Holographic butterfly effect and diffusion in quantum critical region

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ABSTRACT: We investigate the butterfly effect and charge diffusion near the quantum phase transition in holographic approach. We argue that their criticality is controlled by the holographic scaling geometry with deformations induced by a relevant operator at finite temperature. Specifically, in the quantum critical region controlled by a single fixed point, the butterfly velocity decreases when deviating from the critical point. While, in the non-critical region, the behavior of the butterfly velocity depends on the specific phase at low temperature. Moreover, in the holographic Berezinskii-Kosterlitz-Thouless transition, the universal behavior of the butterfly velocity is absent. Finally, the tendency of our holographic results matches with the numerical results of Bose-Hubbard model. A comparison between our result and that in the $O(N)$ nonlinear sigma model is also given.

KEYWORDS: Holography and condensed matter physics (AdS/CMT), AdS-CFT Correspondence, Black Holes

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1 Introduction

1.1 Background

Quantum chaos is a fascinating phenomenon and plays a key role in understanding thermalization in many-body system. Two general processes related to chaos are relaxation and scrambling. As a characteristic quantity describing relaxation, the relaxation time τ_{relax} can be calculated by the local decay of time-order two point function [1]. When temperature T becomes the dominant scale in a system, the relaxation time scales with the temperature as $\tau_{\text{relax}} \sim T^{-1}$, where $\hbar = k_B = 1$. Usually, relaxation is followed by

scrambling, during which the scrambling time t_* measures the time for a system to lose the memory of its initial state [2, 3]. Black hole has the fastest scramble process and performs chaos in the decrease of the mutual information after a perturbation [2, 4–6]. For a gauge theory with rank N , the scrambling time behaves as $t_* \sim \tau_L \log N^2$ [2, 7], where τ_L is the Lyapunov time defined by the reciprocal of Lyapunov exponent λ_L as $\tau_L \equiv 1/\lambda_L$. While Lyapunov exponent λ_L can be extracted from the square of the commutator [4, 7–15]

$$\tilde{C}(t, x) = \langle [W(x, t), V(0)]^\dagger [W(x, t), V(0)] \rangle_\beta \sim A_C e^{\lambda_L(t-t_*-|x|/v_B)}, \quad (1.1)$$

where $W(x, t) = e^{iHt}W(x)e^{-iHt}$ and $W(x)$ and $V(0)$ are local operators at x and 0. $\langle \dots \rangle_\beta \equiv Z^{-1} \text{Tr}\{e^{-\beta H} \dots\}$ denotes the ensemble average at temperature $T = \beta^{-1}$ and A_C is a normalized factor. For a gauge theory with rank N , one has $A_C \sim N^{-2}$. The commutator $\tilde{C}(t, x)$ become significant at scrambling time t_* .

The Lyapunov exponent λ_L [10] characterizes how chaos grow for early time. Similar to the Kovtun-Son-Starinets (KSS) bound for η/s [16], Maldacena et al. [7] conjectured a universal bound on chaos,

$$\lambda_L \leq 2\pi T, \quad (1.2)$$

which is saturated in Einstein gravity and Sachdev-Ye-Kitaev (SYK) model [10]. It is further conjectured that a large- N system will have an Einstein gravity dual in the near horizon region if the bound (1.2) is saturated [7, 10]. Unlike the KSS bound, the bound for λ_L is unchanged even in gravity theories with higher derivative corrections [7, 10].

Butterfly velocity v_B characterizes how chaos spreads in space [4]. One can define a ‘butterfly’ cone, $t - |x|/v_B = t_*$, inside the light cone [9]. For unitary operators $W(x)$ and $V(0)$, the normalized commutator $\tilde{C}(t, x)/(\langle WW \rangle_\beta \langle VV \rangle_\beta)$ is nearly zero outside the butterfly cone, which means that the part of system is not affected by the perturbation of $V(0, 0)$ [14]. Later, when crossing the butterfly cone it exponentially increases. At the final stage, it saturates the value 2 inside the butterfly cone and the exponential behavior in (1.1) breaks down [9].

Out-of-time-order correlation (OTOC) function plays a similar role in the study of chaos, which is defined as [4, 7–15]

$$\tilde{F}(t, x) = \langle W^\dagger(t, x) V^\dagger(0, 0) W(t, x) V(0, 0) \rangle_\beta \sim \alpha_0 - \alpha_1 e^{\lambda_L(t-|x|/v_B)}. \quad (1.3)$$

It is linked to (1.1) by $\tilde{C}(t, x) = 2(1 - \text{Re}[\tilde{F}(t, x)])$ when $W(x)$ and $V(0)$ are unitary operators [9, 17].

Usually, it is rather complicated to calculate OTOC in a many-body system [13, 15]. Thanks to the gauge/gravity duality, recent progress indicates that the holographic nature of gravity may shed light on quantum butterfly effect which can be viewed as a dual of shockwave solutions in an asymptotically AdS black hole background [4, 8–10]. This directly stimulates us to further investigate the butterfly effect in holographic approach in this paper, with a focus on its behavior close to the quantum critical point.

On the other hand, motivated by the charge diffusion bound on incoherent metal [18], Blake [19, 20] recently proposed that v_B may work as the characteristic velocity bounding

diffusion constant D in incoherent transport,¹

$$D \gtrsim v_B^2/T. \quad (1.4)$$

Here, the symbol ‘ $\gtrsim$ ’ means greater up to a constant. The most concerned diffusion constants contain the charge diffusion constant D_c , the energy diffusion constant D_e and the momentum diffusion constant D_p , when their diffusive quantities are conserved. Blake’s conjecture has been tested in many holographic models [19–28] and condensed matter models [29–31]. The bound for D_c is found to be violated in [21, 24, 30]. A possible explanation for the violation is that chaos should be linked to the loss of quantum coherence and energy fluctuations, rather than the transportation of conserved electric charges [22, 30, 31]. Recently, a stronger bound for energy diffusion constant,

$$D_e \gtrsim v_B^2 \tau_L, \quad (1.5)$$

is studied in [31, 33]. When a quantum field theory has a holographic dual [10, 19–21, 23, 31], the bound for λ_L in (1.2) is saturated and then the original bound (1.4) and stronger bound (1.5) are equivalent. However, such stronger bound (1.5) is violated in inhomogeneous SYK chains [31], which raises a puzzle on the relation between transport and quantum chaos in strange metals.

1.2 Butterfly effect near the quantum critical point

Inspired by recent progress in holography, OTOC (1.3) has been studied near quantum phase transition (QPT) in many-body systems [22, 32–34]. In [32], Shen et al. found that both λ_L and v_B reach a maximum near the critical point $g = g_c$ at finite temperature in $(1+1)$ dimensional Bose-Hubbard model (BHM), XXZ model and transverse Ising model. They also conjectured that λ_L would display a maximum around the quantum critical point (QCP), which is equivalent to the minimization of Lyapunov time τ_L .

Critical phenomenon is a very nice area for observing the universality of a system because the microscopic details become irrelevant near the critical point. It is quite nature to expect that the bounds and extremal behaviors mentioned above for butterfly effect and diffusion would also exhibit some universal feature during phase transitions. To better understand this, we intend to briefly review the basic structure in quantum critical phenomenon, which takes place in continuous QPT. A general QPT can be accessed by tuning some coupling constant g crossing a critical point g_c at zero temperature $T = 0$ [35]. If such QPT is continuous, the point g_c is called the QCP, at which the correlation length ξ diverges. In particular, if the QPT is the second order, then from the viewpoint of renormalization group (RG), the QCP corresponds to an unstable fixed point, which enjoys the property of scaling invariance. Here for scale transformations, we remark that time and space may have different scaling dimensions

$$[t] = -z, [x] = -1, \quad (1.6)$$

where z is the dynamical critical exponent.

¹We thank Wei-Jia Li for clarifying the applicability of such bound and drawing our attention to the momentum transport.

There are two important scales near the QCP, namely, the temperature T and the distance $g - g_c$ away from QCP, whose scaling dimensions are separately given as

$$[T] = z, [g - g_c] = 1/\nu, \quad (1.7)$$

where ν is another critical exponent. Both of the scaling dimensions should be positive, ensuring that their deformations to the QCP are relevant.² From the perspective of quantum field theory (QFT), the scale $g - g_c$ can be introduced by deforming the fixed point theory with a relevant operator [35–39]

$$S_{\text{QFT}} = S_{\text{fixed point}} + \kappa \int dx^{d+1} W(\mathcal{O}), \quad (1.8)$$

where $W(\mathcal{O})$ is a function of operator $\mathcal{O}$ and the source κ is identified with $g - g_c$, namely $\kappa \sim g - g_c$. Hence, $[\kappa] = 1/\nu$. In addition, for a QPT there exists an UV scale Λ_{UV} , which is close to the energy of microscopic interaction in a many-body system. When $T \gg \Lambda_{\text{UV}}$, it is called the region of lattice high temperature [35]. Quantum critical phenomenon emerges when $T \ll \Lambda_{\text{UV}}$. The competition between two scales T and κ divides the phase diagram into the quantum critical region and the non-quantum-critical region, as illustrated in the left plot of figure 1.

In the quantum critical region, $T \gg |\kappa|^{z\nu}$, temperature T is the dominant scale. In general, under external perturbations a system will lose local quantum phase coherence and such a process can be characterized by the phase coherent time τ_φ , which can technically be evaluated by the exponential decay of local commutators, which is close to the measurement of relaxation time τ_{relax} . A general feature about the phase coherent time τ_φ is

$$\tau_\varphi \gtrsim T^{-1}, \quad (1.9)$$

which becomes saturated in quantum critical region [35, 38]. Therefore, in this region both of quantum and thermal fluctuations are important, which usually leads to a non-quasiparticle description of dynamics at finite temperature.

In the non-quantum-critical region where $T \ll |\kappa|^{z\nu}$, the specific low temperature phase is controlled by the corresponding IR fixed point of theory (1.8). For a gapped phase there are quasi-particles with energy gap $\Delta_E \sim |\kappa|^{z\nu} \gg T$, leading to sparse excitations and a long phase coherence time $\tau_\varphi \sim e^{\Delta_E/T}$.

The similarity between the bound for τ_L in (1.2) and the bound for τ_φ in (1.9) has been suggested in [38]. The conjecture about the minimization of Lyapunov time τ_L in [32] is also reminiscent of (1.9). Therefore, in this sense it is very desirable to check the scaling behaviors of τ_L in the phase diagram of QPT. Recently, the scaling of τ_L is calculated in the $O(N)$ nonlinear sigma model at large N in [34], where it is found that $\tau_L \sim T^{-1}$ in the quantum critical region, $\tau_L \sim T^{-3}$ in the symmetry-broken region and $\tau_L \sim e^{2\Delta_E/T}$ in the

²So far we have only considered the QCP with hyperscaling symmetry, i.e. the hyperscaling violating exponent $\theta = 0$. So a relevant thermal deformation requires $z > 0$. Actually, even hyperscaling is violated, the region of $z < 0$ is found to be ‘pathological’ from the perspective of the consistent dimensional reduction and entanglement entropy [41–43].

symmetry-unbroken region. However, the value of τ_L obtained from the side of classical gravity always saturates the bound in (1.2). So it is difficult to test the conjecture on the minimization of τ_L in the quantum critical region in holographic approach.

As seen from above, the dominant scale T^{-1} bounds other dynamical time scales in the quantum critical region.³ However, when another important scale κ is involved, one may expect deviations of those time scales from T^{-1} . From (1.9), it is understood that τ_φ has to increase and deviate from T^{-1} when leaving the quantum critical region at fixed temperature T . Nevertheless, (1.9) is just an approximate description, and does not guarantee that τ_φ must reach a minimum at $\kappa = 0$. We will discuss this in subsection 5.2.

After having discussed the time scales in QPT, we turn back to the butterfly effect. What kind of behavior should we expect for v_B near QCP? To answer this question, let us firstly estimate the characteristic velocity v_{qp} of quasi-particles in some weakly coupled many-body system. For example, we assume a relativistic dispersion relation $\epsilon_k^2 = c^2 k^2 + m_{qp}^2$, where c is the speed of light and m_{qp} is the effective mass of quasi-particles. At high temperature, $T \gg m_{qp}$, from the estimation $T \approx \epsilon_k$, we have

$$v_{qp} = \frac{\partial \epsilon_k}{\partial k} \approx c \sqrt{1 - \left(\frac{m_{qp}}{T}\right)^2} \approx c \left(1 - \frac{1}{2} \left(\frac{m_{qp}}{T}\right)^2\right) + \dots, \quad (1.10)$$

where $T \gg m_{qp}$ has been applied in the last approximate equality. One simple but direct interpretation on (1.10) is that the effective mass hinders the spread of quasi-particles. If weak chaos can develop from the weak interaction among quasi-particles, we expect a decrease of v_B similar to (1.10).

Now let us discuss v_B in the quantum critical region. As we have mentioned before, quasi-particle usually is not well defined, let alone its velocity v_{qp} . Nevertheless, based on the spirit in [19], v_B should be able to stand for a characteristic velocity even in the quantum critical region, no matter the QCP is relativistic or not. Unlike (1.10), without quasi-particle scenario, the calculation of v_B will be complicated and the dependence of v_B on two scales T and Δ_E is not clear on field theory side. However, we still expect some similarity between the weakly coupled system and the near critical system. Specifically, m_{qp} should correspond to Δ_E , as m_{qp} is just the energy gap of quasi-particle-like excitations. So the relation $T \gg m_{qp}$ would correspond to $T \gg \Delta_E$, i.e. the condition of the quantum critical region. We will see that such naive correspondences are consistent with the result of v_B (1.13) from holography.

For the estimation of v_B in non-quantum-critical region, the picture of quasi-particle is useful on field theory side, see [34]. While, on gravity side, the strategy is different. In [45], v_B displays distinct scaling behaviors in different phases. Thus a discontinuity of v_B appears close to the critical point at rather low temperature, which leads to a peak of $\partial v_B / \partial g$ as well. Since such phenomenon is controlled by two fixed points separately corresponding to low temperature phases, while v_B in the quantum critical region is controlled by the dynamics of QCP, its behavior in these different regions has no direct connections.

³It is so called ‘Planck time’ $t_{pl} \equiv T^{-1}$ in [40].

1.3 Scaling formula for the butterfly velocity and diffusion constant

Finally, it is interesting to understand the charge diffusion bound (1.4) in the quantum critical region. Based on dimensional analysis, our direct expectation is following. When κ vanishes, v_B can be written as $v_B^2 \sim T^{2-\frac{2}{z}}$ according to its scaling dimension $[v_B] = [x] - [t] = -1 + z$.⁴ When κ is turned on, we expect a scaling formula

$$v_B^2 = T^{2-\frac{2}{z}} \Phi\left(\frac{\kappa}{T^{\frac{1}{z\nu}}}\right), \quad (1.11)$$

where $\Phi(x)$ is a function that should be determined by the details of theory, and is expected to exhibit some universal behavior when x is small. For later convenience, we always write out the expression for v_B^2 rather than v_B itself. Similar consideration can be applied to the charge diffusion bound (1.4), giving rise to a dimensionless “diffusion ratio”

$$\frac{D_c \lambda_L}{v_B^2} = \Psi\left(\frac{\kappa}{T^{\frac{1}{z\nu}}}\right), \quad (1.12)$$

where $\Psi(x)$ is a function to be determined by the theory as well.

In this paper, we are going to derive the specific forms of (1.11) and (1.12) near QCP for a class of holographic models with classical gravity [36–38, 44]. So the dual system under consideration is described by a large N gauge theory with strong couplings. Such kind of field theory at fixed point $S_{\text{fixed point}}$ in (1.8) with dynamical critical exponent z is dual to the Lifshitz spacetime [54–56]. For large N limit and without string corrections, v_B and D_c can be calculated over a classical bulk geometry with the use of the method developed in [19, 20, 23]. Especially, we focus on the quantum critical region, which is dominantly controlled by the dynamics of the QCP. The gravity dual at finite temperature can be a black hole or a thermal gas [57].

Before going into the details of the holographic construction, we demonstrate an intuitive picture for the butterfly velocity v_B^2 over the phase diagram with QPT obtained holographically. As an illustration of the scaling formula in (1.11), we numerically calculate v_B^2 over an AdS-AdS domain wall background which is given in appendix C. Its value over the phase diagram is shown in the right plot of figure 1. One can compare it with the schematic phase diagram of QPT in the left plot by identifying $\kappa \sim g - g_c$. The AdS-AdS domain wall is linked by a scalar ϕ which is dual to the operator $\mathcal{O}_\phi$ in (1.8). The quantum critical region, $T \gg |\kappa|^{z\nu}$, is dual to the UV AdS black hole deformed by scalar field ϕ ; while the non-quantum-critical-region, $T \ll |\kappa|^{z\nu}$, is dual to the IR AdS black hole deformed by scalar field ϕ . The nontrivial IR AdS fixed point can be understood as an example of a gapless phase. However, more common phases in QPT are gapped phases flowing to trivial fixed points [35].

The isolation between the fixed points makes our aim clear. We will mainly focus on the quantum critical region and do not need to care about the IR fixed point to which the deformation $W(\mathcal{O}_\phi)$ will drive the system.

⁴Dimensional analysis does not work in the generalized SYK model and the holographic theories with $AdS_2 \times R^d$ near horizon geometry, since spaces are decoupled from the scaling symmetry. By further considering the spatial irrelevant modes, it is found that $v_B \sim \sqrt{T}$ [23, 29].

Figure 1. Left: the schematic phase diagram of quantum phase transition. Right: density plot of butterfly velocity v_B in the phase diagram of AdS-AdS domain wall. The red dashed line mark the place where scalar field at the horizon reaches half of its value at IR fixed point. Such line indicate the vague boundary between quantum critical region and non-quantum-critical-region, namely T and $|\kappa|^{z\nu}$ are comparable ($z\nu$ in these two plots is equal to 1). One can compare these two plots by identifying $\kappa \sim g - g_c$.

We organize this paper as follows. In section 2, we calculate (1.11) and (1.12) in an AdS space with scalar field ϕ deformation, which is dual to a conformal fixed point with scalar operator deformation $W(\mathcal{O}_\phi)$. In section 3, we numerically calculate (1.11) in a Lifshitz fixed point with scalar deformation. When $W(\mathcal{O}_\phi)$ is a single trace deformation, we derive v_B in quantum critical region as

$$v_B^2 = T^{2-\frac{2}{z}} \Phi(0) \left(1 - \gamma \frac{\kappa^2}{T^{\frac{2}{z\nu}}} + \dots \right), \quad T \gg |\kappa|^{z\nu}, \quad (1.13)$$

where γ is a non-negative constant independent of T and κ . Therefore, this formula exhibits an universal behavior that v_B is always decreasing when the system is deformed away from $\kappa = 0$ at fixed T . In other words, v_B reaches its local peak near the QCP. Surprisingly, the holographic result (1.13) has the similar form as (1.10) at $z = 1$ if we recognize $\kappa^\nu \sim \Delta_E$ and replace m_{qp} by Δ_E in (1.10). We also calculate the charge diffusion constant D_c and the diffusion ratio (1.12) in the AdS case. The result for $d > 1$ is

$$\frac{D_c \lambda_L}{v_B^2} = \frac{d}{d-1} \left(1 + \eta \frac{\kappa^2}{T^{\frac{2}{z\nu}}} + \dots \right), \quad T \gg |\kappa|^\nu, \quad (1.14)$$

where η is a non-negative constant independent of T and κ as well. It shows that D_c is bounded from below by $\frac{d}{d-1} \frac{v_B^2}{\lambda_L}$ in the quantum critical region and the ratio will increase when the system goes away from $\kappa = 0$. A similar increase is found in the ratio $D_p \lambda_L / v_B^2$ as well. In section 4, we turn to discuss v_B in some other low temperature phases in holography. Moreover, a simple holographic Berezinskii-Kosterlitz-Thouless (BKT) phase transition will be studied, as a comparison to the second order QPT in the main text. In section 5, we will compare our holographic results with those from (1+1) Bose-Hubbard model in [32, 33] and (2+1) $O(N)$ non-linear sigma model at large N in [33]. Some similarities and differences are found.

2 AdS-Schwarzschild black hole with scalar deformation

In this section we will investigate how the butterfly velocity v_B (1.11) and the charge diffusion ratio $D_c \lambda_L / v_B^2$ (1.12) change under the scalar deformation $W(\mathcal{O}_\phi)$. As the starting point, we consider a classical Einstein gravity theory with the solution of AdS_{d+2} spacetime, which is dual to a large N and strongly coupled theory at conformal fixed point ($z = 1$) in d dimensional space. The action of the Einstein-Scalar model is given as

$$\mathcal{S} = \frac{1}{16\pi G_N} \int dx^{d+2} \sqrt{-g} \left(R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right), \quad (2.1)$$

whose equations of motion are read as

$$0 = R_{\mu\nu} - \frac{1}{d} g_{\mu\nu} V(\phi) - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi, \quad (2.2a)$$

$$0 = \nabla^2 \phi - V'(\phi). \quad (2.2b)$$

For a constant ϕ_* satisfying $V'(\phi_*) = 0$ and $V(\phi_*) < 0$, there is an AdS_{d+2} solution

$$ds^2 = \frac{L^2}{r^2} (-dt^2 + dr^2 + d\mathbf{x}^2), \quad -V(\phi_*) L^2 = (d+1)d, \quad \phi = \phi_*, \quad (2.3)$$

where $\mathbf{x} = (x_1, x_2, \dots, x_d)$. At finite temperature the bulk geometry can be described by an AdS-Schwarzschild black hole with flat horizon

$$ds^2 = \frac{L^2}{r^2} (-f(r) dt^2 + f(r)^{-1} dr^2 + d\mathbf{x}^2), \quad f(r) = 1 - \left(\frac{r}{r_h} \right)^{d+1}, \quad (2.4)$$

where r_h is the location of the horizon. We have presented the detailed derivation of the holographic butterfly effect and charge diffusion over a general black hole background in appendix B. According to the horizon formula (B.3) and (B.10), the butterfly velocity v_B and temperature T are

$$v_B^2 = \frac{d+1}{2d}, \quad T = \frac{d+1}{4\pi r_h}. \quad (2.5)$$

Now we introduce a deformation of the scalar field by turning on its source, which will back-react to the metric. Suppose the potential $V(\phi)$ can be expanded near $\phi = \phi_*$ as

$$V(\phi) = V(\phi_*) + \frac{m^2}{2} (\phi - \phi_*)^2 + \dots. \quad (2.6)$$

Then the equation of motion for the scalar field leads to the following deformation uniformly

$$\phi = \phi_* + \phi_- r^{\Delta_-} + \dots + \phi_+ r^{\Delta_+} + \dots, \quad (2.7)$$

where $\Delta_\pm = \frac{1}{2} \left(d+1 \pm \sqrt{(d+1)^2 + 4m^2 L^2} \right)$. For later convenience, we define a parameter $\vartheta \equiv \frac{\Delta_-}{d+1}$. As the calculation presented in appendix A, the scalar field equation at $O(\phi - \phi_*)$ give

$$\phi_+ = \phi_- H(\vartheta) r_h^{\Delta_- - \Delta_+} + O(\phi_-^2), \quad (2.8)$$

where the specific expression of function $H(\vartheta)$ is given in (A.11). Such a relation is nothing but the Green function of $\mathcal{O}_\phi$ at the conformal fixed point once the quantization method is specified. The scalar field back-reacts to the metric at order $O((\phi - \phi_*)^2)$, and affects v_B through the horizon formula (B.10). We are mainly concerned with the relevant or ‘weakly’ irrelevant deformation, which is subject to the condition $-\frac{1}{2} < \vartheta < \frac{1}{2}$. The result is

$$v_B^2 = \frac{d+1}{2d} \left(1 - \phi_-^2 r_h^{2\Delta_-} I(\vartheta) \frac{d+1}{2d} \right) + O(\phi_-^3), \quad (2.9)$$

where the specific expression of function $I(\vartheta)$ is given in (A.14). When $-\frac{1}{2} < \vartheta < \frac{1}{2}$, we find $I(\vartheta) \geq 0$ with the equality only for $\vartheta = 0$. It tells us that v_B always decreases up to $O(\phi_-^2)$ except for a marginal deformation. It seems this result coincides with the argument presented in [46], where it is found that AdS-Schwarzschild black hole has the maximal v_B in Einstein-Scalar model (2.1) when entropy is fixed. Nevertheless, we point out in appendix E that one of their hypotheses in [46] is actually violated in our model, following the perturbation analysis presented in appendix A.

According to (1.11), we expect that the second term in (2.9) can be expressed in terms of the dimensionless source κ and the other scale T . Firstly, we point out that T is still related to r_h by (2.5) up to $O(\phi_-)$, since the scalar field does not back-react to the metric at $O(\phi - \phi_*)$. To ensure this, we obtain the variation of T at $O(\phi_-^2)$ for $0 < \vartheta < \frac{1}{2}$ in (A.18), indicating that it is not important enough to correct v_B at $O(\phi_-^2)$, indeed. Secondly, we remark that the identification of κ depends on the choice of quantization method as well as the form of the deformation $W(\mathcal{O}_\phi)$. We present our specific consideration as follows.

We first focus on the standard quantization with $[\mathcal{O}_\phi] = \Delta_+$. According to the asymptotic expansion (2.7), the source κ_s of $W(\mathcal{O}_\phi)$ is identified as [47]

$$\phi_- = \kappa_s W'((\Delta_+ - \Delta_-)\phi_+), \quad (2.10)$$

where the subscript ‘s’ of κ_s denotes the standard quantization.

For single trace deformation $W(\mathcal{O}_\phi) = \mathcal{O}_\phi$, we have $1/\nu = [\kappa_s] = \Delta_-$. Such deformation is relevant. From (2.10), we identify $\kappa_s = \phi_-$ and obtain

$$v_B^2 = \frac{d+1}{2d} \left(1 - \gamma \frac{\kappa_s^2}{T^{2/\nu}} \right) + O(\kappa_s^3), \quad (2.11)$$

where $\gamma = \frac{(d+1)I(\vartheta)}{2d} \left(\frac{d+1}{4\pi} \right)^{\frac{2}{\nu}}$. We check our analytical result in (2.11) by numerically constructing an AdS domain wall at finite temperature in appendix C. Both relevant and weakly irrelevant cases are well testified.

For double trace deformation $W(\mathcal{O}_\phi) = \frac{1}{2}\mathcal{O}_\phi^2$, we have $[\mathcal{O}_\phi^2] = 2\Delta_+ > d+1$ and $1/\nu = [\kappa_s] = \Delta_- - \Delta_+ < 0$. Such deformation is irrelevant. ϕ_- is offset in (2.10) and a critical temperature is obtained as [49]

$$T_c = \frac{d+1}{4\pi} \left(-\frac{H(\vartheta) \kappa_s}{\nu} \right)^\nu. \quad (2.12)$$

We will see later that the physical meaning of T_c becomes more transparent in the alternative quantization.

If $\frac{1}{2} - \frac{1}{d+1} < \vartheta < \frac{1}{2}$, i. e. $\frac{d+1}{2} - 1 < \Delta_- < \frac{d+1}{2}$, we can choose alternative quantization with $[\mathcal{O}_\phi] = \Delta_-$. The source κ_a of $W(\mathcal{O}_\phi)$ is identified by

$$(\Delta_- - \Delta_+)\phi_+ = \kappa_a W'(\phi_-), \quad (2.13)$$

where the subscript ‘ a ’ of κ_a denotes alternative quantization.

For single trace deformation $W(\mathcal{O}_\phi) = \mathcal{O}_\phi$, we have $1/\nu = [\kappa_a] = \Delta_+$. Such deformation is relevant. We identify $\kappa_a = \phi_+$ and obtain

$$v_B^2 = \frac{d+1}{2d} \left(1 - \gamma \frac{\kappa_a^2}{T^{2/\nu}} \right) + O(\kappa_a^3), \quad (2.14)$$

where $\gamma = \frac{(d+1)I(\vartheta)}{2dH(\vartheta)} \left(\frac{d+1}{4\pi} \right)^{\frac{2}{\nu}}$.

For double trace deformation $W(\mathcal{O}_\phi) = \frac{1}{2}\mathcal{O}_\phi^2$, we have $[\mathcal{O}_\phi^2] = 2\Delta_- < d+1$ and $1/\nu = [\kappa_a] = \Delta_+ - \Delta_- > 0$. Such deformation is relevant. Similar to the case of standard quantization, there is a critical temperature

$$T_c = \frac{d+1}{4\pi} \left(-\frac{\nu\kappa_a}{H(\vartheta)} \right)^\nu. \quad (2.15)$$

Eqs. (2.15) and (2.12) become identical as $\kappa_a = -1/\kappa_s$, which is obtained just by exchanging the source and the expectation value.⁵ If $\vartheta > 0$, $H < 0$ always, then T_c is well defined only when $\kappa_a < 0$. When $\kappa_a > 0$, double trace deformation does not affect the metric and v_B , at least at $O(\kappa_a^2)$. It is not surprising since such deformation with $\kappa_a > 0$ drives a RG flow from the UV fixed point with $[\mathcal{O}_\phi] = \Delta_-$ to the IR fixed point with $[\mathcal{O}_\phi] = \Delta_+$ [47], but does not affect the geometry classically [48]. To investigate the possible effects of such flow on v_B , one should further study quantum corrections. However, when one attempts to go to the subleading order with finite N , the first term $\frac{d+1}{2d}$ of v_B may receive corrections as well, which makes the effect of such flow unclear. When $\kappa_a > 0$, Faulkner et al. [49, 50] found that a new instability at $T \leq T_c$, where the scalar field will condensate with a mean-field critical exponent at finite temperature. Then v_B will behave just like undergoing the phase transition in a holographic superconductor model, whose derivative is discontinuous at the phase transition point [51].

Next we turn to investigate the charge diffusion bound (1.12) in this holographic model. We introduce an electromagnetic term into the action and then consider the perturbations of electromagnetic field A

$$\mathcal{S}_c = \frac{1}{16\pi G_N} \int dx^{d+2} \sqrt{-g} \left(-\frac{1}{4} F^2 \right) \quad (2.16)$$

where F is the field strength $F = dA$. We calculate the change of T and D_c for $\vartheta > 0$ in appendix A and then substitute them into the diffusion ratio (1.12). For the case of standard quantization and single trace deformation, the result is

$$\frac{D_c \lambda_L}{v_B^2} = \Psi(0) \left(1 + \eta \frac{\kappa_s^2}{T^{2/\nu}} \right) + O(\kappa_s^3), \quad (2.17)$$

⁵It should be cautious that the definitions of ν in (2.15) and (2.12) are different.

where

$$\eta = J(\vartheta, d) \left(\frac{d+1}{4\pi} \right)^{\frac{2}{d}}, \quad \Psi(0) = \begin{cases} \log \left(\frac{\Lambda_{UV}}{2\pi T} \right), & \text{for } d = 1 \\ \frac{d}{d-1}, & \text{for } d > 1 \end{cases} \quad (2.18)$$

and Λ_{UV} is the UV cutoff, which reflects the UV sensitivity of D_c at $d = 1$ [19]. The expression of $J(\vartheta, d)$ is given in (A.22) or (A.23). We have numerically estimated $J(\vartheta, d)$ for a wide range of parameters (ϑ, d) and found that it is always non-negative. It means that when the system goes away from the quantum critical region, the bound is more solid.

Finally, we may investigate (1.4) for the momentum transport. In a large, neutral and homogeneous system, according to the thermodynamic relation [52, 53]

$$\epsilon = Ts - p, \quad (2.19)$$

the momentum diffusion constant is [19]

$$D_p = \frac{\eta}{\epsilon + p} = \frac{\eta}{Ts} = \frac{1}{4\pi T}, \quad (2.20)$$

where ϵ is the energy density and s is the entropy density, while p is the pressure. The saturated KSS bound $\eta/s = 1/(4\pi)$ in Einstein gravity is also used. Then its diffusion ratio is

$$\frac{D_p \lambda_L}{v_B^2} = \frac{1}{2v_B^2}, \quad (2.21)$$

which increases as well when the system goes away from the quantum critical region.

3 Lifshitz black hole with scalar deformation

In previous section we have analysed v_B and $D_c \lambda_L / v_B^2$ in holographic systems with AdS_{d+2} geometry deformed by scalar field. In this section we show that it can be generalized to the Lifshitz case [54]. Lifshitz spacetime can be found in massive vector model [55, 56]

$$\mathcal{S} = \frac{1}{16\pi G_N} \int dx^{d+2} \sqrt{-g} \left(R - \frac{1}{4} \mathcal{G}^2 - \frac{1}{2} W \mathcal{B}^2 - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right), \quad (3.1)$$

where $\mathcal{B}$ is a one-form field and $\mathcal{G} = d\mathcal{B}$. To study the scalar deformation, we have added a minimally coupled scalar field into the action. It allows a Lifshitz solution

$$\begin{aligned} ds^2 &= L^2 \left(-\frac{dt^2}{r^{2z}} + \frac{dr^2 + d\mathbf{x}^2}{r^2} \right), & \mathcal{B} &= L \sqrt{\frac{2(z-1)}{z}} r^{-z} dt, & \phi &= \phi_*, \\ WL^2 &= dz, & -V(\phi_*)L^2 &= z^2 + z(d-1) + d^2, & V'(\phi_*) &= 0, \end{aligned} \quad (3.2)$$

where z is the dynamical critical exponent. This solution enjoys a scaling symmetry with scaling dimension $[t] = -z$, $[r] = -1$, $[x] = -1$. It is worthwhile to point out that the Lifshitz solution with massive vector (3.2) can smoothly go back to the AdS solution (2.3) if one sets $z = 1$, while the Lifshitz solution with running dilaton can not [55, 56].

To study butterfly effect, one should construct a bulk geometry with finite temperature. While, a general analytical Lifshitz black hole with flat horizon in massive

Figure 2. The two coefficients $\Phi(0)$ and γ in the expansion (3.3) as functions of dynamical exponent z . In the right plot, spatial dimension is $d = 2$.

vector model has not yet been found [56]. But we still expect the scaling dimension of temperature $[T] = -[t] = z$. For the expansion of potential $V(\phi)$ in (2.6), the scalar field has the same expansion as (2.7) but with different scaling dimensions $\Delta_{\pm} = \frac{1}{2} \left(d + z \pm \sqrt{(d+z)^2 + 4m^2 L^2} \right)$. The deformation of the scalar back-react to the metric at $O((\phi - \phi_*)^2)$. According to scaling analysis, we expect

$$v_B^2 = T^{2-\frac{2}{z}} \Phi(0) \left(1 - \gamma \frac{\phi_-^2}{T^{\frac{2\Delta_-}{z}}} \right) + O(\phi_-^3) \quad (3.3)$$

with two undetermined constants $\Phi(0)$ and γ . For the case of single trace deformation, we can identify $\kappa = \phi_-$ and $1/\nu = \Delta_-$. (3.3) should go back to (2.11) if we set $z = 1$.

In appendix D, we numerically build a Lifshitz black hole with or without scalar. When scalar field vanishes, we find that $\Phi(0)$ is not equal to $\frac{d+z}{2d}$, which is the coefficient obtained in Lifshitz black hole with running dilaton in [14, 19]. Such discrepancy is not surprising, since Lifshitz black hole with running dilaton is not a solution of (3.1). When scalar field is turned on, we use standard quantization and impose the boundary conditions on ϕ_- . Indeed, our numerical results match (3.3) at small ϕ_- when $0 < \Delta_- < \frac{d+z}{2}$. The coefficients $\Phi(0)$ and γ are shown in figure 2. Within our observation, $\Phi(0)$ and γ are always positive.

4 Comments on other phases at low temperature and phase transitions

In this section we will focus on the behavior of v_B in some low temperature phases in the non-quantum-critical region $T \ll |\kappa|^{z\nu}$. We demonstrate that no universal behavior is observed in these low temperature phases or holographic BKT transition, which is in contrast with the results for quantum critical region controlled by a single fixed point, as we have investigated in previous sections. In next two subsections, a gapless phase and a gapped phase in the holographic framework with hyperscaling violation (HV) will be investigated. A relevant scalar deformation (2.7) with standard quantization and single trace deformation can drive the AdS solution (2.3) from the UV region to these HV solutions in the IR region. We will identify $\kappa = \phi_-$ and $1/\nu = \Delta_-$, where $\Delta_- > 0$ is required for a relevant deformation. In the last subsection, a holographic model with quantum BKT phase transition is discussed.

4.1 Gapless phase with hyperscaling violation

In the Einstein-Scalar model (2.1), we choose the asymptotic behavior of the potential $V(\phi)$ at $\phi \rightarrow \infty$ as

$$V(\phi \rightarrow \infty) \sim V_0 e^{\alpha\phi} (1 + V_1 e^{\alpha_1\phi}), \quad (4.1)$$

in order to construct a AdS-HV domain wall from $\phi = \phi_*$ to $\phi \rightarrow \infty$. The term V_1 is common in the UV completion of HV geometry where the scalar ϕ serves as the dilaton [58–60]. We can heat up the system to finite but low temperature $T \ll |\kappa|^\nu$ by perturbing a small HV black hole in the IR

$$ds^2 = L_{\text{HV}}^2 r^{\frac{2\theta}{d}-2} (1 + g_1 r^\delta + \dots) \left[- \left(1 - c_T r^{d-\theta+1} + \dots \right) dt^2 + \left(1 + c_T r^{d-\theta+1} + \dots \right) dr^2 + d\mathbf{x}^2 \right],$$

$$e^\phi = r^\epsilon (1 + \phi_1 r^\delta + \dots), \quad (4.2)$$

where

$$\alpha\epsilon = -\frac{2\theta}{d}, \quad \epsilon^2 = -\frac{2\theta}{d}(d-\theta), \quad \delta = \alpha_1\epsilon, \quad -V_0 L_{\text{HV}}^2 = (d-\theta)(d-\theta+1). \quad (4.3)$$

The region of $0 < \theta \leq d+1$ is excluded by the requirement of relevant thermal mode c_T and Null Energy Condition [59]. When $\theta > d+1$, the solution is found to be gapped and thermodynamically unstable [60, 61]. We will discuss this in the next subsection. In this subsection, we focus on the case of $\theta < 0$, where the solution is found to be gapless and thermodynamically stable [41, 60, 61]. The IR is located at $r \xrightarrow{\text{IR}} +\infty$, where the induced line element vanishes. There are two perturbation modes in (4.2), whose coupling is approximately negligible if they are small enough. The mode of $\{g_1, \phi_1\}$ is generated by the second term V_1 in (4.1), where the coefficients $\{g_1, \phi_1\}$ can be solved in the series expansion about V_1 . The mode of c_T corresponds to perturbing a small black hole with horizon $r_h = c_T^{\frac{1}{d-\theta+1}}$ and temperature $T = \frac{|d-\theta+1|}{4\pi r_h}$. We find that above two modes are most important to the variation of v_B . One can consider other modes in (4.2), and will find that their scaling dimensions are 0 or $d-\theta+1$ [59]. Except for the thermal mode of c_T , all the relevant modes should not be stimulated for a stable HV solution in the IR. The marginal modes could be introduced but they only provide secondary contribution to v_B when compared with two modes in (4.2).

Plugging (4.2) into the horizon formula of v_B (B.10), we obtain v_B up to the subleading order

$$v_B^2 \approx \frac{d-\theta+1}{2(d-\theta)} \left(1 + \frac{d\delta}{2(d-\theta)} g_1 r_h^\delta \right) = \frac{d-\theta+1}{2(d-\theta)} \left(1 + \gamma \left(\frac{\kappa^\nu}{T} \right)^\delta \right), \quad T \ll |\kappa|^{z\nu}, \quad (4.4)$$

where the approximate equality is used since we neglect the coupling between these two modes of perturbations. This final result is obtained based on the following analysis. In the UV ($r \rightarrow 0$), ϕ is expanded as (2.7). One can find the scaling relation $\kappa^\nu = \phi_-^{1/\Delta_-} = Z_g g_1^{1/\delta}$ where Z_g is a constant which can not be determined by scaling analysis but relies on the specific form of the potential $V(\phi)$. Then we obtain the final result of (4.4) with constant $\gamma = \frac{d\delta}{2(d-\theta)} \left(\frac{|d-\theta+1|}{4\pi Z_g} \right)^\delta$. The power δ in (4.4) is somehow not a universal quantity, since it relies on the second exponent α_1 of $V(\phi)$ in (4.1), which is a tail of the UV completion process.

4.2 Gapped phase with hyperscaling violation

Now we come to the case of $\theta > d+1$. In this case the IR of (4.2) is located at $r \xrightarrow{IR} 0$. The entropy density s over the background in (4.2) behaves as $s \sim T^{d-\theta}$, leading to a negative specific heat such that the HV black hole is thermodynamically unstable [60, 62, 63]. Here we numerically study the AdS-HV domain wall in $d = 2$ with potential $V(\phi) = -6 - 3 \sinh^2\left(\frac{\phi}{\sqrt{3}}\right)$. In the UV, the domain wall approaches the AdS spacetimes with $L = 1$, $\Delta_- = 1$ and $\Delta_+ = 2$. While in the IR, it approaches the HV geometry with $\theta = 8$.

To study the thermodynamics of the system, we should heat it up and calculate its free energy density f . Firstly, from holographic renormalization [53, 64], one notices that the trace of energy momentum tensor is $-\epsilon + 2p = \langle T^i_i \rangle = \phi_- \phi_+$.⁶ Thus, according to the thermodynamic relation (2.19) and $f = \epsilon - Ts$, we obtain the expression of free energy density as⁷

$$f = -\frac{1}{3}(Ts + \phi_- \phi_+). \quad (4.5)$$

We skip the details of numerical analysis since it is similar to the case of AdS-AdS domain wall as presented in appendix C. We construct dimensionless quantities with the unit of $\kappa = \phi_-$. The temperature dependence of the dimensionless free energy density f/κ^3 and the butterfly velocity v_B^2 are shown in figure 3. We find two branches of black hole solutions and a branch of thermal gas solutions. The branch of big black holes behaves like the AdS-Schwarzschild black hole (2.4) with positive specific heat and $v_B^2 \leq \frac{d+1}{2d} = \frac{3}{4}$. While, the branch of small black holes behaves like the HV black hole (4.2) with negative specific heat and $v_B^2 \geq \frac{d-\theta+1}{2(d-\theta)} = \frac{5}{12}$. It becomes extremal at $T/\kappa \rightarrow \infty$. There is a minimal dimensionless temperature $\tilde{T}_{\min}$ for those branches of black holes. The branch of thermal gases has the same form of the extremal solution but has compact imaginary time $\tau \sim \tau + T^{-1}$ [60, 62]. A critical temperature $\tilde{T}_c$ which is higher than $\tilde{T}_{\min}$ appears at the intersection between the branch of big black holes and the branch of thermal gases in the plot of free energy density. The thermal gas dominates when $T/\kappa < \tilde{T}_c$ while the big black hole dominates when $T/\kappa > \tilde{T}_c$. So a first-order phase transition occurs at $\tilde{T}_c$.

Specifically, a holographic description of chaos is ill-defined in thermal gas phase since horizon is absent. So when T/κ decreases and the system falls into such gapped phase with HV, chaos may disappear and v_B becomes ill-defined when $T/\kappa < \tilde{T}_c$.

4.3 Holographic Berezinskii-Kosterlitz-Thouless transition

Now we consider a holographic BKT transition which goes beyond the quantum criticality discussed above, whose QCP is so-called bifurcating QCP [67–73]. From the perspective of RG flow, holographic BKT transition is the result of the annihilation between two fixed points which are linked by the double-trace flow mentioned in section 2. Consider a scalar field ϕ in the bulk and the source of its dual operator $\mathcal{O}_\phi$ in the boundary theory is set to zero. Consider an AdS_n scaling geometry at zero temperature, when one tunes the effective

⁶An alternative counterterm associated with the scalar field is proposed in [65], which leads to a different expression of the trace anomaly.

⁷It also appears in [66].

Figure 3. Dimensionless free energy density f/κ^2 and butterfly velocity v_B^2 as function of dimensionless temperature T/κ for gapped phase with $d = 2, \theta = 8$. In the left plot, black hole solutions are represented by solid line, thermal gas solution are represented by dashed line.

mass m_{eff} of ϕ to become lower than its BF bound m_{BF} with respect to AdS_n , two fixed points which are respectively dual to the AdS_n with standard quantization and alternative quantization will merge and then annihilate [67]. Then ϕ will condense spontaneously and display a nonzero expectation value $\langle \mathcal{O}_\phi \rangle$. The condensation of ϕ generates an intrinsic IR scale Λ_{IR} which exhibits a BKT scaling [67–69]

$$\Lambda_{\text{IR}} \sim \Lambda_{\text{UV},n} e^{-\pi/\tilde{\alpha}}, \quad \tilde{\alpha} = L_n \sqrt{m_{\text{BF}}^2 - m_{\text{eff}}^2}, \quad (4.6)$$

where $\Lambda_{\text{UV},n}$ and L_n are the UV cutoff and the radius of the AdS_n , respectively. A BKT scaling is found in the condensation $\langle \mathcal{O}_\phi \rangle \sim \Lambda_{\text{UV},n}^\Delta e^{-\pi/(2\tilde{\alpha})}$ as well, where Δ is the scaling dimension of operator $\mathcal{O}_\phi$. So such transition is infinite order and is called holographic BKT transition. The bifurcating QCP is located at $\tilde{\alpha} = 0$.

However, when the system goes to finite temperature T , the transition becomes the second order with critical value $\tilde{\alpha}_c = \pi \log^{-1} \left(\frac{\Lambda_{\text{UV},n}}{T} \right)$ and mean field exponent $\langle \mathcal{O}_\phi \rangle \sim (\tilde{\alpha} - \tilde{\alpha}_c)^{\frac{1}{2}}$ [69].

Usually once a bulk action is given, then the mass m of the bulk field ϕ is fixed. What we can tune in a QPT is the source of the operators in the dual QFT. Actually, the effective mass m_{eff} can be tuned by adjusting the fields which are coupled to ϕ [68, 69, 73]. For instance, one can consider a coupling term $\int d^{d+1}x \sqrt{-g} \frac{1}{2} Y(\psi) \phi^2$ in the bulk action, where ψ is a massless axion field and $Y(\psi)$ is a function. The IR value ψ_{IR} of the axion field ψ could be controlled by the source ψ_s of its dual operator. Such coupling term will contribute to the effective mass as $m_{\text{eff}}^2 = m^2 - Y(\psi_{\text{IR}})$. So the relation between m_{eff} and ψ_s depends on the detail of function $Y(\psi)$. Consequently, by tuning the source ψ_s , one can adjust m_{eff} and trigger a BKT transition according to the mechanism given above. For instance, some BKT transitions can be triggered by tuning magnetic fields which lead to condensation of scalar fields on $AdS_2 \times R^d$ geometries in the IR [69, 71, 73].

To further consider the butterfly velocity v_B near the BKT transition, we should study the system at finite temperature. Firstly, we discuss the behavior of v_B when approaching such bifurcating QCP from the uncondensed phase. According to the horizon formula of v_B (B.10), one should investigate the full backreactions of ψ to the metric. The effects of

such backreactions depend on the potential of ψ and its couplings with other fields. We expect a complicated behavior which is different from the simple form (2.9). Especially, a universal maximization of v_B near QCP will not appear, since one can easily shift the location of QCP by changing $Y(\psi)$ while leaving the metric as well as the butterfly velocity v_B unchanged in the uncondensed phase.

Next we consider the system undergoes the phase transition and then enters the condensed phase. Since the phase transition has mean-field scaling at finite temperature, which is just like a holographic superconductor transition, we expect that a discontinuity of $\partial v_B / \partial \tilde{\alpha}$ will appear at the phase boundary [51].

In addition, when $m_{\text{eff}}^2 > m_{\text{BF}}^2$, there is a double-trace flow between the fixed points with standard quantization and alternative quantization. While, as is discussed in section 2, such flow does not back-react to the metric and v_B classically.

5 Comparisons with the results in many-body system

In this section we are going to compare our results in holographic approach with the ones obtained in many body system, including $d = 1$ Bose-Hubbard model (BHM) and $d = 2$ $O(N)$ nonlinear sigma model with large N [32–34].

5.1 Bose-Hubbard model

With the use of numerical simulation, the OTOC (1.3) near the tip of the Mott insulating lobe with density $\rho = 1$ has been computed in [32, 33]. There is a Mott insulator-superfluid transition at $U/J \approx 3.4$, where U measures the in-site repulsion energy and J measures the nearest-neighbor hopping energy. Mott insulating phase falls in the region with $U/J > 3.4$ while superfluid phase falls in $U/J < 3.4$. It is known that such kind of transition is a BKT type which belongs to the universal class of $O(2)$ model in $d = 1$ [35]. The field theory version of such BKT transition is sine-Gordon model where the mergence and annihilation between two fixed points at the QCP is found [67]. Actually, from the perspective of RG flow, such BKT transition is controlled by a line of fixed points rather than a single fixed point [35]. A gap with BKT scaling (4.6) appears in Mott insulating phase.

Nevertheless, so far a clear duality between the BKT transition in BHM and the holographic BKT transition has yet been found. Some difficulties arise when one attempts to compare the butterfly effect in these two different scenarios. Firstly, the mean-field scaling at finite temperature in holographic BKT is different from that one in the BKT transition of many-body system. Secondly, as discussed in subsection 4.3, before two fixed points annihilate, the double-trace flow continuously changes. But such change does not affect the metric at classical level, let alone the butterfly effect. So, as a preliminary approach, we attempt to provide a phenomenological view on this issue by comparing the results (1.13) and (1.14) in the second order QPT with the ones in above Bose-Hubbard model.

In [32], the authors considered a system with 7 sites and 7 bosons and applied the method of exact diagonalization. The system goes across the QCP by tuning the U/J at fixed and finite temperature. The result of butterfly velocity v_B is shown in figure 4, where a peak of v_B near $U/J \approx 9$ is observed. Such value $U/J \approx 9$ is larger than the critical

Figure 4. Butterfly velocity v_B as a function of coupling U/J at temperature $\beta J = 0.9$. This plot is taken from [32].

Figure 5. Butterfly velocity v_B as a function of temperature T/J for different coupling U/J . This plot is taken from [33].

value $U/J \approx 3.4$ at zero temperature. It was conjectured in [32] that such deviation may be ascribed to finite temperature and finite size. In spite of the deviation, the phenomenon found in this many body system is analogous to our holographic results (1.13) with fixed T .

In [33], the authors perform the numerical simulations based on matrix-product-operators at finite-temperature. The system size is large enough to guarantee the convergence. The result of butterfly velocity v_B is shown in figure 5. Let us focus on the case of $U = 3J$ where the system is close to QCP at zero temperature. When $T \gg J$, as shown in figure 5, the system falls in the region of lattice high temperature and the scaling symmetry does not emerge. When $T \lesssim J$, the system lies in quantum critical region. In such region, as T/J goes down, v_B begins to decrease, which coincides with the tendency of our holographic results (1.13) with fixed κ .

5.2 $O(N)$ nonlinear sigma model

Let us turn to the $d = 2$ $O(N)$ nonlinear sigma model. In [34], the authors studied the chaos with QPT up to $1/N$ order. The QCP at $g = g_c$ and $T = 0$ is controlled by a $z = 1$ CFT. In the quantum critical region, they focus on $g = g_c$ and only consider the dominating scale T . The inverse phase coherent time is $\frac{1}{\tau_\varphi} = 1.152 \frac{T}{N}$, which is suppressed by large N . Then quasi-particles are still well defined up to $1/N$ order [35]. The dispersion relation at $N = \infty$ is

$$\epsilon_{\mathbf{k}}^2 = c^2 \mathbf{k}^2 + \mu^2, \quad (5.1)$$

where the speed of light $c = 1$, the thermal mass $\mu = \Theta T$ and the constant $\Theta = 2 \log \frac{1+\sqrt{5}}{2}$. The thermal mass μ gives a finite correlation length $\xi = 1/\mu$ at $N = \infty$.⁸

To study the quantum chaos, the authors in [34] consider the interaction up to $1/N$. By calculating the square commutator (1.1), they extract the Lyapunov exponent and the butterfly velocity, which are $\lambda_L \approx 3.2 \frac{T}{N}$ and $v_B \approx c$. The Lyapunov exponent λ_L is suppressed by large N and the butterfly velocity v_B is closed to the specific velocity of quasi-particle. Both of λ_L and v_B are different from the ones in gravity. It is reasonable, since the dual field theory of Einstein gravity is considered as large N gauge field theories at strong coupling.

To discuss the behavior of v_B in the quantum critical region, as what is done in above sections, we should deviate the system from the QCP and consider another scale $g - g_c$. It generates the energy gap $\Delta_E^\pm$, where $\Delta_E^- \sim (g_c - g)^\nu$ for $g \leq g_c$ and $\Delta_E^+ \sim (g - g_c)^\nu$ for $g \geq g_c$ and the critical exponent $\nu = 1$. Remind that $\kappa \sim g - g_c$, we have $\Delta_E^\pm \sim |\kappa|^\nu$ as usual. The energy gap $\Delta_E^\pm$ only affects the thermal mass μ with the scaling formulas at $N = \infty$ [35]

$$\mu = \begin{cases} 2T \sinh^{-1} \left[\frac{1}{2} \exp(-2\pi \Delta_E^-/T) \right] & \text{for } g \leq g_c \\ 2T \sinh^{-1} \left[\frac{1}{2} \exp(\Delta_E^+/2T) \right] & \text{for } g \geq g_c \end{cases}, \quad (5.2)$$

both of which match $\mu = \Theta T$ at $g = g_c$. From (5.2), μ/T monotonously decreases when g increases. So the point $g = g_c$ is not an extreme point of μ/T . The energy gap $\Delta_E^\pm$ does not explicitly enter the calculations of τ_φ , λ_L and v_B in [34], except changing the thermal mass μ . The value of $\mu/T = \Theta = 2 \log \frac{1+\sqrt{5}}{2}$ at $g = g_c$ is not special in the calculations of τ_φ , λ_L and v_B , so we do not expect that any extremal behavior of τ_φ , λ_L and v_B would appear at $g = g_c$.

In fact, in the quantum critical region, the saturation of the bound for τ_φ in (1.9) does not guarantee the minimization of τ_φ at $g = g_c$. If the absence of extreme for v_B at $g = g_c$ in such model is actually true, it is different from the result of our holographic calculation. It means that the extreme for v_B may only appear in the model which has quasi-particle description and weakly chaotic behavior. So we expect to find universal behavior of v_B in strong chaotic systems. The complicated dependence of (5.2) on two scales T and $\Delta_E^\pm$ implies that, in a many-body system, the finite temperature effects in quantum critical region may not be so simple as that in a holographic system with black holes deformed by a scalar field.

6 Discussion

In this paper, we have investigated the butterfly velocity v_B and the diffusion ratios $D_{c,p} \lambda_L / v_B^2$ near the quantum phase transition in holographic approach. In the quantum critical region, when the relevant scalar deformation is turned on, butterfly velocity v_B universally decreases such that a local peak of v_B is observed near QCP, whose universal behavior depends on the critical exponents and the dimension of the system. In addition, the diffusion ratios $D_{c,p} \lambda_L / v_B^2$ universally increase, which satisfies the diffusion

⁸A holographic thermal screening effect is also observed in a black hole background [38].

bound (1.4). We have also studied the behavior of v_B in low temperature phases beside the QCP, which is controlled by the IR fixed points. For the gapless phase with running dilaton and hyperscaling violation, the variation of v_B mainly relies on the UV completion process, which is not intrinsic. For the gapped phase, black hole does not dominate at low temperature and chaos disappears. In the holographic BKT transition, the behavior of v_B relies on the details of the coupling term in the bulk.

The numerical results in BHM support our holographic observation that v_B decreases when the system goes away from $g = g_c$ but still within the quantum critical region. So we expect that this decreasing behavior of v_B is not an occasional phenomenon.

It is instructive to understand the universality of v_B in quantum critical region from a theoretical point of view. On gravity side, it seems that one may link it to the well-known holographic C-theorem [74].⁹ But their connections actually are not evident. Firstly, central charge, which is related to the AdS radius L , does not appear in the expression of v_B . Secondly, v_B does not monotonously decrease when deformed away from the QCP in the whole phase diagram of QPT. As illustrated in the context of the AdS-AdS domain wall, v_B decreases firstly and then increases again as T/κ decreases, where the low temperature phase is gapless and controlled by another CFT. Finally, the value of v_B at low T/κ is the same as the one at high T/κ . Similar phenomenon is observed in the $O(N)$ nonlinear sigma model in [34], where the values of v_B in the quantum critical region with $g = g_c$ and in the symmetry broken phase are the same, although the crossover behavior has not been obtained. Another possible reason for the decrease of v_B in quantum critical region is the reverse isoperimetric inequality, which is proposed to be linked to a maximum of v_B in [46]. In appendix E, we find that such inequality is true in our perturbation analysis.

On field theory side, the similarity between (1.10) and (1.13) hints the mechanism that the energy gap Δ_E hinders the spread of the chaos even in a system without quasi-particle. While, the discussion about the $O(N)$ nonlinear sigma model tells us that such decrease of v_B may not be always true when quasi-particle description is still valid and the chaos is suppressed. Based on the calculation in [34], the coupling constant $\kappa \sim g - g_c$ affects the chaos only through the thermal mass μ . So we expect a detailed discussion about the dependence of chaos on general μ .

Essentially, the BKT transition in BHM is controlled by a fixed line rather than a single fixed point with relevant deformation. The butterfly effect in other QPTs deserve to be further studied, such as the Mott insulator-superfluid transition with fixed density in BHM at $d \geq 2$. Perhaps one more direct way of studying v_B under deformation lies on field theory side, such as a generalized SYK model with relevant deformation [11, 12, 29, 39].

Finally, we remark that for simplicity only scalar deformation is considered in this paper. It directly leads to a κ^2 -variation of v_B and $D_{c,p}\lambda_L/v_B^2$ for single trace deformation, since scalar field usually back-reacts to the metric at second order. It is desirable to explore the possible new features of holographic butterfly effect by considering other kinds of deformations in future. Furthermore, in this paper all the holographic setup is considered

⁹Some correspondences between butterfly velocities and central charges are observed in massive gravity theories [77].

only at the classical level, which is dual to a gauge field theory with $N = \infty$. When the subleading order with $1/N$ corrections is taken into account, it is expected that the dependent behavior of v_B on κ^2 would receive corrections as well. Last but not least, only Einstein gravity with minimally coupled scalar field is considered in this paper. More general coupling terms such as $f(\phi)R$ or higher order curvature corrections deserve further investigations [9, 78, 79].

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A Scalar deformation on perturbation

In this appendix we will consider the scalar perturbation over the AdS-Schwarzschild black hole (2.4) with the action (2.1). We start with the ansatz for the metric

$$ds^2 = -E(r)dt^2 + B(r)dr^2 + C(r)d\mathbf{x}^2, \quad \phi = \phi(r). \quad (\text{A.1})$$

Then we obtain the equations of motion and zero-energy constraint

$$0 = \frac{(E')^2}{4BE^2} + \frac{B'E'}{4B^2E} - \frac{dC'E'}{4BCE} - \frac{V(\phi)}{d} - \frac{E''}{2BE}, \quad (\text{A.2a})$$

$$0 = \frac{d(C')^2}{4BC^2} + \frac{dB'C'}{4B^2C} + \frac{(E')^2}{4BE^2} + \frac{B'E'}{4B^2E} - \frac{V(\phi)}{d} - \frac{(\phi')^2}{2B} - \frac{dC''}{2BC} - \frac{E''}{2BE}, \quad (\text{A.2b})$$

$$0 = \frac{dC'\phi'}{2BC} + \frac{E'\phi'}{2BE} - \frac{B'\phi'}{2B^2} + \frac{\phi''}{B} - V'(\phi), \quad (\text{A.2c})$$

$$0 = -\frac{d(C')^2}{4BC^2} - \frac{C'E'}{2BCE} + \frac{(C')^2}{4BC^2} + \frac{(\phi')^2}{2Bd} - \frac{V(\phi)}{d}, \quad (\text{A.2d})$$

where the derivative of $\{E, B, C, \phi\}$ is with respect to r while the derivative of V is with respect to ϕ .

For convenience, we take the coordinate transformation $\zeta = 1 - f(r) = (r/r_h)^{d+1}$ such that in coordinates system $\{t, \zeta, \mathbf{x}\}$ equation (2.4) can be written into the form as (A.1), with components

$$E(\zeta) = (1 - \zeta)C(\zeta) = \frac{L^2}{r_h^2}(1 - \zeta)\zeta^{-\frac{2}{d+1}}, \quad B(\zeta) = \frac{L^2}{(d+1)^2(1 - \zeta)\zeta^2}. \quad (\text{A.3})$$

where the asymptotic boundary is located at $\zeta \rightarrow 0$ and the horizon is at $\zeta = 1$.

Now we turn on the deformation of scalar field as presented in (2.7), then the black hole will be back-reacted by the scalar field. We write such variation into the series expansion of λ

$$E(\zeta) = \frac{L^2}{r_h^2} (1 - \zeta) \zeta^{-\frac{2}{d+1}} (1 + \lambda E_1(\zeta) + \lambda^2 E_2(\zeta) + \dots), \quad (\text{A.4a})$$

$$B(\zeta) = \frac{L^2}{(d+1)^2 (1 - \zeta) \zeta^2} (1 + \lambda B_1(\zeta) + \lambda^2 B_2(\zeta) + \dots), \quad (\text{A.4b})$$

$$C(\zeta) = \frac{L^2}{r_h^2} \zeta^{-\frac{2}{d+1}} (1 + \lambda C_1(\zeta) + \lambda^2 C_2(\zeta) + \dots), \quad (\text{A.4c})$$

$$\phi(\zeta) = \phi_* + \lambda \phi_1(\zeta) + \lambda^2 \phi_2(\zeta) + \dots, \quad (\text{A.4d})$$

with the coordinate relation $\zeta = (r/r_h)^{d+1}$ unchanged. We require that the location of both boundary and horizon should not be changed by the deformation at higher orders. We adopt the gauge

$$E_i(\zeta) = C_i(\zeta), \quad \text{for } i = 1, 2, \dots. \quad (\text{A.5})$$

The variation of the scalar field only appears in Einstein equations at $O(\lambda^2)$. The first order deformation of metric is

$$E_1(\zeta) = C_1(\zeta) = c_1 - \frac{c_2}{(1+d)\zeta}, \quad B_1(\zeta) = c_2 \left(\frac{1}{2} - \frac{1}{\zeta} \right). \quad (\text{A.6})$$

where c_1 and c_2 are two integral constants. c_1 corresponds to rescaling t and $\mathbf{x}$ which should be set to zero to fix the coordinates of the dual field theory. c_2 controls an irrelevant mode, which should be set to zero as well for preserving the AdS in the UV. Then

$$E_1(\zeta) = B_1(\zeta) = C_1(\zeta) = 0. \quad (\text{A.7})$$

The scalar equation at $O(\lambda)$ is

$$(1 - \zeta) \phi_1'' - \phi_1' + \frac{(1 - \vartheta) \vartheta}{\zeta^2} \phi_1 = 0, \quad (\text{A.8})$$

where $\vartheta = \frac{\Delta_-}{d+1}$ and $\Delta_{\pm} = \frac{1}{2} \left(d+1 \pm \sqrt{(d+1)^2 + 4m^2 L^2} \right)$. Here we only study the situation that Breitenlohner-Freedman (BF) bound is satisfied, then $\vartheta < \frac{1}{2}$. The violation of BF bound leads to a holographic Berezinskii-Kosterlitz-Thouless transition in subsection 4.3. ϕ_1 should be regular at the horizon. The solution up to a constant factor is

$$\phi_1(\zeta) = \zeta^{\vartheta} {}_2F_1(\vartheta, \vartheta; 2\vartheta; \zeta) - \zeta^{1-\vartheta} \frac{\Gamma(1-\vartheta)^2 \Gamma(2\vartheta)}{\Gamma(2-2\vartheta) \Gamma(\vartheta)^2} {}_2F_1(1-\vartheta, 1-\vartheta; 2-2\vartheta; \zeta), \quad (\text{A.9})$$

where ${}_2F_1(a, b; c; \zeta)$ is the Gaussian hypergeometric function. Near $\zeta \rightarrow 0$, ϕ_1 is expanded as

$$\phi_1(\zeta) \sim \zeta^{\vartheta} + \dots + H(\vartheta) \zeta^{1-\vartheta} + \dots = \left(\frac{r}{r_h} \right)^{\Delta_-} + \dots + H(\vartheta) \left(\frac{r}{r_h} \right)^{\Delta_+} + \dots, \quad (\text{A.10})$$

where

$$H(\vartheta) = -\frac{\Gamma(1-\vartheta)^2 \Gamma(2\vartheta)}{\Gamma(2-2\vartheta) \Gamma(\vartheta)^2}. \quad (\text{A.11})$$

The mode led by ζ^ϑ is relevant when $\vartheta > 0$ while irrelevant when $\vartheta < 0$. The mode led by $\zeta^{1-\vartheta}$ is always relevant. We have restored the asymptotic expansion of $\phi_1(\zeta)$ with the r coordinate in order to display the r_h dependence. By comparing (A.10) with (2.7), we can identify $\lambda = \phi_- r_h^{-\Delta_-} = \phi_+ H(\vartheta) r_h^{-\Delta_+}$ at $O(\lambda)$ and obtain (2.8). At $\zeta = 1$,

$$\phi_1(1) = \frac{2\pi \cot(\pi\vartheta)\Gamma(2\vartheta)}{\Gamma(\vartheta)^2}. \quad (\text{A.12})$$

Now we consider the perturbations at the subleading order $O(\lambda^2)$. The Einstein equations at $O(\lambda^2)$ are differential equations for metric E_2, B_2, C_2 with source ϕ_1 where ϕ_2 is absent. The deformation of metric is found to be

$$C'_2(\zeta) = E'_2(\zeta) = \frac{1}{d+1} B'_2(\zeta) = \frac{1}{d\zeta^2} \left(\int_\zeta^1 y^2 \phi'_1(y)^2 dy - c_3 \right). \quad (\text{A.13})$$

Similar to the step in (A.6), we demand that the boundary mode $B_2 \sim \#\zeta^{-1}$ vanishes, which determines the constant c_3 to be

$$I(\vartheta) = \int_0^1 y^2 \phi'_1(y)^2 dy, \quad (\text{A.14})$$

which is a function of ϑ and plotted in figure 6. According to the asymptotic expansion (A.10), integral $I(\vartheta)$ diverges when $\vartheta < -\frac{1}{2}$, which makes such cancellation subtle. However, here we only consider the case of relevant or ‘weakly’ irrelevant deformation. So we assume $-\frac{1}{2} < \vartheta < \frac{1}{2}$ from now on. By setting $c_3 = I(\vartheta)$, we obtain

$$C'_2(\zeta) = E'_2(\zeta) = \frac{1}{d+1} B'_2(\zeta) = -\frac{1}{d\zeta^2} I_1(\zeta; \vartheta), \quad (\text{A.15})$$

where $I_1(\zeta; \vartheta) = \int_0^\zeta y^2 \phi'_1(y)^2 dy \sim \frac{\vartheta^2}{2\vartheta+1} \zeta^{2\vartheta+1}$ near $\zeta \rightarrow 0$. Obviously, $I(\vartheta) = I_1(1; \vartheta)$.

Now, we can calculate the variation of v_B according to (B.10).¹⁰ By using (A.5), (A.7) and (A.15), it is enough to derive it as

$$v_B^2 = \frac{d+1}{2d} \left(1 - \lambda^2 I(\vartheta) \frac{(d+1)}{2d} \right) + O(\lambda^3), \quad (\text{A.16})$$

which is (2.9).

On the asymptotic boundary, $\zeta \rightarrow 0$, the deformation of the metric behaves as $\zeta^{2\vartheta}$, which is vanishing when $0 < \vartheta < \frac{1}{2}$ and divergent when $\vartheta < 0$. From now on, we only discuss the case of $0 < \vartheta < \frac{1}{2}$. By applying zero-energy constraint (A.2d) at $O(\lambda^2)$ on $\zeta \rightarrow 0$ and requiring that constant modes of C_2, E_2, B_2 vanish, we can obtain

$$C_2(\zeta) = E_2(\zeta) = \frac{1}{d+1} B_2(\zeta) = -\frac{1}{d} \int_0^\zeta y^{-2} I_1(y; \vartheta) dy \equiv -\frac{1}{d} I_2(\zeta; \vartheta). \quad (\text{A.17})$$

According to (B.3), the Hawking temperature T is

$$T = \frac{d+1}{4\pi r_h} \left(1 + \frac{\lambda^2}{2} I_2(1; \vartheta) \right) + O(\lambda^3). \quad (\text{A.18})$$

¹⁰It should be cautious that the coordinate r in (B.10) is ζ now.

Figure 6. $I(\vartheta)$ as a function of ϑ .

By combining (A.16), (B.3) and (B.4), we obtain

$$\frac{v_B^2}{\lambda_L} = \frac{r_h}{d} - \frac{\lambda^2 r_h ((d+1)I_1(\vartheta, 1) + dI_2(\vartheta, 1))}{2d^2}. \quad (\text{A.19})$$

Charge diffusion constant D_c can be evaluated from (B.12). For $d = 1$, it depends on the UV cutoff Λ_{UV} in the original r coordinate

$$D_c = r_h \log(\Lambda_{\text{UV}} r_h) \left(1 + \frac{1}{2} \lambda^2 \left(I_2(\vartheta, 1) - \frac{I_3(\vartheta, -1)}{\log(\Lambda_{\text{UV}} r_h)} \right) \right). \quad (\text{A.20})$$

For $d > 1$, the result is

$$D_c = \frac{r_h}{d-1} + \frac{\lambda^2 r_h}{2d(d^2-1)} \left((-d^2 + d + 2) I_2(1; \vartheta) - 2(d-1) I_3\left(\vartheta, -\frac{2}{d+1}\right) \right), \quad (\text{A.21})$$

where $I_3(\vartheta, p) = \int_0^1 y^p I_2(y; \vartheta) dy$.

Finally, one can calculate the diffusion ratio $D_c \lambda_L / v_B^2$. For $d = 1$,

$$\begin{aligned} \frac{D_c \lambda_L}{v_B^2} &= \log\left(\frac{\Lambda_{\text{UV}}}{2\pi T}\right) (1 + \lambda^2 J(\vartheta; 1)), \\ J(\vartheta; 1) &= I_1(1; \vartheta) + I_2(1; \vartheta) + \frac{I_2(1; \vartheta) - I_3(\vartheta, -1)}{2 \log\left(\frac{\Lambda_{\text{UV}}}{2\pi T}\right)}, \end{aligned} \quad (\text{A.22})$$

Since $\Lambda_{\text{UV}} \gg T$, the logarithmic terms $\log\left(\frac{\Lambda_{\text{UV}}}{2\pi T}\right)$ is usually large such that the third term of $J(\vartheta; 1)$ is negligible. For $d > 1$,

$$\begin{aligned} \frac{D_c \lambda_L}{v_B^2} &= \frac{d}{d-1} (1 + \lambda^2 J(\vartheta, d)), \\ J(\vartheta, d) &= \frac{(d+1)^2 I_1(1; \vartheta) + 2(d+1) I_2(1; \vartheta) - 2(d-1) I_3\left(\vartheta, -\frac{2}{d+1}\right)}{2d(d+1)}. \end{aligned} \quad (\text{A.23})$$

We numerically evaluate $J(\vartheta, d)$ for a wide range of $\{\vartheta, d\}$ and find it is always non-negative. Those integrals are collected here

$$\begin{aligned}
 I(\vartheta) &= \int_0^1 y^2 \phi_1'(y)^2 dy, \\
 I_1(\zeta; \vartheta) &= \int_0^\zeta y^2 \phi_1'(y)^2 dy \sim \frac{\vartheta^2}{2\vartheta+1} \zeta^{2\vartheta+1}, \\
 I_2(\zeta; \vartheta) &= \int_0^\zeta y^{-2} I_1(y; \vartheta) dy \sim \frac{\vartheta}{2(2\vartheta+1)} \zeta^{2\vartheta}, \\
 I_3(\vartheta, p) &= \int_0^1 y^p I_2(y; \vartheta) dy \sim \frac{\vartheta}{2(2\vartheta+1)(2\vartheta+p+1)} \quad \text{if } 2\vartheta+p+1 > 0.
 \end{aligned} \tag{A.24}$$

B Formula of butterfly velocity and charge diffusion constant

In this appendix we derive the formulas of v_B and D_c , closely following the strategy presented in [9, 14, 19]. Given a black hole metric in $d+2$ dimensions with the form

$$ds^2 = -E(r)dt^2 + B(r)dr^2 + C(r)d\mathbf{x}^2, \tag{B.1}$$

whose components are expanded near the horizon r_h as

$$\begin{aligned}
 E(r) &\sim E'(r_h)(r-r_h) + \dots, \quad B(r) \sim B^{(-1)}(r_h)(r-r_h)^{-1} + \dots, \\
 C(r) &\sim C(r_h) + C'(r_h)(r-r_h) + \dots,
 \end{aligned} \tag{B.2}$$

where $E'(r_h), B^{(-1)}(r_h), C'(r_h)$ are finite and negative. We consider flat horizon and require that the asymptotic boundary is located at $r \rightarrow 0$. Then the black hole temperature and entropy density are

$$\begin{aligned}
 T &= \frac{1}{\beta} = \frac{-E'(r_h)}{4\pi\sqrt{B(r_h)E(r_h)}} = \frac{1}{4\pi} \sqrt{\frac{E'(r_h)}{B^{(-1)}(r_h)}}, \\
 s &= \frac{C(r_h)^{d/2}}{4G_N}.
 \end{aligned} \tag{B.3}$$

Chaos bound (1.2) is saturated in Einstein gravity. So

$$\lambda_L = 2\pi T. \tag{B.4}$$

We are going to derive the butterfly velocity v_B in terms of horizon quantities. Firstly we introduce the tortoise coordinate

$$dr_* = -\sqrt{\frac{B(r)}{E(r)}} dr \tag{B.5}$$

to write the metric into

$$ds^2 = E(r)(-dt^2 + dr_*^2) + C(r)d\mathbf{x}^2, \tag{B.6}$$

where the asymptotic boundary is located at $r_* = 0$ and the horizon is located at $r_* = -\infty$. Then we use the Kruskal coordinates

$$uv = -e^{4\pi r_*/\beta}, \quad u/v = -e^{-4\pi t/\beta}, \quad (\text{B.7})$$

to further give

$$ds^2 = 2P(uv)dudv + Q(uv)d\mathbf{x}^2, \quad (\text{B.8})$$

where $P(uv) = E(r)\frac{2}{uv}\left(\frac{\beta}{4\pi}\right)^2$, $Q(uv) = C(r)$. The horizon is located at $uv = 0$ and the asymptotic boundary is located at $uv = -1$. One can find

$$uv \sim (r - r_h)\theta + \dots, \quad (\text{B.9})$$

where $\theta > 0$. Applying the expression of v_B in [14], we can obtain

$$v_B^2 = \left(\frac{4\pi}{\beta}\right)^2 \frac{P(0)}{2dQ'(0)} = \frac{E'(r_h)}{dC'(r_h)}, \quad (\text{B.10})$$

where $P(0) = \frac{2E'(r_h)}{\theta}\left(\frac{\beta}{4\pi}\right)^2$ and $Q'(0) = \frac{C'(r_h)}{\theta}$ have been used.

We add the Maxwell term $\mathcal{S}_c$ (2.16) into the action to study the charge diffusion. Following the method in [19], we firstly write down the DC conductivity σ and the susceptibility χ^{-1}

$$\sigma = C(r_h)^{\frac{d}{2}-1}, \quad \chi^{-1} = \int_0^{r_h} C(r)^{-\frac{d}{2}} \sqrt{B(r)E(r)} dr. \quad (\text{B.11})$$

Then the charge diffusion constant D_c can be read from the Einstein relation

$$D_c = \frac{\sigma}{\chi} = C(r_h)^{\frac{d}{2}-1} \int_0^{r_h} C(r)^{-\frac{d}{2}} \sqrt{B(r)E(r)} dr. \quad (\text{B.12})$$

If above integration diverges, one can regularize it by introducing a UV cutoff Λ_{UV} into the integral as $\int_{\Lambda_{\text{UV}}^{-1}}^{r_h}$.

C Numerical solutions for AdS-Schwarzschild black hole

Here we work on spatial dimension $d = 2$. To study the $AdS_4 - AdS_4$ domain wall, we choose the potential as

$$V(\phi) = -6 - \phi^2 + \frac{1}{8}\phi^4. \quad (\text{C.1})$$

We adopt the domain wall ansatz

$$ds^2 = \mathcal{F}(r)^{-1}dr^2 + e^{2\mathcal{A}(r)}(-\mathcal{F}(r)dt^2 + dx_1^2 + dx_2^2), \quad \phi = \phi(r). \quad (\text{C.2})$$

The potential (C.1) allows three AdS_4 fixed points. One of them has the larger radius of AdS and stays in the UV ($r \rightarrow +\infty$), which is

$$\mathcal{A} = r/L_{\text{UV}}, \quad \mathcal{F} = 1, \quad \phi = 0 \quad (\text{C.3})$$

with scalar modes

$$\phi = \phi_-^{\text{UV}} e^{\Delta_- \mathcal{A}} + \phi_+^{\text{UV}} e^{\Delta_+ \mathcal{A}}, \quad (\text{C.4})$$

where $L_{\text{UV}} = 1, \Delta_- = 1, \Delta_+ = 2$. The other two have the smaller radius of AdS and stay in the IR ($r \rightarrow -\infty$), one of which is

$$\mathcal{A} = r/L_{\text{IR}}, \quad \mathcal{F} = 1, \quad \phi = \phi_{\text{IR}} \quad (\text{C.5})$$

with scalar modes

$$\phi = \phi_{\text{IR}} + \phi_-^{\text{IR}} e^{\delta_- \mathcal{A}} + \phi_+^{\text{IR}} e^{\delta_+ \mathcal{A}}, \quad (\text{C.6})$$

where $L_{\text{IR}} = \frac{\sqrt{3}}{2}, \phi_{\text{IR}} = 2, \delta_{\pm} = \frac{1}{2}(3 \pm \sqrt{21})$. The other IR fixed point is obtained by the reflection $\phi \rightarrow -\phi$.

Firstly, we study the zero temperature flow. Given a slight deviation from the IR fixed point, the irrelevant IR modes ϕ_-^{IR} will be stimulated and intergraded to the UV fixed point. We find a UV-IR relation

$$(\phi_-^{\text{IR}})^{1/\delta_-} = Z_\phi (\phi_-^{\text{UV}})^{1/\Delta_-}, \quad (\text{C.7})$$

where the coefficient $Z_\phi = 0.178$, which can be understood as a renormalization of operator $\mathcal{O}_\phi$.

Secondly, we study the thermal flow. Notice that three sorts of symmetries are contained in (C.2): the first is $r \rightarrow r + c$, which allows us to set the horizon at $r = 0$; the second is $r \rightarrow rc, t \rightarrow t/c, \mathcal{F} \rightarrow \mathcal{F}c^2$, which allows us to set $\mathcal{F}'(0) = 1$; the third is $t \rightarrow tc^2, x \rightarrow xc^2, \mathcal{A} \rightarrow \mathcal{A} - \log c$, which allows us to set $\mathcal{A}(0) = 0$. Then, under the last boundary condition which sets the value of $\phi(0)$, we can integrate the flow from the horizon to the UV. Be cautious that $\mathcal{F}(+\infty)$ is no longer equal to 1 because of the gauge $\mathcal{F}'(0) = 1$. One should recover $\mathcal{F}(+\infty) = 1$ by using the second symmetry inversely.

Finally, according to (B.3), (B.10) and (B.12), we can numerically determine the temperature T , butterfly velocity v_B and diffusion ratios $D_c \lambda_L / v_B^2$ by using

$$T = \frac{e^{\mathcal{A}(0)} \mathcal{F}'(0)}{4\pi}, \quad v_B^2 = \frac{\mathcal{F}'(0)}{4\mathcal{A}'(0)}, \quad \frac{D_c \lambda_L}{v_B^2} = 2e^{\mathcal{A}(0)} \mathcal{A}'(0) \int_0^\infty e^{-\mathcal{A}(r)} dr. \quad (\text{C.8})$$

Let us employ the standard quantization and consider the single trace deformation $W(\mathcal{O}_\phi) = \mathcal{O}_\phi$. Considering the UV fixed point with relevant deformation, we can identify $\kappa_s = \phi_-^{\text{UV}}$ in (2.11), whose value can be extracted by $\phi = \phi_-^{\text{UV}} e^{\Delta_- \mathcal{A}} + \dots$ in the UV. Considering the IR fixed point with irrelevant deformation, we can identify $\kappa_s = \phi_-^{\text{IR}}$ in (2.11), whose value is given by (C.7).

As $\Delta_- = 1$, T/ϕ_-^{UV} is a dimensionless parameter. In the left plot of figure 7, we plot the numerical result of the quantity $1 - \frac{4}{3}v_B^2$ as a function of T/ϕ_-^{UV} . The analytical results based on (2.11) for the UV fixed point and the IR fixed point are shown as well, where the constant γ of the UV (IR) fixed point is calculated by using Δ_- (δ_-). The numerical result matches well with the analytical results in both the $T \gg \phi_-^{\text{UV}}$ region and $T \ll \phi_-^{\text{UV}}$ region. The value of v_B^2 is also plotted in the phase diagram in figure 1. When $\phi(0) \approx \phi_{\text{IR}}/2 = 1$, the effects of thermodynamic and $\mathcal{O}_\phi$ deformation are commensurate.

Figure 7. Quantities $1 - \frac{4}{3}v_B^2$ and $\frac{D_c \lambda_L}{2v_B^2} - 1$ as functions of T/ϕ_-^{UV} . Points are results of numerical calculation and lines are results of perturbation analysis.

At this time, $T/\phi_-^{\text{UV}} \approx 0.1$ can be understood as the vague boundary between the quantum critical region and the gapless low temperature phase.

In the right plot of figure 7, we plot the numerical result of the quantity $\frac{D_c \lambda_L}{2v_B^2} - 1$. The analytical result based on (2.17) for the UV fixed point is shown as well, where the constant η is calculated by using Δ_- . Note that the quantity $\frac{D_c \lambda_L}{2v_B^2} - 1$ at low T/ϕ_-^{UV} behaves as T/ϕ_-^{UV} rather than $(T/\phi_-^{\text{UV}})^{-2\delta_-}$, which may result from the breakdown of (A.17), because the variation of the metric becomes divergent near asymptotic boundary of the IR region under an irrelevant deformation.

D Numerical solutions for Lifshitz black hole

For numerical calculation, we adopt the following ansatz

$$\begin{aligned} ds^2 &= \frac{L^2}{r^2} \left(-\frac{(1-r)U(r)}{r^{2z-2}} e^{-S(r)} dt^2 + \frac{dr^2}{(1-r)U(r)} + d\mathbf{x}^2 \right), \\ \mathcal{B} &= L \sqrt{\frac{2(z-1)}{z}} r^{-z} \mathcal{B}(r) (1-r) U(r) dt, \quad \phi = r^{\Delta_-} \phi(r), \end{aligned} \quad (\text{D.1})$$

and choose the parameters in the action (3.1) as

$$W = \frac{dz}{L^2}, \quad V(\phi) = -\frac{z^2 + z(d-1) + d^2}{L^2} + \frac{\Delta_- (\Delta_- - (d+z))}{2L^2} \phi^2. \quad (\text{D.2})$$

Then L will not appear in the equation of motions. The asymptotic boundary is located at $r \rightarrow 0$ and the horizon is located at $r = 1$. The boundary conditions at $r = 0$ are $U(0) = 1, S(0) = 0, \mathcal{B}(0) = 1, \phi(0) = \phi_-$, which ensure the asymptotic Lifshitz solution (3.2) with asymptotic behavior (2.7). The boundary conditions at $r = 1$ are regular conditions.

According to (B.3) and (B.10), temperature T and butterfly velocity v_B in above ansatz are separately given by

$$T = \frac{U(1)}{4\pi} e^{-\frac{S(1)}{2}}, \quad v_B^2 = \frac{1}{2d} e^{-S(1)} U(1). \quad (\text{D.3})$$

Firstly, we build up a Lifshitz black hole without scalar deformation by setting $\phi = 0$. Then we calculate $\Phi(0)$ by evaluating $v_B^2/T^{2-\frac{2}{z}}$, whose values, as a function of z in different spatial dimensions d , are shown in the left plot of figure 2.

Secondly, we deform the Lifshitz black hole with scalar field. The value of Δ_- should satisfy $0 < \Delta_- < \frac{d+z}{2}$ to make the deformation relevant. Then we can impose a small perturbation with non-zero ϕ_- and study the variation of v_B . The numerical results match (3.3) when $\phi_- \ll T^{\frac{\Delta_-}{z}}$, where the coefficient γ as a function of z for different Δ_- is shown in the right plot of figure 2.

E Testing the relation between v_{th} and v_E

We consider a neutral black hole with flat horizon and scalar hair. In [46], the authors propose a relation between the thermodynamical volume density v_{th} and the Euclidean bounded volume density v_E as

$$v_{\text{th}} \doteq \sqrt{\frac{g_1}{h_1}} v_E, \quad (\text{E.1})$$

where g_1 and h_1 appear in the near horizon expansion of the black hole metric and the equality with a dot “ $\doteq$ ” marks their supposition. In a ρ coordinate, the metric appears as

$$ds^2 = -h(\rho)dt^2 + \frac{d\rho^2}{g(\rho)} + \frac{\rho^2}{L^2}d\mathbf{x}^2 \quad (\text{E.2})$$

with the horizon expansion

$$h(\rho) \sim h_1(\rho - \rho_h) + h_2(\rho - \rho_h)^2 + \dots, \quad g(\rho) \sim g_1(\rho - \rho_h) + g_2(\rho - \rho_h)^2 + \dots \quad (\text{E.3})$$

and the asymptotic boundary expansion

$$h(\rho) \sim \frac{\rho^2}{L^2} \left(1 + \dots - \frac{16\pi G_N m_{\text{BH}} L^{d+2}}{d\rho^{d+1}} + \dots \right), \quad g(\rho) \sim \frac{\rho^2}{L^2} \left(1 + \dots + \frac{\#}{\rho^{d+1}} + \dots \right). \quad (\text{E.4})$$

The horizon is located at ρ_h and the asymptotic boundary at $\rho \rightarrow \infty$. m_{BH} is the mass density of the black hole. The Euclidean bounded volume density v_E is

$$v_E = \frac{1}{d+1} \rho_h^{d+1} L^{-d}. \quad (\text{E.5})$$

The thermodynamical volume density v_{th} is determined by the general first law of black hole thermodynamics [75, 76]

$$dm_{\text{BH}} = Tds + v_{\text{th}}dP, \quad (\text{E.6})$$

where the thermodynamical pressure of the black hole is

$$P = \frac{d(d+1)}{16\pi G_N L^2}. \quad (\text{E.7})$$

For a neutral black hole with flat horizon and scalar hair, there are two Smarr relations [46]¹¹

$$\begin{aligned} m_{\text{BH}} &= \frac{d}{d+1} Ts, \\ m_{\text{BH}} &= \frac{d}{d-1} Ts - \frac{4}{d-1} v_{\text{th}} P, \end{aligned} \quad (\text{E.8})$$

¹¹Note that the coefficient before $v_{\text{th}}P$ is different from the one in [46]. The reason is that the spatial component of our metric (E.2) is ρ^2/L^2 , while the one in [46] is ρ^2 . Such difference change the definition of v_{th} through the first law (E.6).

which give

$$v_{\text{th}} P = \frac{1}{2} m_{\text{BH}} = \frac{d}{2(d+1)} T s. \quad (\text{E.9})$$

The hypothesis (E.1) relates the butterfly velocity v_B to v_{th} by

$$v_B^2 = \sqrt{\frac{h_1}{g_1} \frac{T s}{4 v_{\text{E}} P}} \doteq \frac{T s}{4 v_{\text{th}} P} = \frac{d+1}{2d}, \quad (\text{E.10})$$

which leads to a constant v_B and is in conflict with our analytical result in (2.9) and numerical result in appendix C where the scalar hair is considered. Such contradiction results from the violation of hypothesis (E.1).¹²

Taking the coordinate transformation $\rho^2/L^2 = C(\zeta)$, we can change (A.4) into (E.2) and check the hypothesis in (E.1). The result is

$$\sqrt{\frac{g_1}{h_1} \frac{v_{\text{E}}}{v_{\text{th}}}} = 1 - \frac{d+1}{2d} \lambda^2 (I_1(1; \vartheta) + 2I_2(1; \vartheta)), \quad (\text{E.11})$$

where (A.7), (A.15) and (A.17) have been used. In general, the last term does not vanish. Especially, for $0 < \vartheta < \frac{1}{2}$, it is non-positive. If further demanding one of the null-energy conditions, $T^\rho_\rho - T^t_t \geq 0$, we will have the inequality $\frac{g_1}{h_1} \geq 1$ and finally obtain

$$v_{\text{th}} \geq \sqrt{\frac{g_1}{h_1}} v_{\text{E}} \geq v_{\text{E}} \quad (\text{E.12})$$

up to $O(\lambda^2)$, which is the reverse isoperimetric inequality [46].

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¹²Thank the authors in [46] for pointing out this.

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