Two-point functions of conformal primary operators in $\mathcal{N} = 1$ superconformal theories

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ABSTRACT: In $\mathcal{N} = 1$ superconformal theories in four dimensions the form of two-point functions of superconformal multiplets is known up to an overall constant. A superconformal multiplet contains several conformal primary operators, whose two-point function coefficients can be determined in terms of the multiplet's quantum numbers. In this paper we work out these coefficients in full generality, i.e. for superconformal multiplets that belong to any irreducible representation of the Lorentz group with arbitrary scaling dimension and R-charge. From our results we recover the known unitarity bounds, and also find all shortening conditions, even in non-unitary theories. For the purposes of our computations we have developed a *Mathematica* package for the efficient handling of expansions in Grassmann variables.

KEYWORDS: Supersymmetry and Duality, Superspaces, Conformal and W Symmetry

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1 Introduction

Conformal field theories (CFTs) in four dimensions are abundant and have been studied extensively. However, the full spectrum of consequences of conformal symmetry has remained elusive. This has been made strikingly clear by the recent developments in the numerical conformal bootstrap approach, an implementation of the conformal bootstrap [1, 2] in higher dimensions initiated in [3]. The numerical bootstrap has uncovered extremely interesting implications of conformal symmetry and unitarity. Its success also signals the possible existence of more hidden constraints of conformal symmetry yet to be discovered.

The main quantities of interest in CFTs, as in any quantum field theory, are correlation functions. More specifically, in CFTs the focus is on correlation functions of primary operators, defined as the operators that are annihilated by the generator of special conformal transformations at the origin. Conformal symmetry places powerful constraints on the form of such correlation functions, see e.g. [4, 5]. For example, two-point functions are fixed up to an overall constant, three-point functions are fixed up to a finite set of constants, and

four-point functions can be expressed as a sum in terms of conformal blocks, whose explicit form has been worked out in some even dimensions in [6, 7]. The bootstrap program uses these expressions, along with crossing symmetry and sophisticated numerical analysis, to produce deep and model-independent results about the spectrum of operators and their scaling dimensions in CFTs, as well as coefficients in the operator product expansion.

The numerical bootstrap has also been applied to four-dimensional $\mathcal{N} = 1$ superconformal theories (SCFTs) [8, 9]. Progress, however, hinges on the understanding of superconformal correlation functions; in particular, superconformal blocks appearing in four-point functions. For cases involving chiral operators, linear multiplets, or general scalar operators, the corresponding superconformal blocks are known [8, 10–12], which opens the door to the application of the conformal bootstrap to these cases.

The form of correlation functions in SCFTs has been studied extensively in [13–15] and also, using superembedding methods, in [16–21], with elegant results for the two- and three-point functions of superfields. These results succinctly encapsulate the correlation functions of all the different components of the superfield, including the conformal primaries. However, for many physical applications it is necessary to work out explicitly these component correlation functions and their relations as imposed by superconformal symmetry. For example, to determine the superconformal blocks as linear combinations of conformal blocks, it is necessary to explicitly work out the relations between the two-point function coefficients of the various conformal primary components of a superconformal multiplet.

From the superconformal two- and three-point functions one can also extract the operator product expansion, and use it to explore the phenomenology of models of supersymmetry breaking with a superconformal hidden sector in the ultraviolet [22–24]. In addition to direct physical applications, the coefficients in the component correlation functions expose rich structures of SCFTs. For example, in the case of the two-point function, one can read off all possible shortening conditions and unitarity bounds on the most general supermultiplet.

There is a systematic way to decompose the known form of a superfield correlation function into component correlation functions. One expands each participating superfield in terms of the Grassmann coordinates $\theta, \bar{\theta}$, yielding, at each order, a linear combination of conformal primary component operators and possible descendants. Since the correlation functions among these are determined by conformal symmetry up to unknown constants, one can reconstruct the superfield correlation function using them, and then compare to the known form as follows from superconformal invariance. This comparison uniquely determines all unknown coefficients in the component correlators in terms of the coefficients in the superfield correlator and the quantum numbers of the superfield. In this work we apply this method to the most general superconformal two-point function. We leave the case of three-point functions for future work.

In practice, the computation outlined in the previous paragraph is rather complicated. In order to compare the known and reconstructed forms of the superconformal two-point function order by order in $\theta, \bar{\theta}$, one first needs to expand its known form. This expansion is already rather involved due to the various relations among Lorentz covariant structures, such as Fierz identities. The contribution from the various descendants to the reconstructed form poses another challenge, as they involve complicated derivative operators acting on

two-point functions of conformal primaries. In order to make the computation manageable, we developed a *Mathematica* package that can perform θ -expansions, simplify expressions with various rules satisfied by four-vectors and spinors, and compute two-point functions involving conformal descendants.

Although in this work we take a direct route in obtaining our results, it is natural to ask if there is an alternative, less computationally challenging way, to obtain the same answers. Although we don't have an answer to this question, we believe that our results may help in the development of a method of achieving the conformal decomposition of the superconformal correlation functions with less computational effort. We should note here that there exists another way to obtain the same results, using radial quantization and the superconformal algebra [25], but it is rather tedious as well.

This paper is organized as follows. In the next section we present the construction of the irreducible Lorentz representations for the superconformal descendants of a general superconformal primary operator $\mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}$. We also remind the reader of basic facts on $\mathcal{N} = 1$ superconformal representation theory. In section 3 we summarize, for the reader's convenience, our results for the various superconformal-descendant but conformal-primary two-point functions. We also make various comments on our results, and rederive the well-known unitarity bounds [26, 27] and multiplet shortening conditions. The latter are obtained in all generality without imposing unitarity. In section 4 we provide further details on the derivation of these results, including all the ingredients necessary for constructing the conformal primary operators in a superconformal multiplet. In section 5 we give equations for the supercurrent supermultiplet, or Ferrara-Zumino multiplet [28], which contains the R-current, the supersymmetry current, and the stress-energy tensor. We summarize in section 6 with comments on possible uses of our results. In appendix A we outline the method we used for our computations, and in appendix B we provide more details on our *Mathematica* package.

We follow the conventions of Wess & Bagger [29].

2 $\mathcal{N} = 1$ superconformal primary operators and their descendants

Local operators in a CFT can be classified into representations of the conformal algebra. One can regard the generator of translations, P_μ , as a raising operator, and the generator of special conformal transformations, K_μ , as a lowering operator. A representation can be constructed by applying P_μ in all possible ways on a conformal primary operator, which is annihilated by K_μ at the origin. The operators obtained by acting with P_μ are called conformal descendants. In four dimensions, a generic conformal primary operator can be characterized by its scaling dimension Δ and its Lorentz representation $(j/2, \bar{j}/2)$, explicitly constructed as $\mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}(x)$. $\mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}$ is assumed totally symmetric in its dotted and, separately, its undotted indices,

$$\mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} = \mathcal{O}_{(\alpha_1 \dots \alpha_j); (\dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}})} \equiv \frac{1}{j! \bar{j}!} \sum_{(I, J) = (1, 1)}^{(j!, \bar{j}!)} \mathcal{P}_{\alpha_1 \dots \alpha_j}^{(I)} \mathcal{P}_{\dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}^{(J)} \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}, \quad (2.1)$$

where $\mathcal{P}_{\alpha_1 \dots \alpha_j}^{(I)}$ runs over all possible permutations of the undotted indices and similarly for $\mathcal{P}_{\dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}^{(J)}$ and the dotted indices. Constructed this way, $\mathcal{O}$ furnishes an irreducible representation (irrep) of the Lorentz group. Obviously, j and $\bar{j}$ are nonnegative integers.

The $\mathcal{N} = 1$ superconformal algebra in four dimensions extends the conformal algebra with the supercharges Q_α , $\bar{Q}_{\dot{\alpha}}$, the superconformal supercharges S_α , $\bar{S}_{\dot{\alpha}}$, and the $U(1)_R$ generator, R . One can regard P_μ , Q_α , and $\bar{Q}_{\dot{\alpha}}$ as raising operators, and K_μ , S_α , $\bar{S}_{\dot{\alpha}}$ as lowering operators. A generic representation can be constructed by acting with raising operators on a superconformal primary operator $\mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}$, characterized by $(j, \bar{j}, q, \bar{q})$, where $(j/2, \bar{j}/2)$ again labels its Lorentz representation, and the conformal weights $q, \bar{q}$ are related to scaling dimension and R -charge by

$$\Delta = q + \bar{q}, \quad R = \frac{2}{3}(q - \bar{q}).$$

$\mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}$ is bosonic if $j + \bar{j}$ is even, and fermionic if $j + \bar{j}$ is odd. If $j = \bar{j}$, then $\mathcal{O}$ furnishes an irreducible (symmetric traceless) integer-spin representation of the Lorentz group with spin $\ell = j = \bar{j}$. An operator with zero R -charge has $q = \bar{q} = \Delta/2$. We will also consider the conjugate of $\mathcal{O}$, $\bar{\mathcal{O}}$, obtained via $(j, \bar{j}, q, \bar{q}) \rightarrow (\bar{j}, j, \bar{q}, q)$.

2.1 Components of $\mathcal{O}$

One can construct the full superconformal multiplet by applying raising operators P_μ , Q_α , and $\bar{Q}_{\dot{\alpha}}$ on $\mathcal{O}$. Since Q and $\bar{Q}$ are nilpotent, such a multiplet only contains a finite number of conformal multiplets. For example, if we apply Q^2 , the result $Q^2 \mathcal{O}$ is a superconformal descendant. But it is still a conformal primary with quantum numbers $(j, \bar{j}, q + 1, \bar{q})$ (or $\Delta = q + \bar{q} + 1$ and $R = \frac{2}{3}(q - \bar{q}) + 2$).

If we apply a single Q_α , then the result would fall into two different irreducible representations, since $(\frac{1}{2}, 0) \otimes (\frac{j}{2}, \frac{\bar{j}}{2}) = (\frac{j-1}{2}, \frac{\bar{j}}{2}) \oplus (\frac{j+1}{2}, \frac{\bar{j}}{2})$, where $(\frac{j+1}{2}, \frac{\bar{j}}{2})$ corresponds to symmetrizing the additional index, i.e. $Q_{(\alpha} \mathcal{O}_{\alpha_1 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}$, and $(\frac{j-1}{2}, \frac{\bar{j}}{2})$ corresponds to antisymmetrizing, i.e. $Q^\alpha \mathcal{O}_{\alpha \alpha_1 \dots \alpha_{j-1}; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}$. More explicitly, with the conventions of (2.1) one can derive the identity

$$Q_\alpha \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} = Q_{(\alpha} \mathcal{O}_{\alpha_1 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} + \frac{j}{j+1} \epsilon_{\alpha(\alpha_1} Q^\beta \mathcal{O}_{\beta \alpha_2 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}. \quad (2.2)$$

Since¹ $[K^\mu, Q_\alpha] = -\sigma_{\alpha\dot{\alpha}}^\mu \bar{S}^{\dot{\alpha}}$, the two operators on the right-hand side are conformal primaries, characterized by quantum numbers $(j \pm 1, \bar{j}, q + \frac{1}{2}, \bar{q})$. They can thus be denoted unambiguously by $(Q\mathcal{O})_{j \pm 1, \bar{j}}$. Note that in the second term in the right-hand side of (2.2) the index β of $\mathcal{O}$ is exchanged with each of $\alpha_2, \dots, \alpha_j$ in the symmetrization, but not with α_1 . Explicitly,

$$\epsilon_{\alpha(\alpha_1} Q^\beta \mathcal{O}_{\beta \alpha_2 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} = \frac{1}{j(j!)} \sum_{(I, i)=(1,1)}^{(j!, j)} \mathcal{P}_{\beta \alpha_1 \dots \alpha_{i-1} \alpha_{i+1} \dots \alpha_j}^{(I)} \epsilon_{\alpha \alpha_i} Q^\beta \mathcal{O}_{\beta \alpha_1 \dots \alpha_{i-1} \alpha_{i+1} \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}.$$

¹For the superconformal algebra we use the conventions of [10, appendix A].

If we apply $Q\bar{Q}$ on $\mathcal{O}$, then we get four different operators characterized by $(j \pm 1, \bar{j} \pm 1, q + \frac{1}{2}, \bar{q} + \frac{1}{2})$:

$$\begin{aligned}
 Q_\alpha \bar{Q}_{\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} &= Q_{(\alpha} \bar{Q}_{\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \\
 &+ \frac{j}{j+1} \epsilon_{\alpha(\alpha_1} Q^{\beta} \bar{Q}_{\dot{\alpha}} \mathcal{O}_{|\beta \alpha_2 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \\
 &+ \frac{\bar{j}}{\bar{j}+1} \epsilon_{\dot{\alpha}(\dot{\alpha}_1} Q_{(\alpha} \bar{Q}^{\dot{\beta}} \mathcal{O}_{\alpha_1 \dots \alpha_j); |\dot{\beta} \dot{\alpha}_2 \dots \dot{\alpha}_{\bar{j}}} \\
 &+ \frac{j\bar{j}}{(j+1)(\bar{j}+1)} \epsilon_{\alpha(\alpha_1} \epsilon_{\dot{\alpha}(\dot{\alpha}_1} Q^{\beta} \bar{Q}^{\dot{\beta}} \mathcal{O}_{|\beta \alpha_2 \dots \alpha_j); |\dot{\beta} \dot{\alpha}_2 \dots \dot{\alpha}_{\bar{j}}),
 \end{aligned} \tag{2.3}$$

where, of course, the dotted indices do not participate in the symmetrization of the undotted ones and vice-versa. These four operators can be denoted by $(Q\bar{Q}\mathcal{O})_{j\pm 1, \bar{j}\pm 1}$. The second, third, and fourth operator in the right-hand side of (2.3) only exist if $j \neq 0$, $\bar{j} \neq 0$, and $j\bar{j} \neq 0$ respectively. The operators in (2.3) are not conformal primaries, since, on a superconformal primary, $[K^\mu, Q_\alpha \bar{Q}_{\dot{\alpha}}] = \frac{1}{2}i(\sigma^\mu \bar{\sigma}^\nu \sigma^\rho)_{\alpha\dot{\alpha}} M_{\nu\rho} + (2iD - 3R)\sigma^\mu_{\alpha\dot{\alpha}}$, where $M_{\mu\nu}$ is the generator of Lorentz transformations and D that of dilatations. Nevertheless, since $[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu})$, conformal primaries can be extracted out of them by subtracting $P\mathcal{O}$ with appropriate coefficients, which we will work out explicitly. After this subtraction we will obtain four conformal primaries with different Lorentz representations. We denote these primaries with a subscript “ p ”, i.e. $(Q\bar{Q}\mathcal{O})_{j\pm 1, \bar{j}\pm 1; p}$.

At higher orders we find $Q^2\mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}$ and $\bar{Q}^2\mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}$, or $(Q^2\mathcal{O})_{j, \bar{j}}$ and $(\bar{Q}^2\mathcal{O})_{j, \bar{j}}$, which are already conformal primary Lorentz irreps. Furthermore, the action of $Q^2\bar{Q}$ on $\mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}$ produces two operators, namely $(Q^2\bar{Q}\mathcal{O})_{j, \bar{j}\pm 1}$, as does the action of $\bar{Q}^2Q$, namely $(\bar{Q}^2Q\mathcal{O})_{j\pm 1, \bar{j}}$. Each of these operators contains a conformal primary. Finally, the operator $Q^2\bar{Q}^2\mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}$, or $(Q^2\bar{Q}^2\mathcal{O})_{j, \bar{j}}$, contains a single conformal primary.

We summarize the structure of an $\mathcal{N} = 1$ superconformal multiplet in figure 1.

Generically, the multiplet contains sixteen different conformal primary operators, but if the quantum numbers $(j, \bar{j}, q, \bar{q})$ obey special conditions, then certain higher components may become null. We will systematically determine all such conditions.

3 Summary of results

For each of the sixteen conformal primary components mentioned above we can write its two-point function with its conjugate as

$$\langle \mathcal{T}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}(x) \bar{\mathcal{T}}_{\beta_1 \dots \beta_{\bar{j}}; \dot{\beta}_1 \dots \dot{\beta}_j}(0) \rangle = C_{\mathcal{T}} \frac{x_{(\alpha_1} \dot{x}_{\dot{\beta}_1} \dots x_{\alpha_j) \dot{\beta}_{\bar{j}}}}{x^{2(q_{\mathcal{T}} + \bar{q}_{\mathcal{T}}) + j_{\mathcal{T}} + \bar{j}_{\mathcal{T}}}}, \tag{3.1}$$

where $x_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} x_\mu$ and the dotted indices do not participate in the symmetrization of the undotted ones. In (3.1), $\mathcal{T}$ may be $\mathcal{O}_{j, \bar{j}}$, $(Q\mathcal{O})_{j\pm 1, \bar{j}}$, etc. from figure 1. In unitary theories, $(-i)^{j_{\mathcal{T}} + \bar{j}_{\mathcal{T}}} C_{\mathcal{T}} > 0$.

The x -dependence in (3.1) is uniquely determined by conformal symmetry, while $C_{\mathcal{T}}$ is determined by supersymmetry in terms of the coefficient of the lowest component in figure 1, $C_{\mathcal{O}}$. The results are summarized in table 1.

Figure 1. A superconformal multiplet consists of a finite number of conformal primary operators, related by supersymmetry. Some of these primaries may become null when the multiplet’s quantum numbers $(j, \bar{j}, q, \bar{q})$ satisfy shortening conditions.

3.1 Comments

As is well-known, in conformal theories two-point functions of primary operators can be arranged in a diagonal basis with all coefficients set equal to one by a proper choice of operator normalization. In superconformal theories, since supersymmetry relates the different conformal primaries in the multiplet, their normalizations are fixed by that of the lowest component. For example, if the lowest component of a scalar multiplet, $\mathcal{O}$, is canonically-normalized, then the operator $Q^2\mathcal{O}$ is generally not. This is because the normalization of Q is fixed by the supersymmetry algebra, $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu$, due to the fact that the normalization of P_μ is fixed by $P_\mu\mathcal{O} = i\partial_\mu\mathcal{O}$. Of course, one can define the operator $\tilde{\mathcal{O}} = 2^{-3}(q(q-1))^{-1/2}Q^2\mathcal{O}$, which is canonically-normalized, but then the required normalization coefficient is given exactly by our results. Furthermore, our results are important when one considers three-point functions of conformal primary operators, where the operator normalizations affect the overall coefficients of the three-point functions. This has been illustrated, for example, in the computations of superconformal blocks we alluded to in the introduction.

From our results we can derive unitarity bounds [26, 27] and all shortening conditions (noticing that $j, \bar{j} \geq 0$). Requiring that all primary operators in the multiplet correspond to states with positive norm results in the well-known unitarity bounds

$$\begin{aligned} q = j = 0, \quad \bar{q} &\geq \bar{j}/2 + 1; \\ \bar{q} = \bar{j} = 0, \quad q &\geq j/2 + 1; \\ j, \bar{j} &\geq 0, \quad q \geq j/2 + 1, \quad \bar{q} \geq \bar{j}/2 + 1, \end{aligned}$$

and also the trivial case $j = \bar{j} = q = \bar{q} = 0$, corresponding to the unit operator. As the quantum numbers saturate the unitarity bounds, the multiplet gets shortened as in table 2.

$\mathcal{T}$	$C_{\mathcal{T}}$
$\mathcal{O}_{j,\bar{j}}$	$C_{\mathcal{O}}$
$(Q\mathcal{O})_{j+1,\bar{j}}$	$2iC_{\mathcal{O}} \frac{j+2q}{(j+1)^2}$
$(Q\mathcal{O})_{j-1,\bar{j}}$	$2iC_{\mathcal{O}} \frac{(j+1)(j-2(q-1))}{j}$
$(\bar{Q}\mathcal{O})_{j,\bar{j}+1}$	$2iC_{\mathcal{O}} \frac{\bar{j}+2\bar{q}}{(\bar{j}+1)^2}$
$(\bar{Q}\mathcal{O})_{j,\bar{j}-1}$	$2iC_{\mathcal{O}} \frac{(\bar{j}+1)(\bar{j}-2(\bar{q}-1))}{\bar{j}}$
$(Q^2\mathcal{O})_{j,\bar{j}}$	$-2^4 C_{\mathcal{O}} (j+2q)(j-2(q-1))$
$(\bar{Q}^2\mathcal{O})_{j,\bar{j}}$	$-2^4 C_{\mathcal{O}} (\bar{j}+2\bar{q})(\bar{j}-2(\bar{q}-1))$
$(Q\bar{Q}\mathcal{O})_{j+1,\bar{j}+1;p}$	$-4C_{\mathcal{O}} \frac{(j+2q)(\bar{j}+2\bar{q})(j+\bar{j}+2(q+\bar{q}+1))}{(j+1)^2(\bar{j}+1)^2(j+\bar{j}+2(q+\bar{q}))}$
$(Q\bar{Q}\mathcal{O})_{j-1,\bar{j}+1;p}$	$-4C_{\mathcal{O}} \frac{(j+1)(j-2(q-1))(\bar{j}+2\bar{q})(j-\bar{j}-2(q+\bar{q}))}{j(\bar{j}+1)^2(j-\bar{j}-2(q+\bar{q}-1))}$
$(Q\bar{Q}\mathcal{O})_{j+1,\bar{j}-1;p}$	$-4C_{\mathcal{O}} \frac{(\bar{j}+1)(\bar{j}-2(\bar{q}-1))(j+2q)(j-\bar{j}+2(q+\bar{q}))}{\bar{j}(j+1)^2(j-\bar{j}+2(q+\bar{q}-1))}$
$(Q\bar{Q}\mathcal{O})_{j-1,\bar{j}-1;p}$	$-4C_{\mathcal{O}} \frac{(j+1)(\bar{j}+1)(j-2(q-1))(\bar{j}-2(\bar{q}-1))(j+\bar{j}-2(q+\bar{q}-1))}{j\bar{j}(j+\bar{j}-2(q+\bar{q}-2))}$
$(Q^2\bar{Q}\mathcal{O})_{j,\bar{j}+1;p}$	$-2^5 iC_{\mathcal{O}} \frac{(j+2q)(j-2(q-1))(\bar{j}+2\bar{q})(j+\bar{j}+2(q+\bar{q}+1))(j-\bar{j}-2(q+\bar{q}))}{(\bar{j}+1)^2(j+\bar{j}+2(q+\bar{q}))(j-\bar{j}-2(q+\bar{q}-1))}$
$(Q^2\bar{Q}\mathcal{O})_{j,\bar{j}-1;p}$	$-2^5 iC_{\mathcal{O}} \frac{(\bar{j}+1)(j+2q)(j-2(q-1))(\bar{j}-2(\bar{q}-1))(j+\bar{j}-2(q+\bar{q}-1))(j-\bar{j}+2(q+\bar{q}))}{\bar{j}(j+\bar{j}-2(q+\bar{q}-2))(j-\bar{j}+2(q+\bar{q}-1))}$
$(\bar{Q}^2Q\mathcal{O})_{j+1,\bar{j};p}$	$-2^5 iC_{\mathcal{O}} \frac{(j+2q)(\bar{j}+2\bar{q})(\bar{j}-2(\bar{q}-1))(j+\bar{j}+2(q+\bar{q}+1))(j-\bar{j}+2(q+\bar{q}))}{(j+1)^2(j+\bar{j}+2(q+\bar{q}))(j-\bar{j}+2(q+\bar{q}-1))}$
$(\bar{Q}^2Q\mathcal{O})_{j-1,\bar{j};p}$	$-2^5 iC_{\mathcal{O}} \frac{(j+1)(\bar{j}+2\bar{q})(j-2(q-1))(\bar{j}-2(\bar{q}-1))(j+\bar{j}-2(q+\bar{q}-1))(j-\bar{j}-2(q+\bar{q}))}{j(\bar{j}+\bar{j}-2(q+\bar{q}-2))(j-\bar{j}-2(q+\bar{q}-1))}$
$(Q^2\bar{Q}^2\mathcal{O})_{j,\bar{j};p}$	$2^8 C_{\mathcal{O}} \frac{(j+2q)(j-2(q-1))(\bar{j}+2\bar{q})(\bar{j}-2(\bar{q}-1))(j-\bar{j}+2(q+\bar{q}))(j-\bar{j}-2(q+\bar{q}))}{(j+\bar{j}+2(q+\bar{q}))(j-\bar{j}+2(q+\bar{q}-1))(j-\bar{j}-2(q+\bar{q}-1))} \times \frac{(j+\bar{j}+2(q+\bar{q}+1))(j+\bar{j}-2(q+\bar{q}-1))}{(j+\bar{j}-2(q+\bar{q}-2))}$

Table 1. The coefficients in (3.1) for the various operators $\mathcal{T}$ from figure 1. In our conventions $C_{\mathcal{O}} = i^{j+\bar{j}} c_{\mathcal{O}}$, with $c_{\mathcal{O}} > 0$ in a unitary theory. j and $\bar{j}$ are non-negative integers. If $j = 0$ (resp. $\bar{j} = 0$), the components labeled by $j - 1$ (resp. $\bar{j} - 1$) do not exist.

Short. Condition	Short Multiplet
$q = j = 0$	$\mathcal{O}_{j,\bar{j}} \quad (\bar{Q}\mathcal{O})_{j,\bar{j}\pm 1} \quad (\bar{Q}^2\mathcal{O})_{j,\bar{j}}$
$\bar{q} = \bar{j} = 0$	$\mathcal{O}_{j,\bar{j}} \quad (Q\mathcal{O})_{j,\bar{j}\pm 1} \quad (Q^2\mathcal{O})_{j,\bar{j}}$
$q = \frac{j}{2} + 1$	$\mathcal{O}_{j,\bar{j}} \quad (Q\mathcal{O})_{j+1,\bar{j}} \quad (\bar{Q}\mathcal{O})_{j,\bar{j}\pm 1} \quad (\bar{Q}^2\mathcal{O})_{j,\bar{j}} \quad (Q\bar{Q}\mathcal{O})_{j+1,\bar{j}\pm 1;p} \quad (\bar{Q}^2Q\mathcal{O})_{j+1,\bar{j};p}$
$\bar{q} = \frac{\bar{j}}{2} + 1$	$\mathcal{O}_{j,\bar{j}} \quad (Q\mathcal{O})_{j\pm 1,\bar{j}} \quad (\bar{Q}\mathcal{O})_{j,\bar{j}+1} \quad (Q^2\mathcal{O})_{j,\bar{j}} \quad (Q\bar{Q}\mathcal{O})_{j\pm 1,\bar{j}+1;p} \quad (Q^2\bar{Q}\mathcal{O})_{j,\bar{j}+1;p}$

Table 2. Shortening conditions on a generic superconformal multiplet in unitary theories and the associated short multiplets. The intersection of short multiplets is taken if two corresponding shortening conditions are satisfied simultaneously.

Short. Condition	Null Components
$q = -\frac{j}{2}$	$(Q\mathcal{O})_{j+1,\bar{j}} \quad (Q^2\mathcal{O})_{j,\bar{j}} \quad (Q\bar{Q}\mathcal{O})_{j+1,\bar{j}\pm 1;p}$ $(Q^2\bar{Q}\mathcal{O})_{j,\bar{j}\pm 1;p} \quad (\bar{Q}^2Q\mathcal{O})_{j+1,\bar{j};p} \quad (Q^2\bar{Q}^2\mathcal{O})_{j,\bar{j};p}$
$\bar{q} = -\frac{\bar{j}}{2}$	$(\bar{Q}\mathcal{O})_{j,\bar{j}+1} \quad (\bar{Q}^2\mathcal{O})_{j,\bar{j}} \quad (Q\bar{Q}\mathcal{O})_{j\pm 1,\bar{j}+1;p}$ $(\bar{Q}^2Q\mathcal{O})_{j\pm 1,\bar{j};p} \quad (Q^2\bar{Q}\mathcal{O})_{j,\bar{j}+1;p} \quad (Q^2\bar{Q}^2\mathcal{O})_{j,\bar{j};p}$
$q = \frac{j}{2} + 1$	$(Q\mathcal{O})_{j-1,\bar{j}} \quad (Q^2\mathcal{O})_{j,\bar{j}} \quad (Q\bar{Q}\mathcal{O})_{j-1,\bar{j}\pm 1;p}$ $(Q^2\bar{Q}\mathcal{O})_{j,\bar{j}\pm 1;p} \quad (\bar{Q}^2Q\mathcal{O})_{j-1,\bar{j};p} \quad (Q^2\bar{Q}^2\mathcal{O})_{j,\bar{j};p}$
$\bar{q} = \frac{\bar{j}}{2} + 1$	$(\bar{Q}\mathcal{O})_{j,\bar{j}-1} \quad (\bar{Q}^2\mathcal{O})_{j,\bar{j}} \quad (Q\bar{Q}\mathcal{O})_{j\pm 1,\bar{j}-1;p}$ $(\bar{Q}^2Q\mathcal{O})_{j\pm 1,\bar{j};p} \quad (Q^2\bar{Q}\mathcal{O})_{j,\bar{j}-1;p} \quad (Q^2\bar{Q}^2\mathcal{O})_{j,\bar{j};p}$
$q + \bar{q} = \frac{j-\bar{j}}{2}$	$(Q\bar{Q}\mathcal{O})_{j-1,\bar{j}-1;p} \quad (Q^2\bar{Q}\mathcal{O})_{j,\bar{j}+1;p} \quad (\bar{Q}^2Q\mathcal{O})_{j-1,\bar{j};p} \quad (Q^2\bar{Q}^2\mathcal{O})_{j,\bar{j};p}$
$q + \bar{q} = \frac{j+\bar{j}}{2}$	$(Q\bar{Q}\mathcal{O})_{j+1,\bar{j}-1;p} \quad (Q^2\bar{Q}\mathcal{O})_{j,\bar{j}-1;p} \quad (\bar{Q}^2Q\mathcal{O})_{j+1,\bar{j};p} \quad (Q^2\bar{Q}^2\mathcal{O})_{j,\bar{j};p}$
$q + \bar{q} = \frac{j+\bar{j}}{2} + 1$	$(Q\bar{Q})_{j-1,\bar{j}-1;p} \quad (Q^2\bar{Q}\mathcal{O})_{j,\bar{j}-1;p} \quad (\bar{Q}^2Q\mathcal{O})_{j-1,\bar{j};p} \quad (Q^2\bar{Q}^2\mathcal{O})_{j,\bar{j};p}$
$q + \bar{q} = -\frac{j+\bar{j}}{2} - 1$	$(Q\bar{Q})_{j+1,\bar{j}+1;p} \quad (Q^2\bar{Q}\mathcal{O})_{j,\bar{j}+1;p} \quad (\bar{Q}^2Q\mathcal{O})_{j+1,\bar{j};p} \quad (Q^2\bar{Q}^2\mathcal{O})_{j,\bar{j};p}$

Table 3. Shortening conditions and the associated null components of a generic superconformal multiplet in a non-unitary SCFT.

Actually, with our results we can obtain all shortening conditions on lowest weight superconformal multiplets in non-unitary theories as well. We list these conditions and the corresponding null components in table 3.

Now, by inspection of the coefficients in table 1, it appears that some two-point functions diverge at certain values of $(j, \bar{j}, q, \bar{q})$ consistent with the unitarity bounds. Further inspection shows that this actually does not happen due to the numerators also becoming zero, faster, in fact, than the denominators. To make this more clear, let us consider a specific example. For a scalar operator $\mathcal{O}$ it can be seen from our results² that

$$\langle (Q^2\bar{Q}^2\mathcal{O})_p(x)(Q^2\bar{Q}^2\mathcal{O})_p^\dagger(0) \rangle = 2^{12}C_{\mathcal{O}} \frac{q\bar{q}(q-1)(\bar{q}-1)(q+\bar{q})(q+\bar{q}+1)}{(q+\bar{q}-1)(q+\bar{q}-2)} \frac{1}{x^{2(q+\bar{q}+2)}}, \quad (3.2)$$

where

$$(Q^2\bar{Q}^2\mathcal{O})_p = Q^2\bar{Q}^2\mathcal{O} - 2^4 \frac{\bar{q}(\bar{q}-1)}{(q+\bar{q}-1)(q+\bar{q}-2)} P^2\mathcal{O} - 2^3 \frac{\bar{q}-1}{q+\bar{q}-2} QP\bar{Q}\mathcal{O}. \quad (3.3)$$

The two-point function (3.2) diverges at $q + \bar{q} = 1$, unless $q = \bar{q} - 1 = 0$ or $q - 1 = \bar{q} = 0$. Additionally, unless $q = \bar{q} = 1$, (3.2) diverges at $q + \bar{q} = 2$. The well-defined cases just mentioned correspond to antichiral, chiral, and linear multiplets respectively. They are the only cases consistent with unitarity for which $q + \bar{q} = 1, 2$, and so the two-point function (3.2) and the primary operator (3.3) are always well-defined in a unitary theory. Note that, in non-unitary theories, two-point functions may actually diverge, but then the primary operator appearing in them is not well-defined. This can be easily seen from the example above, since both (3.2) and (3.3) diverge at the same $q, \bar{q}$. Hence, the divergences found in some primary two-point functions in non-unitary theories are a reflection of the fact that the associated primary operators cannot be defined.

²The case of a scalar $\mathcal{O}$ was also worked out in [24, appendix A].

4 Two-point functions

In this section we explicitly demonstrate the matching procedure we used to compute the two-point function coefficients of component primary operators.

It is convenient to define the supersymmetric interval between points x_i and x_j ,

$$x_{\bar{i}j} = -x_{j\bar{i}} \equiv x_{ij} - i\theta_i\sigma\bar{\theta}_i - i\theta_j\sigma\bar{\theta}_j + 2i\theta_j\sigma\bar{\theta}_i, \quad (4.1)$$

where $x_{ij} = x_i - x_j$. The notation $x_{\bar{i}j}$ indicates that this quantity is antichiral at z_i and chiral at z_j [14], where $z = (x, \theta, \bar{\theta})$ is a point in superspace. The two-point function of a superconformal primary operator with its conjugate can be written down very succinctly with the help of (4.1):

$$\langle \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}(z_1) \bar{\mathcal{O}}_{\beta_1 \dots \beta_{\bar{j}}; \dot{\beta}_1 \dots \dot{\beta}_j}(z_2) \rangle = C_{\mathcal{O}} \frac{x_{1\bar{2}}^{\alpha_1 \dot{\beta}_1} \dots x_{1\bar{2}}^{\alpha_j \dot{\beta}_j} x_{\bar{1}2}^{\beta_1 \dot{\alpha}_1} \dots x_{\bar{1}2}^{\beta_{\bar{j}} \dot{\alpha}_{\bar{j}}}}{x_{1\bar{2}}^{2q+j} x_{\bar{1}2}^{2\bar{q}+\bar{j}}}. \quad (4.2)$$

For the coefficient $C_{\mathcal{O}}$ in (4.2) we may write

$$C_{\mathcal{O}} = i^{j+\bar{j}} c_{\mathcal{O}}, \quad c_{\mathcal{O}} > 0 \text{ in a unitary theory.} \quad (4.3)$$

In a unitary theory we can of course always choose a basis for the nonzero operators $\mathcal{O}$ such that $c_{\mathcal{O}} = 1$. But we do not make this choice here.

The superfield $\mathcal{O}(z)$ can be obtained by applying $e^{i\theta Q + i\bar{\theta} \bar{Q}}$ on its zero component $\mathcal{O}(x)$,

$$\mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}(z) = e^{i\theta Q + i\bar{\theta} \bar{Q}} \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}(x).$$

Note that we use the symbol $\mathcal{O}$ both for the superfield operator and its zero component. The Baker-Campbell-Hausdorff formula and the supersymmetry algebra imply that

$$e^{i\theta Q + i\bar{\theta} \bar{Q}} = e^{i\theta Q} e^{i\bar{\theta} \bar{Q}} e^{\theta P \bar{\theta}},$$

and expanding the exponentials it is straightforward to evaluate

$$\begin{aligned} e^{i\theta Q + i\bar{\theta} \bar{Q}} &= 1 + i\theta Q + i\bar{\theta} \bar{Q} + \frac{1}{2} \theta \sigma^\mu \bar{\theta} (Q \sigma_\mu \bar{Q} + 2P_\mu) + \frac{1}{4} \theta^2 Q^2 + \frac{1}{4} \bar{\theta}^2 \bar{Q}^2 \\ &\quad - \frac{i}{4} \theta^2 \bar{\theta}^{\dot{\alpha}} (Q^2 \bar{Q}_{\dot{\alpha}} - 2Q^\alpha \sigma_{\alpha\dot{\alpha}}^\mu P_\mu) + \frac{i}{4} \bar{\theta}^2 \theta^\alpha (\bar{Q}^2 Q_\alpha + 2\sigma_{\alpha\dot{\alpha}}^\mu \bar{Q}^{\dot{\alpha}} P_\mu) \\ &\quad + \frac{1}{24} \theta^2 \bar{\theta}^2 (Q^2 \bar{Q}^2 - 4P^2 - 4Q \sigma^\mu \bar{Q} P_\mu). \end{aligned} \quad (4.4)$$

Our task is now straightforward: we need to perform the θ -expansion of both sides of (4.2), and read off the various two-point functions of the conformal primary operators that appear in the expansion of the left-hand side, which we can obtain from (4.4). In practice, even obtaining the θ -expansion of the right-hand side of (4.2) is a very cumbersome computation, as can be seen from (4.1), but, fortunately, it can be coded, for example in *Mathematica*. This amounts to implementing spinors, 4-vectors and various relations between them, such as Fierz identities. For more details on this and other aspects of the computation the reader is referred to appendix A.

The other hurdle in this computation is the contamination of descendants starting at order $\theta_1 \bar{\theta}_1 \theta_2 \bar{\theta}_2$. Indeed, the expansion of the right-hand side of (4.2) contains contributions

of two-point functions involving descendants — for example a two-point function of the form $\langle P\mathcal{O}(x)P^\dagger\bar{\mathcal{O}}(0)\rangle$ at order $\theta_1\bar{\theta}_1\theta_2\bar{\theta}_2$. Such contamination has to be appropriately subtracted out, by working out the linear combinations of operators that are conformal primary. This can be done using information from contributions to the right-hand side of (4.2) coming purely from two-point functions involving descendants. An example is the order $\theta_1\bar{\theta}_1$, which is present in (4.2) simply because of two-point functions of the form $\langle P\mathcal{O}(x)\bar{\mathcal{O}}(0)\rangle$.

In the remainder of this section we list our results for the various primary two-point functions. Our method of computation is explained in appendix A.

4.1 Orders $\theta_1\bar{\theta}_2$ and $\bar{\theta}_1\theta_2$

For the symmetric part of $Q\mathcal{O}$ we get

$$\begin{aligned} \langle Q_{(\alpha}\mathcal{O}_{\alpha_1\dots\alpha_j); \dot{\alpha}_1\dots\dot{\alpha}_{\bar{j}}}(x)\bar{Q}_{(\dot{\beta}}\bar{\mathcal{O}}_{\beta_1\dots\beta_{\bar{j}}; \dot{\beta}_1\dots\dot{\beta}_j)}(0)\rangle = \\ 2iC_{\mathcal{O}}(-1)^{j+\bar{j}+1} \frac{j+2q}{(j+1)^2} \frac{x_{(\alpha}\dot{x}_{\alpha_1\dot{\beta}_1}\cdots x_{\alpha_j)\dot{\beta}_j}x_{(\beta_1\dot{\alpha}_1}\cdots x_{\beta_{\bar{j}})\dot{\alpha}_{\bar{j}}}}{x^{2(q+\bar{q}+1)+j+\bar{j}}}, \end{aligned} \quad (4.5)$$

while for the antisymmetric part of $Q\mathcal{O}$ we find

$$\begin{aligned} \langle Q^\alpha\mathcal{O}_{\alpha\alpha_1\dots\alpha_{j-1}; \dot{\alpha}_1\dots\dot{\alpha}_{\bar{j}}}(x)\bar{Q}^{\dot{\beta}}\bar{\mathcal{O}}_{\beta_1\dots\beta_{\bar{j}}; \dot{\beta}_1\dots\dot{\beta}_{j-1}}(0)\rangle = \\ 2iC_{\mathcal{O}}(-1)^{j+\bar{j}+1} \frac{(j+1)(j-2(q-1))}{j} \frac{x_{(\alpha_1\dot{\beta}_1}\cdots x_{\alpha_{j-1})\dot{\beta}_{j-1}}x_{(\beta_1\dot{\alpha}_1}\cdots x_{\beta_{\bar{j}})\dot{\alpha}_{\bar{j}}}}{x^{2(q+\bar{q})+j+\bar{j}}}. \end{aligned} \quad (4.6)$$

As a consistency check on (4.6) we see that for $j=1$ and $\bar{j}=0$ the corresponding two-point function, namely $\langle Q^\alpha\mathcal{O}_\alpha(x)\bar{Q}^{\dot{\beta}}\bar{\mathcal{O}}_{\dot{\beta}}(0)\rangle = \langle Q^\alpha\mathcal{O}_\alpha(x)(Q^\beta\mathcal{O}_\beta)^\dagger(0)\rangle$, is indeed positive in a unitary theory (recall (4.3)).

We also find

$$\begin{aligned} \langle \bar{Q}_{(\dot{\alpha}}\mathcal{O}_{\alpha_1\dots\alpha_j; \dot{\alpha}_1\dots\dot{\alpha}_{\bar{j}}}(x)Q_{(\beta}\bar{\mathcal{O}}_{\beta_1\dots\beta_{\bar{j}}; \dot{\beta}_1\dots\dot{\beta}_j)}(0)\rangle = \\ 2iC_{\mathcal{O}}(-1)^{j+\bar{j}+1} \frac{\bar{j}+2\bar{q}}{(\bar{j}+1)^2} \frac{x_{(\alpha_1\dot{\beta}_1}\cdots x_{\alpha_j)\dot{\beta}_j}x_{(\beta\dot{\alpha}}x_{\beta_1\dot{\alpha}_1}\cdots x_{\beta_{\bar{j}})\dot{\alpha}_{\bar{j}}}}{x^{2(q+\bar{q}+1)+j+\bar{j}}}, \end{aligned} \quad (4.7)$$

and

$$\begin{aligned} \langle \bar{Q}^{\dot{\alpha}}\mathcal{O}_{\alpha_1\dots\alpha_j; \dot{\alpha}\dot{\alpha}_1\dots\dot{\alpha}_{\bar{j}-1}}(x)Q^\beta\bar{\mathcal{O}}_{\beta\beta_1\dots\beta_{\bar{j}-1}; \dot{\beta}_1\dots\dot{\beta}_j)}(0)\rangle = \\ 2iC_{\mathcal{O}}(-1)^{j+\bar{j}+1} \frac{(\bar{j}+1)(\bar{j}-2(\bar{q}-1))}{\bar{j}} \frac{x_{(\alpha_1\dot{\beta}_1}\cdots x_{\alpha_j)\dot{\beta}_j}x_{(\beta_1\dot{\alpha}_1}\cdots x_{\beta_{\bar{j}-1})\dot{\alpha}_{\bar{j}-1}}}{x^{2(q+\bar{q})+j+\bar{j}}}. \end{aligned} \quad (4.8)$$

These, of course, can also be obtained from (4.5) and (4.6) with $(j, \bar{j}, q, \bar{q}) \rightarrow (\bar{j}, j, \bar{q}, q)$.

4.2 Orders $\theta_1^2\bar{\theta}_2^2$ and $\bar{\theta}_1^2\theta_2^2$

At orders $\theta_1^2\bar{\theta}_2^2$ and $\bar{\theta}_1^2\theta_2^2$ we find

$$\begin{aligned} \langle Q^2\mathcal{O}_{\alpha_1\dots\alpha_j; \dot{\alpha}_1\dots\dot{\alpha}_{\bar{j}}}(x)\bar{Q}^2\bar{\mathcal{O}}_{\beta_1\dots\beta_{\bar{j}}; \dot{\beta}_1\dots\dot{\beta}_j)}(0)\rangle = \\ 2^4C_{\mathcal{O}}(j+2q)(j-2(q-1)) \frac{x_{(\alpha_1\dot{\beta}_1}\cdots x_{\alpha_j)\dot{\beta}_j}x_{(\beta_1\dot{\alpha}_1}\cdots x_{\beta_{\bar{j}})\dot{\alpha}_{\bar{j}}}}{x^{2(q+\bar{q}+1)+j+\bar{j}}}, \end{aligned} \quad (4.9)$$

and

$$\begin{aligned} \langle \bar{Q}^2 \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}(x) Q^2 \bar{\mathcal{O}}_{\beta_1 \dots \beta_{\bar{j}}; \dot{\beta}_1 \dots \dot{\beta}_j}(0) \rangle = \\ 2^4 C_{\mathcal{O}}(\bar{j} + 2\bar{q})(\bar{j} - 2(\bar{q} - 1)) \frac{x_{(\alpha_1 \dot{\beta}_1 \dots x_{\alpha_j) \dot{\beta}_j} x_{(\beta_1 \dot{\alpha}_1 \dots x_{\beta_{\bar{j}}) \dot{\alpha}_{\bar{j}}}}}{x^{2(q+\bar{q}+1)+j+\bar{j}}}, \end{aligned} \quad (4.10)$$

respectively. The overall sign here can be checked for a scalar operator, taking into account the relation $\bar{Q}^2 \bar{\mathcal{O}} = -(Q^2 \mathcal{O})^\dagger$, which follows from the fact that the bosonic operator Q^2 acts with the adjoint action, i.e. $Q^2 \mathcal{O} \equiv [Q^2, \mathcal{O}]$.

4.3 Order $\theta_1 \bar{\theta}_1 \theta_2 \bar{\theta}_2$

At order $\theta_1 \bar{\theta}_1 \theta_2 \bar{\theta}_2$ we have to consider conformal descendant contributions to (4.2). More specifically, we can write

$$\begin{aligned} e^{i\theta Q + i\bar{\theta} \bar{Q}} \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \big|_{\theta\bar{\theta}} = -\theta^\alpha \bar{\theta}^{\dot{\alpha}} \left\{ \left[Q_{(\alpha} \bar{Q}_{\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \right]_p - c_1 P_{(\alpha(\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}})} \right\} \\ + \frac{j}{j+1} \theta_{(\alpha_1} \bar{\theta}^{\dot{\alpha}} \left\{ \left[Q^\alpha \bar{Q}_{\dot{\alpha}} \mathcal{O}_{|\alpha \alpha_2 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \right]_p - c_2 P_{(\alpha} \mathcal{O}_{|\alpha \alpha_2 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \right\} \\ + \frac{\bar{j}}{\bar{j}+1} \theta^\alpha \bar{\theta}_{(\dot{\alpha}_1} \left\{ \left[Q_{(\alpha} \bar{Q}^{\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j); |\dot{\alpha} \dot{\alpha}_2 \dots \dot{\alpha}_{\bar{j}}} \right]_p + c_3 P_{(\alpha}^{\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j); |\dot{\alpha} \dot{\alpha}_2 \dots \dot{\alpha}_{\bar{j}}} \right\} \\ - \frac{j\bar{j}}{(j+1)(\bar{j}+1)} \theta_{(\alpha_1} \bar{\theta}_{\dot{\alpha}_1} \left\{ \left[Q^\alpha \bar{Q}^{\dot{\alpha}} \mathcal{O}_{|\alpha \alpha_2 \dots \alpha_j); |\dot{\alpha} \dot{\alpha}_2 \dots \dot{\alpha}_{\bar{j}}} \right]_p - c_4 P^{\alpha \dot{\alpha}} \mathcal{O}_{|\alpha \alpha_2 \dots \alpha_j); |\dot{\alpha} \dot{\alpha}_2 \dots \dot{\alpha}_{\bar{j}}} \right\}, \end{aligned}$$

where $[\cdot]_p$ denotes a conformal primary operator. The $\theta_1 \bar{\theta}_1$ order of (4.2) arises purely because of the descendants above via two-point functions of the form $\langle P\mathcal{O}(x)\bar{\mathcal{O}}(0) \rangle$, and this allows us to compute

$$\begin{aligned} c_1 = \frac{j - \bar{j} + 2(q - \bar{q})}{(j+1)(\bar{j}+1)(j+\bar{j}+2(q+\bar{q}))}, \quad c_2 = \frac{j + \bar{j} - 2(q - \bar{q} - 1)}{j(\bar{j}+1)(j - \bar{j} - 2(q + \bar{q} - 1))}, \\ c_3 = -\frac{j + \bar{j} + 2(q - \bar{q} + 1)}{\bar{j}(j+1)(j - \bar{j} + 2(q + \bar{q} - 1))}, \quad c_4 = \frac{j - \bar{j} - 2(q - \bar{q})}{j\bar{j}(j + \bar{j} - 2(q + \bar{q} - 2))}. \end{aligned} \quad (4.11)$$

With these results and (4.4) it is now easy to find the combinations of the operators $Q\bar{Q}\mathcal{O}$ and $P\mathcal{O}$ that are conformal primaries. For example,

$$\left[Q_{(\alpha} \bar{Q}_{\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \right]_p = Q_{(\alpha} \bar{Q}_{\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} + (c_1 - 1) P_{(\alpha(\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}})},$$

with similar expressions for the other primary operators.

With the results (4.11) and the zero component of (4.2) we can now compute the two-point functions

$$\begin{aligned} \left\langle \left[Q_{(\alpha} \bar{Q}_{\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j); \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \right]_p(x) \left[\bar{Q}_{(\dot{\beta}} Q_{\beta} \bar{\mathcal{O}}_{\beta_1 \dots \beta_{\bar{j}}); \dot{\beta}_1 \dots \dot{\beta}_j} \right]_p(0) \right\rangle = \\ 4C_{\mathcal{O}} \frac{(j+2q)(\bar{j}+2\bar{q})(j+\bar{j}+2(q+\bar{q}+1))}{(j+1)^2(\bar{j}+1)^2(j+\bar{j}+2(q+\bar{q}))} \frac{x_{(\alpha \dot{\beta} x_{\alpha_1 \dot{\beta}_1} \dots x_{\alpha_j) \dot{\beta}_j} x_{(\beta \dot{\alpha} x_{\beta_1 \dot{\alpha}_1} \dots x_{\beta_{\bar{j}}) \dot{\alpha}_{\bar{j}}}}}{x^{2(q+\bar{q}+2)+j+\bar{j}}}, \end{aligned} \quad (4.12)$$

$$\left\langle \left[Q^\alpha \bar{Q}_{(\dot{\alpha}} \mathcal{O}_{\alpha\alpha_1\dots\alpha_{j-1}; \dot{\alpha}_1\dots\dot{\alpha}_{\bar{j}})} \right]_p(x) \left[\bar{Q}^{\dot{\beta}} Q_{(\beta} \bar{\mathcal{O}}_{\beta_1\dots\beta_{\bar{j}}); \dot{\beta}_1\dots\dot{\beta}_{j-1}} \right]_p(0) \right\rangle =$$

$$4C_{\mathcal{O}} \frac{(j+1)(j-2(q-1))(\bar{j}+2\bar{q})(j-\bar{j}-2(q+\bar{q}))}{j(\bar{j}+1)^2(j-\bar{j}-2(q+\bar{q}-1))} \frac{X_{(\alpha_1\dot{\beta}_1}\dots X_{\alpha_{j-1})\dot{\beta}_{j-1}} X_{(\beta\dot{\alpha}} X_{\beta_1\dot{\alpha}_1}\dots X_{\beta_{\bar{j}})\dot{\alpha}_{\bar{j}}}}{x^{2(q+\bar{q}+1)+j+\bar{j}}}, \quad (4.13)$$

$$\left\langle \left[Q_{(\alpha} \bar{Q}^{\dot{\alpha}} \mathcal{O}_{\alpha_1\dots\alpha_j; \dot{\alpha}_1\dots\dot{\alpha}_{\bar{j}-1}} \right]_p(x) \left[\bar{Q}_{\dot{\beta}} Q^\beta \bar{\mathcal{O}}_{\beta\beta_1\dots\beta_{\bar{j}-1}; \dot{\beta}_1\dots\dot{\beta}_j} \right]_p(0) \right\rangle =$$

$$4C_{\mathcal{O}} \frac{(\bar{j}+1)(\bar{j}-2(\bar{q}-1))(j+2q)(j-\bar{j}+2(q+\bar{q}))}{\bar{j}(j+1)^2(j-\bar{j}+2(q+\bar{q}-1))} \frac{X_{(\alpha\dot{\beta}} X_{\alpha_1\dot{\beta}_1}\dots X_{\alpha_j)\dot{\beta}_j} X_{(\beta_1\dot{\alpha}_1}\dots X_{\beta_{\bar{j}-1})\dot{\alpha}_{\bar{j}-1}}}{x^{2(q+\bar{q}+1)+j+\bar{j}}}, \quad (4.14)$$

$$\left\langle \left[Q^\alpha \bar{Q}^{\dot{\alpha}} \mathcal{O}_{\alpha\alpha_1\dots\alpha_{j-1}; \dot{\alpha}_1\dots\dot{\alpha}_{\bar{j}-1}} \right]_p(x) \left[\bar{Q}^{\dot{\beta}} Q^\beta \bar{\mathcal{O}}_{\beta\beta_1\dots\beta_{\bar{j}-1}; \dot{\beta}_1\dots\dot{\beta}_{j-1}} \right]_p(0) \right\rangle =$$

$$4C_{\mathcal{O}} \frac{(j+1)(\bar{j}+1)(j-2(q-1))(\bar{j}-2(\bar{q}-1))(j+\bar{j}-2(q+\bar{q}-1))}{j\bar{j}(j+\bar{j}-2(q+\bar{q}-2))} \times \frac{X_{(\alpha_1\dot{\beta}_1}\dots X_{\alpha_{j-1})\dot{\beta}_{j-1}} X_{(\beta_1\dot{\alpha}_1}\dots X_{\beta_{\bar{j}-1})\dot{\alpha}_{\bar{j}-1}}}{x^{2(q+\bar{q})+j+\bar{j}}}. \quad (4.15)$$

As expected, (4.13) and (4.14) are exchanged under $(j, \bar{j}, q, \bar{q}) \rightarrow (\bar{j}, j, \bar{q}, q)$, while (4.12) and (4.15) are invariant under $(j, \bar{j}, q, \bar{q}) \rightarrow (\bar{j}, j, \bar{q}, q)$. When $j = \bar{j} = 0$, only (4.12) survives, while (4.13), (4.14) and (4.15) are defined when $j \neq 0$, $\bar{j} \neq 0$ and $j\bar{j} \neq 0$ respectively.

4.4 Orders $\theta_1^2 \bar{\theta}_1 \theta_2 \bar{\theta}_2^2$ and $\theta_1 \bar{\theta}_1^2 \theta_2^2 \bar{\theta}_2$

At this order we consider

$$e^{i\theta Q + i\bar{\theta} \bar{Q}} \mathcal{O}_{\alpha_1\dots\alpha_j; \dot{\alpha}_1\dots\dot{\alpha}_{\bar{j}}} \big|_{\theta^2 \bar{\theta}} = -\frac{i}{4} \theta^2 \bar{\theta}^{\dot{\alpha}} \left\{ \left[Q^2 \bar{Q}_{(\dot{\alpha}} \mathcal{O}_{\alpha_1\dots\alpha_j; \dot{\alpha}_1\dots\dot{\alpha}_{\bar{j}})} \right]_p + 2c_5 P^{\dot{\alpha}}_{(\dot{\alpha}} Q_{(\alpha} \mathcal{O}_{\alpha_1\dots\alpha_j); \dot{\alpha}_1\dots\dot{\alpha}_{\bar{j}})} \right.$$

$$\left. - 2c_6 \frac{j}{j+1} P_{(\alpha_1(\dot{\alpha}_1} Q^\alpha \mathcal{O}_{|\alpha\alpha_2\dots\alpha_j); |\dot{\alpha}\dot{\alpha}_2\dots\dot{\alpha}_{\bar{j}})} \right\}$$

$$+ \frac{i}{4} \frac{\bar{j}}{\bar{j}+1} \theta^2 \bar{\theta}_{(\dot{\alpha}_1} \left\{ \left[Q^2 \bar{Q}^{\dot{\alpha}} \mathcal{O}_{\alpha_1\dots\alpha_j; |\dot{\alpha}\dot{\alpha}_2\dots\dot{\alpha}_{\bar{j}})} \right]_p + 2c_7 P^{\dot{\alpha}\alpha} Q_{(\alpha} \mathcal{O}_{\alpha_1\dots\alpha_j); |\dot{\alpha}\dot{\alpha}_2\dots\dot{\alpha}_{\bar{j}})} \right.$$

$$\left. - 2c_8 \frac{j}{j+1} P^{\dot{\alpha}}_{(\alpha_1} Q^\alpha \mathcal{O}_{|\alpha\alpha_2\dots\alpha_j); |\dot{\alpha}\dot{\alpha}_2\dots\dot{\alpha}_{\bar{j}}} \right\}.$$

The $\theta_1^2 \bar{\theta}_1 \bar{\theta}_2$ order of (4.2) arises because of the descendants above via two-point functions of the form $\langle PQ\mathcal{O}(x)\bar{Q}\bar{\mathcal{O}}(0) \rangle$. Using this we can determine

$$c_5 = \frac{j+\bar{j}-2(q-\bar{q}-1)}{(\bar{j}+1)(j-\bar{j}-2(q+\bar{q}-1))}, \quad c_6 = \frac{j-\bar{j}+2(q-\bar{q})}{(\bar{j}+1)(j+\bar{j}+2(q+\bar{q}))}, \quad (4.16)$$

$$c_7 = \frac{j-\bar{j}-2(q-\bar{q})}{\bar{j}(j+\bar{j}-2(q+\bar{q}-2))}, \quad c_8 = \frac{j+\bar{j}+2(q-\bar{q}+1)}{\bar{j}(j-\bar{j}+2(q+\bar{q}-1))}.$$

The results (4.16), as well as (4.5) and (4.6), allow us to determine the two-point functions

$$\begin{aligned} \left\langle \left[Q^2 \bar{Q}_{(\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}})} \right]_p(x) \left[\bar{Q}^2 Q_{(\beta} \bar{\mathcal{O}}_{\beta_1 \dots \beta_{\bar{j}}; \dot{\beta}_1 \dots \dot{\beta}_{\bar{j}})} \right]_p(0) \right\rangle = \\ 2^5 i C_{\mathcal{O}} (-1)^{j+\bar{j}} \frac{(j+2q)(j-2(q-1))(\bar{j}+2\bar{q})(j+\bar{j}+2(q+\bar{q}+1))(j-\bar{j}-2(q+\bar{q}))}{(\bar{j}+1)^2(j+\bar{j}+2(q+\bar{q}))(j-\bar{j}-2(q+\bar{q}-1))} \\ \times \frac{x_{(\alpha_1 \dot{\beta}_1 \dots x_{\alpha_j} \dot{\beta}_{\bar{j}} x_{(\beta \dot{\alpha}} x_{\beta_1 \dot{\alpha}_1 \dots x_{\beta_{\bar{j}}) \dot{\alpha}_{\bar{j}}}}}{x^{2(q+\bar{q}+2)+j+\bar{j}}} \end{aligned} \quad (4.17)$$

and

$$\begin{aligned} \left\langle \left[Q^2 \bar{Q}^{\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha} \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}-1}} \right]_p(x) \left[\bar{Q}^2 Q^{\beta} \bar{\mathcal{O}}_{\beta \beta_1 \dots \beta_{\bar{j}-1}; \dot{\beta}_1 \dots \dot{\beta}_{\bar{j}}} \right]_p(0) \right\rangle = \\ 2^5 i C_{\mathcal{O}} (-1)^{j+\bar{j}} \frac{(\bar{j}+1)(j+2q)(j-2(q-1))(\bar{j}-2(\bar{q}-1))(j+\bar{j}-2(q+\bar{q}-1))}{\bar{j}(\bar{j}+\bar{j}-2(q+\bar{q}-2))} \\ \times \frac{j-\bar{j}+2(q+\bar{q})}{j-\bar{j}+2(q+\bar{q}-1)} \frac{x_{(\alpha_1 \dot{\beta}_1 \dots x_{\alpha_j} \dot{\beta}_{\bar{j}} x_{(\beta_1 \dot{\alpha}_1 \dots x_{\beta_{\bar{j}-1}) \dot{\alpha}_{\bar{j}-1}}}}}{x^{2(q+\bar{q}+1)+j+\bar{j}}}. \end{aligned} \quad (4.18)$$

We can also obtain the two-point functions

$$\left\langle \left[\bar{Q}^2 Q_{(\alpha} \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}})} \right]_p(x) \left[Q^2 \bar{Q}_{(\dot{\beta}} \bar{\mathcal{O}}_{\beta_1 \dots \beta_{\bar{j}}; \dot{\beta}_1 \dots \dot{\beta}_{\bar{j}})} \right]_p(0) \right\rangle$$

and

$$\left\langle \left[\bar{Q}^2 Q^{\alpha} \mathcal{O}_{\alpha \alpha_1 \dots \alpha_{j-1}; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \right]_p(x) \left[Q^2 \bar{Q}^{\dot{\beta}} \bar{\mathcal{O}}_{\beta_1 \dots \beta_{\bar{j}}; \dot{\beta} \dot{\beta}_1 \dots \dot{\beta}_{\bar{j}-1}} \right]_p(0) \right\rangle$$

by letting $(j, \bar{j}, q, \bar{q}) \rightarrow (\bar{j}, j, \bar{q}, q)$ in (4.17) and (4.18) respectively.

4.5 Order $\theta_1^2 \bar{\theta}_1^2 \theta_2^2 \bar{\theta}_2^2$

At this order we have to consider six new descendants:

$$\begin{aligned} e^{i\theta Q + i\bar{\theta} \bar{Q}} \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \big|_{\theta^2 \bar{\theta}^2} = \frac{1}{2^4} \theta^2 \bar{\theta}^2 \left\{ \left[Q^2 \bar{Q}^2 \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \right]_p \right. \\ - 4c_9 P^{\dot{\alpha}\alpha} \left[Q_{(\alpha} \bar{Q}_{\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \right]_p - 4 \frac{j}{j+1} c_{10} P^{\dot{\alpha}}_{(\alpha_1} \left[Q^{\alpha} \bar{Q}_{\dot{\alpha}} \mathcal{O}_{|\alpha \alpha_2 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \right]_p \\ - 4 \frac{\bar{j}}{\bar{j}+1} c_{11} P^{\alpha}_{(\dot{\alpha}_1} \left[Q_{\alpha} \bar{Q}^{\dot{\alpha}} \mathcal{O}_{\alpha_1 \dots \alpha_j; |\dot{\alpha} \dot{\alpha}_2 \dots \dot{\alpha}_{\bar{j}}} \right]_p \\ - 4 \frac{j\bar{j}}{(j+1)(\bar{j}+1)} c_{12} P_{(\alpha_1(\dot{\alpha}_1} \left[Q^{\alpha} \bar{Q}^{\dot{\alpha}} \mathcal{O}_{|\alpha \alpha_2 \dots \alpha_j; |\dot{\alpha} \dot{\alpha}_2 \dots \dot{\alpha}_{\bar{j}}} \right]_p \\ \left. - 2^3 c_{13} P^2 \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} - 2^5 c_{14} P_{(\alpha_1(\dot{\alpha}_1} P^{\dot{\alpha}\alpha} \mathcal{O}_{|\alpha \alpha_2 \dots \alpha_j; |\dot{\alpha} \dot{\alpha}_2 \dots \dot{\alpha}_{\bar{j}}} \right\}. \end{aligned}$$

The first four descendants result in terms $\theta_1^2 \bar{\theta}_1^2 \theta_2^2 \bar{\theta}_2^2$ in the expansion of (4.2), via two-point functions of the form $\langle P[Q\bar{Q}\mathcal{O}]_p(x)[\bar{Q}Q\bar{\mathcal{O}}]_p(0) \rangle$. At this θ -order these are not the only contributions; two-point functions of the form $\langle P^2 \mathcal{O}(x) P^{\dagger} \bar{\mathcal{O}}(0) \rangle$ also need to be taken into account. For these contributions we need to first determine $c_{13,14}$. This can be easily done

since the associated descendants generate the order $\theta_1^2 \bar{\theta}_1^2$ in the expansion of (4.2) via $\langle P^2 \mathcal{O}(x) \bar{\mathcal{O}}(0) \rangle$. Note that in order to compute $c_{13,14}$ we also need the coefficients in (4.11). After $c_{13,14}$ are determined, it is straightforward to find $c_{9,\dots,12}$.

Taking into consideration all the relevant contributions we can compute

$$\begin{aligned} c_9 &= \frac{j - \bar{j} - 2(q - \bar{q})}{j + \bar{j} - 2(q + \bar{q} - 2)}, & c_{10} &= \frac{j + \bar{j} + 2(q - \bar{q} + 1)}{j - \bar{j} + 2(q + \bar{q} - 1)}, \\ c_{11} &= -\frac{j + \bar{j} - 2(q - \bar{q} - 1)}{j - \bar{j} - 2(q + \bar{q} - 1)}, & c_{12} &= \frac{j - \bar{j} - 2(q - \bar{q})}{j + \bar{j} + 2(q + \bar{q})}, \end{aligned} \quad (4.19)$$

and

$$\begin{aligned} c_{13} &= \frac{(j + \bar{j})^2(j + \bar{j} + 2(q + \bar{q})) - 4(q - \bar{q})^2(j + \bar{j} + 2(q + \bar{q} - 2)) + 8(j + q)(\bar{j} + \bar{q}) + 8(jq + \bar{j}\bar{q} + 3q\bar{q}) - 4(j + \bar{j} + 2(q + \bar{q}))}{(j + \bar{j} + 2)(j + \bar{j} + 2(q + \bar{q}))(j - \bar{j} + 2(q + \bar{q} - 1))(j - \bar{j} - 2(q + \bar{q} - 1))}, \\ c_{14} &= \frac{j(j + 2) + \bar{j}(\bar{j} + 2) - 4(q(q - 1) + \bar{q}(\bar{q} - 1))}{(j + \bar{j} + 2(q + \bar{q}))(j + \bar{j} + 2(q + \bar{q} - 1))(j + \bar{j} - 2(q + \bar{q} - 1))(j + \bar{j} - 2(q + \bar{q} - 2))}. \end{aligned} \quad (4.20)$$

Using (4.19) and (4.20), the zero component of (4.2), as well as (4.12)–(4.15), we can finally obtain

$$\begin{aligned} \left\langle \left[Q^2 \bar{Q}^2 \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}} \right]_p(x) \left[\bar{Q}^2 Q^2 \bar{\mathcal{O}}_{\beta_1 \dots \beta_{\bar{j}}; \dot{\beta}_1 \dots \dot{\beta}_j} \right]_p(0) \right\rangle = \\ (j + 2q)(j - 2(q - 1))(\bar{j} + 2\bar{q})(\bar{j} - 2(\bar{q} - 1))(j - \bar{j} + 2(q + \bar{q}))(j - \bar{j} - 2(q + \bar{q})) \\ \times (j + \bar{j} + 2(q + \bar{q} + 1))(j + \bar{j} - 2(q + \bar{q} - 1)) \\ - 2^8 C_{\mathcal{O}} \frac{(j + \bar{j} + 2(q + \bar{q}))(j - \bar{j} + 2(q + \bar{q} - 1))(j - \bar{j} - 2(q + \bar{q} - 1))(j + \bar{j} - 2(q + \bar{q} - 2))}{(j + \bar{j} + 2(q + \bar{q}))(j - \bar{j} + 2(q + \bar{q} - 1))(j - \bar{j} - 2(q + \bar{q} - 1))(j + \bar{j} - 2(q + \bar{q} - 2))} \\ \times \frac{x_{(\alpha_1 \dot{\beta}_1} \dots x_{\alpha_j \dot{\beta}_j} x_{(\beta_1 \dot{\alpha}_1} \dots x_{\beta_{\bar{j}} \dot{\alpha}_{\bar{j}}}}{x^{2(q + \bar{q} + 2) + j + \bar{j}}}. \end{aligned} \quad (4.21)$$

This is the last two-point function to consider.

5 Example: the supercurrent multiplet

The supercurrent multiplet, or Ferrara-Zumino multiplet [28], has $j = \bar{j} = 1$ and $q = \bar{q} = \frac{3}{2}$. The shortening conditions $q = \frac{j}{2} + 1$ and $\bar{q} = \frac{\bar{j}}{2} + 1$ are obviously satisfied, and, thus, as can be seen from table 2, the multiplet can be expanded as

$$\mathcal{J}_\mu(x, \theta, \bar{\theta}) = j_\mu^R - \frac{i}{2} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \theta^\beta Q_{(\beta} j_{\alpha)}^R + \frac{i}{2} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \bar{\theta}^{\dot{\beta}} \bar{Q}_{(\dot{\beta}} j_{\alpha|\dot{\alpha})}^R - \theta \sigma^\nu \bar{\theta} T_{\mu\nu} + \text{descendants}, \quad (5.1)$$

where

$$T_{\mu\nu} = \frac{1}{4} \bar{\sigma}_\mu^{\dot{\alpha}\alpha} \bar{\sigma}_\nu^{\dot{\beta}\beta} \left[Q_{(\beta} \bar{Q}_{\dot{\beta}} j_{\alpha)}^R \right]_p,$$

which is obviously symmetric and traceless. In general, for any operator with integer spin ℓ , we have $\mathcal{O}_{\mu_1 \dots \mu_\ell} = (-\frac{1}{2})^\ell \bar{\sigma}_{\mu_1}^{\dot{\alpha}_1 \alpha_1} \dots \bar{\sigma}_{\mu_\ell}^{\dot{\alpha}_\ell \alpha_\ell} \mathcal{O}_{\alpha_1 \dots \alpha_\ell; \dot{\alpha}_1 \dots \dot{\alpha}_\ell}$.

The lowest component of $\mathcal{J}_\mu$ is the R-current, the $\theta \bar{\theta}$ component is the energy-momentum tensor, while the θ , $\bar{\theta}$ components are the supersymmetry currents. A generic supermultiplet

would contain many more primary component fields, but once $(j, \bar{j}, q, \bar{q}) \rightarrow (1, 1, \frac{3}{2}, \frac{3}{2})$ the multiplet is shortened, and all other conformal primary components become null. We check this by noting that all the component two-point functions vanish in this limit except (4.2), (4.5), (4.7) and (4.12), which become³

$$\langle j_\mu^R(x) j_\nu^R(0) \rangle = \frac{c_{\mathcal{J}}}{2} \frac{1}{x^6} I_{\mu\nu}(x), \quad (5.2)$$

$$\langle Q_{(\alpha} j_{\alpha_1 \dot{\alpha}_1}^R(x) \bar{Q}_{(\dot{\beta}} j_{|\beta_1| \dot{\beta}_1)}^R(0) \rangle = ic_{\mathcal{J}} \frac{(x_{\alpha \dot{\beta}} x_{\alpha_1 \dot{\beta}_1} + x_{\alpha_1 \dot{\beta}} x_{\alpha \dot{\beta}_1}) x_{\beta_1 \dot{\alpha}_1}}{x^{10}}, \quad (5.3)$$

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle = 5c_{\mathcal{J}} \frac{1}{x^8} \left(I_{\mu\rho}(x) I_{\nu\sigma}(x) + I_{\mu\sigma}(x) I_{\nu\rho}(x) - \frac{1}{2} \eta_{\mu\nu} \eta_{\rho\sigma} \right), \quad (5.4)$$

where

$$I_{\mu\nu}(x) = \eta_{\mu\nu} - 2 \frac{x_\mu x_\nu}{x^2}, \quad (5.5)$$

and $c_{\mathcal{J}} > 0$ in unitary theories. These results imply

$$\partial^\mu j_\mu^R = 0, \quad \partial^\mu T_{\mu\nu} = 0, \quad T_\mu^\mu = 0, \quad \partial_\mu (\bar{\sigma}^{\mu\dot{\beta}\beta} Q_{(\alpha} j_{\beta)}^R) = 0, \quad (5.6)$$

which are the correct conservation conditions of $\mathcal{N} = 1$ superconformal symmetry. Note that we did not need to impose these conservation conditions as extra constraints. They follow from the correct choice of quantum numbers of the multiplet.

Conservation conditions like (5.6) hold for any conformal primary with scaling dimension $\Delta = \frac{j+\bar{j}}{2} + 2$. As we just saw, the structure of the superconformal two-point function forces them to hold, but note that for three- and higher-point functions they are non-trivial Ward identities, in the sense that they are not implied by the superconformal symmetry.

6 Summary

Superconformal symmetry imposes powerful constraints on quantum field theories in any dimensions. For four-dimensional $\mathcal{N} = 1$ SCFTs the form of superspace correlation functions consistent with the symmetry can be obtained with various methods. In particular, superspace two-point functions between superfields are determined up to an overall constant, and the form of the three-point function is determined up to a few constants. These results imply relations between the correlation functions involving different components of the supermultiplet. These relations are physically important, but have not been worked out explicitly in full generality. For two-point functions of superconformal multiplets with quantum numbers $(j, \bar{j}, q, \bar{q}) = (\ell, \ell, \Delta/2, \Delta/2)$ they were worked out in [8], and the general results of this paper agree with those obtained there.

In this work we developed a method that systematically computes such relations based on superspace correlation functions. In particular we decomposed the superspace two-point function to contributions from the various conformal primaries and their descendants. Consequently, we determined the relation imposed by the superconformal symmetry among the two-point function coefficients of the conformal primary components in a superconformal

³For two-point functions of operators with four-vector indices see appendix A, section A.1.

multiplet. This result enables us to determine all possible shortening conditions associated with supermultiplets built from any superconformal primary operator. It also gives an alternative derivation of the unitarity bounds. Our results are consistent with existing literature.

The method described in this paper can also be applied to three-point functions, which, together with the results presented here, will systematically determine relations between OPE coefficients of conformal primaries in a supermultiplet. This analysis will yield expressions for superconformal blocks as linear combinations of conformal blocks. Additionally, our formalism can be generalized to theories with more supersymmetry and in other spacetime dimensions.

To make our calculation possible, we developed a *Mathematica* package that automates the simplification, expansion, and differentiation of expressions built with four-vectors and two-component spinors. This package can be useful in other calculations in supersymmetric field theories and beyond.

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A The index-free formalism

The correlators of operators in generic Lorentz representations involve many complicated tensor structures. The index-free formalism is an efficient representation of these tensor structures. In addition, this formalism also exposes the various linear relations between the tensor structures. We thus employ this formalism in the implementation of our calculation. In this section, we introduce the index-free formalism that describes superfields and correlators in $\mathcal{N} = 1$ SCFTs.

The index-free formalism represents a symmetric tensor as a scalar field defined on a space of auxiliary spinors,

$$T_{(\alpha_1 \dots \alpha_n)} \rightarrow \frac{1}{n!} \eta^{\alpha_1} \dots \eta^{\alpha_n} T_{\alpha_1 \dots \alpha_n} \equiv T(\eta, n), \quad (\text{A.1})$$

where

$$T_{(\alpha_1 \dots \alpha_n)} \equiv \frac{1}{n!} \sum_{\mathcal{P}_n} \mathcal{P}_n T_{\alpha_1 \dots \alpha_n} \quad (\text{A.2})$$

is a symmetric tensor with $\mathcal{P}_n$ denoting permutations over the indices $\alpha_1, \dots, \alpha_n$. Contrary to the spinor coordinates of superspace, the auxiliary spinors η commute with each other, implying $\eta^2 = \eta^\alpha \eta_\alpha = \epsilon^{\alpha\beta} \eta_\beta \eta_\alpha = 0$. The index-free field $T(\eta, n)$ constructed above can be mapped back to the traditional form by differentiating with respect to the auxiliary spinors,

$$\partial_{\eta^{\alpha_1}} \dots \partial_{\eta^{\alpha_n}} T(\eta, n) = T_{(\alpha_1 \dots \alpha_n)}, \quad (\text{A.3})$$

where the spinor derivatives are defined in the usual way, i.e. $\partial_{\eta^\alpha} \eta^\beta \equiv \delta_\alpha^\beta$, $\partial_{\eta^\alpha} = -\epsilon_{\alpha\beta} \partial_{\eta_\beta}$. Note the index-free form $T(\eta, n)$ contains exactly the same information as the original traditional form $T_{(\alpha_1 \dots \alpha_n)}$. From now on we will omit the parentheses around totally symmetrized indices.

An operator $\mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}$ in the irrep $(j/2, \bar{j}/2)$ of the Lorentz group is represented by the index-free form

$$\mathcal{O}_{j, \bar{j}}^\eta \equiv \mathcal{O}(\eta, j; \bar{\eta}, \bar{j}) \equiv \frac{1}{j! \bar{j}!} \eta^{\alpha_1} \dots \eta^{\alpha_j} \bar{\eta}^{\dot{\alpha}_1} \dots \bar{\eta}^{\dot{\alpha}_{\bar{j}}} \mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}. \quad (\text{A.4})$$

The two-point function of any such conformal primary operator with its conjugate is given by

$$\langle \mathcal{O}(\eta_1, j; \bar{\eta}_1, \bar{j})(x) \mathcal{O}(\eta_2, j; \bar{\eta}_2, \bar{j})^\dagger(0) \rangle = C_{\mathcal{O}} \frac{(\eta_1 x \bar{\eta}_2)^j (\eta_2 x \bar{\eta}_1)^{\bar{j}}}{x^{2(q+\bar{q})+j+\bar{j}}}. \quad (\text{A.5})$$

We now turn to the supersymmetric case. Applying a generator Q_α on $\mathcal{O}_{j, \bar{j}}^\eta$ will generate a reducible representation of the Lorentz group, which contains two irreducible representations, namely

$$(Q\mathcal{O})_{j+1, \bar{j}}^\eta \equiv \frac{1}{j+1} [\eta Q, \mathcal{O}_{j, \bar{j}}^\eta], \quad (Q\mathcal{O})_{j-1, \bar{j}}^\eta \equiv \frac{1}{j} [Q \partial_\eta, \mathcal{O}_{j, \bar{j}}^\eta], \quad (\text{A.6})$$

where Q acts with a commutator (resp. anticommutator) if $j + \bar{j}$ is even (resp. odd). With these normalizations we have

$$\theta Q \mathcal{O}_{j, \bar{j}}^\eta \equiv [\theta Q, \mathcal{O}_{j, \bar{j}}^\eta] = \theta \partial_\eta (Q\mathcal{O})_{j+1, \bar{j}}^\eta + \frac{j}{j+1} \theta \eta (Q\mathcal{O})_{j-1, \bar{j}}^\eta. \quad (\text{A.7})$$

We can similarly write down each component of the superfield as a linear combination of operators in irreps of the Lorentz group. We will suppress the (anti)commutator and simply write the analogs of $\theta Q \mathcal{O}_{j, \bar{j}}^\eta$ from now on. For example, we have

$$\begin{aligned} \mathcal{O}_{j, \bar{j}}^\eta|_{\theta\bar{\theta}} &= -\theta \partial_\eta \bar{\theta} \partial_{\bar{\eta}} ((Q\bar{Q}\mathcal{O})_{j+1, \bar{j}+1; p}^\eta - ic_1 \eta \partial_x \bar{\eta} \mathcal{O}_{j, \bar{j}}^\eta) \\ &\quad - \frac{j}{j+1} \theta \eta \bar{\theta} \partial_{\bar{\eta}} ((Q\bar{Q}\mathcal{O})_{j-1, \bar{j}+1; p}^\eta - ic_2 \partial_\eta \partial_x \bar{\eta} \mathcal{O}_{j, \bar{j}}^\eta) \\ &\quad + \frac{\bar{j}}{\bar{j}+1} \theta \partial_\eta \bar{\theta} \bar{\eta} ((Q\bar{Q}\mathcal{O})_{j+1, \bar{j}-1; p}^\eta + ic_3 \eta \partial_x \partial_{\bar{\eta}} \mathcal{O}_{j, \bar{j}}^\eta) \\ &\quad + \frac{j\bar{j}}{(j+1)(\bar{j}+1)} \theta \eta \bar{\theta} \bar{\eta} ((Q\bar{Q}\mathcal{O})_{j-1, \bar{j}-1; p}^\eta - ic_4 \partial_\eta \partial_x \partial_{\bar{\eta}} \mathcal{O}_{j, \bar{j}}^\eta), \\ \mathcal{O}_{j, \bar{j}}^\eta|_{\theta^2\bar{\theta}} &= -\frac{i}{4} \theta^2 \bar{\theta} \partial_{\bar{\eta}} \left((Q^2\bar{Q}\mathcal{O})_{j, \bar{j}+1; p}^\eta + 2ic_5 \partial_\eta \partial_x \bar{\eta} (Q\mathcal{O})_{j+1, \bar{j}; p}^\eta - 2ic_6 \frac{j}{j+1} \eta \partial_x \bar{\eta} (Q\mathcal{O})_{j-1, \bar{j}; p}^\eta \right) \\ &\quad + \frac{i}{4} \frac{\bar{j}}{\bar{j}+1} \theta^2 \bar{\theta} \bar{\eta} \left((Q^2\bar{Q}\mathcal{O})_{j, \bar{j}-1; p}^\eta + 2ic_7 \partial_\eta \partial_x \partial_{\bar{\eta}} (Q\mathcal{O})_{j+1, \bar{j}; p}^\eta - 2ic_8 \frac{j}{j+1} \eta \partial_x \partial_{\bar{\eta}} (Q\mathcal{O})_{j-1, \bar{j}; p}^\eta \right), \end{aligned}$$

and

$$\begin{aligned} \mathcal{O}_{j, \bar{j}}^\eta|_{\theta^2\bar{\theta}^2} &= \frac{1}{2^4} \theta^2 \bar{\theta}^2 \left((Q^2\bar{Q}^2\mathcal{O})_{j, \bar{j}; p}^\eta - 4ic_9 \partial_\eta \partial_x \partial_{\bar{\eta}} (Q\bar{Q}\mathcal{O})_{j+1, \bar{j}+1; p}^\eta \right. \\ &\quad - 4\frac{j}{j+1} ic_{10} \eta \partial_x \partial_{\bar{\eta}} (Q\bar{Q}\mathcal{O})_{j-1, \bar{j}+1; p}^\eta - 4\frac{\bar{j}}{\bar{j}+1} ic_{11} \partial_\eta \partial_x \bar{\eta} (Q\bar{Q}\mathcal{O})_{j+1, \bar{j}-1; p}^\eta \\ &\quad \left. - 4\frac{j\bar{j}}{(j+1)(\bar{j}+1)} ic_{12} \eta \partial_x \bar{\eta} (Q\bar{Q}\mathcal{O})_{j-1, \bar{j}-1; p}^\eta + 2^3 c_{13} \partial_x^2 \mathcal{O}_{j, \bar{j}}^\eta + 2^5 c_{14} \eta \partial_x \bar{\eta} \partial_\eta \partial_x \partial_{\bar{\eta}} \mathcal{O}_{j, \bar{j}}^\eta \right), \end{aligned}$$

in accord with corresponding expressions in section 4.

The superfield two-point function of $\mathcal{O}_{j,\bar{j}}^\eta$, following (4.2), is given by

$$\langle \mathcal{O}_{j,\bar{j}}^{\eta_1}(z_1)(\mathcal{O}_{j,\bar{j}}^{\eta_2})^\dagger(z_2) \rangle = C_{\mathcal{O}} \frac{(\eta_1 x_{1\bar{2}} \bar{\eta}_2)^j (\eta_2 x_{\bar{1}2} \bar{\eta}_1)^{\bar{j}}}{x_{1\bar{2}}^{2q+j} x_{\bar{1}2}^{2\bar{q}+\bar{j}}}. \quad (\text{A.8})$$

By conformal symmetry, the two-point function coefficient of all conformal primary operators in a supermultiplet is proportional to $C_{\mathcal{O}}$, with coefficient determined by the quantum numbers $(j, \bar{j}, q, \bar{q})$ of the lowest component. The coefficients c_i of the descendants are also determined by this information. In this paper we use the superfield two-point function (A.8) to explicitly work out all these coefficients.

For example, to determine the two-point function coefficients of $(Q\mathcal{O})_{j+1,\bar{j}}^\eta$ and $(Q\mathcal{O})_{j-1,\bar{j}}^\eta$, we simply expand (A.8) and match to the expected form obtained with (A.5) and (A.7). In particular, we can define

$$\langle (Q\mathcal{O})_{j\pm 1,\bar{j}}^{\eta_1}(x)((Q\mathcal{O})_{j\pm 1,\bar{j}}^{\eta_2})^\dagger(0) \rangle = C_{(Q\mathcal{O})_{j\pm 1,\bar{j}}} \frac{(\eta_1 x \bar{\eta}_2)^{j\pm 1} (\eta_2 x \bar{\eta}_1)^{\bar{j}}}{x^{2(q+\bar{q})+1+j+\bar{j}\pm 1}}, \quad (\text{A.9})$$

and obtain (setting $x_2 = 0$)

$$\begin{aligned} \langle \mathcal{O}_{j,\bar{j}}^{\eta_1}(z_1)(\mathcal{O}_{j,\bar{j}}^{\eta_2})^\dagger(z_2) \rangle \Big|_{\theta_1 \bar{\theta}_2} &= C_{(Q\mathcal{O})_{j+1,\bar{j}}} \theta_1 \partial_{\eta_1} \bar{\theta}_2 \partial_{\bar{\eta}_2} \frac{(\eta_1 x \bar{\eta}_2)^{j+1} (\eta_2 x \bar{\eta}_1)^{\bar{j}}}{x^{2(q+\bar{q}+1)+j+\bar{j}}} \\ &\quad + \left(\frac{j}{j+1} \right)^2 C_{(Q\mathcal{O})_{j-1,\bar{j}}} \theta_1 \eta_1 \bar{\theta}_2 \bar{\eta}_2 \frac{(\eta_1 x \bar{\eta}_2)^{j-1} (\eta_2 x \bar{\eta}_1)^{\bar{j}}}{x^{2(q+\bar{q})+j+\bar{j}}}. \end{aligned}$$

From this relation we can determine the two unknowns $C_{(Q\mathcal{O})_{j\pm 1,\bar{j}}}$. There will be two independent tensor structures appearing on both sides, providing the two necessary equations. The result is

$$C_{(Q\mathcal{O})_{j+1,\bar{j}}} = 2i \frac{j+2q}{(j+1)^2} C_{\mathcal{O}}, \quad C_{(Q\mathcal{O})_{j-1,\bar{j}}} = 2i \frac{(j+1)(j-2(q-1))}{j} C_{\mathcal{O}}. \quad (\text{A.10})$$

We use similar methods to determine coefficients appearing in all components of the general superconformal primary superfield $\mathcal{O}_{j,\bar{j}}^\eta$. In some cases, in order to obtain the primary two-point function coefficients, we need to determine the descendant coefficients first. For example, $c_{1,\dots,4}$ can be determined through the $\theta_1 \bar{\theta}_1$ or $\theta_2 \bar{\theta}_2$ order of (A.8), which is then used as input for determining $C_{(Q\bar{Q}\mathcal{O})_{j\pm 1,\bar{j}\pm 1}}$ in the $\theta_1 \bar{\theta}_1 \theta_2 \bar{\theta}_2$ order of (A.8). In general, the number of independent tensors in a particular order may exceed the number of unknown coefficients. These extra constraints provide non-trivial consistency checks for our results.

A.1 Operators with integer spin

Operators $\mathcal{O}_{\alpha_1 \dots \alpha_j; \dot{\alpha}_1 \dots \dot{\alpha}_{\bar{j}}}$ with $j = \bar{j} = \ell$ form an interger spin representation of the Lorentz group. We can convert the spinor indices to four-vector indices by

$$\mathcal{O}_{\mu_1 \dots \mu_\ell} \equiv \left(-\frac{1}{2}\right)^\ell \bar{\sigma}_{\mu_1}^{\alpha_1 \dot{\alpha}_1} \dots \bar{\sigma}_{\mu_\ell}^{\alpha_\ell \dot{\alpha}_\ell} \mathcal{O}_{\alpha_1 \dots \alpha_\ell; \dot{\alpha}_1 \dots \dot{\alpha}_\ell}. \quad (\text{A.11})$$

Then, $\mathcal{O}_{\mu_1 \dots \mu_\ell}$ is a symmetric traceless tensor. For completeness, we derive in this section the explicit form of the two-point function of $\mathcal{O}_{\mu_1 \dots \mu_\ell}$ from (A.5). We first rewrite $\mathcal{O}$ in an index-free form,

$$\mathcal{O}_\ell^b = \frac{1}{\ell!} b^{\mu_1} \dots b^{\mu_\ell} \mathcal{O}_{\mu_1 \dots \mu_\ell}, \quad (\text{A.12})$$

where b^{μ_i} are auxiliary bosonic four-vectors satisfying $b^\mu b_\mu = 0$, corresponding to the fact that $\mathcal{O}$ is traceless. Then,

$$\mathcal{O}_{\mu_1 \dots \mu_\ell} = \frac{\partial}{\partial b^{\mu_1}} \dots \frac{\partial}{\partial b^{\mu_\ell}} \mathcal{O}_\ell^b - \text{traces}. \quad (\text{A.13})$$

Equation (A.11) then implies the following mapping from (A.4) to (A.12):

$$\mathcal{O}_\ell^b = (-\tfrac{1}{2})^\ell \frac{1}{\ell!} (\partial_\eta b \partial_{\bar{\eta}})^\ell \mathcal{O}_{j=\ell, \bar{j}=\ell}^\eta, \quad b_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^\mu b_\mu. \quad (\text{A.14})$$

We apply this mapping to the two-point function (A.8) and get

$$\langle \mathcal{O}_\ell^{b_1}(x) (\mathcal{O}_\ell^{b_2})^\dagger(0) \rangle = c_{\mathcal{O}} \frac{\ell!}{2^\ell} \frac{(x^2 b_1 \cdot b_2 - 2(b_1 \cdot x)(b_2 \cdot x))^\ell}{x^{2(\Delta_{\mathcal{O}} + \ell)}},$$

where we have used $b_{1,2}^2 = 0$. The coefficient $c_{\mathcal{O}} = (-1)^\ell C_{\mathcal{O}}$ is positive in unitary theories. Finally, using (A.13), we get the familiar result

$$\langle \mathcal{O}_{\mu_1 \dots \mu_\ell}(x) \mathcal{O}_{\nu_1 \dots \nu_\ell}^\dagger(0) \rangle = c_{\mathcal{O}} \frac{(\ell!)^3}{2^\ell} \frac{I_{(\mu_1|\nu_1|}(x) \cdots I_{\mu_\ell)\nu_\ell}(x) - \text{traces}}{x^{2\Delta_{\mathcal{O}}}}. \quad (\text{A.15})$$

B The *Mathematica* package

In this appendix we provide more details on our *Mathematica*⁴ package, which is an efficient tool for expanding functions with Grassmann variables and simplifying expressions with Lorentz structures.

This package handles general expressions built with any number of four-vectors x^μ and two-component spinors θ_α , $\bar{\theta}_{\dot{\alpha}}$ and η_β , $\bar{\eta}_{\dot{\beta}}$, where θ and $\bar{\theta}$ are Grassmann variables, while η and $\bar{\eta}$ are commuting variables. We use the index-free formalism of the previous appendix to represent all expressions with free indices as a Lorentz-invariant scalar function. The standard simplifications for such expressions are automated, employing rules like $\theta_{i\alpha}\theta_{i\beta}\theta_{i\gamma} = 0$, $\theta_\alpha\theta^\beta = -\frac{1}{2}\delta_\alpha^\beta\theta^2$, $x_\mu x_\nu \eta_1 \sigma^\mu \bar{\sigma}^\nu \eta_2 = -x^2 \eta_1 \eta_2$, etc. In the package we follow the conventions of Wess & Bagger [29].

One of the main features of our package is the implementation of the Taylor expansion in the Grassmann variables. For example, given a function f involving two undotted Grassmann spinors $\theta_{1,2\alpha}$, the program decomposes it as follows:

$$f(x, \theta_1, \theta_2) = f^{(0,0)}(x) + f_\alpha^{(1,0)}(x) \theta_1^\alpha + f_\beta^{(0,1)}(x) \theta_2^\beta + f_{\alpha\beta}^{(1,1)}(x) \theta_1^\alpha \theta_2^\beta + f^{(2,0)}(x) \theta_1^2 + f^{(0,2)}(x) \theta_2^2,$$

where x refers to any other bosonic variables f depends on. When more copies of Grassmann variables $\theta_{i\alpha}$ and $\bar{\theta}_{i\dot{\alpha}}$ appear, the size of the computation quickly grows beyond human capability. In this work, we decomposed a generic superconformal two-point function which depends on two pairs of θ , $\bar{\theta}$ and two pairs of η , $\bar{\eta}$. The fully simplified result still contains ~ 100 distinct tensor structures. This process takes about 7 seconds on a laptop computer.

In addition, we implement generic differential operators such as ∂_x^2 , $\partial_{\eta_1} \sigma_\mu \partial_x^\mu \bar{\eta}_2$, $\partial_{\eta_1}^\alpha (\partial_{\eta_2})_\alpha$, etc. They are used to work out the contributions from particular descendant operators to the superconformal two-point function. This involves acting with up to eight such operators on a generic conformal two-point function, which takes about a minute to complete.

We hope that this package will help realize complicated calculations both in supersymmetric field theories and beyond.

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⁴In the development of the package we have used *Mathematica* 9.

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