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Variations on Slavnov's scalar product

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ABSTRACT: We consider the rational six-vertex model on an $L \times L$ lattice with domain wall boundary conditions and restrict N parallel-line rapidities, $N \leq L/2$, to satisfy length- L XXX spin- $\frac{1}{2}$ chain Bethe equations. We show that the partition function is an $(L - 2N)$ -parameter extension of Slavnov's scalar product of a Bethe eigenstate and a generic state, with N magnons each, on a length- L XXX spin- $\frac{1}{2}$ chain. Decoupling the extra parameters, we obtain a third determinant expression for the scalar product, where the first is due to Slavnov [1], and the second is due to Kostov and Matsuo [2]. We show that the new determinant is Casoratian, and consequently that tree-level $\mathcal{N} = 4$ SYM structure constants that are known to be determinants, remain determinants at 1-loop level.

KEYWORDS: Lattice Integrable Models, AdS-CFT Correspondence

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1 Introduction

Scalar products of N -magnon states on a length- L spin chain, play a central role in studies of correlation functions in integrable spin chains [3, 4]. Recently, they have appeared in studies of 3-point functions in 4-dimensional $\mathcal{N}=4$ super Yang-Mills theory, SYM₄ [5–8]. Of particular interest to us is the scalar product of an eigenstate of the spin-chain transfer matrix and a generic state,¹ which in the case of integrable XXX and XXZ spin- $\frac{1}{2}$ chains was evaluated by N Slavnov as an $N \times N$ determinant [1].² Recently, I Kostov and Y Matsuo obtained a second expression for the same object as a $2N \times 2N$ determinant [2].

In this work, we start from Izergin’s determinant expression for the domain wall partition function of the rational six-vertex model on an $L \times L$ lattice [9], and require that the rapidities on N parallel lattice lines, $N \leq L/2$, satisfy the Bethe equations of a length- L XXX spin- $\frac{1}{2}$ chain. We show that the result is an extended version of Slavnov’s scalar product that depends on $(L - 2N)$ extra parameters. Taking these extra parameters to infinity, they decouple from the partition function, and we obtain a third expression for Slavnov’s scalar product as an $L \times L$ determinant. We show that the new determinant is a discrete KP τ -function in the inhomogeneity parameters. This allows to use the results of N Gromov and P Vieira [10, 11] to show that SYM₄ structure constants that are known to be determinants at tree-level [8], remain determinants at 1-loop level.

1.1 Outline of contents

In section 1, we recall basic definitions to make the presentation reasonably self-contained, and review recent results to put this work in context. In 2, we recall basic facts related to the scalar product. In 3, we consider Izergin’s determinant expression for the domain wall partition function on an $L \times L$ lattice, require that the rapidities on N parallel lines, $N \leq L/2$ are Bethe roots, and identify the result as a scalar product with $(L - 2N)$ extra parameters. In 4, we decouple the extra parameters of section 3, to obtain a third determinant expression of Slavnov’s scalar. In 5, we point out that it follows from [12] that the third determinant expression obtained in this work is equal to the second of [2].

¹This is the only scalar product that appears in this work, and we refer to it simply as ‘the scalar product’.

²We refer to the determinant of an $N \times N$ as ‘an $N \times N$ determinant’.

In 6, we give a new proof that the second determinant expression of Kostov and Matsuo is equal to the first of Slavnov. Our proof can be regarded as an alternative to that in [2]. In 7, we show that SYM_4 tree-level structure constants that are determinants [8], remain determinants when 1-loop corrections are included along the lines of [10, 11]. Finally, in 8, we collect a number of remarks.

2 Overview

2.1 Context, notation, etc. used in this work

We restrict our attention to the rational six-vertex model and XXX spin- $\frac{1}{2}$ chain, but our conclusions extend to the trigonometric six-vertex model and XXZ spin- $\frac{1}{2}$ chain, as well as to vertex models based on higher-spin $su(2)$ representations and spin chains. All six-vertex configurations have L vertical lattice lines, and all spin chains are of length L and periodic.

In six-vertex terms, $\{x\}$ is a set of free rapidities that flow in horizontal lattice lines, $\{y\}$ is a set of free rapidities that flow in vertical lattice lines, and $\{b\}$ is a set of rapidities that flow in horizontal lattice lines and satisfy length- L spin- $\frac{1}{2}$ chain Bethe equations. From now on, the rapidities that flow in horizontal lattice lines are ‘rapidities’, those that flow in vertical lattice lines are ‘inhomogeneities’, and those that satisfy Bethe equations are ‘Bethe roots’.

In spin-chain terms, $\{x\}$ is a set of free auxiliary space rapidities, $\{y\}$ is a set of free quantum space rapidities, or inhomogeneities, and $\{b\}$ is a set of auxiliary space rapidities that satisfy Bethe equations. From now on, and similarly to the six-vertex case, we refer to $\{x\}$ as ‘rapidities’, to $\{y\}$ as ‘inhomogeneities’, and to $\{b\}$ as ‘Bethe roots’. We use $|z|$ for the cardinality of a set $\{z\}$.

Six-vertex model configurations and partition functions that depend on Bethe roots are ‘Bethe-restricted’. Those that depend on rapidities such that, a subset of these rapidities are set equal to a subset of the inhomogeneities are ‘inhomogeneity-restricted’.

2.2 Bethe eigenstates and a generic states in XXX spin- $\frac{1}{2}$ chains

Consider a length- L periodic integrable XXX spin- $\frac{1}{2}$ chain. The Hilbert space of states $\mathcal{H}$ is spanned by magnon states. An N -magnon state, $N = 0, 1, 2, \dots$, is created by the action of N Bethe raising-operators $B(x_i)$, where x_i , $i = 1, 2, \dots, N$, are free rapidities, on a pseudo-vacuum state $|vac\rangle$. A dual Hilbert space of states $\mathcal{H}^*$ is analogously created by the action of Bethe lowering-operators $C(x_i)$ on a dual pseudo-vacuum state $\langle vac|$. For more details using the same notation and terms used in this work, see [13].

States characterized by free rapidities $\{x\}$ are not eigenstates of the spin chain transfer matrix. They are ‘generic’, or ‘off-shell’. States characterized by Bethe roots $\{b\}$ are eigenstates of the spin chain transfer matrix. They are ‘Bethe eigenstates’, or ‘on-shell’.

2.3 The scalar product of a Bethe eigenstate and a generic state

Scalar products of two magnon states play an essential role in studies of integrable spin chains. If both states are off-shell, $|x_1\rangle$ and $|x_2\rangle$, then the scalar product $\langle x_1|x_2\rangle = \langle x_2|x_1\rangle$

can be expressed in Izergin-Korepin sum form [3]. If both states are on-shell, $|b_1\rangle$ and $|b_2\rangle$, then the scalar product vanishes unless $\{b_1\} = \{b_2\} = \{b\}$. In that case, $\langle b|b\rangle$ is the Gaudin norm [14, 15].

If one state is on-shell, $|b\rangle$, and the other off-shell, $|x\rangle$, then the scalar product $\langle b|x\rangle = \langle x|b\rangle$ can be evaluated in determinant form. This scalar product is the focus of this work.

2.4 The first determinant expression for the scalar product

For spin chains with $su(2)$ -symmetry, such as the XXX and XXZ spin- $\frac{1}{2}$ chains, and their higher-spin analogues, $\langle b|x\rangle$ was evaluated by Slavnov in determinant form [1]. In light of the results in [2] and in this work, we refer to Slavnov's determinant expression as 'the first determinant expression for the scalar product'. No tractable expression, such as a determinant, is known for the corresponding objects in models based on higher rank algebras. For recent progress on this topic, see [16, 17].

2.5 The scalar product is a discrete KP τ -function

The scalar product $\langle b|x\rangle$ of two N -magnon states is a function of three sets of variables: **1.** A set of Bethe roots $\{b\}$, of cardinality N , **2.** A set of free rapidities $\{x\}$, of cardinality N , **3.** A set of free inhomogeneities $\{y\}$, of cardinality L , where L is the number of sites of the spin chain that the states live on. To simplify the notation, we frequently omit to show the dependence on $\{y\}$.

In [18, 19], it was shown that $\langle b|x\rangle$ is a discrete KP τ -function in $\{x\}$. Since $\langle b|x\rangle$ is symmetric in $\{y\}$ as well, it was conceivable that $\langle b|x\rangle$ is a τ -function in $\{y\}$ as well, but it was not straightforward to show that.

2.6 Inhomogeneity-restricted scalar products

In [4, 20], a class of restricted scalar products, obtained by setting a subset of $\{x\}$ equal to a subset of $\{y\}$, was studied. Since the scalar products are determinants, the inhomogeneity-restricted scalar products are also determinants.

In [20], the six-vertex model configurations whose partition functions are scalar products, and their step-by-step restrictions that lead to inhomogeneity-restricted scalar products were studied in detail.

2.7 Tree-level SYM_4 structure constants that can be expressed in determinant form

In [5–7], a class of tree-level 3-point functions of states that live in scalar $su(2)$ sectors³ were formulated in XXX spin- $\frac{1}{2}$ chain terms, and Izergin-Korepin sum expressions were obtained for their structure constants.

In [8], the tree-level structure constants of [5–7] were identified with the inhomogeneity-restricted scalar products of [4, 20]. The basic idea of [8] was to formulate the structure constants in six-vertex terms, compare the result with the analogous formulation of the

³More than one $su(2)$ sector is involved in these 3-point functions.

inhomogeneity-restricted scalar products in six-vertex terms of [20], and to show that the two objects are equal, up to an overall factor that is easily computed.

In [7], a special case of the 3-point functions of [5, 6] where one operator is a non-BPS, while the other two operators are (essentially) BPS states was studied in detail. It turns out that these objects can also be expressed in terms of determinants that are obtained from inhomogeneity-restricted Slavnov determinants by taking a set of Bethe roots to infinity.

2.8 The ‘theta polynomial’ of Gromov and Vieira

In studies of quantum integrability in weakly-coupled $\mathcal{N} = 4$ supersymmetric Yang-Mills, SYM_4 , gauge-invariant single-trace composite operators are mapped to states in closed spin chains [21]. In particular, tree-level single-trace operators, in $su(2)$ scalar sectors, that are eigenstates of the 1-loop mixing matrix, are mapped to states of periodic XXX spin- $\frac{1}{2}$ chains. So far, these spin chains were homogeneous in the sense that the inhomogeneities were set to the same value, which can be set to zero.

In [10, 11], N Gromov and P Vieira showed that 1-loop radiative corrections can be introduced into the structure constants studied in [5–7], and that were expressed as determinants in [8], by switching on the inhomogeneities, that is, by choosing $y_i = \theta$, $i = 1, \dots, L$, computing the structure constants to lowest non-trivial order in θ , which is $O(\theta^2)$, as there are no order $O(\theta)$ contributions, and setting θ equal to the gauge coupling constant.

2.9 1-loop SYM_4 structure constants that can be expressed in determinant form

In [12], structure constants of the latter 3-point functions were identified with six-vertex model configurations on $(N \times L)$ -rectangular lattices, $N \leq L$, with ‘partial domain wall boundary conditions’, and the corresponding ‘partial domain wall partition functions’, pDWPF’s, were studied in some detail.

In [12], we obtained two expressions for these partition functions: **1.** As a determinant of an $L \times L$ matrix, $Z_{L \times L}$ and **2.** As a determinant of an $N \times N$ matrix, $Z_{N \times N}$. The latter was first obtained by I Kostov [22, 23]. Further, we showed that $Z_{L \times L}$ is a discrete KP τ -function in the inhomogeneities $\{y\}$, and $Z_{N \times N}$ is a discrete KP τ -function in the free rapidities $\{x\}$.

Using the fact that these pDWPF’s are discrete KP τ -functions in the inhomogeneities $\{y\}$, together with the recent results of N Gromov and P Vieira [10, 11] allowed us to show that these structure constants remain determinants in the presence of 1-loop radiative corrections. On the other hand, the fact that we were unable to show that the inhomogeneity-restricted scalar product is a τ -function in the inhomogeneities prevented us from extending the determinant result to 1-loop level for tree-level structure constants with 3 non-BPS operators.

2.10 A second determinant expression for the scalar product

In [24, 25], I Kostov and F Smirnov independently suggested that the scalar product of a Bethe eigenstate and a generic state can be obtained from Izergin’s $(L \times L)$ domain wall

partition function, either by sending $(L - 2N)$ rapidities, $N \leq L/2$, to infinity, and thereby decoupling them so that one ends up with a partial domain wall partition function, then setting N of the remaining rapidities to satisfy appropriate Bethe equations [24], or by re-interpreting Korepin’s domain wall configuration as the scalar product of a Bethe eigenstate that is built on the lowest-weight pseudo-vacuum (all spins down, rather than up) and a generic state, by requiring an appropriate subset of rapidities to satisfy appropriate Bethe equations [25]. The latter Bethe equations should come from ‘beyond the equator’ [26].

In [2], I Kostov and Y Matsuo obtained a realization of the suggestion of [24], starting from $2N \times L$ lattice configurations, $2N \leq L$, called ‘partial domain wall configurations’ in [12], whose partition functions can be written in determinant form in two different ways: As the determinant of an $(2N \times 2N)$ matrix [22, 23], or as the determinant of an $(L \times L)$ matrix. In [2], Kostov and Matsuo start from the $(2N \times 2N)$ determinant, require that N rapidities satisfy an appropriate set of Bethe equations, then show that the result is equal to Slavnov’s $(N \times N)$ determinant expression for the scalar product.

2.11 Bethe-restricted domain wall partition functions as parameter-extended scalar products

In this work, we obtain a parameter-extension of [2, 24], that is not identical to but partially along the lines of [25]. We start from an $(L \times L)$ Korepin domain wall configuration, and the corresponding $(L \times L)$ Izergin determinant expression for the partition function, then we set N rapidities to satisfy appropriate Bethe equations, such that Izergin’s determinant now has $(L - N)$ free rapidities, and split the latter into two subsets $\{x\}$ of cardinality N , and $\{t\}$ of cardinality $(L - 2N) \geq 0$. We interpret the resulting determinant as an $(L - 2N)$ -parameter extension of the scalar product of an N -magnon Bethe eigenstate and an N -magnon generic state, on a length- L XXX spin- $\frac{1}{2}$ chain. Everything we say applies without obstruction to six-vertex models with trigonometric weights and the corresponding XXZ spin- $\frac{1}{2}$ chains.

2.12 A third determinant expression for the scalar product

Taking the $(L - 2N)$ -extension parameters to infinity, having first normalized properly, we obtain a determinant of an $(L \times L)$ matrix, that we interpret as a third determinant expression of the scalar product of a Bethe eigenstate and a generic state, of N magnons each, on a length- L XXX spin- $\frac{1}{2}$ chain.

We prove that by showing that our determinant expression is equal to the second determinant expression in [2]. The fact that the two have different forms follows from the fact that [2] start from the $(2N \times 2N)$ determinant expression of the $(2N \times L)$ partial domain wall partition function, while in this work, we start from the $(L \times L)$ determinant expression of the same object.

2.13 The scalar product as a discrete KP τ -function in the inhomogeneities

In previous works [18, 19], we showed that the scalar product is a discrete KP τ -function in the free rapidities, but we could not obtain the same result in terms of the inhomogeneities, even though the scalar product is symmetric in them.

Figure 1. The six vertices and their associated Boltzmann weights.

As an application of the third determinant form of the scalar product obtained in this paper, we show that the scalar product is a discrete KP τ -function in the inhomogeneities.

2.14 Structure constants that are determinants at tree-level remain determinants at 1-loop level

Given that the scalar product is a discrete KP τ -function in the inhomogeneity variables $\{y\}$, it can be written in Casoratian form in any of the variables y_i , $i \in \{1, 2, \dots, L\}$. This allows us to show that the structure constants that can be expressed in determinant form [8], retain their determinant form in the presence of 1-loop radiative corrections, when the latter are included along the lines proposed in [10, 11]. This is an application of the results of this work to computations in weak-coupling $\mathcal{N} = 4$ supersymmetric Yang-Mills.

3 Slavnov's scalar product

3.1 Six-vertex model

We consider the six-vertex model in the rational parametrization, with the following normalization of the Boltzmann weights

$$a(x, y) = 1, \quad b(x, y) = \frac{x - y}{x - y + 1}, \quad c(x, y) = \frac{1}{x - y + 1} \quad (3.1)$$

The assignment of weights to the vertices of the model is shown in figure 1.

3.2 Scalar product

Let $\{b\}_N = \{b_1, \dots, b_N\}$, $\{x\}_N = \{x_1, \dots, x_N\}$, $\{y\}_L = \{y_1, \dots, y_L\}$ be three sets of rapidities.⁴ Assume that their cardinalities satisfy $2N \leq L$ and that $\{b\}$ satisfy the Bethe

⁴To simplify the notation, we often leave the subscripts off the sets $\{b\}_N$, $\{x\}_N$, $\dots$ when the cardinality is clear from the context.

Figure 2. Lattice representation of $S(\{x\}_N, \{b\}_N | \{y\}_L)$. In the algebraic Bethe Ansatz setting, the top N horizontal lines denote $B(b_i)$ operators, while the bottom N horizontal lines denote $C(x_i)$ operators. The values 1 assigned to the top and bottom of the lattice denote the pseudo-vacuum states $|vac\rangle$ and $\langle vac|$, respectively.

equations

$$\prod_{k=1}^L \left(\frac{b_i - y_k + 1}{b_i - y_k} \right) = - \prod_{j=1}^N \left(\frac{b_i - b_j + 1}{b_i - b_j - 1} \right), \quad \forall 1 \leq i \leq N \quad (3.2)$$

We define the scalar product $S(\{x\}, \{b\} | \{y\})$ to be the partition function of the lattice shown in figure 2. This definition is a completely consistent graphical representation of the algebraic form of the scalar product [3]. The *first* determinant expression for the scalar product was found by Slavnov in [1]. Slavnov's expression is given by⁵

$$S \left(\{x\}, \{b\} | \{y\} \right) = \Delta^{-1}\{x\} \Delta^{-1}\{-b\} \times \det \left(\frac{1}{x_\alpha - b_j} \left(\prod_{k \neq j}^N (b_k - x_\alpha - 1) \prod_{l=1}^L \frac{(x_\alpha - y_l)}{(x_\alpha - y_l + 1)} - \prod_{k \neq j}^N (b_k - x_\alpha + 1) \right) \right)_{\substack{1 \leq \alpha \leq N \\ 1 \leq j \leq N}} \quad (3.3)$$

where we adopt the notation

$$\Delta\{x\} = \prod_{1 \leq i < j \leq N} (x_j - x_i), \quad \Delta\{-x\} = \prod_{1 \leq i < j \leq N} (x_i - x_j) \quad (3.4)$$

for the Vandermonde in a set of N variables $\{x\} = \{x_1, \dots, x_N\}$.

4 A scalar product that depends on extra parameters

We consider a domain wall partition function on an $L \times L$ lattice, whose rapidities are the union of the three sets $\{t\}_{L-2N} = \{t_1, \dots, t_{L-2N}\}$, $\{b\}_N = \{b_1, \dots, b_N\}$, $\{x\}_N =$

⁵Here, and in all subsequent determinants, we use Greek indices to label the rows and Latin indices to label the columns.

Figure 3. Lattice representation of $Z(\{x\}_N, \{b\}_N, \{t\}_{L-2N} | \{y\}_L)$. The lattice is $L \times L$, with domain wall boundary conditions. The horizontal lattice lines are split into three subsets, corresponding to the rapidities $\{x\}_N$, $\{b\}_N$ and $\{t\}_{L-2N}$. The extension parameters are the set $\{t\}_{L-2N}$.

$\{x_1, \dots, x_N\}$ and whose inhomogeneities are $\{y\}_L = \{y_1, \dots, y_L\}$. We denote such a partition function by $Z(\{x\}, \{b\}, \{t\} | \{y\})$. Its graphical version is shown in figure 3.

Using Izergin's determinant formula for the DWPF [9] (applied to the case where the rapidities are of mixed type⁶), we have

$$Z \left(\{x\}, \{b\}, \{t\} \middle| \{y\} \right) = \Delta^{-1}\{x\} \Delta^{-1}\{b\} \Delta^{-1}\{t\} \Delta^{-1}\{-y\} \times$$

$$\frac{\prod_{\alpha,j} (x_\alpha - y_j) \prod_{\beta,j} (b_\beta - y_j) \prod_{\gamma,j} (t_\gamma - y_j)}{\prod_{\alpha,\beta} (b_\beta - x_\alpha) \prod_{\alpha,\gamma} (t_\gamma - x_\alpha) \prod_{\beta,\gamma} (t_\gamma - b_\beta)} \det \left(\begin{array}{c} \frac{1}{(x_\alpha - y_j)(x_\alpha - y_j + 1)} \\ \frac{1}{(b_\beta - y_j)(b_\beta - y_j + 1)} \\ \frac{1}{(t_\gamma - y_j)(t_\gamma - y_j + 1)} \end{array} \right)_{\substack{1 \leq \alpha \leq N \\ 1 \leq \beta \leq N \\ 1 \leq \gamma \leq L-2N \\ 1 \leq j \leq L}} \quad (4.2)$$

where indices in the products range over precisely the same values as in the determinant, namely $1 \leq \alpha, \beta \leq N$, $1 \leq \gamma \leq L - 2N$, $1 \leq j \leq L$. We adopt this convention in all analogous formulae which follow. One of the purposes of this paper is to prove the following result. The proof is given over the course of sections 5–7.

⁶The usual form of Izergin's determinant, where the rapidities and inhomogeneities are labelled uniformly, as $\{x\} = \{x_1, \dots, x_N\}$ and $\{y\} = \{y_1, \dots, y_N\}$ respectively, is

$$Z \left(\{x\} \middle| \{y\} \right) = \Delta^{-1}\{x\} \Delta^{-1}\{-y\} \prod_{\alpha,j} (x_\alpha - y_j) \det \left(\frac{1}{(x_\alpha - y_j)(x_\alpha - y_j + 1)} \right)_{\substack{1 \leq \alpha \leq N \\ 1 \leq j \leq N}} \quad (4.1)$$

Figure 4. Lattice representation of $Z(\{x\}_N, \{b\}_N | \{y\}_L)$. The top boundary segments are summed over both colours $\{1, 2\}$, which is indicated by the dots placed on these segments. This $2N \times L$ lattice is obtained from the DWPF lattice by trivializing the top $L - 2N$ lines, or in other words, by sending the extension parameters which live on those lines to infinity.

Lemma 1. *The domain wall partition function $Z(\{x\}, \{b\}, \{t\} | \{y\})$, given by (4.2), is an $(L - 2N)$ parameter extension of Slavnov's scalar product $S(\{x\}, \{b\} | \{y\})$, given by (3.3). The extension parameters are precisely the variables $\{t_1, \dots, t_{L-2N}\}$.*

5 A third determinant expression for the scalar product

Define the function

$$Z \left(\{x\}, \{b\} | \{y\} \right) = \lim_{t_1, \dots, t_{L-2N} \rightarrow \infty} \left(t_1 \dots t_{L-2N} Z \left(\{x\}, \{b\}, \{t\} | \{y\} \right) \right) \quad (5.1)$$

This limit was studied in [12], and the resulting object was called a ‘partial domain wall partition function’. This is in reference to the fact that $Z(\{x\}, \{b\} | \{y\})$ is the partition function of the rectangular lattice shown in figure 4, with ‘partial domain wall boundary conditions’.

Starting from the determinant expression (4.2), the limits in (5.1) can be taken explicitly. This procedure was explained in detail in [12], so we only quote the result here,

$$Z \left(\{x\}, \{b\} | \{y\} \right) = \frac{\prod_{\alpha, j} (x_\alpha - y_j) \prod_{\beta, j} (b_\beta - y_j)}{\Delta\{x\} \Delta\{b\} \Delta\{-y\} \prod_{\alpha, \beta} (b_\beta - x_\alpha)} \det \left(\begin{array}{c} \frac{1}{(x_\alpha - y_j)(x_\alpha - y_j + 1)} \\ \frac{1}{(b_\beta - y_j)(b_\beta - y_j + 1)} \\ y_j^{L-2N-\gamma} \end{array} \right)_{\substack{1 \leq \alpha \leq N \\ 1 \leq \beta \leq N \\ 1 \leq \gamma \leq L-2N \\ 1 \leq j \leq L}} \quad (5.2)$$

To prove Lemma 1, it is clearly sufficient to prove the following.

Lemma 2. *Let $S(\{x\}, \{b\} | \{y\})$ and $Z(\{x\}, \{b\} | \{y\})$ be Slavnov's scalar product (3.3) and the partial domain wall partition function (5.2), respectively. Assuming $\{b\}$ obey the Bethe*

equations, for all values of the variables $\{x\}, \{y\}$ we have

$$S \left(\{x\}, \{b\} \middle| \{y\} \right) = (-)^N Z \left(\{x\}, \{b\} \middle| \{y\} \right) \quad (5.3)$$

Lemma 2 is another version of the recent result of I Kostov and Y Matsuo [2], who proved that the Slavnov scalar product is equal to a pDWPF. The result of [2] was in the context of the formula (6.2) for the pDWPF, which we discuss in the next section, whereas our result is in the context of equation (5.2) for the pDWPF. We call (6.2) and (5.2) the *second* and *third* expression for the scalar product, respectively.

6 The third determinant expression equals the second

An alternative determinant expression for the pDWPF was found by Kostov in [22]. Kostov's expression for the pDWPF is⁷

$$Z \left(\{x\}, \{b\} \middle| \{y\} \right) = \frac{\Delta^{-1}\{x\}\Delta^{-1}\{b\}}{\prod_{\alpha,\beta}(b_\beta - x_\alpha)} \det \left(\begin{array}{cc} x_\alpha^{j-1} - \prod_{l=1}^L \left(\frac{x_\alpha - y_l}{x_\alpha - y_l + 1} \right) (x_\alpha + 1)^{j-1} \\ b_\beta^{j-1} - \prod_{l=1}^L \left(\frac{b_\beta - y_l}{b_\beta - y_l + 1} \right) (b_\beta + 1)^{j-1} \end{array} \right)_{\substack{1 \leq \alpha \leq N \\ 1 \leq \beta \leq N \\ 1 \leq j \leq 2N}} \quad (6.2)$$

In contrast to the determinant (5.2), which is $L \times L$, the determinant in (6.2) is $2N \times 2N$. A direct proof of the equivalence of the two determinants (5.2) and (6.2) is in [12]. We do not repeat this proof here, and from now on treat (5.2) and (6.2) as interchangeable expressions for the pDWPF.

7 The second and third determinant expressions equal the first

In this section we give an alternative proof of Lemma 2. The original proof was given in [2]. The basis of our proof is to define *inhomogeneity-restricted* versions of both the scalar product (3.3) and the pDWPF (5.2), (6.2). The inhomogeneity-restricted scalar products were defined and calculated previously in [4, 20].

⁷Again, we remark that the usual form of Kostov's determinant is when the rapidities and inhomogeneities are labelled uniformly, as $\{x\} = \{x_1, \dots, x_N\}$ and $\{y\} = \{y_1, \dots, y_L\}$, respectively. In that case, it is given by

$$Z \left(\{x\}_N \middle| \{y\}_L \right) = \Delta^{-1}\{x\} \det \left(x_\alpha^{j-1} - \prod_{l=1}^L \left(\frac{x_\alpha - y_l}{x_\alpha - y_l + 1} \right) (x_\alpha + 1)^{j-1} \right)_{\substack{1 \leq \alpha \leq N \\ 1 \leq j \leq N}} \quad (6.1)$$

In the case $N = L$, the determinant (6.1) becomes an alternative expression for the DWPF.

7.1 Inhomogeneity-restricted scalar products

In this section, we consider variations of the size of the sets $\{x\}, \{b\}, \{y\}$. For that reason, it is necessary to restore subscripts to these sets to indicate their cardinality. For all $0 \leq n \leq N$ we define

$$S \left(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1} \right) = S \left(\{x\}_N, \{b\}_N \middle| \{y\}_L \right) \Big|_{x_{N-i+1}=y_i, \forall 1 \leq i \leq N-n} \quad (7.1)$$

where $\{x\}_n = \{x_1, \dots, x_n\}$, $\{y\}_{N-n} = \{y_1, \dots, y_{N-n}\}$, $\{y\}_L^{N-n+1} = \{y_{N-n+1}, \dots, y_L\}$. We split up the dependence on the inhomogeneities deliberately, to indicate the separate symmetry in these two sets. The case $n = N$ is the scalar product (3.3) itself.

Starting from the formula (3.3), we can explicitly evaluate the function defined in (7.1). The result is the following hybrid determinant

$$S \left(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1} \right) = \frac{\Delta^{-1}\{x\}_n \Delta^{-1}\{-b\}_N \Delta^{-1}\{-y\}_{N-n}}{\prod_{\alpha, \gamma} (y_\gamma - x_\alpha)} \times \det \left(\begin{array}{c} \frac{1}{x_\alpha - b_j} \left(\prod_{k \neq j}^N (b_k - x_\alpha - 1) \prod_{l=1}^L \frac{(x_\alpha - y_l)}{(x_\alpha - y_l + 1)} - \prod_{k \neq j}^N (b_k - x_\alpha + 1) \right) \\ \frac{1}{b_j - y_\gamma} \prod_{k \neq j}^N (b_k - y_\gamma + 1) \end{array} \right) \Big|_{\substack{1 \leq \alpha \leq n \\ N-n \geq \gamma \geq 1 \\ 1 \leq j \leq N}} \quad (7.2)$$

Since this object comes from the first expression for the scalar product (3.3), we call it the first expression for the inhomogeneity-restricted scalar product.

7.2 Properties of the inhomogeneity-restricted scalar product

Considering the inhomogeneity-restricted scalar product (7.2) as a function in x_n , we can show that it has the following properties.

A. It is a meromorphic function in x_n of the form

$$S \left(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1} \right) = \frac{P \left(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1} \right)}{\prod_{i=1}^n \prod_{j=N-n+1}^L (x_i - y_j + 1)} \quad (7.3)$$

where $P(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1})$ is a polynomial of degree $L - N + n - 1$ in x_n .

B. It is symmetric in the set of variables $\{y_{N-n+1}, \dots, y_L\}$.

C. By setting $x_n = y_{N-n+1}$, we obtain the recursion relation

$$S \left(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1} \right) \Big|_{x_n = y_{N-n+1}} = S \left(\{x\}_{n-1}, \{b\}_N \middle| \{y\}_{N-n+1}, \{y\}_L^{N-n+2} \right) \quad (7.4)$$

D. In the case $n = 0$, we have

$$S \left(\{x\}_0, \{b\}_N \middle| \{y\}_N, \{y\}_L^{N+1} \right) = \prod_{i,j=1}^N \frac{(b_i - y_j + 1)}{(b_i - y_j)} Z \left(\{b\}_N \middle| \{y\}_N \right) \quad (7.5)$$

where $Z(\{b\}_N | \{y\}_N)$ is the DWPF with rapidities $\{b\}_N = \{b_1, \dots, b_N\}$ and inhomogeneities $\{y\}_N = \{y_1, \dots, y_N\}$.

Since it is quite straightforward to verify that these statements are true, we only comment briefly on their proof. **A** is proved by showing that all poles in x_n in the denominator of (7.2) are cancelled by a zero resulting from setting two rows of the determinant equal. Similarly, one shows that the determinant itself has only poles at the points specified in the denominator of (7.3). Degree counting establishes the correct degree for the polynomial in the numerator.

B is easy to prove, since the only place where $\{y_{N-n+1}, \dots, y_L\}$ appear are in the factors $\prod_{l=1}^L (x_\alpha - y_l) / (x_\alpha - y_l + 1)$, which are obviously symmetric with respect to these variables. **C** follows from the definition (7.1) of the inhomogeneity-restricted scalar products. **D** comes from comparing the $n = 0$ case of (7.2) with Izergin's determinant representation of the DWPF (4.1).

Importantly, the properties **A–D** uniquely determine the set of scalar products (7.1). If another set of functions obey the same properties, they must be equal to the inhomogeneity-restricted scalar products.

7.3 Inhomogeneity-restricted partial domain wall partition functions

In analogy with section 7.1, we define inhomogeneity-restricted versions of the pDWPF (5.2), (6.2). This is done in precisely the same way, namely, for all $0 \leq n \leq N$ we define

$$Z \left(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1} \right) = Z \left(\{x\}_N, \{b\}_N \middle| \{y\}_L \right) \Big|_{x_{N-i+1}=y_i, \forall 1 \leq i \leq N-n} \quad (7.6)$$

The case $n = N$ is just the pDWPF itself. Explicit formulae for the inhomogeneity-restricted pDWPF can be obtained by starting from the determinants (5.2) or (6.2), and specializing the variables in the way prescribed by (7.6). Doing this in the case of (5.2) gives

$$Z \left(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1} \right) = \frac{\prod_{\alpha,j} (x_\alpha - y_j) \prod_{\beta,j} (b_\beta - y_j)}{\Delta \{x\}_n \Delta \{b\}_N \Delta \{-y\}_L^{N-n+1} \prod_{\alpha,\beta} (b_\beta - x_\alpha)} \det \begin{pmatrix} \frac{1}{(x_\alpha - y_j)(x_\alpha - y_j + 1)} \\ \frac{1}{(b_\beta - y_j)(b_\beta - y_j + 1)} \\ y_j^{L-2N-\gamma} \end{pmatrix}_{\substack{1 \leq \alpha \leq n \\ 1 \leq \beta \leq N \\ 1 \leq \gamma \leq L-2N \\ N-n+1 \leq j \leq L}} \quad (7.7)$$

where the indices of the products range over the same values as in the determinant, namely, $1 \leq \alpha \leq n$, $1 \leq \beta \leq N$, $N - n + 1 \leq j \leq L$.⁸ In the case of (6.2), we get

$$Z \left(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1} \right) = \frac{\Delta^{-1}\{x\}_n \Delta^{-1}\{b\}_N \Delta^{-1}\{y\}_{N-n}}{\prod_{\alpha, \gamma} (x_\alpha - y_\gamma) \prod_{\beta, \gamma} (b_\beta - y_\gamma) \prod_{\alpha, \beta} (b_\beta - x_\alpha)} \det \left(\begin{array}{c} y_\gamma^{j-1} \\ x_\alpha^{j-1} - \prod_{l=1}^L \left(\frac{x_\alpha - y_l}{x_\alpha - y_l + 1} \right) (x_\alpha + 1)^{j-1} \\ b_\beta^{j-1} - \prod_{l=1}^L \left(\frac{b_\beta - y_l}{b_\beta - y_l + 1} \right) (b_\beta + 1)^{j-1} \end{array} \right)_{\substack{1 \leq \gamma \leq N-n \\ 1 \leq \alpha \leq n \\ 1 \leq \beta \leq N \\ 1 \leq j \leq 2N}} \quad (7.8)$$

where we again remark that the indices in the products range over the values $1 \leq \alpha \leq n$, $1 \leq \beta \leq N$, $1 \leq \gamma \leq N - n$. Ultimately, we show that (7.8) and (7.7) are equal to the inhomogeneity-restricted scalar product (7.2). For that reason, hereafter we refer to them as the second and third expression for the inhomogeneity-restricted scalar product, respectively.

7.4 Properties of the inhomogeneity-restricted pDWPF

Consider the determinant representations (7.7) and (7.8) for the inhomogeneity-restricted pDWPF, as a function in x_n . We claim that the inhomogeneity-restricted pDWPF has the following properties.

A. It is a meromorphic function in x_n of the form

$$Z \left(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1} \right) = \frac{P \left(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1} \right)}{\prod_{i=1}^n \prod_{j=N-n+1}^L (x_i - y_j + 1)} \quad (7.9)$$

where $P(\{x\}_n, \{b\}_N | \{y\}_{N-n}, \{y\}_L^{N-n+1})$ is a polynomial of degree $(L - N + n - 1)$ in x_n .

B. It is symmetric in the set of variables $\{y_{N-n+1}, \dots, y_L\}$.

C. By setting $x_n = y_{N-n+1}$, we obtain the recursion relation

$$Z \left(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1} \right) \Big|_{x_n = y_{N-n+1}} = Z \left(\{x\}_{n-1}, \{b\}_N \middle| \{y\}_{N-n+1}, \{y\}_L^{N-n+2} \right) \quad (7.10)$$

D. In the case $n = 0$, we have

$$Z \left(\{x\}_0, \{b\}_N \middle| \{y\}_N, \{y\}_L^{N+1} \right) = Z \left(\{b\}_N \middle| \{y\}_L^{N+1} \right) \quad (7.11)$$

where $Z(\{b\}_N | \{y\}_L^{N+1})$ is the pDWPF with rapidities $\{b\}_N = \{b_1, \dots, b_N\}$ and inhomogeneities $\{y\}_L^{N+1} = \{y_{N+1}, \dots, y_L\}$.

⁸Interestingly, (7.7) does not depend on the inhomogeneities $\{y\}_{N-n} = \{y_1, \dots, y_{N-n}\}$, which we restrict to.

To prove these properties, it is convenient to freely change between the expressions (7.7) and (7.8). **A** is proved using (7.8). One can easily check that all poles in x_n in the denominator of (7.8) are cancelled by a zero from setting two rows of the determinant equal. The only poles which are present are the ones described by (7.9), and degree counting establishes the correct degree for the polynomial in the numerator.

B is proved using (7.7), which is invariant under simultaneously reordering the inhomogeneities in the Vandermonde $\Delta\{-y\}_L^{N-n+1}$ and those in the determinant. **C** follows from the definition (7.6) of the inhomogeneity-restricted pDWPF. **D** is proved by considering the $n = 0$ case of (7.7), when it is identically the pDWPF described in (7.11).

7.5 Returning to proof of Lemma 2

The functions $S(\{x\}_n, \{b\}_N | \{y\}_{N-n}, \{y\}_L^{N-n+1})$ and $Z(\{x\}_n, \{b\}_N | \{y\}_{N-n}, \{y\}_L^{N-n+1})$ satisfy the same set of properties **A–C**. The only apparent difference is their initial condition, property **D**. In the following subsection we show that because $\{b\}_N$ satisfy the Bethe equations, the right hand sides of equations (7.5) and (7.11) are in fact equal up to the sign $(-)^N$. In doing so, we will have proved that

$$S\left(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1}\right) = (-)^N Z\left(\{x\}_n, \{b\}_N \middle| \{y\}_{N-n}, \{y\}_L^{N-n+1}\right) \quad (7.12)$$

for all $0 \leq n \leq N$, due to the fact that the properties **A–D** are uniquely determining. This is of course sufficient to prove Lemma 2, which corresponds to the case $n = N$ of equation (7.12).

7.6 Resolving the initial condition

We begin by adjusting the right hand side of (7.5),

$$\prod_{i,j=1}^N \frac{(b_i - y_j + 1)}{(b_i - y_j)} Z\left(\{b\}_N \middle| \{y\}_N\right) = \Delta^{-1} \{b\}_N \det \left(\prod_{k=1}^N \frac{(b_i - y_k + 1)}{(b_i - y_k)} b_i^{j-1} - (b_i + 1)^{j-1} \right)_{1 \leq i,j \leq N} \quad (7.13)$$

which follows from Kostov's expression for the DWPF (see (6.1) with $N = L$). Hence the right hand side of (7.5) can be written as

$$\prod_{i,j=1}^N \frac{(b_i - y_j + 1)}{(b_i - y_j)} Z\left(\{b\}_N \middle| \{y\}_N\right) = \Delta^{-1} \{b\}_N \det \left(\prod_{k=N+1}^L \frac{(b_i - y_k)}{(b_i - y_k + 1)} \prod_{k \neq i}^N \frac{(b_i - b_k + 1)}{(b_i - b_k - 1)} b_i^{j-1} - (b_i + 1)^{j-1} \right)_{1 \leq i,j \leq N} \quad (7.14)$$

where the last line follows from using the Bethe equations (3.2) to modify every entry of the determinant. We denote the final determinant in (7.14) by

$$D_1(N, \kappa) = \det \left(\kappa_i \prod_{k \neq i}^N \frac{(b_i - b_k + 1)}{(b_i - b_k - 1)} b_i^{j-1} - (b_i + 1)^{j-1} \right)_{1 \leq i, j \leq N} \quad \text{with } \kappa_i \equiv \prod_{k=N+1}^L \frac{(b_i - y_k)}{(b_i - y_k + 1)} \quad (7.15)$$

and from now on, treat it as a linear function in free variables $\{\kappa_1, \dots, \kappa_N\}$. On the other hand, using Kostov's pDWPF formula (6.1), the right hand side of (7.11) is given by

$$Z \left(\{b\}_N \middle| \{y\}_L^{N+1} \right) = \Delta^{-1} \{b\}_N \det \left(b_i^{j-1} - \prod_{k=N+1}^L \frac{(b_i - y_k)}{(b_i - y_k + 1)} (b_i + 1)^{j-1} \right)_{1 \leq i, j \leq N} \quad (7.16)$$

Up to the sign $(-)^N$, which it is necessary for us to introduce at some point, we write the determinant in (7.16) as

$$D_2(N, \kappa) = \det \left(\kappa_i (b_i + 1)^{j-1} - b_i^{j-1} \right)_{1 \leq i, j \leq N} \quad (7.17)$$

where again $\kappa_i \equiv \prod_{k=N+1}^L \frac{(b_i - y_k)}{(b_i - y_k + 1)}$, but we treat these as free variables. We prove that $D_1(N, \kappa) = D_2(N, \kappa)$.

$D_2(N, \kappa)$ is a linear function in κ_N . Evaluating it at $\kappa_N = 0$, we obtain

$$D_2(N, \kappa) \Big|_{\kappa_N=0} = - \begin{vmatrix} (\kappa_1 - 1) & (\kappa_1 \bar{b}_1 - b_1) & \cdots & (\kappa_1 \bar{b}_1^{N-1} - b_1^{N-1}) \\ \vdots & \vdots & & \vdots \\ (\kappa_{N-1} - 1) & (\kappa_{N-1} \bar{b}_{N-1} - b_{N-1}) & \cdots & (\kappa_{N-1} \bar{b}_{N-1}^{N-1} - b_{N-1}^{N-1}) \\ 1 & b_N & \cdots & b_N^{N-1} \end{vmatrix} \quad (7.18)$$

where we have introduced the notation $\bar{b}_i = b_i + 1$. Subtracting (column $j + 1$)/ b_N from (column j) for all $1 \leq j \leq N - 1$, the final row of the determinant is only non-zero in the final entry. This reduces the size of the determinant by 1, and after extraction of common factors from the surviving rows, one obtains

$$\begin{aligned} & D_2(N, \kappa) \Big|_{\kappa_N=0} \\ &= - \prod_{i=1}^{N-1} (b_N - b_i) \begin{vmatrix} (\tilde{\kappa}_1 - 1) & (\tilde{\kappa}_1 \bar{b}_1 - b_1) & \cdots & (\tilde{\kappa}_1 \bar{b}_1^{N-2} - b_1^{N-2}) \\ \vdots & \vdots & & \vdots \\ (\tilde{\kappa}_{N-1} - 1) & (\tilde{\kappa}_{N-1} \bar{b}_{N-1} - b_{N-1}) & \cdots & (\tilde{\kappa}_{N-1} \bar{b}_{N-1}^{N-2} - b_{N-1}^{N-2}) \end{vmatrix} \\ &= - \prod_{i=1}^{N-1} (b_N - b_i) D_2(N-1, \tilde{\kappa}) \end{aligned} \quad (7.19)$$

where we have defined $\tilde{\kappa}_i \equiv \kappa_i \frac{(b_N - \bar{b}_i)}{(\bar{b}_N - b_i)}$. One can also evaluate the derivative with respect to κ_N ,

$$\partial_{\kappa_N} D_2(N, \kappa) = \begin{vmatrix} (\kappa_1 - 1) & (\kappa_1 \bar{b}_1 - b_1) & \cdots & (\kappa_1 \bar{b}_1^{N-1} - b_1^{N-1}) \\ \vdots & \vdots & & \vdots \\ (\kappa_{N-1} - 1) & (\kappa_{N-1} \bar{b}_{N-1} - b_{N-1}) & \cdots & (\kappa_{N-1} \bar{b}_{N-1}^{N-1} - b_{N-1}^{N-1}) \\ 1 & \bar{b}_N & \cdots & \bar{b}_N^{N-1} \end{vmatrix} \quad (7.20)$$

and subtracting (column $j+1$)/ $\bar{b}_N$ from (column j) for all $1 \leq j \leq N-1$, in analogy with above, we find that

$$\begin{aligned} \partial_{\kappa_N} D_2(N, \kappa) &= \prod_{i=1}^{N-1} (\bar{b}_N - b_i) \begin{vmatrix} (\kappa'_1 - 1) & (\kappa'_1 \bar{b}_1 - b_1) & \cdots & (\kappa'_1 \bar{b}_1^{N-2} - b_1^{N-2}) \\ \vdots & \vdots & & \vdots \\ (\kappa'_{N-1} - 1) & (\kappa'_{N-1} \bar{b}_{N-1} - b_{N-1}) & \cdots & (\kappa'_{N-1} \bar{b}_{N-1}^{N-2} - b_{N-1}^{N-2}) \end{vmatrix} \\ &= \prod_{i=1}^{N-1} (\bar{b}_N - b_i) D_2(N-1, \kappa') \end{aligned} \quad (7.21)$$

with the definition $\kappa'_i \equiv \kappa_i \frac{(\bar{b}_N - \bar{b}_i)}{(\bar{b}_N - b_i)}$. Since (7.19) and (7.21) are recursion relations for the linear function $D_2(N, \kappa)$ at two different values of κ_N (namely, $\kappa_N = 0$ and as $\kappa_N \rightarrow \infty$), together with the initial condition $D_2(1, \kappa) = \kappa_1 - 1$, they determine it uniquely.

Following essentially the same procedure discussed above, but applied to the determinant $D_1(N, \kappa)$, one can similarly show that

$$D_1(N, \kappa) \Big|_{\kappa_N=0} = - \prod_{i=1}^{N-1} (b_N - b_i) D_1(N-1, \tilde{\kappa}) \quad (7.22)$$

$$\partial_{\kappa_N} D_1(N, \kappa) = \prod_{i=1}^{N-1} (\bar{b}_N - b_i) D_1(N-1, \kappa') \quad (7.23)$$

Hence $D_1(N, \kappa)$ and $D_2(N, \kappa)$ satisfy the same two recursion relations, and since $D_1(1, \kappa) = \kappa_1 - 1$, they share the same initial condition. Hence they are equal for all $N \geq 1$.

8 One-loop $\mathcal{N}=4$ SYM structure constants as determinants

8.1 The Gromov-Vieira mapping

In [10, 11], Gromov and Vieira define the following mapping on any function $f(\theta_1, \dots, \theta_N)$ of the variables $\{\theta_1, \dots, \theta_N\}$,

$$f \mapsto [f]_\theta = f \Big|_{\theta_1, \dots, \theta_N \rightarrow 0} + \frac{g^2}{2} \sum_{i=1}^N (\partial_{\theta_i} - \partial_{\theta_{i+1}})^2 f \Big|_{\theta_1, \dots, \theta_N \rightarrow 0} + O(g^4) \quad (8.1)$$

where $\partial_{\theta_{N+1}} \equiv \partial_{\theta_1}$. Note that the mapping is defined to $O(g^2)$ in some small expansion parameter g .

8.2 Complete symmetric functions and discrete derivatives

Following [27], the complete symmetric functions $h_i\{y\}$ in the set of variables $\{y\} = \{y_1, \dots, y_L\}$ are defined as coefficients in a generating series,

$$\sum_{i=0}^{\infty} h_i\{y\} z^i = \prod_{l=1}^L \frac{1}{1 - y_l z} \quad (8.2)$$

We define a discrete derivative Δ_l which acts on the complete symmetric functions as

$$\Delta_l h_i\{y\} = \frac{h_i\{y\} - h_i\{\widehat{y}_l\}}{y_l} = h_{i-1}\{y\} \quad (8.3)$$

where the subscript l is used to denote the l -th element of the set $\{y\}$, and $\widehat{y}_l$ denotes the omission of that variable from the set.

8.3 Casoratian determinants

We define a Casoratian matrix Ω to be one whose entries $\omega_{i,j}$ are symmetric with respect to a set of variables $\{y\} = \{y_1, \dots, y_L\}$, and satisfy

$$\omega_{i,j+1}\{y\} = \Delta_l \omega_{i,j}\{y\} \quad (8.4)$$

where Δ_l is the discrete derivative with respect to any variable $y_l \in \{y\}$. The determinant of a Casoratian matrix is a Casoratian determinant, and is a discrete analogue of the Wronskian.

Using the definition (8.3) of the discrete derivative, it is easy to see that the determinant

$$|\Omega| = \det \left(\omega_{i,j} \right)_{1 \leq i,j \leq L} = \det \left(\sum_{k=1}^M c_{ik} h_{k-j}\{y\} \right)_{1 \leq i,j \leq L} \quad (8.5)$$

satisfies (8.4), and is therefore Casoratian, for arbitrary $M \geq L$ and coefficients c_{ik} , $1 \leq i \leq L$, $1 \leq k \leq M$. Using the Jacobi-Trudi identity for Schur functions [27], we can write (8.5) equivalently as

$$|\Omega| = \Delta^{-1}\{-y\} \det \left(\sum_{k=1}^M c_{ik} y_j^{k-1} \right)_{1 \leq i,j \leq L} = \Delta^{-1}\{-y\} \det \left(\sum_{k=1}^M c_{jk} y_i^{k-1} \right)_{1 \leq i,j \leq L} \quad (8.6)$$

where the final equality is just from matrix transposition. We take (8.6) as our generic form of a Casoratian determinant. Casoratian determinants are τ -functions of the discrete KP hierarchy (see, for example, [19] for a more detailed exposition). All results stated in the sequel apply to any determinant of the form (8.6), and all such determinants can also be viewed as discrete KP τ -functions in the $\{y\}$ variables.

8.4 Action of Gromov-Vieira mapping on Casoratian determinants

In [12], we studied the action of the GV mapping on partial domain wall partition functions, which are Casoratian determinants in their inhomogeneities $\{y\}_L$. However the procedure outlined in [12] applies generally to any Casoratian determinant of the form (8.5) (or equivalently, (8.6)). We briefly review these results here.

Let $[[\Omega]]_y$ denote the GV mapping of the determinant (8.6) (the transposed version, for notational convenience), with respect to the variables $\{y_1, \dots, y_L\}$. We claim that

$$[[\Omega]]_y = \left| \begin{array}{c} c_{j,1} \\ \vdots \\ c_{j,L-2} \\ c_{j,L-1} + g^2 L c_{j,L+1} \\ c_{j,L} + g^2 L c_{j,L+2} \end{array} \right|_{1 \leq j \leq L} + O(g^4) \quad (8.7)$$

We do not reproduce the proof of this equation here, since full details can be found in [12]. Equation (8.7) says that, up to higher order corrections in g , the determinant structure of Casoratian determinants is preserved under the GV mapping (8.1).

8.5 Restricted scalar products are Casoratian determinants

Returning to equation (7.7) for the inhomogeneity-restricted scalar product $S(\{x\}_n, \{b\}_N | \{y\}_{N-n}, \{y\}_L^{N-n+1})$, we renormalize it as follows,

$$\begin{aligned} \mathbb{S} \left(\{x\}_n, \{b\}_N | \{y\}_{N-n}, \{y\}_L^{N-n+1} \right) &\equiv \\ \prod_{\alpha=1}^n \prod_{j=N-n+1}^L (x_\alpha - y_j + 1) \prod_{\beta=1}^N \prod_{j=N-n+1}^L (b_\beta - y_j + 1) &S \left(\{x\}_n, \{b\}_N | \{y\}_{N-n}, \{y\}_L^{N-n+1} \right) \end{aligned} \quad (8.8)$$

so that $\mathbb{S}(\{x\}_n, \{b\}_N | \{y\}_{N-n}, \{y\}_L^{N-n+1})$ is a polynomial in $\{y_{N-n+1}, \dots, y_L\}$. For convenience, define the combined set of rapidities $\{X\}_{n+N} = \{X_1, \dots, X_{n+N}\}$, where

$$X_i = x_i, \quad \forall 1 \leq i \leq n, \quad X_{n+i} = b_i, \quad \forall 1 \leq i \leq N \quad (8.9)$$

It is then straightforward to show that

$$\begin{aligned} \mathbb{S} \left(\{x\}_n, \{b\}_N | \{y\}_{N-n}, \{y\}_L^{N-n+1} \right) &= \\ \Delta^{-1} \{X\}_{n+N} \Delta^{-1} \{-y\}_L^{N-n+1} \det \left(\sum_{k=1}^{L+2n} c_{ik} \{X\} y_j^{k-1} \right) &\end{aligned} \quad (8.10)$$

$\begin{matrix} 1 \leq i \leq L-N+n \\ N-n+1 \leq j \leq L \end{matrix}$

where the coefficients $c_{ik}\{X\}$ are given by

$$c_{ik}\{X\} = \begin{cases} e_{2n+2N-k-1} \left(\{-X, -\bar{X}\} \setminus \{-X_i, -\bar{X}_i\} \right), & 1 \leq i \leq n+N \\ e_{3n+N+L-k-i+1} \{-X, -\bar{X}\}, & n+N+1 \leq i \leq L-N+n \end{cases} \quad (8.11)$$

with $\bar{X}_i \equiv X_i + 1$ and where $e_k\{-X, -\bar{X}\}$ and $e_k(\{-X, -\bar{X}\} \setminus \{-X_i, -\bar{X}_i\})$ denote elementary symmetric functions [27], given by the generating series

$$\sum_{k=0}^{2n+2N} e_k\{-X, -\bar{X}\} z^k = \prod_{j=1}^{n+N} (1 - X_j z)(1 - \bar{X}_j z) \quad (8.12)$$

$$\sum_{k=0}^{2n+2N-2} e_k \left(\{-X, -\bar{X}\} \setminus \{-X_i, -\bar{X}_i\} \right) z^k = \prod_{j \neq i}^{n+N} (1 - X_j z)(1 - \bar{X}_j z) \quad (8.13)$$

From equation (8.10), we see that the (renormalized) inhomogeneity-restricted scalar products are Casoratian determinants in the set of inhomogeneities $\{y_{N-n+1}, \dots, y_L\}$. Hence their image under the GV mapping in these variables is given by (8.7). We conclude that the determinant structure of tree-level SYM₄ structure constants between three non-BPS states (which are equal to inhomogeneity-restricted scalar products, [8]) is preserved under 1-loop corrections. This extends our previous result [12] in the context of tree-level SYM₄ structure constants between two BPS and one non-BPS state (which are equal to pDWPF's, [7]). Previously, we were unable to obtain such a result starting from the expression (7.2) for the inhomogeneity-restricted scalar product, which is not manifestly a Casoratian determinant in the inhomogeneities.

9 Remarks

1. Our results extend to the trigonometric six-vertex model and the corresponding XXZ spin- $\frac{1}{2}$ chain without obstruction.

2. We note the different dimensions of the determinants in the three expressions for the inhomogeneity-restricted scalar product, equations (7.2), (7.8) and (7.7). The first expression (7.2) comes directly from Slavnov's determinant for the scalar product, and is $N \times N$. The second expression (7.8) comes from the pDWPF expression for the scalar product in [2], which has $2N$ rapidities. Hence (7.8) is $2N \times 2N$. The third expression (7.7), which is new in this work, comes from a specialization of an $L \times L$ determinant, and is $(L - N + n) \times (L - N + n)$. Since the second and third expressions, (7.8) and (7.7), are larger in size than the first (7.2), it is unclear whether these are computationally advantageous in studies of $\mathcal{N} = 4$ SYM structure constants.

3. It is tempting to conjecture that Slavnov-type scalar products in spin chains based on higher-rank algebras can also be obtained from Bethe-restricted versions of the higher-rank domain wall partition functions (giving a parameter-extended scalar product), in the limit where the extension parameters are decoupled [28].

4. We have no statistical mechanical interpretation for the extra parameters in the parameter-extended Slavnov scalar product. Hopefully, these parameters act as regularization parameters in computations of physical objects, such as correlation functions, to be removed at the end of the computation. It may also be that the parameter-extended scalar product is an expectation value of an operator that is characterized by the extension parameters.

5. We used the fact that the third determinant expression for the inhomogeneity-restricted scalar product is a discrete KP τ -function, and therefore can be explicitly written as a Casoratian determinant, together with the results of N Gromov and P Vieira [10, 11], to show that tree-level SYM_4 structure constants, with three non-BPS states that are known to be determinants [8], remain determinants in the presence of 1-loop corrections. The results of [10, 11] actually extend to 2-loop radiative corrections, and our computations can be straightforwardly extended to 2-loops. In [29], D Serban argued that the results of [10, 11] extend to all loops, at least in the limit where all three operators are represented by asymptotically long spin chain states.

6. The second determinant expression of Kostov and Matsuo is expressed as an expectation value of free charged fermions in [2]. From this expectation value, one can deduce that the second determinant expression is a discrete KP τ -function in the inhomogeneities.⁹

7. While writing the results reported in this work, we became aware of [30], where the rational Gaudin model was studied, and it was shown that the scalar product between a generic state and Bethe eigenstate can be expressed as a domain wall partition function. This observation was based on replacing the original N -magnon Bethe eigenstate (coming from the action of N spin-lowering operators on the highest weight state) with an equivalent $(L - N)$ -magnon eigenstate (coming from the action of $L - N$ spin-raising operators on the lowest weight state). This is a different approach to the one reported in this paper, where we have considered the equivalence of the Slavnov scalar product and domain wall partition function without needing to change the Bethe roots. Hence the results of [30] are essentially unrelated to the results in this work.

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⁹We thank I Kostov for pointing this out to us.

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