) and then exchange $3 \rightarrow 2$. It is clear from that this operation still preserves the constraint (3.14). Therefore we obtain the solution

$$\begin{aligned} X'_0 &= \cosh \rho \cosh \tilde{t}, & X'_2 &= \cosh \rho \sinh \tilde{t}, \\ X'_1 &= \sinh \rho \cosh \tilde{\phi}, & X'_3 &= \sinh \rho \sinh \tilde{\phi}. \end{aligned} \quad (3.20)$$

3. We now perform 2 rotations in the $0-3$ plane and $2-1$ plane, each with angle $\pi/4$. Therefore we obtain the solution

$$\begin{aligned} X_0^w &= \frac{X'_0 + X'_3}{\sqrt{2}}, & X_3^w &= \frac{X'_0 - X'_3}{\sqrt{2}}, \\ X_1^w &= \frac{X'_2 + X'_1}{\sqrt{2}}, & X_2^w &= \frac{X'_2 - X'_1}{\sqrt{2}}. \end{aligned} \quad (3.21)$$

This solution is the Wilson loop in the embedding co-ordinates.

4. Finally we write the solution (3.21) in the Poincaré patch by using the following relations

$$X_3^w - X_2^w = \frac{1}{\Lambda z}, \quad X_0^w = \frac{x_0}{z}, \quad X_1^w = \frac{x_1}{z}, \quad X_3^w + X_2^w = \Lambda \left(z + \frac{x^2}{z} \right). \quad (3.22)$$

where $x^2 = -x_0^2 + x_1^2$. Recall that z, x^0, x^1 are the co-ordinates in the Poincaré patch with metric given in (3.1).

After performing these operations on the spinning string solution given in (2.87) we obtain the following solution in the Poincaré patch.

$$\begin{aligned} \Lambda z &= \sqrt{2} e^{\tau-\sigma} + \frac{\tilde{b}}{\sqrt{2}} \left(e^{2\sigma} (2\sigma - \pi c_1^{(0)}) + 2\pi c_1^{(0)} - 2 \right) e^{\tau-3\sigma} \\ &+ \frac{\tilde{b}^2 e^{\tau-3\sigma}}{4\sqrt{2}} \left\{ 4(1 - \pi c_1^{(0)})(\pi c_1^{(0)} - 2\sigma + 2) \right. \\ &\left. + \cosh(2\sigma) \left(11\pi^2 (c_1^{(0)})^2 + 4\pi c_1^{(0)}(-2\sigma + \tau - 5) + 4(\sigma^2 - \sigma + 3) \right) \right\} \end{aligned} \quad (3.23)$$

$$\begin{aligned}
 & + \sinh(2\sigma) \left(-5\pi^2(c_1^{(0)})^2 + 4\pi c_1^{(0)}(-2\sigma + \tau + 3) + 4(\sigma^2 - \sigma - 1) \right) \Big\}, \\
 \Lambda x_0 = & e^{\tau-\sigma} \cosh(\tau+\sigma) + \frac{1}{2}\tilde{b}e^{-4\sigma} \left\{ e^{2\tau+4\sigma}(2\sigma - \pi c_1^{(0)}) + e^{2\tau+2\sigma}(2\pi c_1^{(0)} - 2) - e^{2\sigma} + \pi c_1^{(0)} - 1 \right\} \\
 & + \frac{1}{4}\tilde{b}^2 e^{-6\sigma} \left\{ 4(\pi c_1^{(0)} - 1)^2 e^{2(\sigma+\tau)} + 4(\pi c_1^{(0)} - 1)(2\sigma - \pi c_1^{(0)})e^{4\sigma+2\tau} \right. \\
 & + (4 - 4\pi c_1^{(0)})e^{2\sigma} + 2(\pi c_1^{(0)} - 1)^2 + e^{6\sigma+2\tau} \left((\pi c_1^{(0)} - 2\sigma)^2 + 2\pi c_1^{(0)}\tau \right) (\pi c_1^{(0)} - 1)^2 \\
 & \left. + e^{4\sigma}(\pi c_1^{(0)}(\pi c_1^{(0)} - 2\sigma - 2) - 2\sigma + 2) \right\}, \\
 \Lambda x_1 = & e^{\tau-\sigma} \sinh(\tau+\sigma) + \frac{1}{2}\tilde{b}e^{-4\sigma} \left\{ e^{2\tau+4\sigma}(2\sigma - \pi c_1^{(0)}) + e^{2\tau+2\sigma}(2\pi c_1^{(0)} - 2) + e^{2\sigma} - \pi c_1^{(0)} + 1 \right\} \\
 & + \frac{1}{4}\tilde{b}^2 e^{-6\sigma} \left(4(\pi c_1^{(0)} - 1)^2 e^{2(\sigma+\tau)} + 4(\pi c_1^{(0)} - 1)(2\sigma - \pi c_1^{(0)})e^{4\sigma+2\tau} \right. \\
 & + 4(\pi c_1^{(0)} - 1)e^{2\sigma} - 2(\pi c_1^{(0)} - 1)^2 + e^{6\sigma+2\tau} \left((\pi c_1^{(0)} - 2\sigma)^2 + 2\pi c_1^{(0)}\tau \right) \\
 & \left. + e^{4\sigma}(\pi c_1^{(0)}(-\pi c_1^{(0)} + 2\sigma + 2) + 2(\sigma - 1)) \right\}.
 \end{aligned}$$

One can explicitly verify that the expressions for x , x_0 and x_1 , given in (3.23) are solutions to the equations of motion of the Nambu-Goto action (3.3) to order b^2 .

The solution given in (3.23) seems complicated, but we can see that it ends on a light like cusp at the AdS_3 boundary. To show this we compute

$$\frac{z^2}{x_0^2 - x_1^2} = G = 2\tilde{b}^2 e^{-4\sigma} \left(-\pi c_1^{(0)} + e^{2\sigma} + 1 \right)^2 + \tilde{b} \left((2 - 2\pi c_1^{(0)})e^{-2\sigma} + 2 \right) + 2. \quad (3.24)$$

Since $b \ll 1$, the quantity G does not vanish,⁹ therefore at $z = 0$, the boundary of AdS, the minimal surface in (3.23) ends on the light cone $x_0 = \pm x_1$. To make the resemblance with the minimal surface corresponding to the light like cusp more apparent we can perform a world sheet re-parametrization. Let us first equate the solution in (3.23) to the following general ansatz.

$$\Lambda z = f(\tau, \sigma)g(\tau, \sigma), \quad \Lambda x_0 = f(\tau, \sigma) \cosh(X(\tau, \sigma)), \quad \Lambda x_1 = f(\tau, \sigma) \sinh(X(\tau, \sigma)). \quad (3.25)$$

After a bit of tedious algebra we can extract the functions $f(\tau, \sigma)$, $g(\tau, \sigma)$ and $X(\tau, \sigma)$ to order b^2 . These are given by

$$\begin{aligned}
 f(\tau, \sigma) = & e^{\tau-\sigma} + \frac{1}{2}\tilde{b} \left(e^{2\sigma}(-\pi c_1^{(0)} + 2\sigma - 1) + 3\pi c_1^{(0)} - 3 \right) e^{\tau-3\sigma} + \\
 & \frac{1}{8}\tilde{b}^2 e^{\tau-5\sigma} \left\{ 2(\pi c_1^{(0)} - 1)e^{2\sigma}(-3\pi c_1^{(0)} + 6\sigma - 5) + 11(\pi c_1^{(0)} - 1)^2 + \right. \\
 & \left. e^{4\sigma} \left(3\pi^2(c_1^{(0)})^2 - 2\pi c_1^{(0)}(4\sigma - 2\tau + 1) + 4(\sigma - 2)\sigma + 3 \right) \right\}, \\
 g(\tau, \sigma) = & \sqrt{2} + \frac{\tilde{b}}{\sqrt{2}} e^{-2\sigma} \left(-\pi c_1^{(0)} + e^{2\sigma} + 1 \right) + \frac{3\tilde{b}^2}{4\sqrt{2}} e^{-4\sigma} \left(-\pi c_1^{(0)} + e^{2\sigma} + 1 \right)^2, \\
 X(\tau, \sigma) = & \sigma + \tau + \tilde{b}e^{-\sigma} \left\{ (-\pi c_1^{(0)} + \sigma + 1) \sinh(\sigma) + \sigma \cosh(\sigma) \right\} + \frac{1}{2}\tilde{b}^2 e^{-2\sigma} \left\{ \pi c_1^{(0)} - 1 \right. \\
 & \left. + \sinh(2\sigma)(\pi c_1^{(0)}\sigma + \pi c_1^{(0)}\tau + \sigma) + \cosh(2\sigma) \left[\pi c_1^{(0)}(-\pi c_1^{(0)} + \sigma + \tau + 2) + \sigma - 1 \right] \right\}.
 \end{aligned} \quad (3.26)$$

⁹For G to vanish it can be shown that $e^{2\sigma}$ has to become complex.

Then we re-parametrize the world sheet co-ordinates by introducing co-ordinates which satisfy the condition

$$\tau' + \sigma' = X(\tau, \sigma), \quad \tau' - \sigma' = \log[f(\tau, \sigma)]. \quad (3.27)$$

From these equations it is possible express the co-ordinates τ, σ in terms τ', σ' perturbatively in b . This change of variables is given by

$$\begin{aligned} \tau &= \tau' + \tilde{b} \left\{ (1 - \pi c_1^{(0)}) e^{-2\sigma'} + \frac{\pi c_1^{(0)}}{2} - \sigma' \right\} + \\ &\quad + \frac{1}{2} \tilde{b}^2 \left\{ 2(\pi c_1^{(0)} - 1)^2 e^{-4\sigma'} + (3 - 3\pi c_1^{(0)}) e^{-2\sigma'} - \pi c_1^{(0)} \tau' + 1 \right\}, \\ \sigma &= \sigma' - \frac{1}{2} \tilde{b} e^{-2\sigma'} \left(-\pi c_1^{(0)} + e^{2\sigma'} + 1 \right) \\ &\quad + \frac{1}{4} \tilde{b}^2 e^{-4\sigma'} \left\{ e^{4\sigma'} \left((\pi c_1^{(0)} - 1)^2 - 2(\pi c_1^{(0)} + 1)\sigma' \right) - (\pi c_1^{(0)} - 1)^2 \right\}. \end{aligned} \quad (3.28)$$

After this reparametrization, we can write the solutions as

$$\Lambda z = e^{\tau' - \sigma'} g(\tau', \sigma'), \quad \Lambda x_0 = e^{\tau' - \sigma'} \cosh(\tau' + \sigma'), \quad \Lambda x_1 = e^{\tau' - \sigma'} \sinh(\tau' + \sigma'), \quad (3.29)$$

where

$$\begin{aligned} g(\tau', \sigma') &= \sqrt{2} + \frac{\tilde{b}}{\sqrt{2}} e^{-2\sigma'} \left(-\pi c_1^{(0)} + e^{2\sigma'} + 1 \right) + \\ &\quad + \frac{\tilde{b}^2}{4\sqrt{2}} e^{-4\sigma'} \left(-7\pi c_1^{(0)} + 3e^{2\sigma'} + 7 \right) \left(-\pi c_1^{(0)} + e^{2\sigma'} + 1 \right). \end{aligned} \quad (3.30)$$

To verify all the manipulations performed we have checked that the minimal surface given in (3.29) solves the Nambu-Goto equations of motion. Since the ansatz in (3.4) is of the same form given in (3.29), it can be verified that the expression for g given in (3.30) solves the equations of motion (3.6) to order b^2 . Note that the solution given in (3.29) and (3.30) is a solution for any arbitrary $c_1^{(0)}$ and it is interesting to note that it reduces to the ‘uniform’ solution $g = c$ with c given in (3.8) for $c_1^{(0)} = \frac{1}{\pi}$. Therefore this solution is a one parameter generalization of the ‘uniform’ solution.

Let us now go over to the ρ and ξ co-ordinates introduced in (3.9) to evaluate the Euclidean action of the solution. We define

$$\tau' = \frac{1}{2}(\xi + \log(\rho)), \quad \sigma' = \frac{1}{2}(\xi - \log(\rho)), \quad (3.31)$$

in terms of these co-ordinates, the solutions become

$$\begin{aligned} \Lambda z &= \sqrt{2}\rho + \frac{\tilde{b}}{\sqrt{2}} e^{-\xi} \rho \left(-\pi c_1^{(0)} \rho + e^\xi + \rho \right) + \\ &\quad + \frac{\tilde{b}^2}{4\sqrt{2}} e^{-2\xi} \rho \left(-7\pi c_1^{(0)} \rho + 3e^\xi + 7\rho \right) \left(-\pi c_1^{(0)} \rho + e^\xi + \rho \right), \\ \Lambda x_0 &= \rho \cosh(\xi), \quad \Lambda x_1 = \rho \sinh(\xi). \end{aligned} \quad (3.32)$$

We now further scale

$$\rho = \Lambda \rho', \quad \Lambda \rightarrow 0, \quad \rho' : \text{finite}. \quad (3.33)$$

Then it is easy to see that solution for z reduces to $z = c\rho'$ where c is given in (3.8) and $x_0 = \rho' \cosh(\xi)$, $x_1 = \rho' \sinh(\xi)$. Thus the Euclidean action of the solution in this scaling limit will be given by (3.13). Therefore the coefficient of the b^2 term in the area of the Wilson surface obtained from the spinning string solution by conformal transformations and re-parametrizations exhibits the same behaviour as the $O(b^2(\log S)^2)$ term in the dispersion relation of the spinning string.

One might wonder what would be the result if one were not to perform the scaling in (3.33). We have verified that the coefficient of the log divergence in the area of the Wilson loop still remains the same. This can be seen by evaluating Nambu-Goto action for the solution given in (3.32)

$$\begin{aligned} S_{NG} &= \frac{\sqrt{\lambda}}{2\pi} \int_{-\gamma/2}^{\gamma/2} \int_{\epsilon}^L \left(\sqrt{G} + \frac{\tilde{b}}{z^2} (x'_0 x_1 - x'_1 x_0) \right) d\rho d\xi \\ &= \frac{\sqrt{\lambda}}{2\pi} \left[\frac{1}{2} \log \frac{L}{\epsilon} + \tilde{b} \left(\frac{\gamma}{2} \log \frac{L}{\epsilon} - (c_1^{(0)} \pi - 1)(L - \epsilon) \sinh \frac{\gamma}{2} \right) + \right. \\ &\quad \left. + \frac{\tilde{b}^2}{4} \left(\gamma \log \frac{L}{\epsilon} + (\pi c_1^{(0)} - 1) \left((\pi c_1^{(0)} - 1) \sinh(\gamma)(L^2 - \epsilon^2) + 4 \sinh \left(\frac{\gamma}{2} \right) (\epsilon - L) \right) \right) \right] \\ &\quad + O(\tilde{b}^3) \end{aligned} \quad (3.34)$$

where x_i and x'_i represent derivative with respect to ρ and ξ respectively. We have to reinstate $\tilde{b} = -ib$ to obtain the answer. Thus there are non-universal quadratic divergent terms in the area that depends on the parameter $c_1^{(0)}$. The coefficients of these terms vary if one changes the integration constants in (2.87), but the coefficient of the log-divergence is invariant and universal. This observation together with the fact that the classical solution of the spinning string is related to the cusp minimal surface suggests that the $O(b^2)$ coefficient of logarithmic divergence of the area of the Wilson loop is related to the coefficient of $O(b^2 \log^2 S)$ in the dispersion relation of the spinning string.

4 Conclusions

We have studied classical spinning strings and their dispersion relation in the AdS_3 with mixed 3-form fluxes. We have shown that the dispersion relation acquires the term $-\frac{\sqrt{\lambda} b^2}{2\pi} \log^2 S$ in addition to the usual $\log S$ term. We have observed that the coefficient of the b^2 term in the logarithmic divergence of the area of the minimal surface corresponding to the cusp-Wilson line is identical to the correction in the dispersion relation of the folded spinning string. This observation together with the fact that the spinning string in the presence of the NS-flux can be mapped to the minimal surface suggests that the coefficient of this term can be derived to all orders in the coupling λ . It will be interesting to study this observation further.

There has been progress in writing down the S-matrix of strings in $AdS_3 \times S^3 \times \mathcal{M}$ [11, 35–43]. This has been extended to the case with mixed 3-form fluxes in [22–24]. The results obtained in this paper will serve as tests of these proposals. For the case of $\mathcal{N} = 4$ Yang-Mills a crucial step in understanding of the S-matrix was the derivation of the cusp-anomalous dimension to all order in the coupling [9]. Our results suggest that the cusp-anomalous dimension in $AdS_3 \times S^3$ has an interesting deformation parametrized by the NS-B flux. Deriving the cusp anomalous dimension from the S-matrix to all orders in the coupling as well its deformation in the NS-B flux is an important direction to pursue in this subject. It will lead to crucial insights for the strings in $AdS_3 \times S^3$ and its holographic dual.

Note added. While this paper was being written up we noticed [44] on the arXiv where classical strings in $AdS_3 \times S^3$ was studied with an emphasis on the giant magnon solution.

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