The anomalous transport of axial charge: topological vs non-topological fluctuations

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Abstract: Axial charge imbalance is an essential ingredient in novel effects associated with chiral anomaly such as chiral magnetic effects (CME). In a non-Abelian plasma with chiral fermions, local axial charge can be generated a) by topological fluctuations which would create domains with non-zero winding number b) by conventional non-topological thermal fluctuations. We provide a holographic evaluations of medium's response to dynamically generated axial charge density in hydrodynamic limit and examine if medium's response depends on the microscopic origins of axial charge imbalance. We show a local domain with non-zero winding number would induce a non-dissipative axial current due to chiral anomaly. We illustrate holographically that a local axial charge imbalance would be damped out with the damping rate related to Chern-Simon diffusive constant. By computing chiral magnetic current in the presence of dynamically generated axial charge density, we found that the ratio of CME current over the axial charge density is independent of the origin of axial charge imbalance in low frequency and momentum limit. Finally, a stochastic hydrodynamic equation of the axial charge is formulated by including both types of fluctuations.

Contents

1 Introduction

The parity-odd response of a medium with chiral fermions and its deep relationship to topology and quantum anomalies have attracted significant interest. One such effect under extensive study is the chiral magnetic effect (CME) [1–4] , which is the appearance of a vector current along the direction of an external magnetic field in the presence of axial charge imbalance (see refs. [5, 6] for a recent review). The CME has been demonstrated in various theoretical frameworks, such as in hydrodynamics [7–13], kinetic theories [14–19], perturbative theories [4, 20, 21], effective theories [22–26] and in the AdS/CFT correspondence [27–32]. A closely related effect is the chiral vortical effects (CVE), which is the appearance of a current along the direction of vorticity. CVE and its relation to mixed anomalies has been studied in [33–37]. Interesting properties of chiral media have been discussed in [38–40].

Those anomalous effects are not only theoretically well-motivated, but also phenomenologically important. In a heavy ion collision, a very strong magnetic field, on the order of $eB \sim m_\pi^2$, is created from the incoming nuclei that are positively charged and move at nearly the speed of light. Therefore, CME will convert axial charge fluctuations generated in heavy-ion collisions into (vector) charge-dependent correlation which could be

potentially detected by experimental observables. Recently, there have been significant experimental efforts in searching for CME and other anomalous transport effects (see [41] for a review) in heavy ion collision experiments [42–45].

One essential ingredient in those anomaly-related effects is the presence of axial charge imbalance. For example, in terms of chiral charge imbalance parametrized by the axial chemical potential μ_A, CME can be expressed as:

$$\boldsymbol{j}_V^{\text{CME}} = C_A \mu_A e \boldsymbol{B}\,, \qquad C_A = \frac{N_c}{2\pi^2}\,. \tag{1.1}$$

Previously, most studies were based on introducing axial charge asymmetry by hand, after which the response of the medium to a magnetic field is investigated (see ref. [46, 47] for exceptional cases). However, axial charge density $n_A(t, \boldsymbol{x})$ is a local and dynamical quantity depending on space and time. The medium's response to time-dependent, in-homogeneous axial charge density and the connections of this response to anomaly have been rarely studied before. One motivation of this paper is to fill this gap.

A distinctive feature of local axial charge density (in contrast to vector charge density) is that there are microscopically two different mechanisms for local generation of axial charge imbalance. The first one is by *topological fluctuations* of gluonic fields which would create domains with non-zero winding number. The resulting topological charge will in turn convert into axial charge density via anomaly relation:

$$\partial_\mu j_A^\mu = -2q\,, \qquad q \equiv \frac{g^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} \text{Tr} G_{\mu\nu} G_{\rho\sigma}\,, \tag{1.2}$$

where q is topological charge density (Pontryagin density) and $G_{\mu\nu}$ denotes field strength of gluonic fields. Such topological fluctuations would in general create both global axial charge and local axial charge imbalance. The second mechanism is through *thermal fluctuations*. In the absence of topological fluctuations, while the whole system is a grand canonical ensemble (of axial charge), each fluid cell of that system can be considered as a canonical ensemble (of axial charge). Therefore axial charge inside a fluid cell still fluctuates due to thermal fluctuations. For a non-Abelian plasma with chiral fermions, both mechanisms would contribute to local axial charge density fluctuations. Would response of the medium depend on the way that axial charge density is generated?

In this paper, we will consider a de-confined non-Abelian plasma with chiral fermions at finite temperature. We will begin by studying the medium's response to the interplay between local axial charge density $n_A(t, \boldsymbol{x})$ and $\boldsymbol{j}_A(t, \boldsymbol{x})$, $q(t, \boldsymbol{x})$ in long time and long wavelength limit (i.e. in hydrodynamic regime). In particular, we will examine if the relation among those one point functions depends on the microscopic origin of local axial charge density. As a basis for this study, we will work in a top-down holographic model, namely Sakai-Sugimoto model [48, 49]. The Sakai-Sugimoto model is considered to be close to the large N_c QCD with massless chiral quarks in quenched approximation. It has been widely applied to study anomaly-related effects (e.g. [27, 50, 51]). To model gluonic fluctuations and implement anomaly relation (1.2), we will consider the dynamics of C_7 Ramond-Ramond field and its Wess-Zumino coupling to flavor sector. As we are working in the long time, long wave length limit, the results presented in this paper are analytic.

We found axial current $\boldsymbol{j}_A(t,\boldsymbol{x})$ can be created by in-homogeneity of topological domains. It is well known that in-homogeneity of axial charge densities leads to diffusion: $\boldsymbol{j}_A(t,\boldsymbol{x}) = -D\nabla n_A(t,\boldsymbol{x})$ where D is the conventional diffusive constant. However, if the axial charge density results from a local topological domain which will be represented by an effective field "$\theta(t,\boldsymbol{x})$" in this paper, such topological domain will also induce an axial charge density and non-dissipative axial current in addition to diffusive current:

$$n_A = \frac{\Gamma_{\mathrm{CS}}}{T}\theta(t,x)\,, \qquad \boldsymbol{j}_A^{\mathrm{new}} = \kappa_{\mathrm{CS}}\nabla\theta\,. \tag{1.3}$$

Here, Γ_{CS} and κ_{CS} is related to the behavior of retarded Green's function $G_R^{qq}(t,\boldsymbol{x}) \sim \langle[q(t,\boldsymbol{x}),q(0,0)]\rangle$ in hydrodynamic regime:

$$G_R^{qq}(\omega,k) = \frac{1}{2}\left[-i\frac{\Gamma_{\mathrm{CS}}}{T}\omega - \kappa_{\mathrm{CS}}k^2\right]\,, \tag{1.4}$$

and T denotes temperature. It is worthy noting that κ_{CS} term in (1.3) is opposite to the direction of diffusive current and non-dissipative. One way to understand current (1.3) is that a topological domain also carries kinetic energy which would be transfered to chiral fermions via anomaly relation (1.2). In ref. [52] by us, we have derived (1.3) based on a generic setting and presented a brief verification of (1.3) in the Sakai-Sugimoto model. We provide more details on this calculation in section 3.1 and elucidate how axial current (1.3) is generated from gravity side of the duality.

Our calculation in section 3.2 confirms that a local chiral charge imbalance $n_A(t,\boldsymbol{x})$ will induced a non-zero $q(t,\boldsymbol{x})$ and they are related by

$$q(t,\boldsymbol{x}) = \frac{n_A(t,\boldsymbol{x})}{2\tau_{\mathrm{sph}}}\,. \tag{1.5}$$

τ_{sph} here can be interpreted as the axial charge damping time. Indeed, substituting (1.5) into (1.2), one has: $\partial_\mu J_A^\mu = -n_A/\tau_{\mathrm{sph}}$. Eqs. (1.5) hence implies that a non-zero axial charge density will eventually be damped out by inducing a non-zero q. Furthermore, we verify by holographic computations that τ_{sph} is related to Chern-Simon diffusive rate Γ_{CS} and susceptibility χ:

$$\tau_{\mathrm{sph}} = \frac{\chi T}{2\Gamma_{\mathrm{CS}}}\,. \tag{1.6}$$

Previously, relation (1.6) has been derived based on the standard fluctuations-dissipation argument [53] (see also section 5). Very recently, τ_{sph} has also been computed numerically in a bottom-up holographic model [54]. To best of our knowledge, current work is the first direct verification of relation (1.6) in strong coupling regime.

As we find that the axial current in response to axial charge density depend on how such axial charge imbalance is generated, it is natural to ask if chiral magnetic current (1.1) also depends on the origin of axial charge imbalance. To be quantitative, we consider the ratio between CME current and axial charge density in low frequency, small momentum limit in the presence of constant magnetic field:

$$(\chi_{\mathrm{dyn}})^{-1} \equiv \lim_{\omega,k\to 0}\left[\frac{\mu_A(\omega,k)}{n_A(\omega,k)}\right]\,, \qquad \mu_A(\omega,k) \equiv \frac{j_V^{CME}(\omega,k)}{C_A eB}\,. \tag{1.7}$$

For a system with constant axial charge density n_A, the ratio $\boldsymbol{j}_V^{CME}/(CeB)$ equals to axial chemical potential μ_A due to (1.1). However, if n_A is space-time dependent, the definition of axial chemical potential μ_A is ambiguous. If one takes the ratio $\boldsymbol{j}_V^{CME}/(CeB)$ as the generalized definition of axial chemical potential, the ratio $n_A(\omega, k)/\mu_A(\omega, k)$ can be interpreted as susceptibility. For this reason, we will call χ_{dyn} the "dynamical axial susceptibility".

We would like to emphasis that χ_{dyn} (1.7) is conceptually different from chiral magnetic conductivity [55] which is the proportionality coefficient of CME current to the *time-dependent* magnetic field for a medium with *homogeneous, time-independent* axial chemical potential. In (1.7) however, magnetic field is constant while n_A is space-time dependent. It is worthy noting that in realistic situations such as quark-gluon plasma (QGP) created in heavy-ion collisions, χ_{dyn} would be a relevant measure of CME as in those situations, the axial charge density is always generated dynamically. Previously, chiral magnetic conductivity has been calculated for plasma in equilibrium at both weak coupling limit [20, 21, 55] and strong coupling [50, 56, 57] and for plasma out-of-equilibrium [58]. However, we are not aware any existing literature discussing the "dynamic axial susceptibility" χ_{dyn} and its universality.

We have computed χ_{dyn} with $n_A(\omega, k)$ generated by topological and thermal (non-topological) fluctuations in Sakai-Sugimoto model. We found such ratio χ_{dyn} (1.7) is *independent of* the origin of axial charge imbalance and equals to static susceptibility χ. Moreover, we derive a simple analytic expression (4.15) relating the (integration of) gravity metric to χ_{dyn} which applies to a large class of holographic model. From such expression, we obtain a condition on the universality of χ_{dyn}.

Having established the fact that axial charge generate by both topological fluctuations and thermal fluctuations would contribution to CME current, it is then important to incorporate both fluctuations in the framework of stochastic hydrodynamics. Recently, there are encouraging progress on applying anomalous hydrodynamics to simulate charge seperation effects [59, 60] and chiral magnetic wave effects [61, 62] in heavy-ion collisions. In those studies, axial charge density enters as the initial conditions while the fluctuations of axial charge density during hydrodynamic evolution have been neglected. Motivated by findings in this paper, we formulate a stochastic hydrodynamic equation of axial charge density in section 5. Such hydrodynamic equation includes stochastic noise from both topological fluctuations and thermal fluctuations. While it is a direct generalization of the general framework [63–65], to our knowledge, stochastic equation (5.8) is new in literature. We hope our theory would be applied to simulate phenomenology of anomalous transport in the future.

The paper is organized as follows. We will begin with a brief review of pertinent ingredients of Sakai-Sugimoto model and realization of anomaly relation (1.2) in section 2. Section 3 is devoted to studying medium's response to axial charge density. The computation of χ_{dyn} (1.7) is presented in section 4. The stochastic hydrodynamic equation for axial charge is formulated in section 5. We conclude in section 6.

2 Sakai-Sugimoto model and chiral anomaly

2.1 Set-up of the model and realization of anomaly

In this paper, we will work in Sakai-Sugimoto model [48, 49]. In this model, the de-confined phase is given by $D4$ black-brane metric, which is a warped product of a $5d$ black hole and $S^1 \times S^4$ [66, 67]. The $D4$ brane background is given by [66]:

$$ds^2 = \left(\frac{U}{R}\right)^{3/2} \left(-f(U)dt^2 + d\boldsymbol{x}^2 + dx_4^2\right) + \left(\frac{R}{U}\right)^{3/2} \left(U^2 d\Omega_4^2 + \frac{dU^2}{f(U)}\right), \tag{2.1}$$

$$F_{(4)}^{\mathrm{RR}} = \frac{2\pi N_c \epsilon_4}{V_4}, \quad e^{\phi} = g_s \left(\frac{U}{R}\right)^{3/4}, \quad R^3 = \pi g_s N_c l_s^3, \quad f(U) = 1 - \left(\frac{U_T}{U}\right)^3. \tag{2.2}$$

Here x_4 is the coordinates of S_1 and ϵ_4 is the volume form of the four sphere S_4. In addition, $V_4 = 8\pi^2/3$ is the volume of S_4 and g_s, l_s are string coupling and string length respectively. The location of the horizon U_T is related to the inverse temperature:

$$\frac{4\pi}{3} \frac{R^{3/2}}{U_T^{1/2}} = \frac{1}{T}. \tag{2.3}$$

The periodicity of x_4 is given by

$$\delta\tau = 2\pi R_4 = \frac{2\pi}{M_{\mathrm{KK}}}. \tag{2.4}$$

The background (2.1) is stable for $T > 1/(2\pi R_4) = M_{KK}/2\pi$ [67]. Finally, 't Hooft coupling λ is given by:

$$\lambda = N_c \frac{(2\pi)^2 g_s l_s}{2\pi R_4} = 2\pi N_c g_s l_s M_{KK}. \tag{2.5}$$

To model gluonic fluctuations, we will consider the dynamics of C_7 Ramond-Ramond form. The kinetic energy of C_7 are given by

$$S_{RR} = -\frac{(2\pi l_s)^6}{4\pi} \int_{10} dC_7 \wedge (*dC_7). \tag{2.6}$$

In Sakai-Sugimoto model, right and left handed quarks are introduced by N_f $D8$ branes and N_f $\bar{D}8$ branes [48, 49]. Right-handed (left-handed) U(1) gauge filed A_R (A_L) lives on $D8$ ($\bar{D}8$) and is dual to right-handed (left-handed) current J_R^{μ} (J_L^{μ}) on the boundary. The $D8/\bar{D}8$ branes are separated along the x_4 direction, with $D8$ branes located at $x_4 = 0$ and $\bar{D}8$ branes located at $x_4 = \pi R_4$. In this work will consider $N_f = 1$ though generalization to the case of multi-flavors is straightforward.

The action of bulk gauge field $A_{R,L}$ or its field strength $F_{R,L}$ is given by the summation of Dirac-Born-Infeld (DBI) term

$$S_{DBI}^{R,L} = -\frac{1}{(2\pi)^8 l_s^9} \int d^9 x e^{-\phi} \sqrt{-det \left(g_{MN} + (2\pi\alpha') F_{MN}^{R,L}\right)}, \tag{2.7}$$

and Wess-Zumino (WZ) term which couples $A_{R,L}$ to Ramond-Ramond form:

$$S_{WZ}^{R,L} = \pm \int_{\Sigma 9} \Sigma_q C_{q+1} \wedge tr e^{\frac{F^{R,L}}{2\pi}}. \tag{2.8}$$

We normalize the RR forms C as in [48] and use hermitian worldvolume gauge field A. The DBI action is identical for A_R and A_L. In WZ term, plus/minus sign is corresponding to A_R/A_L (i.e. F_{MN}^R/F_{MN}^L) respectively. Axial gauge field and vector gauge field are related to right-handed and left-handed gauge field by:

$$A = \frac{A^R - A^L}{2}, \qquad V = \frac{A^R + A^L}{2}. \tag{2.9}$$

The total action we will study then becomes:

$$S = S_{RR} + S_{\text{DBI}} + S_{\text{WZ}}. \tag{2.10}$$

It is instructive to show how axial anomaly relation (1.2) is realized in the current holographic model. Following holographic dictionary, the axial current j_A^μ is given by the variation of holographic on-shell action S_{holo} with respect the boundary value of $a_\mu \equiv A_\mu(U \to \infty)$ and q is given by the variation of S_{holo} with respect to θ:

$$j_A^\mu = \frac{\delta S_{\text{holo}}}{\delta a_\mu}, \qquad q = \frac{\delta S_{\text{holo}}}{\delta \theta}. \tag{2.11}$$

Here θ is determined by the holonomy of C_1 on the compactified x_4 direction [68]:

$$\theta(t, \boldsymbol{x}) = \lim_{U \to \infty} \int dx_4 (C_1^{(4)}). \tag{2.12}$$

One may note that the normalization in (2.12) is consistent with the one found by considering action of multiple probe color branes.

C_1 is related to C_7 and $A_{R,L}$ field [69]:

$$dC_1 = (2\pi l_s)^6 * dC_7 - A^R \wedge (\delta(x_4 - \pi R_4)dx_4) + A^L \wedge (\delta(x_4)dx_4). \tag{2.13}$$

It is clear from (2.13) that C_7 is invariant under axial gauge transformation:

$$\delta_\Lambda A^R = d\Lambda, \qquad \delta_\Lambda A^L = -d\Lambda, \qquad \delta_\Lambda C_1 = -\Lambda\delta(x_4)dx_4 - \Lambda\delta(x_4 - \pi R_4)dx_4, \tag{2.14}$$

where Λ is an arbitrary scalar function. At boundary and after integrating out x_4 dependence, the axial gauge transformation is reduced to

$$\delta_\Lambda a_\mu(t, \boldsymbol{x}) = \partial_\mu \Lambda(t, \boldsymbol{x}), \qquad \delta_\Lambda \theta(t, \boldsymbol{x}) = -2\Lambda(t, \boldsymbol{x}). \tag{2.15}$$

Since the action (2.10) expressed in terms of C_7 and field strength F_R/F_L are manifestly invariant under axial gauge transformation (2.14), S_{holo} would also be invariant under (2.15). We therefore have that for an infinitesimal transformation $\delta\Lambda$:

$$\begin{aligned}
\delta_\Lambda S_{\text{holo}} &= \int d^4x \left[\frac{\delta S_{\text{holo}}}{\delta a_\mu(t, \boldsymbol{x})} \partial_\mu \left(\delta\Lambda(t, \boldsymbol{x})\right) + \frac{\delta S_{\text{holo}}}{\delta \theta(t, \boldsymbol{x})}(-2\delta\Lambda(t, \boldsymbol{x})) \right] \\
&= \int d^4x \left[j_A^\mu(t, \boldsymbol{x}) \left(\partial_\mu \delta\Lambda(t, \boldsymbol{x})\right) - 2q(t, \boldsymbol{x})\delta\Lambda(t, \boldsymbol{x}) \right] \\
&= -\int d^4x \left[\partial_\mu j_A^\mu(t, \boldsymbol{x}) + 2q(t, \boldsymbol{x}) \right] \delta\Lambda(t, \boldsymbol{x}) = 0.
\end{aligned} \tag{2.16}$$

The anomaly relation (1.2) then follows from the requirement that (2.16) holds for arbitrary $\delta\Lambda$, This is the holographic realization of axial anomaly in the current model. Realization of axial anomaly in general Dp/Dq brane can be found in [70].

2.2 Fluctuations of bulk fields

We wish to study medium's response to local axial charge imbalance. To model that process in holography, we need to introduce sources on the boundary. Those sources will excite bulk fields which in turn would generate one point functions such as axial current j_A^μ and q on the boundary. As we discussed in the introduction, local axial charge imbalance can be generated by a gluonic configuration with non-zero winding number and by thermal fluctuations. Correspondingly, we will create axial charge imbalance by putting a non-zero $\theta(t, \boldsymbol{x})$ and by putting a non-zero axial gauge field $a_\mu(t, \boldsymbol{x})$ on the boundary. In this work, we will restrict ourselves to the longitudinal fluctuations, i.e., the source on the boundary are $\theta(t, x), a_t(t, x), a_x(t, x)$ where x corresponds to the direction of non-vanishing current and sources will only depend on t, x. Consequently, non-zero bulk fields are A_t, A_x, A_U, C_7 and they would only depend on t, x, U.[1] For later convenience, we introduce a dimensionless radial coordinate:

$$u \equiv U/U_T. \tag{2.17}$$

We will also rescale all other dimensionful quantities by

$$\tilde{T} \equiv \sqrt{\frac{U_T}{R^3}} = \frac{4\pi}{3} T = KT, \qquad K \equiv \frac{4\pi}{3}. \tag{2.18}$$

where we have used (2.3).

We take following ansartz for C_7:

$$C_7 = B_M(t, x; u)\, dx^M \wedge d\sigma_{12} \wedge \epsilon_4 = [B_t(t, x; u)dt + B_x(t, x; u)dx + B_u(t, x; u)du] \wedge d\sigma_{12} \wedge \epsilon_4. \tag{2.19}$$

where $d\sigma_{12} = dx_1 \wedge dx_2$. Here, M, N run over t, x, u. Adopting the convention $\epsilon_{tx_1 x_2 x_3 x_4 \theta_1 \theta_2 \theta_3 \theta_4 u} = 1$ with $\theta_i (i = 1, 2, 3, 4)$ being angular variables on S^4, we obtain the explicit expressions for dC_7 and $*dC_7$:

$$dC_7 = [G_{tx}(t, x; u)dt \wedge dx + G_{ut}(t, x; u)du \wedge dt + G_{ux}(t, x; u)du \wedge dx] \wedge d\sigma_{12} \wedge \epsilon_4, \tag{2.20a}$$

$$*dC_7 = -u^{-1} \left[G^{xu}(t, x; u)dt + G^{ut}(t, x; u)dx + G^{tx}(t, x; u)du \right] \wedge dx_4, \tag{2.20b}$$

where $G_{MN} = \partial_M B_N - \partial_N B_M$ is the field strength of B. Indices are raised/lowered by metric (2.1), i.e.:

$$g_{tt}(u) = -u^{3/2} f(u), \qquad g_{xx}(u) = u^{3/2}, \qquad g_{uu}(u) = u^{-3/2} f(u)^{-1}, \tag{2.21}$$

and $f(u) = 1 - u^{-3}$. Action (2.10), after performing trivial integration of x_4, can then be written in a compact form:

$$S = \int d^4 x\, du \left(-\frac{1}{4} N_c \sqrt{-\gamma} F^{MN} F_{MN} - \frac{1}{4} \sqrt{-g} G^{MN} G_{MN} - K \epsilon^{LMN} B_L F_{MN} \right), \tag{2.22}$$

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[1]Note that C_7 can also depend on x_4. This can happen when we consider backreaction of $D8$ brane. We will restric ourselves to lowest mode on S^1 with no x_4 dependence in this work.

where $F_{MN} \equiv \partial_N A_M - \partial_M A_N$ is the field strength of A. Three terms in (2.22) are corresponding to S_{DBI}, S_{RR}, S_{WZ} respectively and the sign convention for Levi-Civita symbol is $\epsilon^{txu} = 1$. In (2.22),

$$\sqrt{-g} = C_g u^{-1}\,. \qquad \sqrt{-\gamma} = C_\gamma u^{5/2}\,. \tag{2.23}$$

The dimensionless parameter C_g, C_γ can be expressed in terms of M_{KK} and T as

$$C_g = \frac{729\pi K^5 M_{KK}^2}{4\lambda^3 T^2}\,, \qquad C_\gamma = \frac{2\lambda T}{27\pi K M_{KK}}\,. \tag{2.24}$$

Varying (2.22) with respect to A and B, we obtain equation of motion for G^{MN}, F^{MN}:

$$\partial_M(\sqrt{-g}G^{MN}) = K\epsilon^{QMN}F_{QM}\,, \tag{2.25a}$$

$$N_c\left(\sqrt{-\gamma}\partial_M F^{MN}\right) = K\epsilon^{QMN}G_{QM}\,. \tag{2.25b}$$

2.3 The prescription for computing one point function

In this section, we will derive the explicit holographic prescription for computing one point function n_A, j_A, q from definition (2.11). For this purpose, we need to first obtain holographic on-shell action and then perform the variation with respect to the sources, θ, a_t, a_x. It is therefore more natural to rewrite the holographic action in terms of A and C_1 field instead of C_7. The reason is that C_1 is directly related to the boundary source θ as seen in eq. (2.12). From (2.13) and (2.19), we have explicitly:

$$\sqrt{-g}\,\epsilon_{LMN}G^{MN} = 2K\left[\partial_L M + 2A_L\right]\,. \tag{2.26}$$

where M is the x_4 component of C_1, integrated over the x_4 circle:

$$M \equiv \int dx_4 C_1^{(4)}\,. \tag{2.27}$$

The boundary value of M is θ. Note that G^{MN} and $(\partial_N M + 2A_N)$ are invariant under both flavor and C_7 gauge transformations (2.14). Using (2.26), (2.25b) can be written in terms of M as

$$N_c\partial_M\left(\sqrt{-\gamma}F^{MN}\right) = 2K^2\sqrt{-g'}\left(\partial^N M + 2A^N\right)\,, \tag{2.28}$$

where we have defined:

$$\sqrt{-g'} \equiv \frac{(-g_{tt}g_{xx}g_{uu})}{\sqrt{-g}}\,. \tag{2.29}$$

Moreover, the Bianchi identity of G_{MN} reads

$$\epsilon^{LMN}\partial_L G_{MN} = 0\,, \tag{2.30}$$

together with relation (2.26) gives

$$\partial_N\left[\sqrt{-g'}\left(2A^N + \partial^N M\right)\right] = 0. \tag{2.31}$$

It is easy to see the action in terms of the M and A_M fields, which would lead to equation of motion (2.28) and (2.31) is

$$S = \int d^4x du \left[-\frac{1}{4} N_c \sqrt{-\gamma} F^{MN} F_{MN} - \frac{K^2}{2} \sqrt{-g'} \left(\partial_Q M + 2A_Q \right) \left(\partial^Q M + 2A^Q \right) \right], \quad (2.32)$$

as it is analyzed in [69, 70]. The action has the same form as eq. (9) in [71] from a bottom-up model. However our action differs in the N_c dependence of different terms in (2.32), which is absent in [71].

In order to compute the one point functions of q, n_A and j_A we use the action (2.32) instead of (2.22), since it is expressed in terms of the bulk fields which are directly related to the sources of the boundary operators. To obtain the on-shell action, we do variation of (2.32):

$$\delta S = \int d^4x du \{ \partial_M \left[N_c \sqrt{-\gamma} F^{MN} - 2K^2 \sqrt{-g'} \left(\partial^N M + 2A^N \right) \right] \delta A_N$$
$$+ K^2 \sqrt{-g'} \partial_N \left(\partial^N M + 2A^N \right) \delta M \}$$
$$+ \int d^4x \left[N_c \sqrt{-\gamma} F^{NU} \delta A_N - K^2 \sqrt{-g'} (\partial^u M + 2A^u) \delta M \right]. \quad (2.33)$$

The bulk term vanishes by eqs. (2.28), (2.31), and the boundary term gives the on-shell action. We therefore have from (2.11)

$$q(t,x) = \lim_{u \to \infty} \left[-K^2 \sqrt{-g'} \left(\partial^u M(t,x;u) + 2A^u(t,x;u) \right) \right]_{\text{Ren}} = \lim_{u \to \infty} \left[-K G_{tx}(t,x;u) \right]_{\text{Ren}}, \quad (2.34)$$

and

$$n_A(t,x) = N_c \lim_{u \to \infty} \left[\sqrt{-\gamma} F^{tu}(t,x;u) \right]_{\text{Ren}}, \qquad j_A(t,x) = N_c \lim_{u \to \infty} \left[\sqrt{-\gamma} F^{xu}(t,x;u) \right]_{\text{Ren}}. \quad (2.35)$$

In (2.35), we have introduced a "bulk axial current" [72]:

$$J_A^\mu(u) \equiv N_c \sqrt{-\gamma} F^{\mu u}(u). \quad (2.36)$$

As in general bulk current $J_A^\mu(u)$ and bulk field $G_{tx}(u)$ might be divergent near the boundary $u \to \infty$. we use the subscript "Ren" in (2.35) and (2.34) to denote the subtraction of such divergences in (2.34). The correspondence: (2.35) and (2.34) has been used in ref. [52]. It is also interesting to note that from the u-component of the first equation in (2.25a):

$$N_c \partial_\mu \left[\sqrt{-\gamma} F^{\mu u} \right] = 2K G_{tx}, \quad (2.37)$$

the anomaly relation (1.2) will be reproduced by taking $u \to \infty$ limit on both side of (2.37) using (2.35) and (2.34).

We would like to comment on the nature of current j_A. Naively j_A obtained by a functional derivative is by definition a consistent current with respect to flavor gauge. In fact, it is also the covariant current. We can confirm this by noting that boundary source entering the bulk field strength only through boundary values of E_A and G, which are

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axial gauge invariant, therefore j_A is manifestly axial gauge invariant. The agreement of consistent and covariant currents may appear odd: this is because the QCD anomaly studied in this section is realized with an on-shell action that is manifestly axial gauge invariant. Therefore, the current obtained from functional derivative is also invariant, as if it were an ordinary current. In contrast, the QED anomaly is realized with an anomalous on-shell action under axial gauge transform. In this case, we can not have a current which is both conserved (consistent) and invariant (covariant), in the presence of external axial field. As an example, we will see that the covariant current is not conserved in section. 4, where we expand our study to include QED anomaly. Furthermore, because holography has access to gauge invariant (with respect to $\mathrm{SU}(N_c)$ gauge) quantities only, j_A is also covariant current with respect to $\mathrm{SU}(N_c)$ gauge transform. We stress that the action (2.32) is very different from what we would have obtained by a naive substitution of (2.26) into (2.22). In particular, the kinetic term of M would have an opposite sign, which would lead to a wrong sign for q, [48, 69, 70].

2.4 Computing one point function

We now ready to compute one point function. We will work in Fourier space: $\partial_t \to -i\omega$, $\partial_x \to ik$. Then using (2.37) and Bianchi identity, one would express bulk current J_A^t, J_A^x as:

$$J_A^t(u;\omega,k) = \left[\frac{N_c\sqrt{-\gamma}(ik)fE_A'(u;\omega,k) - 2K(i\omega)G(u;\omega,k)}{\omega^2 - k^2f} \right], \tag{2.38a}$$

$$J_A^x(u;\omega,k) = \left[\frac{N_c\sqrt{-\gamma}(i\omega)fE_A'(u;\omega,k) - 2K(ik)fG(u;\omega,k)}{\omega^2 - k^2f} \right]. \tag{2.38b}$$

Here we have introduced short-handed notations:

$$E_A(u;\omega,k) \equiv -F_{tx}(u;\omega,k), \qquad G(u;\omega,k) \equiv -G_{tx}(u;\omega,k). \tag{2.39}$$

Here and hereafter, we use prime to denote the derivative with respect to u.

From (2.38), we observe that we only need to solve equations for $G(u;\omega,k), E_A(u;\omega,k)$ to obtain one point function n_A, j_A, q. From (2.25), one finds:

$$G'' + \left(-\frac{1}{u} + \frac{\omega^2 f'}{f(\omega^2 - k^2f)} \right) G' + \frac{(\omega^2 - k^2f)}{u^3f^2}G =$$
$$\left(\frac{1}{N_c} \right) \left[\frac{4K^2}{\sqrt{-g}\sqrt{-\gamma}} \right] G + \left[\frac{2K\omega kf'}{\sqrt{-g}f(\omega^2 - k^2f)} \right] E_A, \tag{2.40a}$$

$$E_A'' + \left(\frac{5}{2u} + \frac{\omega^2 f'}{f(\omega^2 - k^2f)} \right) E_A' + \frac{(\omega^2 - k^2f)}{u^3f^2}E_A =$$
$$\left(\frac{1}{N_c} \right) \left[\frac{4K^2}{\sqrt{-g}\sqrt{-\gamma}} \right] E_A + \left(\frac{1}{N_c} \right) \left[\frac{2K\omega kf'}{\sqrt{-\gamma}f(\omega^2 - k^2f)} \right] G. \tag{2.40b}$$

It is understood that the back-reaction of the flavor branes will induce $1/N_c$ correction to the black-brane metric. Analysis shows that the correction to the metric could induce

terms $\sim G/N_c$ and $\sim E_A/N_c$ to (2.40a) and terms $\sim E_A/N_c$ and $\sim G/N_c^2$ to (2.40b). we will seek solutions to the leading nontrivial order in power series of $1/N_c$:

$$G = G^{(0)} + \frac{1}{N_c}G^{(1)} + \dots, \qquad E_A = E_A^{(0)} + \frac{1}{N_c}E_A^{(1)} + \dots. \qquad (2.41)$$

To the leading nontrivial order, the solutions are not affected by the back-reaction. Accordingly, we will compute one point function such as q, n_A, j_A to first non-trivial order in N_c. From (2.25) and (2.41), we found that $E^{(0)}$ satisfies the homogeneous equation:

$$E_A^{(0)\prime\prime} + \left(\frac{5}{2u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)}\right)E_A^{(0)\prime} + \frac{(\omega^2 - k^2 f)}{u^3 f^2}E_A^{(0)} = 0, \qquad (2.42a)$$

while $G^{(0)}$ satisfies in-homogeneous equation:

$$G^{(0)\prime\prime} + \left(-\frac{1}{u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)}\right)G^{(0)\prime} + \frac{(\omega^2 - k^2 f)}{u^3 f^2}G^{(0)} = \frac{2Kk\omega u f'}{C_g f(\omega^2 - k^2 f)}E_A^{(0)}, \quad (2.42b)$$

At order $1/N_c$, we further have:

$$E_A^{(1)\prime\prime} + \left(\frac{5}{2u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)}\right)E_A^{(1)\prime} + \frac{(\omega^2 - k^2 f)R^3}{u^3 f^2}E_A^{(1)} = \frac{2Kk\omega f'}{C_\gamma u^{5/2}f(\omega^2 - k^2 f)}G^{(0)}, \quad (2.42c)$$

and a similar equation for $G^{(1)}$.

Behavior of $G^{(0)}(u; \omega, k)$, $E_A^{(0)}(u; \omega, k)$ and $E_A^{(1)}(u; \omega, k)$ near boundary can be determined directly from (2.42):

$$G^{(0)}(u; \omega, k) = a_2(\omega, k)u^2(1 + \cdots) + b_0(\omega, k)(1 + \cdots), \qquad (2.43a)$$

$$E_A^{(0)}(u; \omega, k) = E_0(\omega, k)(1 + \cdots) + E_1^{(0)}(\omega, k)\left(u^{-3/2} + \cdots\right), \qquad (2.43b)$$

$$E_A^{(1)}(u; \omega, k) = E_1^{(1)}(\omega, k)\left(u^{-3/2} + \cdots\right), \qquad (2.43c)$$

where $\dots$ denote terms higher order in $1/u$. In (2.43c) we have defined the solution to in-homogeneous function (2.42c) $E_A^{(1)}$ in such a way that $E_A^{(1)}(u \to \infty) = 0$. a_2 here is related to a_t, a_x, θ by

$$a_2 = \frac{K}{2C_g}\left[(\omega^2 - k^2)\theta + 2i\omega a_t + 2ika_x\right]. \qquad (2.44)$$

In deriving (2.44), we have used the relation:

$$u^{-1}\partial_u G = -\frac{K}{2C_g}\left[f^{-1}\partial_t\left(\partial_t M + 2A_t\right) - \partial_x\left(\partial_x M + 2A_x\right)\right], \qquad (2.45)$$

which can be derived from (2.30) and (2.31). It is useful to note that E_0, a_2 are invariant under transformation (2.15).

As usual, we impose the infalling wave condition at the black hole horizon for (2.42):

$$\lim_{u \to u_H} G^{(0)}(u; \omega, k), E_A^{(0)}(u; \omega, k), E_A^{(1)}(u; \omega, k) \to (u - 1)^{-i\omega/3}. \qquad (2.46)$$

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Here $u_H = 1$ denotes the location of horizon. Consequently, with given boundary value a_2, E_0, (2.42) can be solved and $b_0(\omega, k), E_1^{(0)}(\omega, k), E_1^{(1)}(\omega, k)$ will be determined from the resulting solutions. They are related to one point function q, n_A, j_A via (2.38) and definition (2.35), (2.34). We therefore have

$$q(\omega, k) = K\, b_0(\omega, k)\,, \tag{2.47a}$$

$$n_A(\omega, k) = \frac{1}{\omega^2 - k^2}\left[-\frac{3C_\gamma}{2}(ik)\left(N_c E_1^{(0)}(\omega, k) + E_1^{(1)}(\omega, k)\right) - 2K(i\omega)b_0(\omega, k)\right], \tag{2.47b}$$

$$j_A(\omega, k) = \frac{1}{\omega^2 - k^2}\left[-\frac{3C_\gamma}{2}(i\omega)\left(N_c E_1^{(0)}(\omega, k) + E_1^{(1)}(\omega, k)\right) - 2K(ik)b_0(\omega, k)\right]. \tag{2.47c}$$

3 Medium's response to chiral charge imbalance

In section, we will solve (2.42). with two different boundary conditions and consider the relation between j_A, q and n_A. Physically, we would like to use those two different boundary conditions to model two different mechanisms for the generation of axial charge imbalance. In particular, we consider:

Case 1 **axial charge imbalance is generated by a domain with non-zero winding number.** To model this situation, we set axial gauge field to be zero at boundary, i.e., $a_t, a_x = 0$ but turn on a non-zero $\theta(\omega, k)$. Consequently, boundary condition for (2.42) becomes:

$$a_2(\omega, k) = \frac{K}{2C_g}\left[(\omega^2 - k^2)\theta(\omega, k)\right], \qquad E_0(\omega, k) = 0\,. \tag{3.1}$$

Case 2 **axial charge imbalance is generated by non-topological fluctuations.** To model this situation, we instead set $a_2 = 0$ and consider a non-zero axial electric field on the boundary:

$$a_2(\omega, k) = 0, \qquad E_0(\omega, k) \neq 0\,. \tag{3.2}$$

3.1 Medium's response to axial charge imbalance generated by topological fluctuations

In this section, we will study medium's response to axial charge imbalance generated by topological fluctuations. As we discussed previously, this amounts to solve (2.42) with boundary condition (3.1) and (2.43c). As in this case, there is no source term for (2.42a), $E^{(0)}(u; \omega, k) = 0$ trivially satisfies (2.42a) and consequently (2.42b) becomes an homogeneous equation:

$$G^{(0)\prime\prime} + \left(-\frac{1}{u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)}\right)G^{(0)\prime} + \frac{(\omega^2 - k^2 f)}{u^3 f^2}G^{(0)} = 0\,. \tag{3.3}$$

We will seek the in-falling solution for (3.3) in hydrodynamic regime $\omega, k \ll 1$. In this regime, the solution can be obtained analytically by first solving (3.3) order by order in power of ω, k away from horizon and then determining integration constants by matching

with in-falling wave boundary conditions near the hrozion. Away from the horizon, we can drop the third term in (3.3) to obtain the following solution

$$g_h(u) = 1 - \left(\frac{i\omega}{3}\right) \left[\int_{u_H}^u du' \left(\frac{3C_g(1-s^2 f)}{\sqrt{-g}f} - \frac{1}{u'-1} \right) + \log(u-1) \right], \qquad (3.4)$$

where we have defined

$$s \equiv \frac{k}{\omega}, \qquad (3.5)$$

to save notations. It is easy to check that behavior of (3.4) near the horizon $u \to 1$ can be matched to the infalling wave behavior in small ω limit:

$$(u-1)^{-i\frac{\omega}{3}} = 1 - \frac{i\omega}{3} \log(u-1) + \mathcal{O}(\omega^2). \qquad (3.6)$$

It is also worthy mentioning that the integral over u' in (3.4) is convergent as we have explicitly taken the $\log(u-1)$ outside the integral.

From boundary condition (3.1), we then fix the normalization of $G^{(0)}$:

$$G^{(0)}(u;\omega,k) = (\frac{iK\omega}{2C_g}\theta)g_h(u;\omega,k). \qquad (3.7)$$

Expanding (3.7) near the boundary, we obtain

$$q(\omega,k) = 2Kb_0(\omega,k) = \left(\frac{K^2}{C_g}\right) \{i\omega - k^2 \lim_{u\to\infty} \left[\int_{u_H}^u du' \frac{C_g}{\sqrt{-g}} - \frac{u^2}{2} \right] + \mathcal{O}(\omega^2)\}\theta(\omega,k). \qquad (3.8)$$

In (3.8), the subtraction is necessary to remove the divergence near the boundary. As the ratio $-q(\omega,k)/\theta(\omega,k)$ should be matched to the behavior of retarded Green's function (1.4), we identify $\Gamma_{\text{CS}}, \kappa_{\text{CS}}$ in the present model:

$$\frac{\Gamma_{\text{CS}}}{T} = \frac{2K^2}{C_g} = \frac{2K^2}{C_g}\tilde{T}^3 = \frac{8\lambda^3 T^6}{729\pi M_{KK}^2}, , \qquad (3.9)$$

$$\kappa_{\text{CS}} = -2\left(\frac{K^2}{C_g}\right) \lim_{u\to\infty} \left[\int_{u_H}^u du' u' - \frac{u^2}{2} \right] = \frac{1}{2}\left(\frac{\Gamma_{\text{CS}}}{T}\right)\tilde{T}^{-1} = \frac{\lambda^3 T^4}{243\pi^2 M_{KK}^2}, \qquad (3.10)$$

where in the last step, we recover the units and used (2.24) and definition (2.18). The Γ_{CS} in Sakai-Sugimoto model was computed previously in [73][2] (see ref. [74] for Γ_{CS} in other holographic models).

To compute n_A, j_A, one needs to solve in-homogenous equation (2.42c):

$$E_A^{(1)\prime\prime} + \left(\frac{5}{2u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)} \right) E_A^{(1)\prime} + \frac{(\omega^2 - k^2 f)R^3}{u^3 f^2} E_A^{(1)} = \frac{2Kk\omega f'}{C_\gamma u^{5/2} f(\omega^2 - k^2 f)} G^{(0)}, \qquad (3.11)$$

with $G^{(0)}(u;\omega,k)$ given by (3.7). At leading order in ω, k, the solution reads

$$E_A^{(1)}(u;\omega,k) = -\left(\frac{2ikK^2}{3C_\gamma}C_g\right)\theta(\omega,k)u^{-3/2}\left(1 + \mathcal{O}(\omega,k)\right), \qquad (3.12)$$

---

[2]Our results (3.9) has a different normalization from [73].

which can be easily verified by substituting it into (2.42c) and comparing results at leading order in ω, k. Now substituting (3.7) and (3.12) into (2.47), we have obtained the axial charge density generated by θ,

$$n_A = \left(\frac{2K^2}{C_g}\right)\theta(\omega, k)\left[1 + \mathcal{O}(\omega, k)\right] = \left(\frac{\Gamma_{\text{CS}}}{T}\right)\theta(\omega, k), \tag{3.13}$$

and j_A vanishes at this order. To obtain j_A, we consider t, x components of (2.25b) in the presence of $G^{(0)}$ given by (3.7). We then obtain flow equations of bulk current $J_A^t(u; \omega, k), J_A^x(u; \omega, k)$ along radial direction u:

$$\partial_u J_A^t = -2KG_{ux}^{(0)} + \left(ik\sqrt{-\gamma}g^{tt}g^{xx}\right)E_A^{(1)}, \quad \partial_u J_A^x = 2KG_{ut}^{(0)} + \left(i\omega\sqrt{-\gamma}g^{tt}g^{xx}\right)E_A^{(1)}. \tag{3.14}$$

Now using the relation between $G_{tx}^{(0)}$ and $G_{ux}^{(0)}, G_{ut}^{(0)}$ which can be obtained from (2.25a) in the absence of F,

$$G_{ut}^{(0)} = \frac{-ikf\partial_u G_{tx}^{(0)}}{\omega^2 - k^2 f}, \qquad G_{ux}^{(0)} = \frac{i\omega\partial_u G_{tx}^{(0)}}{\omega^2 - k^2 f}. \tag{3.15}$$

we have from (3.7):

$$\partial_u J_A^t = -\frac{2i\omega K^2\theta}{\sqrt{-g}f} + \left(ik\sqrt{-\gamma}g^{tt}g^{xx}\right)E_A^{(1)} = \frac{-i\omega K^2\theta}{\sqrt{-g}f}\left[1 + \mathcal{O}(\omega, k)\right]. \tag{3.16a}$$

$$\partial_u J_A^x = -\frac{2ikK^2\theta}{\sqrt{-g}} + \left(i\omega\sqrt{-\gamma}g^{tt}g^{xx}\right)E_A^{(1)} = -\frac{2ikK^2\theta}{\sqrt{-g}}\left[1 + \mathcal{O}(\omega, k)\right], \tag{3.16b}$$

By integrating over u, we therefore have:

$$n_A(\omega, k) = n_A^H(\omega, k) + \Delta n_A(\omega, k), \qquad j_A(\omega, k) = j_A^H(\omega, k) + \Delta j_A(\omega, k). \tag{3.17}$$

Here n_A^H, j_A^H are values of bulk current $J_A^t(u; \omega, k), J_A^x(u; \omega, k)$ at horizon $u = u_H$. We already know $n_A \sim \mathcal{O}(1)$ from (3.13) and

$$\Delta n_A = (-2i\omega K^2\theta)\left[\int_{u_H}^{\infty}du'\frac{1}{\sqrt{-g}f}\right] \sim \mathcal{O}(\omega, k). \tag{3.18}$$

Therefore we must have

$$n_A^H = \left(\frac{2K^2}{C_g}\right)\theta(\omega, k)\left[1 + \mathcal{O}(\omega, k)\right]. \tag{3.19}$$

On the other hand,

$$\Delta j_A = (-2ikK^2\theta)\left[\int_{u_H}^{\infty}du'\frac{1}{\sqrt{-g}}\right]_{\text{Ren}} = \left(\frac{-2ikK^2\theta}{C_g}\right)\lim_{u\to\infty}\left[\int_{u_H}^{\infty}du'u' - u^2/2\right]. \tag{3.20}$$

By comparing (3.20) with (3.10), we have

$$\Delta j_A = \kappa_{\text{CS}}(ik)\theta(\omega, k). \tag{3.21}$$

The current on the horizon j_A^H can be determined by substituting (3.12) into (2.38) and taking $u \to u_H$ limit:

$$j_A^H = \lim_{u \to u_H} [J_A^x(u; \omega, k)] = \lim_{u \to u_H} \left[\frac{\sqrt{-\gamma}(i\omega)f\partial_u E_A^{(1)}(u; \omega, k) - 2K(ik)fG^{(0)}(u; \omega, k)}{\omega^2 - k^2 f} \right]$$

$$= \lim_{u \to u_H} \left[\sqrt{-\gamma} E_A^{(1)}(u; \omega, k) \right] = -\frac{2K^2}{3C_g}(ik\theta). \quad (3.22)$$

Here we have used a property of any function satisfying in-falling wave boundary condition, say $Z_{\rm in}(u)$ that

$$\lim_{u \to u_H} (f\partial_u Z_{\rm in}) = -i\omega \lim_{u \to u_H} Z_{\rm in}. \quad (3.23)$$

Comparing (3.19) and (3.22), we found that on the horizon, j_A^H and n_A^H are related by Fick's law:

$$j_A^H = -D\nabla n_A^H. \quad (3.24)$$

To establish (3.24), we also used the value of diffusive constant D (3.30) in current model.

To sum up, in this subsection, we have studied axial current in response to axial charge imbalance created by topological fluctuations. To represent a domain with non-zero winding number, we first turn on a non-zero $\theta(\omega, k)$ and found that it will induce a non-zero q (3.8) and consequently a non-zero axial charge density $n_A(\omega, k)$. The axial charge density $n_A(\omega, k)$ and $\theta(\omega, k)$ are related by (3.13). Furthermore, the induced axial current can be divided into two part. The first part is due to the diffusion of n_A while the second part is in the opposite direction to the diffusive current and is proportional to $\kappa_{\rm CS}$, which quantifies the kinetic energy carried by a topological domain. We verified relation (1.3) as first proposed by us in ref. [52]. It is interesting to note that holographically, the diffusive (dissipative) current coincides with the current on the horizon (3.24) while the non-dissipative current (1.3) is given by the integration from horizon to the boundary (cf. (3.20) and (3.21)).

3.2 Medium's response to axial charge fluctuations generated by non-topological fluctuations

We now consider medium's response to axial charge imbalance generated by non-topological fluctuations. Following our discussion in section 2.4 we will solve (2.42) with boundary condition (3.1) and (2.43c). We first need to solve the homogeneous solution (2.42a). Similarly to (3.7), the infalling wave solution to (2.42a) reads:

$$e_h(u; \omega, k) = 1 - \left(\frac{i\omega}{3} \right) \left[\int_{u_H}^u du' \left(\frac{3C_\gamma(1 - s^2 f)}{\sqrt{-\gamma}f} - \frac{1}{u' - 1} \right) + \log(u - 1) \right]. \quad (3.25)$$

Consequently from boundary condition (3.2), we have:

$$E_A^{(0)}(u; \omega, k) = \left[\frac{E_0(\omega, k)}{c_E(\omega, k)} \right] e_h(u; \omega, k), \quad (3.26)$$

where $c_E(\omega, k)$ is defined by the value of $e_h(u; \omega, k)$ at boundary:

$$c_E(\omega, k) \equiv e_h(u \to \infty; \omega, k) = 1 + i\omega \left[s^2 \int_{u_H}^{\infty} du' \frac{C_\gamma}{\sqrt{-\gamma}} + \mathcal{O}(1) \right] = 1 + \frac{2i\omega}{3} \left(s^2 + \mathcal{O}(1) \right) . \tag{3.27}$$

Plug (3.26) into (2.47), we obtain:

$$j_A(\omega, k) = N_c C_\gamma \frac{E_0(\omega, k)}{c_E(\omega, k)} , \qquad n_A(\omega, k) = N_c s \, C_\gamma \frac{E_0(\omega, k)}{c_E(\omega, k)} . \tag{3.28}$$

The conductivity σ, diffusive constant D and susceptibility χ can be extracted from (3.28) as follows. First of all, in the homogeneous limit $s \to 0$ of (3.28), we will reproduce Ohm's law $j_A = \sigma E_A$. Therefore:

$$\sigma = N_c C_\gamma = \frac{2 N_c \lambda T^2}{27 \pi M_{KK}} . \tag{3.29}$$

On the other hand, eqs. (3.28) must have a hydrodynamic pole corresponding to diffusive mode at $\omega = -iDk^2$. This implies that $c_E(\omega = -iDk^2, k) = 0$ hence:

$$D = \int_{u_H}^{\infty} du' \frac{C_\gamma}{\sqrt{-\gamma}} = = \int_{u_H}^{\infty} du' (u')^{-5/2} = \frac{2}{3} \tilde{T}^{-1} = \frac{1}{2\pi T} . \tag{3.30}$$

In (3.29) and (3.30), we have used expression (3.28) and have recovered the units at the last step. Finally, using the Einstein relation $\sigma = \chi D$, we obtain the expression for χ from (3.29) and (3.30):

$$\chi^{-1} = \frac{D}{\sigma} = \frac{1}{N_c} \int_{u_H}^{\infty} \frac{du'}{\sqrt{-\gamma}} . \tag{3.31}$$

This relation between χ and the bulk integration over $\sqrt{-\gamma}$ is in agreement with general expression in ref. [72].

field E_0 would generate a local axial density and axial current. It would also create a non-zero q, which can be determined by solving (2.42b):

$$G^{(0)\prime\prime} + \left(-\frac{1}{u} + \frac{\omega^2 f'}{f(\omega^2 - k^2 f)} \right) G^{(0)\prime} + \frac{(\omega^2 - k^2 f)}{u^3 f^2} G^{(0)} = \frac{2Kk\omega u f'}{C_g f(\omega^2 - k^2 f)} E_A^{(0)} , \tag{3.32}$$

with $E_A^{(0)}$ given by (3.26). At leading order in ω, k, the in-homogeneous solution reads:

$$G^{(0)}(u; \omega, k) = \left(\frac{sK}{C_g} \right) \left[u^2 \left(1 + \mathcal{O}(\omega, k) \right) + \frac{2 g_h(u; \omega, k)}{i\omega(1 - s^2)} \right] \frac{E_0(\omega, k)}{c_E(\omega, k)} . \tag{3.33}$$

As one can check, the first term, i.e., sKu^2/C_g term is a special solution to in-homogeneous equation (3.32) at leading order in ω, k. $g_h(u)$ (3.4), the solution to homogeneous equation, is introduced to guarantee boundary condition (3.2). As a result, we have:

$$q(\omega, k) = \left(\frac{2K^2}{C_g} \right) \left[\frac{-is}{(1 - s^2)\omega} \right] \frac{E_0(\omega, k)}{c_E(\omega, k)} = \left(\frac{2K^2}{C_g} \right) \left[\frac{-is}{-s^2 \omega} \right] \frac{E_0(\omega, k)}{c_E(\omega, k)} . \tag{3.34}$$

In the last step, we dropped the 1 in the bracket. This is justified in the diffusive regime where $k^2 \sim \omega$. To find the response of q to n_A, we first note that q in (3.34) contains

responses to both n_A and axial electric field E_0. In fact, we can exclude the response to the latter by considering n_A induced by a normalizable mode. This occurs when $i\omega s^2 = -3/2$. According to (3.27) and (3.28), it implies that n_A remains finite while both c_E and E_0 vanish. Physically n_A in this case is induced by a diffusion wave. The response of q to n_A is then given by

$$\frac{q}{n_A} = \frac{2K^2}{N_c C_g C_\gamma} \frac{1}{-i\omega s^2} = \frac{4K^2}{3 N_c C_g C_\gamma}, \tag{3.35}$$

where in the last step we used $i\omega s^2 = -3/2$ and dropped higher order terms in ω. We note that (3.35) is precisely (1.5) as we advocated in the introduction.

We now show that the intuitive argument above could be established more rigourously in real space. For this purpose, it is convenient to perform the Fourier transform over ω and directly consider $n_A(t,k), j_A(t,k)$ and $q(t,k)$. By definition and (3.28), we have:

$$\frac{j_A(t,k)}{N_c} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} j_A(\omega,k) = -C_\gamma \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \left[\frac{\omega E_0(\omega,k)}{\omega + iDk^2} \right]$$
$$= -C_\gamma \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{\omega}{\omega + iDk^2} \int dt' e^{i\omega t'} E_0(t',k). \tag{3.36}$$

The above integral is nonvanishing for $t > t'$ when we can pick up the diffusive pole in the lower half plane. We then obtain

$$\frac{j_A(t,k)}{N_c} = -C_\gamma Dk^2 I_0(t,k). \tag{3.37}$$

$I_0(k)$ here is defined by:

$$I_0(k,t) \equiv e^{-iDk^2 t} E_0(\omega = -iDk^2, k) = \int_{-\infty}^{t} dt' e^{-Dk^2(t-t')} E(t',k). \tag{3.38}$$

A similar computation gives:

$$\frac{n_A(t,k)}{N_c} = (-iC_\gamma) k I_0(t,k), \tag{3.39}$$

$$q(t,k) = -\left(\frac{2K^2}{C_g} \right) (-iDk) I_0(t,k). \tag{3.40}$$

In (3.40), we have neglected a contribution higher order in k.

We assume the external field E_0 only exists in a finite time window. At sufficiently late time t, we can regard j_A and q as responses to n_A. Comparing (3.36) and (3.39) and return to real space, we obtain Fick's law:

$$j_A(t,x) = -D\nabla n_A(t,x). \tag{3.41}$$

Moreover, (3.39), (3.40) lead to the relation:

$$q(t,x) = \left(\frac{2K^2}{N_c C_g} \right) \left(\frac{D}{C_\gamma} \right) n_A(t,x) = \left(\frac{\Gamma_{\text{CS}}}{T} \right) \left(\frac{D}{\sigma} \right) n_A(t,x) = \left(\frac{\Gamma_{\text{CS}}}{\chi T} \right) n_A(t,x) = \frac{n_A(t,x)}{2\tau_{\text{sph}}}, \tag{3.42}$$

where we have used (3.9), (1.6).

The results of this section can be summarized by the following response matrix, characterizing the response of q, n_A and j_A to θ and E_A:

$$\begin{pmatrix} q \\ n_A \\ j_A \end{pmatrix} = \begin{pmatrix} \frac{i\omega\Gamma_{\rm CS}}{2T} & \frac{\Gamma_{\rm CS}}{T}\frac{i\omega}{k(\omega+iDk^2)} \\ \frac{\Gamma_{\rm CS}}{T} & \frac{k\sigma}{\omega+iDk^2} \\ ik\left(\kappa_{\rm CS} - D\frac{\Gamma_{\rm CS}}{T}\right) & \frac{\omega\sigma}{\omega+iDk^2} \end{pmatrix} \begin{pmatrix} \theta \\ E_0 \end{pmatrix}. \tag{3.43}$$

We keep only terms lowest order in small ω and k limit. We further restrict ourselves to the regime $\omega \sim k$ for the case with source θ, and to diffusive regime $k^2 \sim \omega$ for the case with source E_0. The transport coefficients we presented are their first nonvanishing order in N_c: the responses of n_A and j_A to E_A are at order $O(N_c)$, while the rest responses are at order $O(1)$.

Before closing this section, we would like to comment on the $O(1)$ correction to the responses of n_A and j_A to E_0 above. This requires us to go beyond leading order in $1/N_c$ and compute $E_A^{(1)}$ from (2.40b) with $E_A^{(0)}$ given by (3.26). Similar analysis shows that near the boundary, $E_A^{(1)} \sim u^{1/2} + \cdots$ as $u \to \infty$, which would give divergent contributions to n_A and j_A. This is because the mixing of the bulk fields changes the dimension of the operator. We note that the change of operator dimension occurs immediately with mixing in bottom-up model in [71], while in our case it occurs from the subleading order in $1/N_c$. As we explained earlier that the solution this order in $1/N_c$ is incomplete without including back-reaction of the flavor branes. It is curious to see if including such backreaction would remove the potential divergence. Although these higher order corrections do not affect the results of our paper, we hope that we could revisit the puzzle in future.

4 Chiral Magnetic Effect and universality

In the previous section, we have considered two different situations where axial charge imbalance is generated. we now want to study whether the CME current would depend on the microscopic origin of axial charge density. In particular, we will compute the ratio between CME current $j_V^{\rm CME}$ and axial charge density n_A in low frequency and momentum limit as defined in (1.7). If axial charge density is static and homogeneous and CME current is universally given by (1.1), one would have:

$$\chi_{\rm dyn} = \chi, \tag{4.1}$$

due to linearized equation of state $\delta n_A = \chi \delta \mu_A$. However, it is not obvious if (4.1) would still hold if n_A is generated dynamically as considered in this paper. Of particular interest is the case considered in section 3.1 that axial charge imbalance is created by topological fluctuations.

To compute (1.7), we turn on a small background magnetic field $F_{yz}^V = eB$, i.e. magnetic field is longitudinal to the direction of in-homogeneity as considered in the previous section. Then in the presence of bulk axial field F_{tu}^A, F_{xu}^A and F_{tx}^A, F_{tu}^V, F_{xu}^V, F_{tu}^V and F_{tx}^V components of vector field strength will be excited due to Wess-Zumino term:

$$S_{\rm WZ} = \int C_3 \wedge {\rm tr} e^{F_R/2\pi} - \int C_3 \wedge {\rm tr} e^{F_L/2\pi}, \tag{4.2}$$

with $F_4 = dC_3$ given in (2.1). The action for vector field consists of DBI term, which has the same form as that of axial gauge field A and WZ term from (4.2):

$$S_V = -N_c \int d^4x dU \left[\frac{1}{4} \sqrt{-\gamma} F_V^{MN} F_{MN}^V + K_B \epsilon^{QMN} A_Q F_{MN}^V \right] , \qquad (4.3)$$

where

$$K_B \equiv \left(-\frac{C_A eB}{2N_c} \right) . \qquad (4.4)$$

Here, we use subscript/superscript V for vector gauge field strength F_{MN}^V (below we will also use subscript/superscript A for axial gauge field).

The variation of S_V gives the equation of motion:

$$\partial_M \left(\sqrt{-\gamma} F_V^{MN} \right) = K_B \epsilon^{NMQ} F_{MQ}^A . \qquad (4.5)$$

F_{MQ}^A will be taken from solutions obtained in previous solutions. To compute (1.7), it is sufficient to work at linear order in eB. We therefore could neglect back-reaction due to eB to the holographic background and solution E_A obtained in the previous section.

As before, we define the bulk vector current

$$J_V^\mu(t, x; u) = N_c \sqrt{-\gamma} F_V^{\mu U}(t, x; u) , \qquad (4.6)$$

One point functions $n_V(t, x), j_V(t, x)$ are similarly given by the boundary values of bulk current:

$$n_V(t, x) \equiv \lim_{u \to \infty} J_V^t(t, x; u) , \qquad j_V(t, x) \equiv \lim_{u \to \infty} J_V^x(t, x; u) . \qquad (4.7)$$

As before, the vector current defined here is a covariant current. Similar to (2.36), it would be convenient to express vector current in terms of E_V and E_A as

$$J_V^t(u) = N_c \frac{(ik)\sqrt{-\gamma} f E_V'(u) - (2i\omega K_B) E_A(u)}{\omega^2 - k^2 f(u)} , \qquad (4.8a)$$

$$J_V^x(u) = N_c \frac{(-i\omega)\sqrt{-\gamma} f E_V'(u) - (2ik K_B) f E_A(u)}{\omega^2 - k^2 f(u)} , \qquad (4.8b)$$

where we have introduced the short-handed notation for "bulk electric field":

$$E_V(u; \omega, k) \equiv -F_{tx}^V(u; \omega, k) . \qquad (4.9)$$

We can easily verify using (4.8) that the covariant current is not conserved in the presence of external axial field E_A. The equation for $E_V(u; \omega, k)$ reads

$$E_V'' + \left(\frac{5}{2u} + \frac{\omega^2 f'}{\omega^2 - k^2 f} \right) E_V' + \frac{(\omega^2 - k^2 f)}{u^3 f^2} E_V =$$
$$\frac{2K_B}{C_\gamma} \left\{ \left[\frac{\omega k f'}{u^{5/2} f (\omega^2 - k^2 f)} \right] E_A + \frac{1}{N_c} \left[\frac{2K\omega k f'}{C_\gamma u^5 f (\omega^2 - k^2 f)} \right] G \right\} + \mathcal{O} \left((eB)^2 \right) . \quad (4.10)$$

We will solve (4.10) with E_A, G determined in the previous section. To concentrate on vector current induced by axial charge imbalance, we will not turn on any source for vector

field, i.e., imposing $E_V(u \to \infty) = 0$ on the boundary and use the standard in-falling wave boundary condition on the horizon.

To compute the vector current, it is also convenient to write down "flow equation" for bulk vector current by taking t, x components of (4.5) and using definition (4.6):

$$\partial_u J_V^t(u) = \frac{K_B}{\sqrt{-\gamma} f} J_A^x(u) - \left(ik\sqrt{-\gamma} g^{tt} g^{xx} \right) E_V(u) = \frac{K_B}{\sqrt{-\gamma} f} J_A^x(u) \left[1 + \mathcal{O}(\omega, k) \right]. \quad (4.11a)$$

$$\partial_u J_V^x(u) = \frac{2K_B}{\sqrt{-\gamma}} J_A^t(u) - \left(i\omega\sqrt{-\gamma} g^{tt} g^{xx} \right) E_V(u) = \frac{2K_B}{\sqrt{-\gamma}} J_A^t(u) \left[1 + \mathcal{O}(\omega, k) \right], \quad (4.11b)$$

On the R.H.S of (4.11b), we have used the fact that $J_V^{t,x}$ term is always dominated over E_V term in small ω, k limit due to additional gradients in front of E_V. This is because from (4.10) we observe that E_V is the same order as E_A and from (2.36), $J_A^{t,x}$ is at least the same order as E_A.

Now integrating (4.11a) over u and using definition (4.7), we have:

$$n_V(\omega, k) = \Delta n_V(\omega, k) + n_V^H(\omega, k), \qquad j_V(\omega, k) = \Delta j_V(\omega, k) + j_V^H(\omega, k). \quad (4.12)$$

Here n_V^H and j_V^H are values of bulk current $J_V^t(u; \omega, k)$ and $J_A^x(u; \omega, k)$ at horizon $u = u_H$ and

$$\Delta n_V(\omega, k) \equiv K_B \left[\int_{u_H}^\infty du' \frac{J_A^x(u; \omega, k)}{f\sqrt{-\gamma}} \right], \quad \Delta j_V(\omega, k) \equiv 2K_B \left[\int_{u_H}^\infty du' \frac{J_A^t(u; \omega, k)}{\sqrt{-\gamma}} \right]. \quad (4.13)$$

We now claim that CME current should be identified with Δj_V, i.e.,

$$j_V^{CME} \equiv 2K_B \left[\int_{u_H}^\infty du' \frac{J_A^t(u; \omega, k)}{\sqrt{-\gamma}} \right]. \quad (4.14)$$

The physical motivation behind identification (4.14) is that generically in a holographic set-up, the current on the horizon is dissipative (see also example below). On the other hand, the CME current is non-dissipative. Therefore one should exclude the horizon current from the total current when identifying CME current holographically.

We now consider the implication of (4.14). With (4.14) and (3.31), χ_{dyn} becomes:

$$\chi_{\text{dyn}} = \lim_{\omega, k \to 0} \left\{ n_A(\omega, k) \Big/ \left[\int_{u_H}^\infty du' \frac{J_A^t(u; \omega, k)}{\sqrt{-\gamma}} \right] \right\}. \quad (4.15)$$

It is clear that if in small ω, k limit, bulk axial current is constant, i.e.,

$$J_A^t(u; \omega, k) = n_A(\omega, k) \left[1 + \mathcal{O}(\omega, k) \right], \quad (4.16)$$

it follows one can replace $J_A^t(u; \omega, k)$ with n_A in (4.15). Consequently, one will arrive at (4.1) by noting (3.31). Therefore (4.16) can be interpreted as a condition for the validity of (4.1).

In both cases considered in section 3, the condition (4.16) is indeed satisfied, we therefor have (4.1) for those cases.

For completeness, we will calculate total j_V for both cases. For the first case (cf. section 3.1), it is straightforward to check that $j_V^H \sim \mathcal{O}(\omega, k)\theta$, which is sub-leading compared with j_V^{CME}, we therefore have:

$$j_V(\omega, k) = j_V^{CME}(\omega, k) = \left(\frac{K_B}{\chi} \right) \left(\frac{T\theta}{\Gamma_{\text{CS}}} \right) = C_A \left(\frac{\theta}{2\tau_{\text{sph}}} \right) eB, \quad (4.17)$$

where we have used (1.6). By comparing (4.17) with (1.1), it is tempting to make the identification:

$$\mu_A = \frac{\theta}{2\tau_{\text{sph}}}.$$

(4.18)

In ref. [4], μ_A is identified with $\partial_t\theta$. Eq. (4.18) corresponds to replace ∂_t in $\partial_t\theta$ with the inverse of characteristic time scale of sphaleron transition $1/\tau_{\text{sph}}$.

The situation is different for the second case (cf. section 3.2). In this case, j_V^H is the same order as j_V^{CME} in small ω, k limit. From (3.28) and (4.15), we have:

$$j_V^{\text{CME}}(\omega, k) = s\left(\frac{2N_cK_B}{\chi}\right)\left(\frac{E_0(\omega, k)}{c_E(\omega, k)}\right) = \frac{4N_cK_B}{3}\left(\frac{kE_0(\omega, k)}{\omega c_E(\omega, k)}\right).$$

(4.19)

On the other hand, to obtain n_V, j_V, we need to solve (4.10), at leading order in N_c, that

$$E_V'' + \left(\frac{5}{2u} + \frac{\omega^2 f'}{\omega^2 - k^2 f}\right)E_V' + \frac{(\omega^2 - k^2 f)}{u^3 f^2}E_V = \frac{2K_B\omega k f'}{C_\gamma u^{5/2} f(\omega^2 - k^2 f)}E_A,$$

(4.20)

where E_A is given by (3.26). The leading order solution to (4.20) reads:

$$E_V = -\frac{4K_B E_0(\omega, k)}{3c_E(\omega, k)C_\gamma}\left[u^{-3/2} + \mathcal{O}(\omega, k)\right]s.$$

(4.21)

Again, (4.21) can be easily verified by direct substitution. Now substituting (4.21) into (4.8), we have:

$$n_V = -2iN_cK_B\frac{E_0(\omega, k)}{\omega c_E(\omega, k)},$$

(4.22)

and j_V vanishes at this order. Since $\Delta n_V \sim \mathcal{O}(1)$, we thus have

$$n_V^H(\omega, k) = n_V(\omega, k) = -2iN_cK_B\frac{E_0(\omega, k)}{\omega c_E(\omega, k)}.$$

(4.23)

j_V^H is obtained in the same way as j_A^H in the previous section:

$$\begin{aligned}
j_V^H(\omega, k) &= \lim_{u \to u_H}\left[J_V^x(u; \omega, k)\right] \\
&= \lim_{u \to u_H} N_c\left[\frac{(-i\omega)\sqrt{-\gamma}f\partial_u E_V(u; \omega, k) - 2(ikK_B)fE_A(u; \omega, k)}{\omega^2 - k^2 f}\right] \\
&= \lim_{u \to u_H} -N_c\left[\sqrt{-\gamma}E_V(u; \omega, k)\right] = -\frac{4N_cK_B}{3}\frac{kE_0(\omega, k)}{\omega c_E(\omega, k)} = -iDkn_V^H(\omega, k).
\end{aligned}$$

(4.24)

Again we see that j_V^H can be interpreted as a diffusive current. As in this example, both diffusive current j_V^H (4.24) and CME current (4.19) would contribute to the total vector current in small ω, k limit, to compute CME coefficient hence χ_{dyn} properly, it is crucial to identify CME contribution, i.e., (4.14).

To close this section, we would like to comment that while in this paper, we are working in a specific holographic model, the relation (4.12) still holds for holographic action for bulk vector field of the form (4.3). Consequently, assuming the identification of CME current (4.14), the condition (4.16) would warrant that $\chi_{\text{dyn}} = \chi$. Moreover, the violation of condition (4.16) would also break the relation (4.1).

5 Stochastic hydrodynamic equations for axial charge density

We now formulate a hydrodynamic theory for axial charge density by including stochastic noise from both topological fluctuations and thermal fluctuations. We will focus on the dynamics of axial charge thus setting temperature and fluid velocity u^μ to be homogeneous and time-independent. We could then work in the frame that the fluid is at rest: $u^\mu = (1, 0, 0, 0)$. To close anomaly:

$$\partial_t n_A(t, \boldsymbol{x}) + \nabla \cdot \boldsymbol{j}_A(t, \boldsymbol{x}) = -2q(t, \boldsymbol{x}). \tag{5.1}$$

we want to express q and $\boldsymbol{j}_A$ in terms of noise and n_A (or its gradients). The constitute relation, which relates axial current $\boldsymbol{j}_A$ to n_A, is of the conventional form:

$$\boldsymbol{j}_A(t, \boldsymbol{x}) = -D\nabla n_A(t, \boldsymbol{x}) + \boldsymbol{\xi}(t, \boldsymbol{x}), \tag{5.2}$$

where $\boldsymbol{\xi}(t, \boldsymbol{x})$ encodes axial charge generated by thermal fluctuations:

$$\langle \boldsymbol{\xi}(t, \boldsymbol{x}) \rangle = 0, \qquad \langle \xi_i(t, \boldsymbol{x})\xi_j(t, \boldsymbol{x}') \rangle = 2\sigma T \delta_{ij}\delta(t - t')\delta^3(\boldsymbol{x} - \boldsymbol{x}'). \tag{5.3}$$

Here $\langle \ldots \rangle$ denotes the average over noise and $i, j = 1, 2, 3$ run over spatial coordinates. The magnitude of $\boldsymbol{\xi}(t, \boldsymbol{x})$ is given by the standard fluctuation-dissipation relation. Furthermore, $q(t, \boldsymbol{x})$ can be related to $n_A(t, \boldsymbol{x})$ using (1.5):

$$q(t, \boldsymbol{x}) = \frac{n_A(t, \boldsymbol{x})}{2\tau_{\text{sph}}} + \xi_q(t, \boldsymbol{x}). \tag{5.4}$$

$\xi_q(t, \boldsymbol{x})$ is the noise due to topological fluctuations:

$$\langle \xi_q(t, \boldsymbol{x}) \rangle = 0, \qquad \langle \xi_q(t, \boldsymbol{x})\xi_q(t, \boldsymbol{x}') \rangle = \Gamma_{\text{CS}}\delta(t - t')\delta^3(\boldsymbol{x} - \boldsymbol{x}'), \tag{5.5}$$

and we will assume that there is no cross correlation between two different types of fluctuations:

$$\langle \xi_q(t, \boldsymbol{x})\xi_i(t, \boldsymbol{x}') \rangle = 0. \tag{5.6}$$

This completely specifies our stochastic anomalous hydrodynamic equations. The noise due to topological fluctuations (5.5) have been considered previously in refs. [75]. On the other hand, for a conserved current (such as vector $\boldsymbol{j}_V$), the noise of the form (5.3) is standard. Incorporating both fluctuations in (5.8) is new to the extent of our knowledge.

As an application, we will consider equal time axial charge correlation function:

$$C_{nn}(t, \boldsymbol{x}) = \langle [n_A(t, \boldsymbol{x}) - n_A(0, \boldsymbol{x})] [n_A(t, \boldsymbol{x}) - n_A(0, \boldsymbol{x})] \rangle. \tag{5.7}$$

We start with equation for n_A readily followed from the stochastic hydrodynamic equations (5.1):

$$\left[\partial_t - D\nabla^2 + \tau_{\text{sph}}^{-1} \right] n_A(t, \boldsymbol{x}) = -\nabla \boldsymbol{\xi}(t, \boldsymbol{x}) + 2\xi_q(t, \boldsymbol{x}). \tag{5.8}$$

Under the initial condition $n_A(0, \boldsymbol{x}) = 0$, $C_{nn}(t, \boldsymbol{x})$ characterize the magnitude of axial charge fluctuations at time t and location $\boldsymbol{x}$ due to (both) fluctuations.

To compute (5.7), we first Fourier transform (5.8) into $\boldsymbol{k}$ space (but keep t-dependence):

$$\left[\partial_t + Dk^2 + \tau_{\text{sph}}^{-1}\right] n_A(t, \boldsymbol{k}) = [-i\boldsymbol{k} \cdot \boldsymbol{\xi}(t, \boldsymbol{k})] + 2\xi_q(t, \boldsymbol{k}). \tag{5.9}$$

The solution of (5.9) under initial condition $n_A(0, \boldsymbol{x}) = 0$ reads:

$$n_A(t, \boldsymbol{k}) = \int_0^t dt' e^{-\left(Dk^2 + \tau_{\text{sph}}^{-1}\right)(t-t')} [(-i\boldsymbol{k}) \cdot \boldsymbol{\xi}(t, \boldsymbol{k}) + 2\xi_q(t, \boldsymbol{k})]. \tag{5.10}$$

We therefore have:

$$\langle n_A(t, \boldsymbol{k}) n_A(t, \boldsymbol{k}')\rangle = \int_0^t dt_1 \int_0^t dt_2\, e^{-\left(Dk^2 + \tau_{\text{sph}}^{-1}\right)(t-t_1) - \left(Dk'^2 + \tau_{\text{sph}}^{-1}\right)(t-t_2)}$$
$$\times \left[k_i k_j' \langle \xi_i(t_1, \boldsymbol{k}) \xi_j(t_2, \boldsymbol{k}')\rangle + 4\langle \xi_q(t_1, \boldsymbol{k}) \xi_q(t_2, \boldsymbol{k}')\rangle\right]. \tag{5.11}$$

Using (5.5) and (5.3) in Fourier space,

$$\langle \boldsymbol{\xi}(t, \boldsymbol{k})\rangle = 0, \qquad \langle \xi_i(t, \boldsymbol{k}) \xi_j(t, \boldsymbol{k}')\rangle = 2\sigma T \delta_{ij} \delta(t - t') \delta^3(\boldsymbol{k} + \boldsymbol{k}'), \tag{5.12a}$$

$$\langle \xi_q(t, \boldsymbol{k})\rangle = 0, \qquad \langle \xi_q(t, \boldsymbol{k}) \xi_q(t, \boldsymbol{k}')\rangle = \Gamma_{\text{CS}} \delta(t - t') \delta^3(\boldsymbol{k} - \boldsymbol{k}'), \tag{5.12b}$$

and performing the average over noise, we have:

$$\langle n_A(t, \boldsymbol{k}) n_A(t, \boldsymbol{k}')\rangle = 2\left(\sigma T k^2 + 2\Gamma_{\text{CS}}\right) \int_0^t dt_1 e^{-2\left(Dk^2 + \tau_{\text{sph}}^{-1}\right)(t-t_1)} \delta^3(\boldsymbol{k} + \boldsymbol{k}') \tag{5.13}$$

$$= \frac{\sigma T k^2 + 2\Gamma_{\text{CS}}}{Dk^2 + \tau_{\text{sph}}^{-1}} \left[1 - e^{-2\left(Dk^2 + \tau_{\text{sph}}^{-1}\right)t}\right] \delta^3(\boldsymbol{k} + \boldsymbol{k}') = \chi T \left[1 - e^{-2\left(Dk^2 + \tau_{\text{sph}}^{-1}\right)t}\right] \delta^3(\boldsymbol{k} + \boldsymbol{k}').$$

In the last step, we have used Einstein relation $\sigma = \chi D$ and (1.6). Now returning to real space, we have:

$$C_{nn}(t, \boldsymbol{x}) = \chi T \int \frac{d^3 k}{(2\pi)^2} e^{i\boldsymbol{x}\cdot\boldsymbol{k}} \left[1 - e^{-2\left(Dk^2 + \tau_{\text{sph}}^{-1}\right)t}\right] = (\chi T) \left[\delta^3(\boldsymbol{x}) - \frac{1}{(8\pi Dt)^{3/2}} e^{-\frac{2t}{\tau_{\text{sph}}}} e^{-\frac{|\boldsymbol{x}|^2}{8Dt}}\right]. \tag{5.14}$$

It is worthy noting that as we are in hydrodynamic regime, $n_A(\boldsymbol{x})$ here should be understood as the coarse-grained axial charge density inside a fluid cell and $\boldsymbol{x}$ is the spatial coordinates labeling the corresponding fluid cell.

We now discuss the implication of (5.14). At very early time that $Dt \ll L_{\text{cell}}^2$ (therefore $t \ll \tau_{\text{sph}}$) where L_{cell} is the size of a fluid cell, the Gaussian appearing in (5.14) essentially becomes a delta function and we then have:

$$C_{nn}(t, \boldsymbol{x}) \approx \chi T \left[1 - e^{-\frac{2t}{\tau_{\text{sph}}}}\right] \delta^3(\boldsymbol{x}) \approx \frac{2\chi T}{\tau_{\text{sph}}} t\, \delta(\boldsymbol{x}) = 4\Gamma_{\text{CS}} t\, \delta(\boldsymbol{x}), \tag{5.15}$$

where in the last step we have used (1.6). At this stage, there is no correlation among axial charge in each fluid cell (cf. the delta function in (5.15)). Integrating (5.15) over volume $\int d^3\boldsymbol{x}$, we further recover relation between the fluctuation of axial charge and Chern-Simon diffusive constant:

$$\langle Q_5^2 \rangle = 4\Gamma_{\text{CS}} V t, \tag{5.16}$$

where V is the volume of the system. While it has been widely used in literature to estimate the fluctuation of axial charge, (5.16) is no longer valid at the stage that $Dt \sim L_{\rm cell}^2$. In this stage, the spatial dependence of axial charge fluctuations in (5.14) become important. The diffusion generates additional spatial correlation among axial charge density. Finally, in the long time limit $t \gg \tau_{\rm sph}$, the second term in (5.14) are suppressed exponentially and axial charge fluctuations are given by:

$$C_{nn}(t \to \infty, \boldsymbol{x}) \to (\chi T)\, \delta^3(\boldsymbol{x}) . \tag{5.17}$$

and

$$\lim_{t \to \infty} \langle Q_5^2 \rangle \to \chi T V , \tag{5.18}$$

This is of course expected as in long time limit, the $\langle Q_5^2 \rangle$ should approach its thermal equilibrium values (5.18).

Finally, we remark that to obtain (5.18), we have used the relation between Chern-Simon diffusive constant $\Gamma_{\rm CS}$ and sphaleron damping rate $\tau_{\rm sph}$ (1.6). Therefore like Einstein relation $\sigma = \chi D$ connecting conductivity σ and diffusive constant D, the relation between $\Gamma_{\rm CS}$ and $\tau_{\rm sph}$ is also fixed by the requirement based on thermodynamics (5.18). It is reassuring that the relation (1.6) is also realized, as we discussed in section 3.2, in the holographic model studied in this work.

6 Summary and outlook

We have analyzed the anomalous transport of a non-Abelian plasma in a de-confined phase with dynamically generated axial charge using a top-down holographic model. In particular, we consider two separate cases in which the axial charge is generated due to a) topological b) non-topological thermal fluctuations. When the axial charge is generated by topological gluonic fluctuations, we show a non-dissipative current (1.3) is induced due to chiral anomaly in section 3.1. We also illustrate holographically the damping of axial charge due to the interplay between flavor sector and gluonic sector. Furthermore, we consider the ratio of the CME current to the axial charge density at small ω and k (cf. (1.7)). We interpret such ratio as (the inverse of) "dynamical axial susceptibility" $\chi_{\rm dyn}$ (cf. the discussion below (1.7)). We found in the context of current holographic model, dynamical susceptibility $\chi_{\rm dyn}$ is independent of the microscopic origin of the axial charge and coincide with the static susceptibility χ. One phenomenological implications of our work, in particular, section 4 is that axial charge generated by topological fluctuations and non-topological thermal fluctuations would contribute to CME signature in heavy-ion collisions. For this reason, we propose a stochastic hydrodynamic equation of the axial charge where we incorporate noise both from both fluctuations in section 5. We found that the magnitude of axial charge fluctuations depend on the time scale that such fluctuations are measured (c.f (5.16), (5.18)) and as (5.14) indicates, the diffusive mode would induce spatial correlations among axial charges.

There are several issues that can be further studied based on the above analysis. The axial charge response to gluonic fluctuations can be studied when the flavor degrees of

freedom back-react to the glue part of the theory. In this case, the coupling of the axial current and the gluonic topological operator, q, is not suppressed.

In this work, we found in hydrodynamic limit, the "dynamical axial susceptibility" χ_{dyn} (1.7) is universal. It is both theoretically interesting and phenomenologically important to extend the definition of χ_{dyn} and study its independence on finite ω, k. In this case, it is possible that the resulting χ_{dyn} would depend on the origin of axial charge imbalance.

In computing axial charge density correlation function C_{nn} (5.7) from stochastic hydrodynamic equation formulated in section 5, we consider a system in the absence of magnetic field. Once there is magnetic field, axial charge would also be transported by chiral magnetic wave [51]. Furthermore, a new diffusive model would emerge due to the interplay between chiral magnetic wave and sphaleron damping [71, 76]. It is interesting to see how those new modes would contribute to correlation among axial charge densities within the framework of stochastic hydrodynamics.

Acknowledgments

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