Supersymmetric $E_{7(7)}$ exceptional field theory

Hadi Godazgar,[a] **Mahdi Godazgar,**[a] **Olaf Hohm,**[b] **Hermann Nicolai**[c] **and Henning Samtleben**[d]

[a] *DAMTP, Centre for Mathematical Sciences, University of Cambridge,*
  Wilberforce Road, Cambridge, CB3 0WA, U.K.

[b] *Center for Theoretical Physics, Massachusetts Institute of Technology,*
  Cambridge, MA 02139, U.S.A.

[c] *Max-Planck-Institut für Gravitationsphsyik, Albert-Einstein-Institut (AEI),*
  Mühlenberg 1, D-14476 Potsdam, Germany

[d] *Université de Lyon, Laboratoire de Physique, UMR 5672, CNRS,*
  École Normale Supérieure de Lyon,
  46, allée d'Italie, F-69364 Lyon cedex 07, France

  E-mail: H.M.Godazgar@damtp.cam.ac.uk, M.M.Godazgar@damtp.cam.ac.uk,
  ohohm@mit.edu, Hermann.Nicolai@aei.mpg.de,
  henning.samtleben@ens-lyon.fr

ABSTRACT: We give the supersymmetric extension of exceptional field theory for $E_{7(7)}$, which is based on a $(4+56)$-dimensional generalized spacetime subject to a covariant constraint. The fermions are tensors under the local Lorentz group $SO(1,3) \times SU(8)$ and transform as scalar densities under the $E_{7(7)}$ (internal) generalized diffeomorphisms. The supersymmetry transformations are manifestly covariant under these symmetries and close, in particular, into the generalized diffeomorphisms of the 56-dimensional space. We give the fermionic field equations and prove supersymmetric invariance. We establish the consistency of these results with the recently constructed generalized geometric formulation of $D = 11$ supergravity.

Contents

1 Introduction

Ever since the discovery of 'hidden' exceptional symmetries in maximal $N = 8$ supergravity [1] a recurring theme has been the question of whether these symmetries are specifically tied to dimensional reduction on tori, or whether they reflect more general properties of the underlying *uncompactified* maximal theories, possibly even providing clues towards a better understanding of M-theory. Starting from $D = 11$ supergravity [2] clear evidence for the existence of hidden structures beyond those of standard differential geometry was already given in the early work of refs. [3, 4], a line of development which was continued in [5] and taken up again in [6–8]. Somewhat independently of these developments, an important insight has been the emergence of generalized geometric concepts in string and M-theory, which enable a duality-covariant formulation of the low-energy effective space-time theories, as manifested in double field theory [9–13], and in the recently constructed

'exceptional field theory' (EFT) [14, 15]. See also refs. [16–19] for a generalized geometric approach in the sense of refs. [20, 21]. The purpose of this paper, then, is to bring together these strands of development: first we complete the construction of the $E_{7(7)}$ EFT by giving the fully supersymmetric extension by fermions; second, we relate the resulting theory to the formulation of [3, 6–8]. As one of our main results we will demonstrate the compatibility of these two formulations, and explain the subtleties involved in making a detailed comparison.

The approach of [3], which has been extended and completed in [7, 8] to also take into account aspects of the $E_{7(7)}$-based exceptional geometry, takes $D = 11$ supergravity as the starting point and reformulates it in order to make a local $SO(1, 3) \times SU(8)$ tangent space symmetry manifest. To this end the fields and coordinates are decomposed in a $(4 + 7)$ splitting, as in Kaluza-Klein compactifications, but keeping the full coordinate dependence of all fields (however, unlike in EFT, no extra coordinates beyond those of the original theory are introduced). The fermions transform under the local $SU(8)$ subgroup, and their supersymmetry transformations, already given in [3], are manifestly $SU(8)$ covariant. Moreover, those parts of the bosonic sector which lead to scalar and vector fields in the dimensionally reduced maximal supergravity can then be assembled into $E_{7(7)}$ objects, namely a 56-bein encoding the internal field components and a 56-plet of vectors combining the 28 electric and 28 magnetic vectors of $N = 8$ supergravity; their supersymmetry transformations can be shown to take the precise form of the four-dimensional maximal gauged supergravity. While in this approach the fermions are included from the beginning (with the supersymmetry variations constituting the starting point of the analysis) and the on-shell equivalence with $D = 11$ supergravity is thus guaranteed at each step of the construction, a proper understanding of the role of $E_{7(7)}$ in *eleven dimensions* (as well as of the $E_{7(7)}$-covariant dynamics of the bosonic sector) was lacking in the original work of [3], and has only emerged with the recent advances. Nevertheless it is remarkable that the combinations of $SU(8)$ connections in the supersymmetry variations of the fermions found 'empirically' in ref. [3] are precisely the ones required by $E_{7(7)}$-covariance as identified here.

The results of ref. [5] suggest that a formulation that is properly covariant under the exceptional groups should include extended coordinates transforming under this group, an idea that also appears in the proposal of ref. [22]. Such an extended spacetime has later been implemented for $E_{7(7)}$ in a particular truncation of $D = 11$ supergravity that retains only the internal coordinates and field components of the $(4 + 7)$ splitting [23]. More recently, similar reformulations of $D = 11$ supergravity have been given for the analogous truncations, casting the theory and their residual gauge transformations into a covariant form [18, 19, 24]. In contrast to the original approach of ref. [3], however, these formulations are not immediately applicable to the untruncated $D = 11$ supergravity. By contrast, the construction of refs. [7, 8], the recent construction of complete EFTs in refs. [14, 15] and finally, the present work extend the formulation of ref. [3] to a fully $E_{7(7)}$-covariant theory.

The $E_{7(7)}$ EFT, which is a natural extension of double field theory, is based on a 4+56-dimensional generalized spacetime, with fields in $E_{7(7)}$ representations initially depending on all coordinates x^μ and Y^M (with fundamental indices $M = 1, \ldots, 56$). The theory is given by an action along with non-abelian twisted self-duality equations for the 56 vector

fields. The fields transform appropriately under $E_{7(7)}$-generalized diffeomorphisms. Crucially, the theory is subject to an $E_{7(7)}$-covariant section condition [18] that implies that the fields depend only on a subset of coordinates. In order to compare with the usual $D = 11$ supergravity, and thus with the results of refs. [3, 7], one has to pick a particular solution of this constraint, which reduces the spacetime to 4+7 dimensions. After solving the section constraint, the various components of the generalized diffeomorphisms can be interpreted as conventional diffeomorphisms and tensor gauge transformations. In addition, and in analogy to type II double field theory [25, 26], the section constraint has two inequivalent solutions: $D = 11$ supergravity and type IIB supergravity. After solving the section constraint, the $E_{7(7)}$ EFT also encodes, as 7 components among the 56 gauge vectors, dual gravity degrees of freedom. This description is consistent by virtue of a covariantly constrained compensating two-form gauge field $\mathcal{B}_{\mu\nu M}$ [15, 27]. The status of this field may appear somewhat mysterious, but its appearance is already implied by consistency of the EFT gauge symmetries. In this paper we will give further credibility to this field by showing that it has consistent supersymmetry variations.

In this paper we introduce the fermions of the $E_{7(7)}$ EFT and give the supersymmetry variations of all fields in a manifestly $E_{7(7)} \times SU(8)$-covariant form, showing that they close, in particular, into the external and internal generalized diffeomorphisms. This is in analogy with the supersymmetrization of DFT [28–30]. Importantly, we find that the supersymmetry transformations of all fields can be written solely in terms of the fields of EFT, in particular the 56-bein, without recourse to the $D = 11$ fields that can be thought of as parametrising these structures in a GL(7) decomposition. Furthermore, we determine the fermionic field equations and verify supersymmetric on-shell invariance. To this end we have to further develop the generalized exceptional geometry underlying the $E_{7(7)}$ covariant formulation by introducing connections and invariant curvatures generalizing the geometry of double field theory [9, 31–34]. For the internal, 56-dimensional sub-sector, such a geometry is to a large extent already contained in the literature [18, 19, 35, 36]. In particular, refs. [18, 19] give the full dynamics and supersymmetry transformation rules for the truncated theory, where the fields and parameters are independent of the four-dimensional external coordinates, in terms of such geometrical objects. We use the opportunity to give a complete and self-contained presentation of this geometry. We give compact and $E_{7(7)}$-covariant expressions for the internal connections in terms of the 56-bein and other covariant objects. One of the main results of this paper then is the formulation including external and internal connection components $\mathcal{Q}_\mu$ and $\mathcal{Q}_M$ for the local SU(8), respectively, and similarly external and internal connection components ω_μ and ω_M for the local SO(1, 3), with all geometric objects being also covariant under $E_{7(7)}$-generalized diffeomorphisms. The various connection components are summarized in the following scheme

$$
\begin{array}{c|c}
\boxed{\omega_\mu} & \boxed{\mathcal{Q}_\mu} \\
\Gamma_{[\mu\nu]}{}^\rho = 0 & \mathcal{D}_\mu \mathcal{V}_M{}^{AB} \equiv \mathcal{P}_\mu{}^{ABCD} \mathcal{V}_{MCD} \\
\hline
\boxed{\omega_M} & \boxed{\mathcal{Q}_M} \\
\mathcal{D}_M e_\mu{}^\alpha \equiv \pi_M{}^{\alpha\beta} e_{\mu\beta} & \Gamma_{MN}{}^K|_{\mathbf{912}} = 0
\end{array}
\qquad . \tag{1.1}
$$

Here we also indicate the corresponding covariant torsion-type constraints satisfied by the connections. The precise definitions of the various tensors and our conventions will be given in the main text. The formulation is manifestly covariant under all gauge symmetries except for the external diffeomorphisms of x^μ that depend also on the 'internal' $E_{7(7)}$ coordinates. The structure of the various diagonal and off-diagonal connection components in (1.1) hints at a larger geometrical framework in which they would emerge from a single 'master connection', whose introduction would finally render all gauge symmetries manifest.

A distinctive feature of generalized geometries is that, in contrast to conventional geometry, the connections are not completely determined by imposing covariant constraints, necessarily featuring undetermined connections that are not given in terms of the physical fields, as first discussed in the geometry of double field theory [9, 28, 31–33] and later extended to exceptional groups [18, 35, 36]. As in double field theory, however, this is consistent with the final form of the (two-derivative) theory depending only on the physical fields, as the undetermined connections drop out of the action and all (supersymmetry) variations, as shown in [19] for the truncated theory. We also clarify the relation of these geometrical structures to the formulation of [3, 7, 8], in which connections carry 'non-metricities' that can be absorbed, as we will show, into SU(8) connections once we include components along the $E_{7(7)}$-extended directions.

One obvious question concerns the precise significance of the term 'symmetry' in the present context. The $E_{7(7)}$ identified here is analogous to the $GL(D)$ that appears in general relativity, and is 'spontaneously broken' when one picks a particular non-trivial solution to the section constraint $(t_{\boldsymbol{\alpha}})^{MN}\partial_M \otimes \partial_N = 0$.[1] However, the new structures exhibited here do *not* imply that $D = 11$ supergravity or IIB supergravity have any new local symmetries beyond the ones already known.[2] Nevertheless it is remarkable and significant that the internal diffeomorphisms can be combined with the tensor gauge transformations of the form fields and their duals in an $E_{7(7)}$-covariant form. Evidently, the true advantage of the reformulation would only become fully apparent if solutions of the section constraint, besides those corresponding to $D = 11$ or IIB supergravity, exist. Such solutions would give genuinely *new* theories (but see below). Although such solutions are somewhat unlikely to exist for the case at hand, the situation may become more interesting when one considers infinite dimensional extensions of the E-series.

A second question concerns the utility of the supersymmetric EFT constructed here in a more general perspective. Here we see two main possible applications and extensions. The first application concerns the non-linear consistency of Kaluza-Klein compactifications other than torus compactifications. These can be investigated along the lines of [40–42], exploiting the present formalism and the fact that it casts the higher-dimensional theory in a form adapted to (gauged) lower dimensional supergravity. Indeed, the full non-linear Kaluza-Klein ansätze for those higher-dimensional fields (including dual fields) yielding

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[1]It is an old idea to interpret the graviton as a Goldstone boson of spontaneously broken GL(4) symmetry [37–39], but the present scheme should not be viewed as a realization of this idea.

[2]The only new local symmetry would be the one associated with the seven 'dual' internal diffeomorphisms, but the corresponding transformation parameters 'miraculously' drop out in all relevant formulae, as shown in ref. [8]. In the formulation of ref. [15] this fact is explained by the 'Stückelberg-like' gauge invariance associated with the two-form field $\mathcal{B}_{\mu\nu M}$.

scalar or vector fields in the compactification have already been obtained in this way for the $AdS_4 \times S^7$ compactification [6, 42–44], as well as for general Scherk-Schwarz compactifications with fluxes [45].[3] Apart from the non-linear ansätze for higher rank tensors, which can now also be deduced in a straightforward fashion, and beyond the extension to other non-trivial compactifications of $D = 11$ supergravity, the main outstanding problem here is to extend these results to the compactification of IIB supergravity on $AdS_5 \times S^5$, for which either the supersymmetric extension of $E_{6(6)}$ EFT [47] or the present version with the IIB solution of the section constraint might be employed. Indeed, a study of the ambiguities inherent in defining generalized connections and how the supersymmetry transformations (and hence the theory) remain invariant under such redefinitions in this paper has lead to an understanding of the hook-type ambiguities observed in the $D = 11$ theory in ref. [41].

Secondly, the fact that the supersymmetric EFT has a structure very similar to four-dimensional maximal gauged supergravity [48] may lead to a higher-dimensional understanding of the new SO(8) gauged supergravities of ref. [49], obtained by performing an electromagnetic U(1) rotation of the 56 electric and magnetic vectors, which is *not* in $E_{7(7)}$. Partial evidence presented in refs. [6, 44], as well as a more explicit argument based on the higher-dimensional embedding tensor in ref. [8], show that these gaugings cannot originate from the $D = 11$ supergravity of ref. [2]. Specifically, the deformed theories can be obtained from the standard SO(8) gauged supergravity by 'twisting' the 56-bein relative to the vectors [6], that is, by making the replacement

$$\mathcal{V}(x) \rightarrow \mathcal{V}(x; \omega) \equiv \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \mathcal{V}(x) \tag{1.2}$$

in all formulae, where each element of the U(1) rotation matrix acts on a 28×28 subblock of the 56×56 matrix $\mathcal{V}$, in precise analogy with the deformation of the four-dimensional theory [49].[4] The present reformulation naturally suggests that a higher-dimensional ancestor of the deformed SO(8) gauged supergravities might thus be obtained by performing an analogous 'twist' of the 56-bein of EFT (see also ref. [45]), $\mathcal{V}(x, Y) \rightarrow \mathcal{V}(x, Y; \omega)$, relative to all vectors and tensors, where the 56-bein is now taken to also depend on the 56 extra coordinates Y^M. Because of the inequivalence of the corresponding gauged SO(8) supergravities in four dimensions, it is clear that such a theory would no longer be on-shell equivalent to the $D = 11$ supergravity of ref. [2], and hence would correspond to a non-trivial deformation of that theory. In fact, this would be the first example of a genuinely new maximal supergravity in the maximal space-time dimension $D = 11$ since the discovery of ref. [2] in 1978, and it would be a remarkable vindication of the present scheme if such a theory could be shown to exist. Equally important, there would be no way to reconcile this deformed theory with $D = 11$ diffeomorphism and Lorentz invariance; in other words, the four-dimensional ω-deformation of ref. [49] would lift to an analogous deformation of $D = 11$ supergravity that is encoded in a suitably generalized geometric framework transcending conventional supergravity.

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[3]See also ref. [46], where uplift ansätze for sphere reductions of the $D = 11$ and type IIB theories are conjectured using similar ideas.

[4]In fact, in the context of four-dimensional maximal gauged theories, the U(1) rotation above is to be understood as part of a more general $SL(2, \mathbb{R})$ symplectic deformation [50].

The outline of the paper is as follows. In section 2 we review the bosonic $E_{7(7)}$-covariant exceptional field theory, of refs. [14, 15]; in section 3 we construct its supersymmetric completion upon introducing the proper fermion connections and working out the supersymmetry algebra. In section 4, we discuss how this theory relates to the reformulation [3, 7, 8] of the full (untruncated) $D = 11$ supergravity after an explicit solution of the section constraint is chosen.

We refer the reader to appendix A for a summary of index notations and conventions.

2 Bosonic $E_{7(7)}$ exceptional field theory

In this section we give a brief review of the bosonic sector of the $E_{7(7)}$-covariant exceptional field theory, constructed in refs. [14, 15] (to which we refer for details) and translate it into the variables appropriate for the coupling to fermions, in particular the 56-bein parametrizing the coset space $E_{7(7)}/SU(8)$. To begin with, all fields in this theory depend on the four external variables x^μ, $\mu = 0, 1, \ldots, 3$, and the 56 internal variables Y^M, $M = 1, \ldots, 56$, transforming in the fundamental representation of $E_{7(7)}$, however the latter dependence is strongly restricted by the section condition

$$(t_{\boldsymbol\alpha})^{MN} \partial_M \partial_N A = 0, \qquad (t_{\boldsymbol\alpha})^{MN} \partial_M A \, \partial_N B = 0, \qquad \Omega^{MN} \partial_M A \, \partial_N B = 0, \qquad (2.1)$$

for any fields or gauge parameters A, B. Here, Ω^{MN} is the symplectic invariant matrix which we use for lowering and raising of fundamental indices according to $X^M = \Omega^{MN} X_N$, $X_N = X^M \Omega_{MN}$. The tensor $(t_{\boldsymbol\alpha})_M{}^N$ is the representation matrix of $E_{7(7)}$ in the fundamental representation. These constraints admit (at least) two inequivalent solutions, in which the fields depend on a subset of seven or six of the internal variables, respectively, according to the decompositions

$$\mathbf{56} \longrightarrow \boxed{\mathbf{7}_{+3}} + \mathbf{21}'_{+1} + \mathbf{21}_{-1} + \mathbf{7}'_{-3}, \qquad (2.2a)$$

$$\mathbf{56} \longrightarrow \boxed{(\mathbf{6}, \mathbf{1})_{+2}} + (\mathbf{6}', \mathbf{2})_{+1} + (\mathbf{20}, \mathbf{1})_0 + (\mathbf{6}, \mathbf{2})_{-1} + (\mathbf{6}', \mathbf{1})_{-2}, \qquad (2.2b)$$

of the fundamental representation of $E_{7(7)}$ with respect to the maximal subgroups $GL(7)$ and $GL(6) \times SL(2)$, respectively. The resulting theories are the full $D = 11$ supergravity and the type IIB theory, respectively. The bosonic field content of the $E_{7(7)}$-covariant exceptional field theory is given by

$$\left\{ e_\mu{}^\alpha, \, \mathcal{V}_M{}^{AB}, \, \mathcal{A}_\mu{}^M, \, \mathcal{B}_{\mu\nu\,\boldsymbol\alpha}, \, \mathcal{B}_{\mu\nu\,M} \right\}, \qquad (2.3)$$

which we describe in the following. The field $e_\mu{}^\alpha$ is the vierbein, from which the external (four-dimensional) metric is obtained as $g_{\mu\nu} = e_\mu{}^\alpha e_{\nu\alpha}$. Its analogue in the internal sector is the complex 56-bein

$$\mathcal{V}_M{}^{\underline{N}} = \{\mathcal{V}_M{}^{AB}, \mathcal{V}_{MAB}\}, \qquad (2.4)$$

satisfying

$$\mathcal{V}_M{}^{AB} = \mathcal{V}_M{}^{[AB]}, \qquad \mathcal{V}_{MAB} = \left(\mathcal{V}_M{}^{AB}\right)^*, \qquad (2.5)$$

with SU(8) indices $A, B, \cdots = 1, \ldots, 8$, in the fundamental **8** representation and collective index $\underline{N}$ labelling the $\mathbf{28} + \overline{\mathbf{28}}$.[5] The fact that the 56-bein is an $E_{7(7)}$ group-valued matrix is most efficiently encoded in the structure of its infinitesimal variation,

$$\delta \mathcal{V}_M{}^{AB} = -\delta q_C{}^{[A} \mathcal{V}_M{}^{B]C} + \delta p^{ABCD} \mathcal{V}_{MCD} \,, \tag{2.6}$$

with

$$\delta q_A{}^B = -\delta q^B{}_A \,, \qquad \delta p^{ABCD} = \frac{1}{24} \epsilon^{ABCDEFGH} \delta p_{EFGH} \,. \tag{2.7}$$

This is equivalent to

$$\mathcal{V}_{MAB}\, \delta \mathcal{V}_N{}^{CD}\, \Omega^{MN} = \frac{2}{3}\, \delta_{[A}{}^{[C}\, \mathcal{V}_{MB]E}\, \delta \mathcal{V}_N{}^{D]E}\, \Omega^{MN} \,,$$
$$\mathcal{V}_{MAB}\, \delta \mathcal{V}_{NCD}\, \Omega^{MN} = \mathcal{V}_{M[AB}\, \delta \mathcal{V}_{NCD]}\, \Omega^{MN} \,,$$
$$\mathcal{V}_M{}^{AB}\, \delta \mathcal{V}_N{}^{CD}\, \Omega^{MN} = -\frac{1}{24} \varepsilon^{ABCDEFGH}\, \mathcal{V}_{M\,EF}\, \delta \mathcal{V}_{N\,GH}\, \Omega^{MN} \,. \tag{2.8}$$

A particular consequence of the group property is

$$\mathcal{V}_M{}^{AB}\, \mathcal{V}_{N\,AB} - \mathcal{V}_{M\,AB}\, \mathcal{V}_N{}^{AB} = \mathrm{i}\, \Omega_{MN} \,,$$
$$\Omega^{MN}\, \mathcal{V}_M{}^{AB}\, \mathcal{V}_{N\,CD} = \mathrm{i}\, \delta^{AB}_{CD} \,,$$
$$\Omega^{MN}\, \mathcal{V}_M{}^{AB}\, \mathcal{V}_N{}^{CD} = 0 \,. \tag{2.9}$$

The analogue of the external metric $g_{\mu\nu}$ in the internal sector is the positive definite symmetric real matrix

$$\mathcal{M}_{MN} \equiv \mathcal{V}_{M\,AB} \mathcal{V}_N{}^{AB} + \mathcal{V}_{N\,AB} \mathcal{V}_M{}^{AB} \,, \tag{2.10}$$

in terms of which the bosonic sector in ref. [15] has been constructed.

The 56 gauge fields $\mathcal{A}_\mu{}^M$ in (2.3) are subject to the first order duality equations given by[6]

$$\mathcal{F}^-_{\mu\nu\,AB} \equiv \frac{1}{2}\, \mathcal{F}_{\mu\nu\,AB} - \frac{1}{4}\, e\, \varepsilon_{\mu\nu\rho\sigma}\, \mathcal{F}^{\rho\sigma}{}_{AB} = 0 \,. \tag{2.11}$$

Here, the 56 non-abelian field strengths are defined as

$$\mathcal{F}_{\mu\nu\,AB} \equiv \mathcal{F}_{\mu\nu}{}^M\, \mathcal{V}_{M\,AB} \,, \tag{2.12}$$

$$\mathcal{F}_{\mu\nu}{}^M \equiv 2\partial_{[\mu} \mathcal{A}_{\nu]}{}^M - 2\, \mathcal{A}_{[\mu}{}^N \partial_N \mathcal{A}_{\nu]}{}^M - \frac{1}{2}\left(24\,(t_{\boldsymbol{\alpha}})^{MN}(t^{\boldsymbol{\alpha}})_{KL} - \Omega^{MN}\Omega_{KL}\right) \mathcal{A}_{[\mu}{}^K \partial_N \mathcal{A}_{\nu]}{}^L$$
$$- 12\,(t^{\boldsymbol{\alpha}})^{MN} \partial_N \mathcal{B}_{\mu\nu\,\boldsymbol{\alpha}} - \frac{1}{2}\, \Omega^{MN}\, \mathcal{B}_{\mu\nu\,N} \,, \tag{2.13}$$

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with the 2-forms $\mathcal{B}_{\mu\nu\,\boldsymbol{\alpha}}$, $\mathcal{B}_{\mu\nu\,N}$ from (2.3), transforming in the adjoint and the fundamental representation of $E_{7(7)}$, respectively. The latter form is a covariantly constrained tensor field, i.e. it is constrained by algebraic equations analogous to (2.1)

$$
\begin{aligned}
(t_{\boldsymbol{\alpha}})^{MN}\,\mathcal{B}_M\mathcal{B}_N = 0\,, \qquad & (t_{\boldsymbol{\alpha}})^{MN}\,\mathcal{B}_M\,\partial_N A = 0\,, \qquad (t_{\boldsymbol{\alpha}})^{MN}\,\partial_M\,\mathcal{B}_N = 0\,, \\
\Omega^{MN}\,\mathcal{B}_M\,\mathcal{B}_N = 0\,, \qquad & \Omega^{MN}\,\mathcal{B}_M\,\partial_N A = 0\,.
\end{aligned}
\tag{2.14}
$$

Its presence is necessary for consistency of the hierarchy of non-abelian gauge transformations and can be inferred directly from the properties of the Jacobiator of generalized diffeomorphisms [15]. In turn, after solving the section constraint it ensures the correct and duality covariant description of those degrees of freedom that are on-shell dual to the 11-dimensional gravitational degrees of freedom.

Using (2.9) and (2.10), equations (2.11) take the form of the twisted self-duality equations[7]

$$
\mathcal{F}_{\mu\nu}{}^M = \frac{1}{2}\,i\,e\varepsilon_{\mu\nu\rho\sigma}\,\Omega^{MN}\mathcal{M}_{NK}\,\mathcal{F}^{\rho\sigma K}\,.
\tag{2.15}
$$

The bosonic exceptional field theory is invariant under generalized diffeomorphisms in the internal coordinates, acting via [18, 51]

$$
\mathbb{L}_\Lambda U^M \equiv \Lambda^K \partial_K U^M - 12\,\mathbb{P}^M{}_N{}^K{}_L\,\partial_K\Lambda^L\,U^N + \lambda(U)\,\partial_P\Lambda^P\,U^M\,,
\tag{2.16}
$$

on a fundamental vector U^M of weight $\lambda(U)$. The projector on the adjoint representation

$$
\mathbb{P}^K{}_M{}^L{}_N \equiv (t_{\boldsymbol{\alpha}})_M{}^K(t^{\boldsymbol{\alpha}})_N{}^L = \frac{1}{24}\,\delta_M^K\delta_N^L + \frac{1}{12}\,\delta_M^L\delta_N^K + (t_{\boldsymbol{\alpha}})_{MN}(t^{\boldsymbol{\alpha}})^{KL} - \frac{1}{24}\,\Omega_{MN}\Omega^{KL}\,,
\tag{2.17}
$$

ensures that the action (2.16) is compatible with the $E_{7(7)}$ group structure. The generalized diffeomorphisms also give rise to the definition of covariant derivatives

$$
D_\mu = \partial_\mu - \mathbb{L}_{\mathcal{A}_\mu}\,,
\tag{2.18}
$$

whose commutator precisely closes into the field strength (2.13). The full bosonic theory is invariant under the vector and tensor gauge symmetries

$$
\begin{aligned}
\delta_\Lambda e_\mu{}^\alpha &= \mathbb{L}_\Lambda e_\mu{}^\alpha\,, \\
\delta_\Lambda \mathcal{V}_M{}^{AB} &= \mathbb{L}_\Lambda \mathcal{V}_M{}^{AB}\,, \\
\delta_{\Lambda,\Xi}\mathcal{A}_\mu{}^M &= D_\mu\Lambda^M + 12\,(t^{\boldsymbol{\alpha}})^{MN}\,\partial_N\Xi_{\mu\,\boldsymbol{\alpha}} + \frac{1}{2}\,\Omega^{MN}\,\Xi_{\mu\,N}\,, \\
\delta_{\Lambda,\Xi}\mathcal{B}_{\mu\nu\,\boldsymbol{\alpha}} &= 2\,D_{[\mu}\Xi_{\nu]\boldsymbol{\alpha}} + (t_{\boldsymbol{\alpha}})_{KL}\,\Lambda^K\mathcal{F}_{\mu\nu}{}^L - (t_{\boldsymbol{\alpha}})_{KL}\,\mathcal{A}_{[\mu}{}^K\,\delta\mathcal{A}_{\nu]}{}^L\,, \\
\delta_{\Lambda,\Xi}\mathcal{B}_{\mu\nu M} &= 2\,D_{[\mu}\Xi_{\nu]M} + 48\,(t^{\boldsymbol{\alpha}})_L{}^K\,\big(\partial_K\partial_M\mathcal{A}_{[\mu}{}^L\big)\,\Xi_{\nu]\boldsymbol{\alpha}} \\
&\quad + \Omega_{KL}\,\big(\mathcal{A}_{[\mu}{}^K\partial_M\delta\mathcal{A}_{\nu]}{}^L - \partial_M\mathcal{A}_{[\mu}{}^K\delta\mathcal{A}_{\nu]}{}^L - \mathcal{F}_{\mu\nu}{}^K\partial_M\Lambda^L + \partial_M\mathcal{F}_{\mu\nu}{}^K\Lambda^L\big)\,, \tag{2.19}
\end{aligned}
$$

with parameters Λ^M, $\Xi_{\mu\,\boldsymbol{\alpha}}$, $\Xi_{\mu\,M}$, the latter constrained according to (2.14). The Λ-weights of the various bosonic fields and parameters are collected in table 1, where we have also

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[7]See footnote 6.

field	$e_\mu{}^\alpha$	$\mathcal{V}_M{}^{AB}$	$\mathcal{A}_\mu{}^M$, Λ^M	$\mathcal{B}_{\mu\nu\,\boldsymbol{\alpha}}$, $\Xi_{\mu\,\boldsymbol{\alpha}}$	$\mathcal{B}_{\mu\nu\,M}$, $\Xi_{\mu\,M}$	χ_{ABC}	ψ_μ^A, ϵ^A
λ	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$

Table 1. Λ-weights for the bosonic and fermionic fields and parameters.

included the Λ-weights of the fermionic fields to be introduced later. Note that $\mathcal{B}_{\mu\nu\,\boldsymbol{\alpha}}$ and $\mathcal{B}_{\mu\nu\,M}$ appear in the field strength (2.13) only via the combination/projection

$$- 12\,(t^{\boldsymbol{\alpha}})^{MN}\,\partial_N \mathcal{B}_{\mu\nu\,\boldsymbol{\alpha}} - \frac{1}{2}\,\Omega^{MK}\,\mathcal{B}_{\mu\nu\,K}\,. \tag{2.20}$$

As a result, we observe the following additional gauge transformations that leave the field strengths invariant

$$\delta_\Omega \mathcal{B}_{\mu\nu\,\boldsymbol{\alpha}} = \partial_M \Omega_{\mu\nu}{}^M{}_{\boldsymbol{\alpha}} + (t_{\boldsymbol{\alpha}})_M{}^N \Omega_{\mu\nu N}{}^M\,,$$
$$\delta_\Omega \mathcal{B}_{\mu\nu\,M} = -\partial_M \Omega_{\mu\nu N}{}^N - 2\,\partial_N \Omega_{\mu\nu M}{}^N\,, \tag{2.21}$$

where $\Omega_{\mu\nu}{}^M{}_{\boldsymbol{\alpha}}$ is a parameter living in the **912** of E$_{7(7)}$, i.e.

$$(t^{\boldsymbol{\alpha}})^{(KL}\Omega_{\mu\nu}{}^{M)}{}_{\boldsymbol{\alpha}} = 0\,, \tag{2.22}$$

and $\Omega_{\mu\nu N}{}^M$ is a parameter constrained in the index $_N$ just as the N index in partial derivatives ∂_N, see equations (2.1), and the two-form $\mathcal{B}_{\mu\nu\,N}$, see equations (2.14). The shift transformations (2.21) should be understood as the tensor gauge transformations of the three-form gauge potentials of the theory (which we have not explicitly introduced) that also act on the two-forms due to the Stückelberg couplings of their field strengths. They precisely drop out in the projection (2.20) which is the one appearing in the vector field strengths.

Other than the first-order duality equations (2.11), the remaining equations of motion of the bosonic theory are most compactly described by a Lagrangian[8]

$$\mathcal{L}_{\text{EFT}} = e\,\widehat{R} + \frac{1}{48}\,e\,g^{\mu\nu}\,D_\mu \mathcal{M}^{MN}\,D_\nu \mathcal{M}_{MN} - \frac{1}{8}\,e\,\mathcal{M}_{MN}\,\mathcal{F}^{\mu\nu M}\mathcal{F}_{\mu\nu}{}^N$$
$$+ \mathcal{L}_{\text{top}} - e\,V(\mathcal{M}_{MN}, g_{\mu\nu})\,. \tag{2.23}$$

Let us present the different terms. The modified Einstein Hilbert term carries the Ricci scalar $\widehat{R}$ obtained from contracting the modified Riemann tensor

$$\widehat{R}_{\mu\nu}{}^{\alpha\beta} \;\equiv\; R_{\mu\nu}{}^{\alpha\beta}[\omega] + \mathcal{F}_{\mu\nu}{}^M e^{\alpha\rho}\partial_M e_\rho{}^\beta\,, \tag{2.24}$$

with the spin connection $\omega_\mu{}^{\alpha\beta}$ obtained from the covariantized vanishing torsion condition

$$0 = \mathcal{D}_{[\mu} e_{\nu]}{}^\alpha \;\equiv\; \partial_{[\mu} e_{\nu]}{}^\alpha - \mathcal{A}_{[\mu}{}^K \partial_K e_{\nu]}{}^\alpha - \frac{1}{2}\,\partial_K \mathcal{A}_{[\mu}{}^K e_{\nu]}{}^\alpha + \omega_{[\mu}{}^{\alpha\beta} e_{\nu]\beta}\,. \tag{2.25}$$

---

[8]Due to the self-duality (2.15) of the vector fields, this is understood as a "pseudo-Lagrangian" in the sense of a democratic action [52] such that the duality equations (2.15) are to be imposed *after* varying the Lagrangian.

The scalar kinetic term can be equivalently expressed as

$$\frac{1}{48} D^\mu \mathcal{M}^{KL} D_\mu \mathcal{M}_{KL} = -\frac{1}{6} \mathcal{P}^\mu{}_{ABCD} \mathcal{P}_\mu{}^{ABCD} , \qquad (2.26)$$

where we have introduced the coset currents $\mathcal{P}_\mu{}^{ABCD}$ as follows

$$\mathcal{D}_\mu \mathcal{V}_M{}^{AB} \equiv D_\mu \mathcal{V}_M{}^{AB} + \mathcal{Q}_{\mu C}{}^{[A} \mathcal{V}_M{}^{B]C} = \mathcal{P}_\mu{}^{ABCD} \mathcal{V}_{MCD} \qquad (2.27)$$

according to the decomposition (2.6) and where D_μ refers to the covariant derivative defined in equation (2.18). This moreover defines the composite SU(8) connection

$$\mathcal{Q}_{\mu A}{}^B = \frac{2i}{3} \mathcal{V}^{NBC} D_\mu \mathcal{V}_{NCA} , \qquad (2.28)$$

indicating that the 56-bein transforms under local SU(8) transformations. Thus, we will in the following use $\mathcal{D}_\mu \equiv D_\mu + \mathcal{Q}_\mu$ to denote the resulting SU(8)-covariant derivatives. The vector kinetic term in (2.23)

$$-\frac{1}{8} e \mathcal{M}_{MN} \mathcal{F}^{\mu\nu M} \mathcal{F}_{\mu\nu}{}^N = -\frac{1}{4} e \mathcal{F}_{\mu\nu}{}^{AB} \mathcal{F}^{\mu\nu}{}_{AB} , \qquad (2.29)$$

simply contracts the non-abelian field strengths (2.13) with the internal metric (2.10), while the topological term is most compactly given as the boundary contribution of a five-dimensional bulk integral

$$\int_{\partial \Sigma_5} d^4 x \int d^{56} Y \, \mathcal{L}_{\text{top}} = \frac{i}{24} \int_{\Sigma_5} d^5 x \int d^{56} Y \, \varepsilon^{\mu\nu\rho\sigma\tau} \, \mathcal{F}_{\mu\nu}{}^M \, \mathcal{D}_\rho \mathcal{F}_{\sigma\tau M} . \qquad (2.30)$$

Finally, the last term in (2.23) is given by

$$V(\mathcal{M}_{MN}, g_{\mu\nu}) = -\frac{1}{48} \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_N \mathcal{M}_{KL} + \frac{1}{2} \mathcal{M}^{MN} \partial_M \mathcal{M}^{KL} \partial_L \mathcal{M}_{NK} \qquad (2.31)$$
$$- \frac{1}{2} g^{-1} \partial_M g \, \partial_N \mathcal{M}^{MN} - \frac{1}{4} \mathcal{M}^{MN} g^{-1} \partial_M g \, g^{-1} \partial_N g - \frac{1}{4} \mathcal{M}^{MN} \partial_M g^{\mu\nu} \partial_N g_{\mu\nu} ,$$

in terms of the internal and external metric. For later use, we note that in terms of the 56-bein and modulo a total derivative $e^{-1} \partial_M (e K^M)$, the potential takes the form

$$V(\mathcal{V}_M{}^{AB}, g_{\mu\nu}) = 4 \mathcal{V}^M{}_{[AB} \mathcal{V}^N{}_{CD]} \left(\partial_M p_N{}^{ABCD} - \frac{1}{2} q_{M E}{}^A p_N{}^{EBCD} \right)$$
$$+ \frac{1}{6} \mathcal{M}^{MN} p_M{}^{ABCD} p_{N ABCD} + 4 \mathcal{V}^M{}_{AB} \mathcal{V}^{N CD} p_M{}^{ABEF} p_{N CDEF}$$
$$- \frac{1}{4} \mathcal{M}^{MN} g^{-1} \partial_M g \, g^{-1} \partial_N g - \frac{1}{4} \mathcal{M}^{MN} \partial_M g^{\mu\nu} \partial_N g_{\mu\nu} , \qquad (2.32)$$

expressed via the standard decomposition of the Cartan form $\mathcal{V}^{-1} \partial_M \mathcal{V}$ along the compact and non-compact parts of the $E_{7(7)}$ Lie algebra

$$q_{M A}{}^B \equiv \frac{2i}{3} \mathcal{V}^{NBC} \partial_M \mathcal{V}_{NCA} , \quad p_M{}^{ABCD} \equiv i \mathcal{V}^{NAB} \partial_M \mathcal{V}_N{}^{CD} . \qquad (2.33)$$

Written in the form of (2.32), it is easy to observe that the first two lines of the potential reproduce the corresponding terms in equation (7.5) of ref. [3].

All five terms in (2.23) are separately gauge invariant under generalized diffeomorphisms (2.19) in the internal coordinates. In addition, the full set of equations of motion is invariant under generalized diffeomorphisms in the external coordinates acting as

$$\delta_\xi e_\mu{}^\alpha = \xi^\nu D_\nu e_\mu{}^\alpha + D_\mu \xi^\nu e_\nu{}^\alpha \,, \tag{2.34}$$

$$\delta_\xi \mathcal{M}_{MN} = \xi^\mu D_\mu \mathcal{M}_{MN} \,,$$

$$\delta_\xi \mathcal{A}_\mu{}^M = \xi^\nu \mathcal{F}_{\nu\mu}{}^M + \mathcal{M}^{MN} g_{\mu\nu} \partial_N \xi^\nu \,,$$

$$\delta_\xi \mathcal{B}_{\mu\nu\,\boldsymbol{\alpha}} = \xi^\rho \mathcal{H}_{\mu\nu\rho\,\boldsymbol{\alpha}} - (t_{\boldsymbol{\alpha}})_{KL}\,\mathcal{A}_{[\mu}{}^K \delta_\xi \mathcal{A}_{\nu]}{}^L \,,$$

$$\delta_\xi \mathcal{B}_{\mu\nu\,M} = \xi^\rho \mathcal{H}_{\mu\nu\rho\,M} - 2ie\,\varepsilon_{\mu\nu\rho\sigma} g^{\sigma\tau} D^\rho \left(g_{\tau\lambda} \partial_M \xi^\lambda \right) - \left(\mathcal{A}_{[\mu}{}^K \partial_M \delta_\xi \mathcal{A}_{\nu]\,K} - \partial_M \mathcal{A}_{[\mu}{}^K \delta_\xi \mathcal{A}_{\nu]\,K} \right) .$$

When $\partial_M = 0$, this reduces to the action of standard four-dimensional diffeomorphisms. Remarkably, the invariance of the theory under (2.34) fixes all relative coefficients in (2.23) and thus uniquely determines all equations of motion.

Variation of (2.23) gives the field equations for the scalar fields parametrizing $\mathcal{M}_{MN}$ and the Einstein field equations for $g_{\mu\nu}$. Variation with respect to the two-forms $\mathcal{B}_{\mu\nu\,\boldsymbol{\alpha}}$ and $\mathcal{B}_{\mu\nu\,M}$ yields projections of the first-order vector field equations (2.15). Finally, the variation of the action with respect to the vector fields leads to second order field equations

$$D_\nu \left(e\,\mathcal{M}_{MN}\,\mathcal{F}^{\mu\nu N} \right) = e \left(\widehat{J}^\mu{}_M + \mathcal{J}^\mu{}_M \right) \tag{2.35}$$

after combining with the derivative of (2.15), and where the gravitational and matter currents are defined by the respective contributions from the Einstein-Hilbert and the scalar kinetic term

$$\widehat{J}^\mu{}_M \equiv -2e_\alpha{}^\mu e_\beta{}^\nu \left(\partial_M \omega_\nu{}^{\alpha\beta} - \mathcal{D}_\nu \left(e^{\rho[\alpha} \partial_M e_\rho{}^{\beta]} \right) \right) \,,$$

$$\mathcal{J}^\mu{}_M \equiv 2i\,e^{-1}\,\partial_N \left(e\,\mathcal{P}^{\mu\,ABCD} \mathcal{V}^N{}_{AB} \mathcal{V}_{MCD} - \text{c.c.} \right) - \frac{1}{24} \mathcal{D}^\mu \mathcal{M}^{KL} \partial_M \mathcal{M}_{KL} \,. \tag{2.36}$$

Equation (2.35) may be compared to the second order field equations obtained from combining the derivative of (2.15) with the Bianchi identities

$$3\,\mathcal{D}_{[\mu} \mathcal{F}_{\nu\rho]}{}^M = -12\,(t^{\boldsymbol{\alpha}})^{MN} \partial_N \mathcal{H}_{\mu\nu\rho\,\boldsymbol{\alpha}} - \frac{1}{2}\,\Omega^{MN}\,\mathcal{H}_{\mu\nu\rho\,N} \,, \tag{2.37}$$

where $\mathcal{H}_{\mu\nu\rho\,\boldsymbol{\alpha}}$ and $\mathcal{H}_{\mu\nu\rho\,M}$ denote the non-abelian field strengths of the two-forms

$$\mathcal{H}_{\mu\nu\rho\,\boldsymbol{\alpha}} = 3\,\mathcal{D}_{[\mu} \mathcal{B}_{\nu\rho]\,\boldsymbol{\alpha}} - 3\,(t_{\boldsymbol{\alpha}})_{KL}\,\mathcal{A}_{[\mu}{}^K \partial_\nu \mathcal{A}_{\rho]}{}^L + \dots$$

$$\mathcal{H}_{\mu\nu\rho\,M} = 3\,\mathcal{D}_{[\mu} \mathcal{B}_{\nu\rho]\,M} - 3 \left(\mathcal{A}_{[\mu}{}^N \partial_M \partial_\nu \mathcal{A}_{\rho]\,N} - \partial_M \mathcal{A}_{[\mu}{}^N \partial_\nu \mathcal{A}_{\rho]\,N} \right) + \dots \,. \tag{2.38}$$

Combining (2.15), (2.35), and (2.37) gives rise to the first-order duality equations describing the dynamics of the two-forms

$$i\widehat{J}^\mu{}_M + \frac{1}{3}\,\mathcal{D}^\mu \mathcal{V}^{NAB}\,\partial_M \mathcal{V}_{NAB} = \frac{1}{12}\,e^{-1}\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{H}_{\nu\rho\sigma\,M} \,,$$

$$(t_{\boldsymbol{\alpha}})_N{}^M \left(\mathcal{P}^{\mu\,ABCD} \mathcal{V}^N{}_{AB} \mathcal{V}_{MCD} - \mathcal{P}^\mu{}_{ABCD} \mathcal{V}^{NAB} \mathcal{V}_M{}^{CD} \right) = e^{-1}\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{H}_{\nu\rho\sigma\,\boldsymbol{\alpha}} \,. \tag{2.39}$$

Strictly speaking, the second equation only holds under projection with $(t^{\boldsymbol{\alpha}})^{KL}\partial_L$. The first-order equations (2.39) show that the two-form fields do not bring in additional degrees of freedom to the theory.

3 SU(8) × E$_{7(7)}$ exceptional geometry

3.1 Connections

In this section we set up the E$_{7(7)}$-covariant geometrical formalism for defining derivatives that are simultaneously covariant with respect to generalized internal diffeomorphisms, local SU(8), and SO(1, 3) Lorentz transformations. This will allow us to couple the bosonic E$_{7(7)}$-covariant exceptional field theory to fermions and to establish the link with the 'ground up' approach to be described in the next section. From the representation content of maximal $N = 8$ supergravity, or equivalently from an appropriate decomposition of the $D = 11$ gravitino, it follows that the fermionic fields of the theory are SO(1, 3) spinors, and transform in the **8** (the gravitini ψ_μ^A) and in the **56** (the matter fermions χ^{ABC}) of SU(8), respectively.[9] The main new feature is that, like the bosonic fields (2.3), the fermions are here taken to depend on $4 + 56$ coordinates modulo the section condition (2.1). Under 'internal' generalized diffeomorphisms (2.16) they transform as scalar densities with weights as given in table 1.

For the external derivatives, the relevant connections have been introduced in the previous section. On a spinorial object in the fundamental representation of E$_{7(7)}$ × SU(8), the covariant derivative is defined as

$$\mathcal{D}_\mu X_{AN} = D_\mu X_{AN} + \frac{1}{4} \omega_\mu{}^{\alpha\beta} \gamma_{\alpha\beta} X_{AN} + \frac{1}{2} \mathcal{Q}_{\mu A}{}^B X_{BN} \,, \tag{3.1}$$

with the E$_{7(7)}$-covariant derivative D_μ from (2.18), and the spin- and SU(8)-connections defined by (2.25) and (2.28), respectively. By construction, these connections ensure covariance of $\mathcal{D}_\mu X_{AN}$. As usual, for covariant derivatives on four-dimensional space-time tensors we may also introduce the covariant derivative ∇_μ which in addition to (3.1) carries the Christoffel connection defined by the standard (though covariantized) vierbein postulate

$$\mathcal{D}_\mu e_\nu{}^\alpha - \Gamma_{\mu\nu}{}^\rho \, e_\rho{}^\alpha = 0 \,. \tag{3.2}$$

For the internal sector, we similarly define a covariant derivative in the internal variables Y^M. The most general such derivative (denoted by ∇_M) acts on Lorentz indices, SU(8) indices and E$_{7(7)}$ indices, and has the form

$$\nabla_M X_{AN} = \partial_M X_{AN} + \frac{1}{4} \omega_M{}^{\alpha\beta} \gamma_{\alpha\beta} X_{AN}$$
$$+ \frac{1}{2} \mathcal{Q}_{MA}{}^B X_{BN} - \Gamma_{MN}{}^K X_{AK} - \frac{2}{3} \lambda(X) \Gamma_{KM}{}^K X_{AN} \,, \tag{3.3}$$

if X is a generalized tensor of weight $\lambda(X)$ under generalized diffeomorphisms (2.16). Likewise, we use

$$\mathcal{D}_M X_{AN} = \partial_M X_{AN} + \frac{1}{4} \omega_M{}^{\alpha\beta} \gamma_{\alpha\beta} X_{AN} + \frac{1}{2} \mathcal{Q}_{MA}{}^B X_{BN} \,, \tag{3.4}$$

---

[9]We use spinor conventions from ref. [48], i.e. in particular $\gamma^{\mu\nu\rho\sigma} = e^{-1} \epsilon^{\mu\nu\rho\sigma} \gamma^5$ and $\gamma^5 \epsilon_A = -\epsilon_A$.

for the derivative without the Christoffel connection $\Gamma_{MN}{}^{K}$. The required transformation rules for the connections are determined by covariance. Under generalized diffeomorphisms (2.16), the non-covariant variation of the first term in (3.3) is given by

$$\Delta_\Lambda^{\mathrm{nc}}\big(\partial_M X_{AN}\big) = 12\,\mathbb{P}^K{}_N{}^P{}_Q\,\partial_M\partial_P\Lambda^Q\,X_{AK}\,, \tag{3.5}$$

where we recall that the covariant terms carry a weight of $-\frac{1}{2}$ [15]. Thus, $\Gamma_{MN}{}^{P}$ also carries a weight of $-\frac{1}{2}$ and has the inhomogeneous transformation

$$\delta_\Lambda\Gamma_{MK}{}^{N} = \mathbb{L}_\Lambda\Gamma_{MK}{}^{N} + 12\,\mathbb{P}^N{}_K{}^P{}_Q\,\partial_M\partial_P\Lambda^Q\,. \tag{3.6}$$

This implies in particular,

$$\delta_\Lambda\Gamma_{MK}{}^{M} = \mathbb{L}_\Lambda\Gamma_{MK}{}^{M} + \frac{3}{2}\partial_K\partial_P\Lambda^P\,, \tag{3.7}$$

explaining the factor $\frac{2}{3}$ in the last term of (3.3). In the following, we will discuss the definition of the internal spin- and SU(8) connection.

The internal spin connection $\omega_M{}^{\alpha\beta}$ is defined by analogy with (2.27) by demanding that

$$\mathcal{D}_M e_\mu{}^{\alpha} = \pi_M{}^{\alpha\beta}\,e_{\mu\beta}\,, \tag{3.8}$$

with $\pi_M{}^{\alpha\beta} = \pi_M{}^{(\alpha\beta)}$ living on the coset $\mathrm{GL}(4)/\mathrm{SO}(1,3)$. As a consequence,

$$\omega_M{}^{\alpha\beta} = e^{\mu[\alpha}\partial_M e_\mu{}^{\beta]}\,, \tag{3.9}$$

and

$$e^{\mu[\alpha}\mathcal{D}_M e_\mu{}^{\beta]} = 0 = e_{\alpha[\mu}\mathcal{D}_M e_{\nu]}{}^{\alpha}\,. \tag{3.10}$$

Later, it will turn out to be convenient to also introduce a modified spin connection $\widehat{\omega}_M{}^{\alpha\beta}$

$$\widehat{\omega}_M{}^{\alpha\beta} \equiv \omega_M{}^{\alpha\beta} - \frac{1}{4}\,\mathcal{M}_{MN}\,\mathcal{F}_{\mu\nu}{}^{N}\,e^{\mu\,\alpha}e^{\nu\,\beta}\,, \tag{3.11}$$

including the non-abelian field strengths $\mathcal{F}_{\mu\nu}{}^{N}$ in a fashion reminiscent of Kaluza-Klein theory, whereby we view fields $e_\mu{}^{\alpha}$, $\mathcal{V}_M{}^{AB}$, and $A_\mu{}^{M}$ as parts of a single big vielbein. We will denote the corresponding covariant derivatives by $\widehat{\mathcal{D}}$ and $\widehat{\nabla}$, respectively.

In order to discuss the remaining connections in (3.3), let us first require that the internal SU(8) connection and the Christoffel connection are related by a generalized vielbein postulate (or 'GVP', for short)

$$0 \equiv \nabla_M\mathcal{V}_N{}^{AB} \;=\; \partial_M\mathcal{V}_N{}^{AB} + \mathcal{Q}_{MC}{}^{[A}\mathcal{V}_N{}^{B]C} - \Gamma_{MN}{}^{K}\,\mathcal{V}_K{}^{AB}\,, \tag{3.12}$$

which is the analogue of (3.2) for the internal sector. In analogy with standard differential geometry one would now like to solve this relation for both the SU(8) connection $\mathcal{Q}_{MA}{}^{B}$ and the generalized affine connection $\Gamma_{MN}{}^{P}$ in terms of the 56-bein $\mathcal{V}$ and its derivatives $\partial_M\mathcal{V}$. While in ordinary differential geometry, a unique such answer can be obtained by imposing vanishing torsion, here there remain further ambiguities. In addition one would like the resulting expressions to satisfy all requisite covariance properties, to wit:

$Q_{MA}{}^B$ should transform as a proper connection under local SU(8) and as a generalized vector under generalized diffeomorphisms, while $\Gamma_{MN}{}^P$ should transform as a generalized affine connection under generalized diffeomorphisms and as a singlet under local SU(8). However, parallel to DFT it is not possible to express a connection satisfying these combined covariance requirements as a function of only $\mathcal{V}$ and $\partial_M \mathcal{V}$ in a covariant way, as we will also confirm in terms of a simplified example in appendix D, and in terms of an explicit calculation for the SU(8) connection in appendix E.

The first step in reducing the ambiguities is to constrain the connections by requiring the generalized torsion to vanish; this amounts to the constraint [18]

$$\mathcal{T}(V,W)^M = \mathcal{T}^M{}_{NK}V^N W^K \equiv \mathbb{L}^\nabla_V W^M - \mathbb{L}_V W^M \equiv 0 \tag{3.13}$$

for vectors V, W of weight $\frac{1}{2}$ where $\mathbb{L}^\nabla$ denotes the generalized Lie derivative with all partial derivatives replaced by covariant derivatives. Explicit evaluation of this condition yields

$$\mathcal{T}_{NK}{}^M = \Gamma_{NK}{}^M - 12\,\mathbb{P}^M{}_K{}^P{}_Q\,\Gamma_{PN}{}^Q + 4\,\mathbb{P}^M{}_K{}^P{}_N\Gamma_{QP}{}^Q \,, \tag{3.14}$$

with $\mathbb{P}$ the adjoint projector defined in equation (2.17). Indeed, it is a straightforward computation to show that this combination transforms covariantly under generalized diffeomorphisms. From (3.6) and using the cubic identity (A.3) of ref. [15]

$$\begin{aligned}\Delta^{\mathrm{nc}}_\Lambda\big(\Gamma_{PM}{}^N - 12\,\mathbb{P}^N{}_M{}^K{}_L\,\Gamma_{KP}{}^L\big) &= -6\,(t^{\boldsymbol{\alpha}})_P{}^R(t_{\boldsymbol{\alpha}})_M{}^N\partial_R\partial_K\Lambda^K \\ &= -4\,\mathbb{P}^N{}_M{}^R{}_P\,\Delta^{\mathrm{nc}}_\Lambda\Gamma_{KR}{}^K \,,\end{aligned} \tag{3.15}$$

where we have used equation (3.7) and the fact that all other terms in (A.3) vanish by the section constraint. The last term is of the form of the non-covariant variation of the final term in (3.14), with the opposite sign. Hence, the generalized torsion transforms as a generalized tensor. The fact that the generalized torsion is gauge covariant means that it can be set consistently to zero.

From equation (3.12), the last two indices in the generalized Christoffel connection $(\Gamma_M)_N{}^K$ take values in the adjoint of $\mathrm{E}_{7(7)}$. Hence, the generalized connection lives in the $\mathrm{E}_{7(7)}$ representations

$$\mathbf{56} \otimes \mathbf{133} = \mathbf{56} + \mathbf{912} + \mathbf{6480} \,. \tag{3.16}$$

Using the explicit form of the corresponding projectors given in ref. [53], one can verify that the vanishing torsion constraint (3.13) translates into [18, 35, 36]

$$\Gamma_{MN}{}^K\Big|_{\mathbf{912}} = 0 \,. \tag{3.17}$$

In addition, requiring density compatibility of the internal derivatives according to

$$\nabla_M e \equiv 0 \,, \tag{3.18}$$

fixes

$$\frac{3}{4}\,e^{-1}\partial_M e = \Gamma_{KM}{}^K = -\Omega_{MN}\Omega^{PQ}\Gamma_{PQ}{}^N \,, \tag{3.19}$$

where the second equality is obtained from contraction of (3.17). As we will explain below, this trace must drop out in all relevant expressions involving the fermions.

Next, we work out the most general SU(8) connection compatible with vanishing generalized torsion. Using equation (3.12), the condition (3.17) is equivalent to the following conditions on the internal SU(8) connection $\mathcal{Q}_M$:

$$\mathcal{V}^{K\,AB}\,\mathcal{D}_P\mathcal{V}_K{}^{CD} = 6\,\mathcal{V}^{K\,[AB}\left(\mathcal{D}_K\mathcal{V}_P{}^{CD]}\right) - \frac{1}{4}\,\epsilon^{ABCDEFGH}\,\mathcal{V}^K{}_{EF}\left(\mathcal{D}_K\mathcal{V}_{P\,GH}\right)$$
$$- 2\,\Gamma_{QK}{}^Q\left(\mathcal{V}^{K\,[AB}\mathcal{V}_P{}^{CD]} - \frac{1}{24}\,\epsilon^{ABCDEFGH}\,\mathcal{V}^K{}_{EF}\mathcal{V}_{P\,GH}\right), \quad (3.20)$$

$$\mathcal{V}^K{}_{AC}\,\mathcal{D}_P\mathcal{V}_K{}^{BC} = 6\left(\mathcal{V}^K{}_{AC}\,\mathcal{D}_K\mathcal{V}_P{}^{BC} + \mathcal{V}^{KBC}\,\mathcal{D}_K\mathcal{V}_{PAC}\right)$$
$$- \frac{3}{4}\,\delta_A^B\left(\mathcal{V}^K{}_{CD}\,\mathcal{D}_K\mathcal{V}_P{}^{CD} + \mathcal{V}^{KCD}\,\mathcal{D}_K\mathcal{V}_{PCD}\right)$$
$$- 2\,\Gamma_{QK}{}^Q\left(\mathcal{V}^K{}_{AC}\,\mathcal{V}_P{}^{BC} + \mathcal{V}^{KBC}\mathcal{V}_{PAC} - \frac{1}{8}\,\delta_A^B\,\mathcal{M}_P{}^K\right), \quad (3.21)$$

which constitute the analogue of (2.25) in the internal sector. Unlike in the external sector and standard geometry, the vanishing torsion conditions (3.20), (3.21) are not sufficient to fully determine the internal SU(8) connection [18, 36], but rather constrain it to the following form

$$\mathcal{Q}_{MA}{}^B = q_{MA}{}^B + R_{MA}{}^B + U_{MA}{}^B + W_{MA}{}^B\,. \quad (3.22)$$

Here

$$q_{MA}{}^B \equiv \frac{2i}{3}\,\mathcal{V}^{NBC}\,\partial_M\mathcal{V}_{NCA}\,, \quad p_M{}^{ABCD} \equiv i\mathcal{V}^{NAB}\,\partial_M\mathcal{V}_N{}^{CD} \quad (3.23)$$

are obtained in the standard way from the decomposition of the Cartan form $\mathcal{V}^{-1}\partial_M\mathcal{V}$ along the compact and non-compact parts of the $E_{7(7)}$ Lie algebra. We note that $q_{M\,A}{}^B$ transforms properly as a connection while $p_M{}^{ABCD}$ transforms covariantly under local SU(8), but neither transforms as a vector under generalized diffeomorphisms. The remaining pieces in (3.22) are given by

$$R_{M\,A}{}^B \equiv \frac{4i}{3}\left(\mathcal{V}^{NBC}\mathcal{V}_M{}^{DE}\,p_{NACDE} + \mathcal{V}^N{}_{AC}\mathcal{V}_{MDE}\,p_N{}^{BCDE}\right)$$
$$+ \frac{20i}{27}\left(\mathcal{V}^{NDE}\mathcal{V}_M{}^{BC}\,p_{NACDE} + \mathcal{V}^N{}_{DE}\mathcal{V}_{MAC}\,p_N{}^{BCDE}\right)$$
$$- \frac{7i}{27}\,\delta_A^B\left(\mathcal{V}^{NCD}\mathcal{V}_M{}^{EF}\,p_{NCDEF} + \mathcal{V}^N{}_{CD}\mathcal{V}_{MEF}\,p_N{}^{CDEF}\right),$$

$$W_{MA}{}^B \equiv \frac{8i}{27}\left(\mathcal{V}_{MAC}\mathcal{V}^{NBC} + \mathcal{V}_M{}^{BC}\mathcal{V}^N{}_{AC} - \frac{1}{8}\,\delta_A^B\,\mathcal{M}_{MK}\Omega^{NK}\right)\Gamma_{LN}{}^L\,, \quad (3.24)$$

and by

$$U_{MA}{}^B = \mathcal{V}_{MCD}\,u^{CD,B}{}_A - \mathcal{V}_M{}^{CD}\,u_{CD,A}{}^B\,, \quad (3.25)$$

where the SU(8) tensor $u_{CD,A}{}^B$ satisfies

$$u^{[CD,B]}{}_A \equiv 0\,, \quad u^{CA,B}{}_C \equiv 0\,, \quad (3.26)$$

and thus belongs to the **1280** of SU(8). It is now straightforward to check that $u_{CD,A}{}^B$ drops out of the vanishing torsion conditions (3.20), (3.21) and thus remains undetermined. An explicit form of $\mathcal{Q}_{MA}{}^B$ in terms of the GL(7) components of $\mathcal{V}_M{}^{AB}$ has been given in ref. [18]. With $\mathcal{Q}_{M\,A}{}^B$ given by (3.22), it is now straightforward to solve (3.12) for the affine connection

$$\Gamma_{MN}{}^P\left(\mathcal{V}, \partial\mathcal{V}, \mathcal{Q}\right) = i\left(\mathcal{V}^{P\,AB}\mathcal{D}_M(\mathcal{Q})\mathcal{V}_{N\,AB} - \mathcal{V}^P{}_{AB}\mathcal{D}_M(\mathcal{Q})\mathcal{V}_N{}^{AB}\right) \quad (3.27)$$

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using (2.9). This, then, is the most general expression for a *torsion-free* affine connection, where the part $U_{MA}{}^B$ of the connection (3.22) corresponding to the **1280** representation of SU(8) represents the irremovable ambiguity that remains even after imposition of the zero torsion constraint [18, 36]. In appendix E we will derive the unique expression for $U_{MA}{}^B$ in terms of only $\mathcal{V}$ and $\partial_M \mathcal{V}$ that makes $\mathcal{Q}_{MA}{}^B$ a generalized vector, but the resulting connection will no longer transform as a proper SU(8) connection, and as a consequence the affine connection would no longer be an SU(8) singlet.[10]

In view of these subtleties it is therefore all the more remarkable how the supersymmetric theory manages to sidestep these difficulties and ambiguities. Namely, in all relevant expressions the internal covariant derivatives $\mathcal{D}_M$ appear only in combinations in which the undetermined part $U_{MA}{}^B$ of the connection is projected out and for which the covariance under generalized diffeomorphisms is manifest. We illustrate this with a number of explicit expressions that will be useful in the following. Using the explicit expression for $\mathcal{Q}_{MA}{}^B$, equation (3.22), in equation (3.4), we have, for example[11]

$$
\mathcal{V}^{MAB} \mathcal{D}_M \Xi_B = \mathcal{V}^{MAB} \partial_M \Xi_B + \frac{1}{2} \mathcal{V}^{MAB} q_{MB}{}^C \Xi_C + \frac{1}{2} \mathcal{V}^M{}_{CD} p_M{}^{ABCD} \Xi_B
$$
$$
+ \frac{1}{2} \Gamma_{KM}{}^K \mathcal{V}^{MAB} \Xi_B ,
$$
$$
\mathcal{V}^{M[AB} \mathcal{D}_M \Xi^{C]} = \mathcal{V}^{M[AB} \partial_M \Xi^{C]} - \frac{1}{2} \mathcal{V}^{M[AB} q_{MD}{}^{C]} \Xi^D - \frac{2}{3} \mathcal{V}^M{}_{ED} p_M{}^{ABCD} \Xi^E
$$
$$
+ \frac{1}{2} \mathcal{V}^M{}_{DE} p_M{}^{DE[AB} \Xi^{C]} + \frac{1}{6} \Gamma_{KM}{}^K \mathcal{V}^{M[AB} \Xi^{C]} , \qquad (3.28)
$$

where the piece involving the trace of the affine connection comes from $W_{MA}{}^B$ (we have ignored the possible appearance of the internal spin connection $\omega_M{}^{\alpha\beta}$). Indeed, $U_{MA}{}^B$ does not survive in any of these combinations, as can be explicitly verified using equations (3.25). In other words, despite the non-covariance of the Cartan form, and thus of q_M and p_M, under generalized diffeomorphisms, the above combinations *are* covariant under generalized diffeomorphisms because under generalized diffeomorphisms all terms with second derivatives of Λ^M cancel out. Modulo density contributions resulting from the non-vanishing weights of the fermions (see below), the particular contractions (3.28) of covariant derivatives with the 56-bein turn out to be precisely those appearing in the supersymmetry transformation rules and fermionic field equations. More specifically, now also allowing for a non-trivial weight λ, and with fully covariant derivatives, we have

$$
\mathcal{V}^{MAB} \nabla_M \Xi_B = \mathcal{V}^{MAB} \partial_M \Xi_B + \frac{1}{2} \mathcal{V}^{MAB} q_{MB}{}^C \Xi_C + \frac{1}{2} \mathcal{V}^M{}_{CD} p_M{}^{ABCD} \Xi_B
$$
$$
+ \left(\frac{1}{2} - \frac{2}{3} \lambda(\Xi) \right) \Gamma_{KM}{}^K \mathcal{V}^{MAB} \Xi_B ,
$$
$$
\mathcal{V}^{M[AB} \nabla_M \Xi^{C]} = \mathcal{V}^{M[AB} \partial_M \Xi^{C]} - \frac{1}{2} \mathcal{V}^{M[AB} q_{MD}{}^{C]} \Xi^D - \frac{2}{3} \mathcal{V}^M{}_{ED} p_M{}^{ABCD} \Xi^E
$$
$$
+ \frac{1}{2} \mathcal{V}^M{}_{DE} p_M{}^{DE[AB} \Xi^{C]} + \left(\frac{1}{6} - \frac{2}{3} \lambda(\Xi) \right) \Gamma_{KM}{}^K \mathcal{V}^{M[AB} \Xi^{C]} . \quad (3.29)
$$

---

[10]By contrast, the connections to be derived directly from $D = 11$ supergravity in the following section do satisfy the required covariance properties, but the corresponding $U_{MA}{}^B$ can then no longer be expressed in a covariant way in terms of $\mathcal{V}$ and $\partial_M \mathcal{V}$ alone.

[11]Such projections onto the **8** and **56** of SU(8) were shown to be insensitive to the ambiguity $U_{MA}{}^B$ in ref. [18].

As we will see in the following section, and as originally shown in ref. [3], there is no term proportional to $e^{-1}\partial_M e$ (cf. (3.19)) in the supersymmetry variations of the fermions. Consequently, the density terms proportional to $\Gamma_{KM}{}^K$ must cancel. This fixes the weight of the corresponding spinors in (3.29) uniquely, and in agreement with the weight assignments given in the table. In summary, the above expressions are indeed fully covariant under both local SU(8) and generalized diffeomorphisms. We will furthermore show in the following section that these expressions do agree with the ones already obtained in ref. [3], upon imposition of the section constraint.

Similar 'miracles' occur in the bosonic sector. For instance, in the bosonic field equations, we find after some computation that the scalar contribution to the vector field equations from (2.36) can be expressed as

$$
\begin{aligned}
\mathcal{J}^\mu{}_M &= -\frac{1}{24}\,\mathcal{D}^\mu \mathcal{M}^{KL}\partial_M \mathcal{M}_{KL} + 2i\,e^{-1}\,\partial_N\left(e\,\mathcal{P}^{\mu\,ABCD}\mathcal{V}^N{}_{AB}\mathcal{V}_{MCD}\ -\ \text{c.c.}\right) \\
&= -2i\,\mathcal{V}_M{}^{AB}\,\mathcal{V}^{N\,CD}\,\nabla_N\left(g^{\mu\nu}\mathcal{P}_{\nu\,ABCD}\right) + \text{c.c.}\,,
\end{aligned}
\tag{3.30}
$$

with the undetermined connection $U_{MA}{}^B$ again dropping out from this contraction of covariant derivatives.

We summarize the structure and definitions of the various components (external and internal, SO$(1,3)$ and SU(8)) of the full spin connection as follows

$$
\begin{array}{c|c}
\boxed{\omega_\mu} & \boxed{\mathcal{Q}_\mu} \\
\Gamma_{[\mu\nu]}{}^\rho = 0 & \mathcal{D}_\mu \mathcal{V}_M{}^{AB} \equiv \mathcal{P}_\mu{}^{ABCD}\,\mathcal{V}_{MCD} \\
\hline
\boxed{\omega_M} & \boxed{\mathcal{Q}_M} \\
\mathcal{D}_M e_\mu{}^\alpha \equiv \pi_M{}^{\alpha\beta}e_{\mu\beta} & \Gamma_{MN}{}^K|_{\mathbf{912}} = 0
\end{array}
\tag{3.31}
$$

The various components of its generalized curvature contain the building blocks for the bosonic field equations (2.15), (2.23) as we shall discuss in section 3.3 below.

3.2 The supersymmetry algebra

A nice illustration of the properties of the full spin connection (3.31) is the algebra of supersymmetry transformations. In particular, the closure of the algebra on the 56-bein hinges on the vanishing of the generalized torsion (3.13) in the very same way as the closure on the vierbein requires the vanishing of the external torsion (2.25). The supersymmetry transformations of the bosonic fields (2.3) take the same structural form as in the four-dimensional theory

$$
\begin{aligned}
\delta_\epsilon e_\mu{}^\alpha &= \bar{\epsilon}^A \gamma^\alpha \psi_{\mu A} + \bar{\epsilon}_A \gamma^\alpha \psi_\mu{}^A\,, \\
\delta_\epsilon \mathcal{V}_M{}^{AB} &= 2\sqrt{2}\,\mathcal{V}_{MCD}\left(\bar{\epsilon}^{[A}\chi^{BCD]} + \frac{1}{24}\varepsilon^{ABCDEFGH}\,\bar{\epsilon}_E\chi_{FGH}\right), \\
\delta_\epsilon \mathcal{A}_\mu{}^M &= -i\sqrt{2}\,\Omega^{MN}\mathcal{V}_N{}^{AB}\left(\bar{\epsilon}^C\,\gamma_\mu\,\chi_{ABC} + 2\sqrt{2}\,\bar{\epsilon}_A\,\psi_{\mu B}\right) + \text{c.c.}\,, \\
\delta_\epsilon \mathcal{B}_{\mu\nu\,\boldsymbol{\alpha}} &= -\frac{2}{3}\sqrt{2}\,(t_{\boldsymbol{\alpha}})^{PQ}\left(\mathcal{V}_{P\,AB}\mathcal{V}_{Q\,CD}\,\bar{\epsilon}^{[A}\gamma_{\mu\nu}\,\chi^{BCD]} + 2\sqrt{2}\,\mathcal{V}_{P\,BC}\mathcal{V}_Q{}^{AC}\,\bar{\epsilon}_A\,\gamma_{[\mu}\psi_{\nu]}{}^B + \text{c.c.}\right) \\
&\quad - (t_{\boldsymbol{\alpha}})_{MN}\,\mathcal{A}_{[\mu}{}^M\,\delta_\epsilon \mathcal{A}_{\nu]}{}^N\,.
\end{aligned}
\tag{3.32}
$$

The supersymmetry variation of the constrained two-form $\mathcal{B}_{\mu\nu\,M}$ which is invisible in the four-dimensional theory can be deduced from closure of the supersymmetry algebra and yields

$$
\begin{aligned}
\delta_\epsilon \mathcal{B}_{\mu\nu\,M} &= \frac{16}{3} \mathcal{V}^{KAB} \mathcal{D}_M \mathcal{V}_{KBC}\, \bar{\epsilon}^C \gamma_{[\mu} \psi_{\nu]A} - \frac{4\sqrt{2}}{3} \mathcal{V}^P{}_{AB} \mathcal{D}_M \mathcal{V}_{PCD}\, \bar{\epsilon}^{[A} \gamma_{\mu\nu} \chi^{BCD]} \\
&\quad - 8i\left(\bar{\epsilon}^A \gamma_{[\mu} \mathcal{D}_M \psi_{\nu]A} - \mathcal{D}_M \bar{\epsilon}^A \gamma_{[\mu} \psi_{\nu]A} \right) + 2i\, e \varepsilon_{\mu\nu\rho\sigma}\, g^{\sigma\tau} \mathcal{D}_M \left(\bar{\epsilon}^A \gamma^\rho \psi_{\tau\,A} \right) + \text{c.c.} \\
&\quad + \Omega_{KL} \left(\mathcal{A}_{[\mu}{}^K \partial_M \delta_\epsilon \mathcal{A}_{\nu]}{}^L - \partial_M \mathcal{A}_{[\mu}{}^K \delta_\epsilon \mathcal{A}_{\nu]}{}^L \right) ,
\end{aligned}
\tag{3.33}
$$

as we show explicitly in appendix C. Note, that all SU(8) connections cancel in the variation (3.33), such that the external index is carried by ∂_M and this variation is indeed compatible with the constraint (2.14) on $\mathcal{B}_{\mu\nu\,M}$. In particular, the variation (3.33) consistently vanishes when $\partial_M = 0$.

In terms of the full spin connection (3.11), (3.31), introduced in the previous section, the fermionic supersymmetry transformation rules take a very compact form given by

$$
\begin{aligned}
\delta_\epsilon \psi_\mu^A &= 2\, \mathcal{D}_\mu \epsilon^A - 4i\, \mathcal{V}^{M\,AB} \widehat{\nabla}_M \left(\gamma_\mu \epsilon_B \right) , \\
\delta_\epsilon \chi^{ABC} &= -2\sqrt{2}\, \mathcal{P}_\mu{}^{ABCD} \gamma^\mu \epsilon_D - 12\sqrt{2}i\, \mathcal{V}^{M[AB} \widehat{\nabla}_M \epsilon^{C]} .
\end{aligned}
\tag{3.34}
$$

It is then straightforward to verify closure of the supersymmetry algebra. The algebra takes the same structural form as in the four-dimensional theory,

$$
\begin{aligned}
[\delta(\epsilon_1), \delta(\epsilon_2)] &= \xi^\mu \mathcal{D}_\mu + \delta_{\text{Lorentz}} \left(\Omega^{\alpha\beta} \right) + \delta_{\text{susy}}(\epsilon_3) + \delta_{\text{SU}(8)} \left(\Lambda^A{}_B \right) + \delta_{\text{gauge}} \left(\Lambda^M \right) \\
&\quad + \delta_{\text{gauge}}(\Xi_{\mu\,\boldsymbol{\alpha}}, \Xi_{\mu\,M}) + \delta_{\text{gauge}} \left(\Omega_{\mu\nu}{}^M{}_{\boldsymbol{\alpha}}, \Omega_{\mu\nu\,M}{}^N \right) .
\end{aligned}
\tag{3.35}
$$

The first term refers to a covariantized general coordinate transformation with diffeomorphism parameter

$$
\xi^\mu = 2\, \bar{\epsilon}_2{}^A \gamma^\mu \epsilon_{1\,A} + 2\, \bar{\epsilon}_{2\,A} \gamma^\mu \epsilon_1{}^A .
\tag{3.36}
$$

The last three terms refer to generalized diffeomorphisms and gauge transformations (2.19), (2.21), with parameters

$$
\begin{aligned}
\Lambda^N &= -8i\, \Omega^{NP} \left(\mathcal{V}_P{}^{AB} \bar{\epsilon}_{2\,A} \epsilon_{1\,B} - \mathcal{V}_{P\,AB} \bar{\epsilon}_2^A \epsilon_1^B \right) \equiv \mathcal{V}^{-1N}{}_{AB} \Lambda^{AB} + \mathcal{V}^{-1N\,AB} \Lambda_{AB} , \\
\Xi_{\mu\,\boldsymbol{\alpha}} &= \frac{8}{3} \left(t_{\boldsymbol{\alpha}} \right)^{PQ} \mathcal{V}_{P\,AC} \mathcal{V}_Q{}^{BC} \left(\bar{\epsilon}_2{}^A \gamma_\mu \epsilon_{1B} + \bar{\epsilon}_{2B} \gamma_\mu \epsilon_1{}^A \right) ,
\end{aligned}
\tag{3.37}
$$

again, as specified by the four-dimensional theory [48]. The remaining (constrained) gauge parameters $\Xi_{\mu\,M}$, $\Omega_{\mu\nu}{}^M{}_{\boldsymbol{\alpha}}$, $\Omega_{\mu\nu\,M}{}^N$ are not present in the four-dimensional theory and will be specified below.

Closure of the supersymmetry algebra on the vierbein $e_\mu{}^\alpha$ is confirmed by a standard calculation:

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\, e_\mu{}^\alpha &= \left(2\, \bar{\epsilon}_{2\,A} \gamma^\alpha \mathcal{D}_\mu \epsilon_1^A - 4i\, \mathcal{V}^{M\,AB} \bar{\epsilon}_{2\,A} \gamma^\alpha \widehat{\nabla}_M \left(\gamma_\mu \epsilon_{1B} \right) + \text{c.c.} \right) - (1 \leftrightarrow 2) \\
&= 2\, \mathcal{D}_\mu \left(\bar{\epsilon}_{2\,A} \gamma^\alpha \epsilon_1^A \right) - 4i\, \widehat{\nabla}_M \left(\mathcal{V}^{M\,AB} \bar{\epsilon}_{2\,A} \epsilon_{1\,B} \right) e_\mu{}^\alpha - 8i\, \mathcal{V}^{M\,AB} \bar{\epsilon}_{2\,A} \epsilon_{1\,B} \widehat{\nabla}_M e_\mu{}^\alpha \\
&\quad - 4i\, e_{\mu\,\beta} \mathcal{V}^{M\,AB} \left(\bar{\epsilon}_{2\,A} \gamma^{\alpha\beta} \widehat{\nabla}_M \epsilon_{1B} - \widehat{\nabla}_M \bar{\epsilon}_{2\,A} \gamma^{\alpha\beta} \epsilon_{1B} \right) + \text{c.c.} \\
&= \mathcal{D}_\mu \left(\xi^\nu e_\nu{}^\alpha \right) + \Lambda^M \partial_M e_\mu{}^\alpha + \frac{1}{2} \partial_M \Lambda^M\, e_\mu{}^\alpha + \tilde{\Omega}^{\alpha\beta}\, e_{\mu\,\beta} ,
\end{aligned}
\tag{3.38}
$$

with parameters from (3.36) and (3.37), and Lorentz transformation given by

$$\tilde{\Omega}^{\alpha\beta} = -8i \, \mathcal{V}^{M \, AB} \bar{\epsilon}_{2 \, A} \gamma^{\alpha\beta} \, \widehat{\nabla}_M \epsilon_{1 \, B} + \text{c.c.} \,. \tag{3.39}$$

The Λ^M terms in (3.38) reproduce the transformation of $e_\mu{}^\alpha$ under generalized diffeomorphisms as scalar densities of weight $\frac{1}{2}$, cf. table 1. Furthermore, the first term in (3.38) can be rewritten in the standard way

$$\mathcal{D}_\mu \left(\xi^\nu e_\nu{}^\alpha \right) = e_\nu{}^\alpha \, \mathcal{D}_\mu \xi^\nu + \xi^\nu \mathcal{D}_\nu e_\mu{}^\alpha + 2 \, \xi^\nu \mathcal{D}_{[\mu} e_{\nu]}{}^\alpha \,, \tag{3.40}$$

into a sum of (covariantized) diffeomorphism and additional Lorentz transformation, upon making use of the vanishing torsion condition (2.25) in the four-dimensional geometry.

An analogous calculation shows closure of the supersymmetry algebra on the 56-bein. We concentrate on the projection of the algebra-valued variation $\mathcal{V}^{-1}\delta\mathcal{V}$ onto the **70** of SU(8), since the remaining part will entirely be absorbed into a local SU(8) transformation. Using transformations (3.34), we obtain

$$\mathcal{V}^{-1 \, M \, AB} \left[\delta_{\epsilon_1}, \delta_{\epsilon_2} \right] \mathcal{V}_M{}^{CD} = \xi^\mu \, \mathcal{P}_\mu{}^{ABCD} + 6i \, \mathcal{V}^{N \, [AB} \nabla_N \Lambda^{CD]} - \frac{i}{4} \, \epsilon^{ABCDEFGH} \mathcal{V}^N{}_{EF} \nabla_N \Lambda_{GH} \,.$$

While the first term is the action of the covariantized diffeomorphism, the remaining terms can be rewritten in complete analogy to (3.40) with the vanishing torsion condition in (3.40) replaced by the corresponding condition (3.20) in the internal space. Specifically,

$$\begin{aligned}
\mathcal{V}^{-1 \, M \, AB} \left[\delta_{\epsilon_1}, \delta_{\epsilon_2} \right] \mathcal{V}_M{}^{CD} &= \xi^\mu \, \mathcal{P}_\mu{}^{ABCD} + 12 \, \mathcal{V}_P{}^{[AB} \mathcal{V}^{-1 \, CD] \, Q} \, \mathbb{P}^P{}_Q{}^N{}_{\underline{L}} \, \nabla_N \left(\mathcal{V}_K{}^{\underline{L}} \Lambda^K \right) \\
&= \xi^\mu \, \mathcal{P}_\mu{}^{ABCD} + 12 \, \mathcal{V}_P{}^{[AB} \mathcal{V}^{-1 \, CD] \, M} \, \mathbb{P}^P{}_M{}^N{}_K \, \partial_N \Lambda^K \\
&\quad + \Lambda^K \left(\nabla_K \mathcal{V}_M{}^{[AB} \right) \mathcal{V}^{-1 \, CD] \, M} \\
&= \xi^\mu \, \mathcal{P}_\mu{}^{ABCD} + \mathcal{V}^{-1 \, M \, AB} \, \delta_\Lambda \, \mathcal{V}_M{}^{CD} \,, \tag{3.41}
\end{aligned}$$

where we have used (3.20) in the second equality. The second line of (3.20) has been absorbed by the weight term associated with the non-trivial $E_{7(7)}$ weight $\frac{1}{2}$ of Λ^K.

Closure of the supersymmetry algebra on the vector and two-form fields can be verified by similar but more lengthy computations, which we relegate to appendix C. Remarkably (and necessarily for consistency), closure on the two-forms $\mathcal{B}_{\mu\nu \, M}$ reproduces not only the action of generalized diffeomorphisms (2.19) but also the shift transformation (2.21) with parameter $\Omega_{\mu\nu \, M}{}^N$ and finally their rather unconventional transformation behaviour (2.34) under external diffeomorphisms. Consistency of the algebra thus confirms the above supersymmetry transformation rules and determines the remaining gauge parameters on the right hand side of (3.35):

$$\begin{aligned}
\Xi_{\mu \, M} &= 8i \left(\bar{\epsilon}_2^A \, \gamma_\mu \mathcal{D}_M \epsilon_{1 \, A} + \mathcal{D}_M \bar{\epsilon}_{2A} \, \gamma_\mu \, \epsilon_1^A \right) - \frac{16}{3} \, \mathcal{V}^K{}_{BC} \, \mathcal{D}_M \mathcal{V}_K{}^{AB} \, \bar{\epsilon}_2^C \gamma_\mu \epsilon_{1 \, A} + \text{c.c.} \,, \\
\Omega_{\mu\nu}{}^M{}_{\boldsymbol{\alpha}} &= -\frac{32}{3} \, i (t_{\boldsymbol{\alpha}})^{PQ} \mathcal{V}_P{}^{AB} \mathcal{V}_{QCB} \mathcal{V}^M{}_{AD} \, \bar{\epsilon}_2^{(C} \gamma_{\mu\nu} \epsilon_1^{D)} + \text{c.c.} \,, \tag{3.42} \\
\Omega_{\mu\nu \, M}{}^N &= -32 \, \mathcal{V}^N{}_{AB} \bar{\epsilon}_{[2}^A \, \gamma_{[\mu} \nabla_M \left(\gamma_{\nu]} \epsilon_{1]}^B \right) - \frac{32i}{3} \, \mathcal{V}^N{}_{AC} \mathcal{V}^{P \, AB} \mathcal{D}_M \mathcal{V}_{P \, BD} \, \bar{\epsilon}_2^{(C} \gamma_{\mu\nu} \epsilon_1^{D)} + \text{c.c.} \,.
\end{aligned}$$

As required for consistency, the parameter $\Omega_{\mu\nu}{}^M{}_\alpha$ lives in the **912**, i.e. satisfies (2.22). Moreover, the parameters $\Xi_{\mu M}$ and $\Omega_{\mu\nu\,M}{}^N$ satisfy the required algebraic constraints analogous to those given in (2.14): one can verify that all SU(8) connection terms above (which would obstruct these constraints) mutually cancel.

3.3 Supersymmetric field equations

In this section we employ the formalism set up in the previous sections to spell out the fermionic field equations and sketch how under supersymmetry they transform into the bosonic field equations of the $E_{7(7)}$ EFT (2.15), (2.23). The Rarita-Schwinger equation is of the form

$$0 = (\mathcal{E}_\psi)^\mu{}_A \equiv -e^{-1}\varepsilon^{\mu\nu\rho\sigma}\gamma_\nu\mathcal{D}_\rho\psi_{\sigma\,A} - \frac{\sqrt{2}}{6}\gamma^\nu\gamma^\mu\chi^{BCD}\,\mathcal{P}_{\nu\,BCDA}$$
$$- 2\,i\,e^{-1}\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{V}^M{}_{AB}\,\gamma_\nu\widehat{\nabla}_M\left(\gamma_\rho\psi_\sigma^B\right) - i\sqrt{2}\,\mathcal{V}^{NBC}\,\widehat{\nabla}_N\left(\gamma^\mu\chi_{ABC}\right), \quad (3.43)$$

where the first two terms can be read off from the dimensionally reduced theory and the second line captures the dependence on the internal variables and can be derived from verifying the supersymmetry transformation of (3.43). It is straightforward to check that the contractions of covariant derivatives in (3.43) are such that the undetermined part from the internal SU(8) connection $\mathcal{Q}_M$ precisely drops out, cf. (3.28) and [19]. Hence, equation (3.43) is fully defined via (3.1) and (3.24).

Under supersymmetry (3.34), and upon using the first order duality equation (2.15), a somewhat lengthy computation confirms that the Rarita-Schwinger equation (3.43) transforms as

$$\delta_\epsilon(\mathcal{E}_\psi)^\mu{}_A = (\mathcal{E}_{\text{Einstein}})^{\mu\nu}\,\gamma_\nu\epsilon_A - 2\,(\mathcal{E}_{\text{vector}})^\mu{}_{AB}\,\epsilon^B\ , \quad (3.44)$$

into the Einstein and the second order vector field equations of motion obtained from varying the action (2.15). It is instructive to give a few details of this computation as it illustrates the embedding of the bosonic equations of motion into the components of the curvature associated to the various blocks of the internal and external spin connections (3.31).

Let us first collect all terms in the variation (3.44) that contain an even number of γ-matrices acting on ϵ^A, which should combine into the second-order vector field equation. These are the terms that carry precisely one internal derivative $\widehat{\nabla}_M$. After some calculation, using in particular (2.11) and (3.10), we find

$$\delta_\epsilon(\mathcal{E}_\psi)^\mu{}_A\Big|_{\text{even }\#\gamma} = 4\,i\,e^{-1}\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{V}^M{}_{AB}\,\gamma_\nu[\nabla_M,\mathcal{D}_\rho]\left(\gamma_\sigma\epsilon^B\right) + 4i\,\mathcal{V}^{MCD}\,\gamma^{\mu\nu}\nabla_M\mathcal{P}_{\nu\,ABCD}\epsilon^B$$
$$+ 4i\,\mathcal{V}^{MCD}\epsilon^B\,\nabla_M\mathcal{P}^\mu_{ABCD} + 2\,\mathcal{P}_{\nu\,ABCD}\,\mathcal{F}^{\mu\nu\,CD}\,\epsilon^B\ . \quad (3.45)$$

The commutator of covariant derivatives can be evaluated as

$$\mathcal{V}^{MAB}\,[\nabla_M,\mathcal{D}_\rho]X^C = -\frac{1}{2}\,\nabla_M\mathcal{P}_\rho{}^{ABDE}\,\mathcal{V}^M{}_{DE}\,X^C + \frac{1}{4}\,\mathcal{V}^{MAB}\widehat{R}_{M\rho}{}^{\alpha\beta}\,\gamma_{\alpha\beta}\,X^C\ , \quad (3.46)$$

where the first term describes the mixed SU(8) curvature, and the second term refers to the 'mixed' curvature of the spin connections

$$\widehat{R}_{M\rho}{}^{\alpha\beta} \equiv \partial_M\,\omega_\rho{}^{\alpha\beta} - \mathcal{D}[\omega]_\rho\,\widehat{\omega}_M{}^{\alpha\beta}\ . \quad (3.47)$$

Evaluating this curvature in particular gives rise to the components

$$\widehat{R}_{M[\nu\,\rho\sigma]} = \frac{1}{4}\,\mathcal{D}_{[\nu}\left(\mathcal{F}_{\rho\sigma]}{}^{N}\,\mathcal{M}_{NM}\right)\ ,$$

$$\widehat{R}_{M\nu}{}^{\mu\nu} = -\frac{1}{2}\,\widehat{J}^{\mu}{}_{M} + \frac{1}{4}\,e_{\alpha}{}^{\mu}e_{\beta}{}^{\nu}\,\mathcal{D}_{\nu}\left(\mathcal{M}_{MN}\mathcal{F}^{\alpha\beta N}\right)\ , \tag{3.48}$$

with the current $\widehat{J}^{\mu}{}_{M}$ from (2.36). Putting everything together, we find for the variation (3.45)

$$\delta_{\epsilon}(\mathcal{E}_{\psi})^{\mu}{}_{A}\Big|_{\text{even }\#\gamma} = -2\,\mathcal{D}_{\nu}\left(\mathcal{F}^{\nu\mu\,+}{}_{AB}\right)\epsilon^{B} - 2\,\mathcal{P}_{\nu\,ABCD}\,\mathcal{F}^{\nu\mu-\,CD}\,\epsilon^{B} + 2i\,\widehat{J}^{\mu}{}_{M}\mathcal{V}^{M}{}_{AB}\,\epsilon^{B}$$

$$+ 4i\,\mathcal{V}^{MCD}\,\nabla_{M}\left(g^{\mu\nu}\mathcal{P}_{\nu\,ABCD}\right)\epsilon^{B} \equiv -2\,(\mathcal{E}_{\text{vector}})^{\mu}{}_{AB}\,\epsilon^{B}\ , \tag{3.49}$$

reproducing the second-order vector field equation obtained from varying the action (2.23), cf. (3.30).

It remains to collect the remaining terms with odd number of γ-matrices in the variation (3.44) which should combine into the Einstein field equations. Many of these terms arrange precisely as in the dimensionally reduced theory. Here we just focus on the additional terms carrying internal derivatives ∇_{M} and combining into

$$\delta_{\epsilon}(\mathcal{E}_{\psi})^{\mu}{}_{A}\Big|_{\nabla\nabla} = 16\,\mathcal{V}^{MBC}\mathcal{V}^{N}{}_{AB}\,\nabla_{M}\left(\gamma^{\mu}\nabla_{N}\epsilon_{C}\right) + 8\,\mathcal{V}^{MBC}\mathcal{V}^{N}{}_{BC}\,\nabla_{M}\left(\gamma^{\mu}\nabla_{N}\epsilon_{A}\right)$$

$$- 8\,e^{-1}\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{V}^{M}{}_{AB}\mathcal{V}^{N\,BC}\,\gamma_{\nu}\nabla_{M}\left(\gamma_{\rho}\nabla_{N}\left(\gamma_{\sigma}\epsilon_{C}\right)\right). \tag{3.50}$$

Collecting all $\nabla_{M}\nabla_{N}\epsilon_{A}$ terms in this variation gives rise to

$$2\left(8\,\mathcal{V}^{[M}{}_{AC}\mathcal{V}^{N]\,CB} + i\,\Omega^{MN}\delta_{A}^{B}\right)\gamma^{\mu}\left[\nabla_{M},\nabla_{N}\right]\epsilon_{B}$$

$$+ 4\left(16\,\mathcal{V}^{(M}{}_{AC}\mathcal{V}^{N)\,CB} + \mathcal{M}^{MN}\delta_{A}^{B}\right)\gamma^{\mu}\nabla_{M}\nabla_{N}\epsilon_{B}\ , \tag{3.51}$$

showing that all double derivatives $\partial_{M}\partial_{N}\epsilon_{A}$ vanish due to the section condition (B.5). We evaluate the full expression (3.51) using the fact that the following combination of covariant derivatives [19]

$$\left(6\,\mathcal{V}^{M}{}_{AC}\mathcal{V}^{N\,CB} + 2\,\mathcal{V}^{N}{}_{AC}\mathcal{V}^{M\,CB} + \mathcal{V}^{MCD}\,\mathcal{V}^{N}{}_{CD}\,\delta_{A}^{B}\right)\nabla_{M}\nabla_{N}\,\epsilon_{B}$$

$$\equiv \left(\frac{1}{16}\mathcal{R}\,\delta_{A}^{B} - \frac{1}{4}\mathcal{V}^{M}{}_{AC}\mathcal{V}^{N\,CB}\gamma^{\nu\rho}g^{\sigma\tau}\nabla_{M}g_{\nu\sigma}\nabla_{N}g_{\rho\tau}\right)\epsilon_{B},\ \ (3.52)$$

gives rise to the definition of the curvature $\mathcal{R}$

$$\mathcal{R} \equiv -4\,\mathcal{V}^{M}{}_{[AB}\mathcal{V}^{N}{}_{CD]}\left(\partial_{M}p_{N}{}^{ABCD} - \frac{1}{2}\,q_{M\,E}{}^{A}\,p_{N}{}^{EBCD}\right) - \frac{1}{6}\,\mathcal{M}^{MN}\,p_{M}{}^{ABCD}p_{N\,ABCD}$$

$$- 4\,\mathcal{V}^{M}{}_{AB}\mathcal{V}^{N\,CD}\,p_{M}{}^{ABEF}p_{N\,CDEF} - \frac{3}{2}\,\mathcal{M}^{MN}\,e^{-1}\partial_{M}\partial_{N}e + \frac{3}{4}\,\mathcal{M}^{MN}\,e^{-2}\partial_{M}e\,\partial_{N}e$$

$$- 6\,\mathcal{V}^{M}{}_{AB}\mathcal{V}^{N}{}_{CD}\,e^{-1}\partial_{M}e\,p_{N}{}^{ABCD}\ , \tag{3.53}$$

which is invariant under generalized internal diffeomorphisms. Comparing the explicit expression for the curvature to the scalar potential V (2.32), we see that they are related by

$$e\,V = -e\,\mathcal{R} - \frac{1}{4}\,e\,\mathcal{M}^{MN}\nabla_{M}g^{\mu\nu}\nabla_{N}g_{\mu\nu} + \text{total derivative}\ , \tag{3.54}$$

in a form analogous to the O(d,d) DFT case discussed in ref. [54]. The operator on the left hand side of (3.52) is such that the double derivatives $\partial_M \partial_N \epsilon_A$ as well as the single derivatives $\partial_M \epsilon_A$ disappear by virtue of the section constraint, and also all ambiguities drop out [19].

The remaining terms in expression (3.50) can be written as

$$4 \left(16\, \mathcal{V}^{(M}{}_{AC} \mathcal{V}^{N)\,CB} + \mathcal{M}^{MN} \delta_A^B \right) \nabla_M \gamma^\mu \nabla_N \epsilon_B - 8\, \mathcal{V}^M{}_{AC} \mathcal{V}^{N\,CB} \gamma^{\mu\nu\rho} \nabla_M \gamma_\nu \nabla_N \gamma_\rho \epsilon_B$$
$$+ 16\, \mathcal{V}^M{}_{AC} \mathcal{V}^{N\,CB} \gamma^{\mu\nu} \nabla_M \nabla_N \gamma_\nu \epsilon_B \,, \tag{3.55}$$

showing that $\partial_M \epsilon$ terms are also absent in these terms. These terms, which are independent of the ambiguities, can be further evaluated to give

$$-\frac{1}{2}\, \partial_M g^{\mu\nu} \partial_N \mathcal{M}^{MN} \gamma_\nu \epsilon_A - \frac{1}{4} e^{-1} \partial_M e\, \partial_N \mathcal{M}^{MN} \gamma^\mu \epsilon_A + 2\, \mathcal{V}^M{}_{AC} \mathcal{V}^{N\,CB} \gamma^{\mu\nu\rho} g^{\sigma\tau} \partial_M g_{\nu\sigma} \partial_N g_{\rho\tau} \epsilon_B$$
$$+\frac{1}{8}\, \mathcal{M}^{MN} \gamma^\mu \left(\partial_M g^{\rho\sigma}\, \partial_N g_{\rho\sigma} - 2\, e^{-1} \partial_M \partial_N e + e^{-2} \partial_M e\, \partial_N e \right) \epsilon_A$$
$$+\frac{1}{2}\, \mathcal{M}^{MN} g^{\mu\sigma} g^{\nu\rho} \left(\partial_M \partial_N g_{\rho\sigma} - g^{\tau\eta}\, \partial_M g_{\rho\tau} \partial_N g_{\sigma\eta} + e^{-1} \partial_M e\, \partial_N g_{\rho\sigma} \right) \gamma_\nu \epsilon_A \,. \tag{3.56}$$

Together, using equation (3.52) and the expression above, the variation (3.50) reduces to

$$\frac{1}{2}\, \mathcal{R}\, \gamma^\mu \epsilon_A - \frac{1}{2}\, \partial_M g^{\mu\nu} \partial_N \mathcal{M}^{MN} \gamma_\nu \epsilon_A - \frac{1}{4} e^{-1} \partial_M e\, \partial_N \mathcal{M}^{MN} \gamma^\mu \epsilon_A \tag{3.57}$$
$$+\frac{1}{8}\, \mathcal{M}^{MN} \gamma^\mu \left(\partial_M g^{\rho\sigma}\, \partial_N g_{\rho\sigma} - 2\, e^{-1} \partial_M \partial_N e + e^{-2} \partial_M e\, \partial_N e \right) \epsilon_A$$
$$+\frac{1}{2}\, \mathcal{M}^{MN} g^{\mu\sigma} g^{\nu\rho} \left(\partial_M \partial_N g_{\rho\sigma} - g^{\tau\eta}\, \partial_M g_{\rho\tau} \partial_N g_{\sigma\eta} + e^{-1} \partial_M e\, \partial_N g_{\rho\sigma} \right) \gamma_\nu \epsilon_A \equiv \mathcal{T}^{\mu\nu}\, \gamma_\nu \epsilon_A \,,$$

and gives part of the scalar matter contributions to the Einstein field equations, cf. (3.44). Indeed, ignoring the first term in the expression above, the remaining terms in $\mathcal{T}^{\mu\nu}$ precisely come from a variation of

$$\frac{1}{4}\, e\, \mathcal{M}^{MN} \nabla_M g^{\mu\nu} \nabla_N g_{\mu\nu} \tag{3.58}$$

with respect to the metric $g_{\mu\nu}$. Together with (3.54), and noting that the variation

$$e\, \delta \mathcal{R} = -\frac{3}{2}\, \partial_M \left(e\, \mathcal{M}^{MN} \partial_N \left(e^{-1} \delta e \right) \right)$$

is a total derivative, we find that the variation of the potential (2.32) with respect to the external metric is given by

$$\delta(-e\, V) = \mathcal{R}\, \delta e + \frac{1}{4}\, \delta \left(e\, \mathcal{M}^{MN} \nabla_M g^{\mu\nu} \nabla_N g_{\mu\nu} \right) = \mathcal{T}^{\mu\nu}\, \delta g_{\mu\nu} \,, \tag{3.59}$$

and precisely coincides with (3.57). In summary, the supersymmetry variation of the gravitino equation (3.43) correctly reproduces the full Einstein equations from (2.23).

Finally, a similar discussion can be repeated for the field equation of the spin-1/2 fermions χ^{ABC}, which under supersymmetry transforms into vector and scalar field equations from (2.23). Rather than going through the details of this computation, we present

the final result in the compact form of the full fermionic completion of the bosonic Lagrangian (2.23), given by

$$\mathcal{L}_{\text{ferm}} = -\varepsilon^{\mu\nu\rho\sigma}\,\bar{\psi}_\mu{}^A\gamma_\nu\mathcal{D}_\rho\psi_{\sigma A} - \frac{1}{6}e\,\bar{\chi}^{ABC}\gamma^\mu\mathcal{D}_\mu\chi_{ABC} - \frac{1}{3}\sqrt{2}\,e\,\bar{\chi}^{ABC}\gamma^\nu\gamma^\mu\psi_\nu^D\,\mathcal{P}_{\mu\,ABCD}$$
$$- 2i\,\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{V}^M{}_{AB}\,\bar{\psi}_\mu^A\gamma_\nu\,\widehat{\nabla}_M\left(\gamma_\rho\psi_\sigma^B\right) - 2\sqrt{2}i\,e\mathcal{V}^{NAB}\,\bar{\psi}_\mu^C\,\widehat{\nabla}_N\left(\gamma^\mu\chi_{ABC}\right)$$
$$- \frac{i}{18}\,e\,\epsilon_{ABCDEFGH}\mathcal{V}^{MAB}\,\bar{\chi}^{CDE}\,\widehat{\nabla}_M\chi^{FGH} + \text{c.c.}\,, \tag{3.60}$$

up to terms quartic in the fermions. The latter can be directly lifted from the dimensionally reduced theory [55], for dimensional reasons they are insensitive to ∇_M corrections. We have thus obtained the complete supersymmetric extension of the bosonic $\mathrm{E}_{7(7)}$ EFT (2.15), (2.23). In the rest of this paper, we shall discuss in detail how this theory after the explicit solution (2.2a) of the section constraint relates to the reformulation [3, 7, 8] of the full (untruncated) $D = 11$ supergravity.

4 Exceptional geometry from $D = 11$ supergravity

Independently of the construction of a field theory based on a particular duality group in ref. [15] and other references alluded to earlier, and described in detail in the two foregoing sections, there is the reciprocal ('ground up') approach of reformulating the higher-dimensional theory in such a way that makes the role of duality groups directly manifest in higher dimensions. This approach goes back to the early work of refs. [3, 4], and has been taken up again recently in a series of papers [6–8], which have succeeded in providing an on-shell equivalent generalized geometric reformulation of the $D = 11$ theory in which the bosonic degrees of freedom are assembled into $\mathrm{E}_{7(7)}$ objects and where the supersymmetry transformations of the bosons assume a manifestly $\mathrm{E}_{7(7)}\times \mathrm{SU}(8)$ covariant form.[12] This reformulation is achieved by starting from the known supersymmetry variations of $D = 11$ supergravity, and then rewriting the theory in such a way that the $\mathrm{E}_{7(7)}$ and $\mathrm{SU}(8)$ structures become manifest (following the work of Cremmer and Julia [1], where this strategy was applied first in the restricted context of the dimensionally reduced theory). One main advantage of this procedure is that the on-shell equivalence of the reformulation with the original $D = 11$ supergravity is guaranteed at each step of the construction; the detailed comparison between the $\mathrm{E}_{7(7)}$-covariant expressions and those originating from $D = 11$ supergravity is also an essential prerequisite for deriving non-linear Kaluza-Klein ansätze for all fields.[13] In this section, we briefly review these developments, and show how they tie up with the results of the two foregoing sections, eventually establishing the equivalence of the two approaches. As we will see, the full identification is subtle, not only because it involves various redefinitions, but also because the ambiguities exhibited in the foregoing sections play a key role in establishing the precise relation.

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[12]There exist partial results along similar lines for the case of the $\mathrm{E}_{8(8)}$ duality group [4, 5, 7]; the full bosonic $\mathrm{E}_{8(8)}$-covariant EFT is constructed in ref. [56].

[13]While the section constraint does admit a solution corresponding to IIB theory (with only six internal dimensions), the full consistency of the $\mathrm{AdS}_5 \times S^5$ reduction remains to be established; this would in fact require a detailed analysis of supersymmetric $\mathrm{E}_{6(6)}$ theory similar to the one presented in this section.

4.1 56-bein and GVP from eleven dimensions

The first step is to identify an $E_{7(7)}$ 56-bein $\mathcal{V}_{MAB}$[14] *in eleven dimensions* with the bosonic degrees of freedom that reduce to scalars under a reduction of the $D = 11$ theory to four dimensions; this 56-bein will be eventually identified with the one introduced in the previous sections. Decomposing the **56** of $E_{7(7)}$ under its SL(8) and GL(7) subgroups

$$\mathbf{56} \to \mathbf{28} \oplus \overline{\mathbf{28}} \; \to \; \mathbf{7} \oplus \mathbf{21} \oplus \overline{\mathbf{21}} \oplus \overline{\mathbf{7}}, \tag{4.1}$$

we have the following decomposition of the 56-bein

$$\mathcal{V}_{M\,AB} \equiv \left(\mathcal{V}^m{}_{AB}, \mathcal{V}_{mn\,AB}, \mathcal{V}^{mn}{}_{AB}, \mathcal{V}_{m\,AB} \right), \tag{4.2}$$

where we will often employ the simplifying notation $\mathcal{V}^m{}_{AB} \equiv \mathcal{V}^{m8}{}_{AB} = -\mathcal{V}^{8m}{}_{AB}$, when considering the embedding of GL(7) into SL(8). The main task is then to directly express this 56-bein in terms of components of eleven-dimensional fields along the seven-dimensional directions, *viz.*

$$\mathcal{V}_{M\,AB} \equiv \mathcal{V}_{M\,AB}\left(e_m{}^a, A_{mnp}, A_{mnpqrs} \right), \tag{4.3}$$

where $e_m{}^a$ is the siebenbein, A_{mnp} are the internal components of the three-form field, and A_{mnpqrs} the internal components of the dual six-form field. In other words, the 56-bein whose existence in eleven dimensions was postulated on the basis of symmetry considerations in the previous section is here given concretely in terms of certain components of the $D = 11$ fields and their duals. The calculation [7] yields the explicit formulae

$$\mathcal{V}^m{}_{AB} = \frac{1}{8} \Delta^{-1/2} \Gamma^m_{AB}, \tag{4.4}$$

$$\mathcal{V}_{mn\,AB} = \frac{1}{8} \Delta^{-1/2} \left(\Gamma_{mn\,AB} + 6\sqrt{2} A_{mnp} \Gamma^p_{AB} \right), \tag{4.5}$$

$$\mathcal{V}^{mn}{}_{AB} = \frac{1}{4 \cdot 5!} \eta^{mnp_1 \cdots p_5} \Delta^{-1/2} \Bigg[\Gamma_{p_1 \cdots p_5\,AB} + 60\sqrt{2} A_{p_1 p_2 p_3} \Gamma_{p_4 p_5\,AB}$$
$$- 6!\sqrt{2} \left(A_{q p_1 \cdots p_5} - \frac{\sqrt{2}}{4} A_{q p_1 p_2} A_{p_3 p_4 p_5} \right) \Gamma^q_{AB} \Bigg], \tag{4.6}$$

$$\mathcal{V}_{m\,AB} = \frac{1}{4 \cdot 7!} \eta^{p_1 \cdots p_7} \Delta^{-1/2} \Bigg[(\Gamma_{p_1 \cdots p_7} \Gamma_m)_{AB} + 126\sqrt{2}\, A_{m p_1 p_2} \Gamma_{p_3 \cdots p_7\,AB} \tag{4.7}$$
$$+ 3\sqrt{2} \times 7! \left(A_{m p_1 \cdots p_5} + \frac{\sqrt{2}}{4} A_{m p_1 p_2} A_{p_3 p_4 p_5} \right) \Gamma_{p_6 p_7\,AB}$$
$$+ \frac{9!}{2} \left(A_{m p_1 \cdots p_5} + \frac{\sqrt{2}}{12} A_{m p_1 p_2} A_{p_3 p_4 p_5} \right) A_{p_6 p_7 q} \Gamma^q_{AB} \Bigg],$$

where Δ is the determinant of the siebenbein $e_m{}^a$. In particular, it can be explicitly verified that the 56-bein defined by the components above satisfies the identities (2.9), and thus is

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[14]The notations and conventions used here are slightly different to those used in [3, 7].

indeed an element of the most general duality group $\mathrm{Sp}(56,\mathbb{R})$. To show that that it is more specifically an $\mathrm{E}_{7(7)}$-valued matrix one either verifies (2.6) directly, or invokes eqs. (14), (17) and (18) of ref. [8] where it is shown that $\mathcal{V}$ transforms as a generalized $\mathrm{E}_{7(7)}$ covector. From the point of view of refs. [3, 7], this matrix corresponds to an element of the coset space $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ in a specific gauge (where the local $\mathrm{SU}(8)$ is taken to act in the obvious way on the indices $A, B, \ldots$), such that after a local $\mathrm{SU}(8)$ rotation the direct identification as given above is lost. Note also the appearance of components of the six-form potential in the expressions, as a consequence of whose presence the identification of the EFT formulated in the previous section and the $D = 11$ supergravity can only be achieved at the level of the equations of motion (which, of course, does not preclude the existence of suitable actions for either formulation).

In the same manner, one identifies a **56**-plet of $\mathrm{E}_{7(7)}$ vector fields $\mathcal{A}_\mu{}^M$ that incorporate the degrees of freedom corresponding to vectors under a reduction to four dimensions, combining the 28 electric and the 28 magnetic vectors of maximal supergravity into a single representation that now live in eleven dimensions. As before, the components in a GL(7) decomposition of the **56** of $\mathrm{E}_{7(7)}$ can be explicitly written in terms of eleven-dimensional fields

$$\mathcal{A}_\mu{}^m = \frac{1}{2} B_\mu{}^m, \qquad\qquad \mathcal{A}_{\mu\, mn} = 3\sqrt{2}\left(A_{\mu mn} - B_\mu{}^p A_{pmn}\right), \tag{4.8}$$

$$\mathcal{A}_\mu{}^{mn} = 6\sqrt{2}\,\eta^{mnp_1\cdots p_5}\left(A_{\mu p_1\cdots p_5} - B_\mu{}^q A_{qp_1\cdots p_5} - \frac{\sqrt{2}}{4}\left(A_{\mu p_1 p_2} - B_\mu{}^q A_{qp_1 p_2}\right)A_{p_3 p_4 p_5}\right),$$

$$\mathcal{A}_{\mu\, m} = 36\,\eta^{n_1\ldots n_7}\Bigg(A_{\mu n_1\ldots n_7, m} + (3\tilde{c} - 1)\left(A_{\mu n_1\ldots n_5} - B_\mu{}^p A_{pn_1\ldots n_5}\right)A_{n_6 n_7 m}$$

$$+ \tilde{c} A_{n_1\ldots n_6}\left(A_{\mu n_7 m} - B_\mu{}^p A_{pn_7 m}\right) + \frac{\sqrt{2}}{12}\left(A_{\mu n_1 n_2} - B_\mu{}^p A_{pn_1 n_2}\right)A_{n_3 n_4 n_5}A_{n_6 n_7 m}\Bigg).$$

The components of the six-form potential appear again in the expression above. However, in the $\mathcal{A}_{\mu\, m}$ component, there appears a new field $A_{\mu n_1\ldots n_7, m}$ (as well as an undetermined constant $\tilde{c}$), related to the *dual graviphoton*.

These $\mathrm{E}_{7(7)}$ objects are found by analysing the $D = 11$ supersymmetry transformations, which in the $\mathrm{SU}(8)$ invariant reformulation were found to take the precise form [3, 6–8]

$$\delta e_\mu{}^\alpha = \bar{\epsilon}^A \gamma^\alpha \psi_{\mu A} + \bar{\epsilon}_A \gamma^\alpha \psi_\mu{}^A \,,$$

$$\delta \mathcal{V}_M{}^{AB} = 2\sqrt{2}\,\mathcal{V}_{MCD}\left(\bar{\epsilon}^{[A}\chi^{BCD]} + \frac{1}{24}\varepsilon^{ABCDEFGH}\,\bar{\epsilon}_D \chi_{EFG}\right),$$

$$\delta \mathcal{A}_\mu{}^M = -i\,\sqrt{2}\,\Omega^{MN}\mathcal{V}_N{}^{AB}\left(\bar{\epsilon}^C \gamma_\mu\,\chi_{ABC} + 2\sqrt{2}\,\bar{\epsilon}_A\,\psi_{\mu B}\right) + \text{c.c.}\,, \tag{4.9}$$

where a compensating $\mathrm{SU}(8)$ rotation has been discarded in the variation $\delta\mathcal{V}_M{}^{AB}$, as explained in refs. [3, 7]. Strictly speaking, the supersymmetry transformations of the last seven components of the vectors cannot be derived from $D = 11$ supergravity, due to the absence of a non-linear formulation of dual gravity, but are here obtained by '$\mathrm{E}_{7(7)}$-covariantization'. The supersymmetry transformations of the last seven components of the vector field instead determine the supersymmetry transformation of the new field $A_{\mu n_1\ldots n_7, m}$ as discussed in ref. [7]. While $A_{\mu n_1\ldots n_7, m}$, which is introduced to complete the **56** of $\mathrm{E}_{7(7)}$

for the vectors, is clearly related to dual gravity degrees of freedom from a four-dimensional tensor hierarchy point of view, its direct relation to the eleven-dimensional fields cannot be determined. This is in stark contrast to the six-form potential that is related to the three-form potential via an explicit duality relation. Nevertheless, our ignorance regarding this field is compensated by the fact that it does not appear in the GVPs (see below).

While the agreement in the supersymmetry variations of the boson fields as derived above and the exceptional field theory approach of the foregoing sections is thus manifest, the agreement in the fermionic variations is much more subtle. This is because the latter depend on the connections, and a detailed comparison would thus require an analysis of the connection (3.22) in terms of the $D = 11$ fields. Of course, ignoring the ambiguity (3.25) for the moment, we could simply try to work out the expressions (3.23) and (3.24) by substituting the explicit formulae (4.4)–(4.7). However, this would lead to extremely cumbersome expressions (but see appendix D for a simplified calculation), whose relation with the ones given below would be far from obvious. We will therefore proceed differently by starting 'from the other end'. The supersymmetry transformations of the fermions were already derived in [3], *viz.*

$$\delta\psi_\mu^A = 2\left(\partial_\mu - B_\mu{}^m\partial_m - \frac{1}{4}\partial_m B_\mu{}^m\right)\epsilon^A + \frac{1}{2}\omega_\mu{}^{\alpha\beta}\gamma_{\alpha\beta}\epsilon^A + \mathcal{Q}_\mu{}^A{}_B\epsilon^B$$

$$+ 2\mathcal{G}_{\alpha\beta}^{-AB}\gamma^{\alpha\beta}\gamma_\mu\epsilon_B - \frac{1}{4}e^{m\,AB}e_{\nu\,\beta}\partial_m e_\rho{}^\beta\gamma^{\nu\rho}\gamma_\mu\epsilon_B$$

$$+ e^{m\,AB}\partial_m\left(\gamma_\mu\epsilon_B\right) + \frac{1}{2}e^{m\,AB}Q'_{m\,B}{}^C\gamma_\mu\epsilon_C - \frac{1}{2}e^m{}_{CD}P'^{m\,ABCD}\gamma_\mu\epsilon_B,$$

$$\delta\chi^{ABC} = -2\sqrt{2}\mathcal{P}_\mu{}^{ABCD}\gamma^\mu\epsilon_D + 6\sqrt{2}\mathcal{G}_{\alpha\beta}^{-[AB|}\gamma^{\alpha\beta}\epsilon^{|C]} - \frac{3}{2\sqrt{2}}e_{\mu\beta}\partial_m e_\nu{}^\beta e^{m[AB}\gamma^{\mu\nu}\epsilon^{C]}$$

$$+ 3\sqrt{2}e^{m[AB}\partial_m\epsilon^{C]} - \frac{3\sqrt{2}}{2}e^{m[AB}Q'_{m\,D}{}^{C]}\epsilon^D - \frac{3\sqrt{2}}{2}e^m{}_{DE}P'^{m\,DE[AB}\epsilon^{C]}$$

$$- 2\sqrt{2}e^m{}_{DE}P'^{m\,ABCD}\epsilon^E, \tag{4.10}$$

where

$$e^m{}_{AB} = e^{mAB} = i\Delta^{-1/2}\Gamma^m_{AB} \tag{4.11}$$

is just part of the 56-bein $\mathcal{V}^M{}_{AB}$ given above in (4.4), and

$$\mathcal{G}_{\alpha\beta AB} \equiv -\frac{i}{8}\Delta^{1/2}e_{[\alpha}{}^\mu e_{\beta]}{}^\nu(\partial_\mu - B_\mu{}^m\partial_m)B_\nu{}^n\Gamma_{nAB} + \frac{\sqrt{2}}{32}i\Delta^{-1/2}F_{\alpha\beta mn}\Gamma^{mn}_{AB} \tag{4.12}$$

comprises the contribution from the spin one degrees of freedom. The link of the particular expressions involving the Kaluza-Klein vectors $B_\mu{}^m$ with those of the previous two sections is easily seen by noting that

$$\partial_\mu - B_\mu{}^m\partial_m \equiv \partial_\mu - \mathcal{A}_\mu{}^M\partial_M \tag{4.13}$$

upon taking the canonical solution of the section constraint. Furthermore, the direct comparison with the fermion transformations of $D = 11$ supergravity yields the expressions

$$Q'_{m\,A}{}^B = \frac{1}{2}\,q_{m\,ab}\,\Gamma^{ab}_{AB} + \frac{\sqrt{2}}{48}\,F_{mabc}\,\Gamma^{abc}_{AB} + \frac{\sqrt{2}}{14\cdot 6!}F_{mabcdef}\,\Gamma^{abcdef}_{AB}, \tag{4.14}$$

$$P'_{mABCD} = -\frac{3}{4}\,p_{m\,ab}\,\Gamma^a_{[AB}\Gamma^b_{CD]} + \frac{\sqrt{2}}{32}\,F_{mabc}\Gamma^a_{[AB}\Gamma^{bc}_{CD]} - \frac{\sqrt{2}}{56\cdot 5!}F_{mabcdef}\,\Gamma^a_{[AB}\Gamma^{bcdef}_{CD]},$$

where

$$q_{m\,ab} \equiv e_{[a}{}^{n}\partial_{|m}e_{n|b]}\,, \quad p_{m\,ab} \equiv e_{(a}{}^{n}\partial_{|m}e_{n|b)} \tag{4.15}$$

are the components of the GL(7) Cartan form, with analogous notation as in the previous section. These objects transform properly under local SU(8): $Q'_{m\,A}{}^{B}$ is the SU(8) connection, while $P'_{m\,ABCD}$ transforms covariantly in the complex self-dual **35** representation of SU(8). However, as written, these connections are not fully covariant under internal diffeomorphisms, because $q_{m\,ab}$ and $p_{m\,ab}$ do *not* transform as proper vectors under internal diffeomorphisms. For this reason we will switch to a slightly different choice below, see (4.17) and (4.18), which satisfies all covariance requirements.

The other important feature of the reformulation [3, 7, 8] is the so-called generalized vielbein postulate (GVP). When evaluated on the different components of $\mathcal{V}^{M}{}_{AB}$ this consists of certain differential equations satisfied by the 56-bein which are analogous to the usual vielbein postulate in differential geometry. The GVPs are equations satisfied by the 56-bein and in the approach of [3, 7, 8] they can be checked explicitly on a component by component basis, while they appear as genuine postulates in the approach of the previous section. Moreover, the direct comparison with $D = 11$ supergravity allows for a direct understanding of four-dimensional maximal gauged theories and the embedding tensor [8, 42, 45] that defines them from a higher-dimensional perspective as well as providing generalized geometric structures that can be interpreted as generalized connections and used to construct a generalized curvature tensor.

The external GVP, which gives the dependence of the 56-bein on the four-dimensional coordinates is given by equation (2.27) (see refs. [7, 8]), where the explicit expressions for $\mathcal{Q}_{\mu}$ and $\mathcal{P}_{\mu}$ in terms of the $D = 11$ fields were already given in ref. [3]. Here we concentrate on the internal part of the GVP which was given in [7, 8] in the form

$$\partial_{m}\mathcal{V}_{M\,AB} - \mathbf{\Gamma}_{mM}{}^{N}\mathcal{V}_{N\,AB} + Q^{C}_{m[A}\mathcal{V}_{M\,B]C} = P_{m\,ABCD}\mathcal{V}_{M}{}^{CD}, \tag{4.16}$$

where[15]

$$Q_{m\,A}{}^{B} = \frac{1}{2}\,\omega_{m\,ab}\,\Gamma^{ab}_{AB} + \frac{\sqrt{2}}{48}\,F_{mabc}\,\Gamma^{abc}_{AB} + \frac{\sqrt{2}}{14\cdot 6!}F_{mabcdef}\,\Gamma^{abcdef}_{AB}, \tag{4.17}$$

$$P_{m\,ABCD} = \frac{\sqrt{2}}{32}\,F_{mabc}\Gamma^{a}_{[AB}\Gamma^{bc}_{CD]} - \frac{\sqrt{2}}{56\cdot 5!}F_{mabcdef}\,\Gamma^{a}_{[AB}\Gamma^{bcdef}_{CD]}. \tag{4.18}$$

Notice that $Q'_{m\,A}{}^{B}$ and $P'_{m\,ABCD}$ defined in equations (4.14) and $Q_{m\,A}{}^{B}$ and $P_{m\,ABCD}$ defined above, (4.18), differ in their components relating to the siebenbein since we have replaced $q_{m\,ab}$ by the spin connection $\omega_{m\,ab}$ and $p_{m\,ab}$ by zero. As explained in ref. [8] this change is required if the connections are to satisfy all the requisite covariance properties, as is indeed the case for (4.17) and (4.18). However, there appears to be no way to reproduce these covariant expressions in terms of the 56-bein $\mathcal{V}$ and its internal derivatives $\partial_{m}\mathcal{V}$ without 'breaking up' the matrix $\mathcal{V}$, and this is one of the main difficulties in establishing agreement between the above expressions and the ones obtained in the previous section.

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[15]Note that in this paper our conventions are such that Cartan's first structure equation takes the form $T^{a} = \mathrm{d}e^{a} + \omega^{a}{}_{b} \wedge e^{b}$.

Fortunately, the apparent discrepancy turns out to reside in the **1280** part of the SU(8) connection (see (3.25)) and the hook ambiguity described in section 4.3 and will thus drop out in all relevant expressions.

The internal GVP as given in (3.12) and (4.16) (and also (4.24), see below) differ in two respects. First of all, and prior to imposing the section constraint, (3.12) involves all 56 components, whereas (4.16) involves only the seven internal dimensions with index $M = m$. The second distinctive feature is the appearance of a non-zero term proportional to P_m on the right-hand side of the GVP. As we will explain in more detail below, this term corresponds to a *generalized non-metricity*.[16] We will show below how to absorb this non-metricity, and thereby bring the GVP into the same form as (3.12). Finally, the connection coefficients $\mathbf{\Gamma}_m$ can appear in the supersymmetry transformations of the fermions only via their traces, because the fermions, while transforming as densities, are otherwise only sensitive to the local SU(8).

Given the coefficients $Q_{mA}{}^B$ and $P_m{}^{ABCD}$ we can solve for the affine connection coefficients $\mathbf{\Gamma}_{mM}{}^N$ in terms of the fields of $D = 11$ supergravity; we use boldface letters here to indicate that these coefficients are *different from the ones identified in* (3.27) of the previous section. With (4.17) and (4.18), $\mathbf{\Gamma}_{mM}{}^N$ takes values in the Lie algebra of $E_{7(7)}$

$$\mathbf{\Gamma}_{mM}{}^N = \mathbf{\Gamma}_m{}^{\boldsymbol{\alpha}}(t_{\boldsymbol{\alpha}})_M{}^N. \tag{4.19}$$

The comparison with $D = 11$ supergravity allows to solve for the components of $\mathbf{\Gamma}_m{}^{\boldsymbol{\alpha}}$ directly in terms of $D = 11$ fields; the non-vanishing components are

$$(\mathbf{\Gamma}_m)_n{}^p \equiv -\Gamma_{mn}^p + \frac{1}{4}\delta_n^p\Gamma_{mq}^q, \qquad (\mathbf{\Gamma}_m)_8{}^8 = -\frac{3}{4}\Gamma_{mn}^n,$$

$$(\mathbf{\Gamma}_m)_8{}^n = \sqrt{2}\eta^{np_1\cdots p_6}\Xi_{m|p_1\cdots p_6}, \qquad (\mathbf{\Gamma}_m)^{n_1\cdots n_4} = \frac{1}{\sqrt{2}}\eta^{n_1\cdots n_4 p_1 p_2 p_3}\Xi_{m|p_1p_2p_3}, \tag{4.20}$$

where $\Gamma_{mn}{}^p$ is the usual Christoffel symbol, and where

$$\Xi_{p|mnq} \equiv D_p A_{mnq} - \frac{1}{4!}F_{pmnq}, \tag{4.21}$$

$$\Xi_{p|m_1\cdots m_6} \equiv D_p A_{m_1\cdots m_6} + \frac{\sqrt{2}}{48}F_{p[m_1m_2m_3}A_{m_4m_5m_6]}$$
$$- \frac{\sqrt{2}}{2}\left(D_p A_{[m_1m_2m_3} - \frac{1}{4!}F_{p[m_1m_2m_3}\right)A_{m_4m_5m_6]} - \frac{1}{7!}F_{pm_1\ldots m_6}. \tag{4.22}$$

One notices that these objects, like the usual Christoffel symbol, indeed transform with second derivatives of the tensor gauge parameters, as would be expected for a generalized affine connection (see ref. [8] for details). Another noteworthy feature is that they vanish under full antisymmetrization:

$$\Xi_{[p|mnq]} = 0, \qquad \Xi_{[p|m_1\ldots m_6]} = 0. \tag{4.23}$$

Therefore, they correspond to hook-type Young tableaux diagrams, and thus encapsulate the non-gauge invariant part of the derivatives of the three-form and the six-form fields.

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[16]We would like to thank Malcolm Perry for pointing this out to us.

In terms of SL(7) these Ξ's correspond to the **210** and **48** representations, respectively; when further decomposed into SO(7) representations, these will become the **21** $\oplus$ **189** and **21** $\oplus$ **27** of SO(7), all of which appear in the **1280** of SU(8). We will also see below that the irreducibility property (4.23) is crucial for the absence of torsion in the sense of generalized geometry.

As given above, the connection coefficients $Q_{m\,A}{}^B$, $P_m{}^{ABCD}$ and $\mathbf{\Gamma}_{mN}{}^P$ have all the desired transformation properties with respect to local SU(8) and generalized diffeomorphisms, as can be verified explicitly from their definitions (see ref. [8]). That is, $Q_{m\,A}{}^B$ transforms as an SU(8) connection (as is obvious from the way the local SU(8) has been introduced in ref. [3] as a Stückelberg-type symmetry), while $P_m{}^{ABCD}$ transforms covariantly under SU(8) transformations. Both $Q_{m\,A}{}^B$ and $P_m{}^{ABCD}$ transform as generalized vectors under generalized diffeomorphisms (for the natural truncation of generalized Lie derivatives to vectors with only seven vector indices). Furthermore, the generalized affine connection $\mathbf{\Gamma}$ is invariant under SU(8) transformations, and transforms as a generalized connection (with a second derivative of the gauge parameters).

A distinctive feature of the internal GVP as given here, to be contrasted with the one given in (3.12), is that, at this point, the connections have non-zero components only along the seven internal dimensions, but vanish otherwise — just like the partial derivative ∂_M after imposition of the section constraint. Nevertheless, we can formally write the internal GVP as

$$\partial_M \mathcal{V}_{N\,AB} - \mathbf{\Gamma}_{MN}{}^P \mathcal{V}_{P\,AB} + Q^C_{M[A} \mathcal{V}_{N\,B]C} = P_{M\,ABCD} \mathcal{V}_N{}^{CD} \tag{4.24}$$

by trivially promoting the GL(7) index m to part of a **56** of E$_{7(7)}$. Hence, taking

$$\partial_M = \begin{cases} \partial_m & \text{if } M = m8, \\ 0 & \text{otherwise} \end{cases} \tag{4.25}$$

and identifying the m components of the connection coefficients with those that appear in equation (4.16), with all other components vanishing, gives back (4.16). In this form the internal GVP can be compared to equation (3.12), *with the proviso* that the section constraint also applies to the connections. However, in view of the derivation given in the foregoing section, a natural question that arises at this point is why all other components of the connection coefficients should vanish. Would it not be more "natural" from a generalized geometric point of view if the connection coefficients had non-trivial components in the other directions of the **56** representation, as has been assumed in section 3 and, for example, ref. [18]? Indeed, we will see below that the introduction of non-vanishing connection components along the other directions will actually be required if we want to recast the supersymmetry variations of the fermions in order to achieve full agreement with the formalism of the preceding section.

We now proceed to reformulate these structures in order to exhibit their precise relationship to those constructed in section 3. However, given that vanishing torsion is taken to be an important ingredient for defining generalized connections in section 3, we will first consider the generalized torsion associated to the generalized affine connection $\mathbf{\Gamma}$.

4.2 Generalized torsion

In ref. [8], the generalized torsion $\mathbf{T}_{MN}{}^P$ is defined as follows

$$[\nabla_M, \nabla_N]S = \mathbf{T}_{MN}{}^P \partial_P S \tag{4.26}$$

for some scalar S and where ∇_M is defined using the connection $\mathbf{\Gamma}_{MN}{}^P$. The generalized torsion as defined above vanishes [8]. An alternative (and *a priori* independent) definition of the torsion is given in equation (3.13) of section 3, which leads to the formula (3.14). While the above definition of torsion and that defined in (3.13) are equivalent in usual differential geometry, this is not the case in generalized geometry. Here we will evaluate the generalized torsion (3.14) explicitly in terms of the connection coefficients $\mathbf{\Gamma}_{mN}{}^P$ given in ref. [8] and above. A simple component-wise calculation using the components of $\mathbf{\Gamma}_{mN}{}^P$ identified above now shows that the generalized torsion does indeed vanish. For example, consider

$$\mathcal{T}_{m8\,n8}{}^{p8} = \mathbf{\Gamma}_{m8\,n8}{}^{p8} - 48\,\mathbb{P}^{p8}{}_{n8}{}^{q8}{}_{r8}\,\mathbf{\Gamma}_{q8\,m8}{}^{r8} + 16\,\mathbb{P}^{p8}{}_{n8}{}^{q8}{}_{m8}\mathbf{\Gamma}_{r8\,q8}{}^{r8}. \tag{4.27}$$

Using the fact that

$$\mathbb{P}^{p8}{}_{r8}{}^{q8}{}_{s8} = \frac{1}{96}\left(2\delta_s^p\delta_r^q + \delta_r^p\delta_s^q\right), \tag{4.28}$$

the above equation reduces to

$$\mathcal{T}_{m8\,n8}{}^{p8} = 2\mathbf{\Gamma}_{[m\,n]}{}^p - \frac{2}{3}\mathbf{\Gamma}_{r\,[m}{}^r\delta_{n]}^p. \tag{4.29}$$

However, the right hand side of the above equation vanishes by substituting the relevant components of $\mathbf{\Gamma}$ from (4.20). Hence,

$$\mathcal{T}_{m8\,n8}{}^{p8} = 0. \tag{4.30}$$

Next consider, for example,

$$\mathcal{T}_{m8\,pq\,r8} = \mathbf{\Gamma}_{m8\,pq\,r8} - 24\mathbb{P}_{pq\,r8}{}^{st\,u8}\mathbf{\Gamma}_{u8\,m8\,st}. \tag{4.31}$$

Using the fact that

$$\mathbb{P}_{pq\,r8}{}^{st\,u8} = \frac{1}{8}\delta_{pqr}^{stu}, \tag{4.32}$$

the above equation reduces to

$$\mathcal{T}_{m8\,pq\,r8} = 4\mathbf{\Gamma}_{[m\,pq\,r]}. \tag{4.33}$$

However,

$$\mathbf{\Gamma}_{[m\,pq\,r]} \sim \Xi_{[m|pqr]} = 0 \tag{4.34}$$

by equation (4.23). Finally, consider the following components

$$\mathcal{T}_{m8\,n8}{}^{pq} = \mathbf{\Gamma}_{m8\,n8}{}^{pq} - 24\mathbb{P}^{pq}{}_{n8}{}^{r8}{}_{st}\mathbf{\Gamma}_{r8\,m8}{}^{st}. \tag{4.35}$$

Using the fact that

$$\mathbb{P}^{pq}{}_{n8}{}^{r8}{}_{st} = -\frac{1}{12}\delta_{n[s}^{pq}\delta_{t]}^r, \tag{4.36}$$

we obtain

$$
\begin{aligned}
\mathcal{T}_{m8\,n8}{}^{pq} &= \mathbf{\Gamma}_{m8\,n8}{}^{pq} + 2\mathbf{\Gamma}_{r8\,m8}{}^{r[p}\delta_n^{q]} \\
&= 3\sqrt{2}\eta^{pqt_1\ldots t_5}\left(\Xi_{m|nt_1\ldots t_5} - \Xi_{n|mt_1\ldots t_5} + 5\Xi_{t_1|mnt_2\ldots t_5}\right) \\
&= 21\sqrt{2}\eta^{pqt_1\ldots t_5}\Xi_{[m|nt_1\ldots t_5]} = 0\,,
\end{aligned}
\tag{4.37}
$$

where we have used the expression for $\mathbf{\Gamma}_{m8\,n8}{}^{pq}$ in the second equality and equation (4.23) in the final equality. All other components of the generalized torsion can be similarly shown to be zero. It should be emphasized that the fact that the full antisymmetrization of the Ξ quantities is zero, equation (4.23), is crucial for this argument.

In summary, the generalized torsion, as defined by equation (3.14) is zero

$$
\mathcal{T}_{MN}{}^P = 0\,.
\tag{4.38}
$$

Let us emphasize again the remarkable feature that the vanishing of the generalized torsion, as originally defined on the basis of very different considerations based on generalized geometry, here follows from the direct comparison with $D = 11$ supergravity.

4.3 Hook ambiguity

As we have already mentioned, the supersymmetry transformations are insensitive to the generalized affine connection, modulo density contributions involving the trace of the affine connection, because the fermions transform only under the chiral SU(8). With the connections as originally given in ref. [3], or equivalently from equations (4.10), the supersymmetry variations of the eight gravitini and the 56 dilatini contain the following combinations of Q'_m and P'_m

$$
\begin{aligned}
\delta\psi_\mu^A &\propto \ldots + \left(e^{m\,AC}Q'_{m\,C}{}^B - e^m{}_{CD}P'_m{}^{ABCD}\right)\gamma_\mu\varepsilon_B, \\
\delta\chi^{ABC} &\propto \ldots + \left(3\,e^{m\,[AB}Q'_m{}^{C]}{}_D + 3\,e^m{}_{EF}P'_m{}^{EF[AB}\delta_D^{C]} + 4\,P'_m{}^{ABCE}e^m{}_{ED}\right)\varepsilon^D.
\end{aligned}
\tag{4.39}
$$

An important property of the expressions appearing here on the right hand side, is that they are actually insensitive to certain modifications of the connections. We first recognize that these are exactly the same combinations that appear in the two first equations of (3.28). Secondly, the expressions on the right hand side of (4.39) admit a non-trivial kernel which is found by looking for solutions of

$$
\begin{aligned}
0 &= \Gamma_{AC}^m\,\delta Q'_{m\,C}{}^B - \Gamma_{CD}^m\,\delta P'_m{}^{ABCD}, \\
0 &= 3\,\Gamma^{m[AB}\,\delta Q'_m{}^{C]}{}_D - 3\,\Gamma_{EF}^m\,\delta P'_m{}^{EF[AB}\delta_D^{C]} - 4\,\delta P'_m{}^{ABCE}\,\Gamma_{ED}^m.
\end{aligned}
\tag{4.40}
$$

Let us proceed with the following ansätze

$$
\begin{aligned}
\delta Q'_{m\,A}{}^B &= X^{(3)}_{m|ab}\,\Gamma_{AB}^{ab} + X^{(4)}_{m|abc}\,\Gamma_{AB}^{abc} + X^{(7)}_{m|abcdef}\,\Gamma_{AB}^{abcdef}, \\
\delta P'_m{}^{ABCD} &= Y^{(3)}_{m|ab}\,\Gamma_{[AB}^a\Gamma_{CD]}^b + Y^{(4)}_{m|abc}\,\Gamma_{[AB}^{[a}\Gamma_{CD]}^{bc]} + Y^{(7)}_{m|abcdef}\,\Gamma_{[AB}^{[a}\Gamma_{CD]}^{bcdef]},
\end{aligned}
\tag{4.41}
$$

where the slash | simply indicates that no *a priori* symmetry conditions are imposed on the X's and Y's other than the obvious ones (to wit, anti-symmetry in $[ab]$, $[abc]$ and $[abcdef]$,

respectively). For the form field contributions it was already shown in ref. [41] that the GVP remains valid if

$$Y^{(4)}_{m|abc} = \frac{3}{2} \, X^{(4)}_{m|abc} \,, \quad Y^{(7)}_{m|abcdef} = -\frac{3}{2} \, X^{(7)}_{m|abcdef} \tag{4.42}$$

with no further restrictions on the X's and Y's. Notice that both $X^{(4)}$ and $X^{(7)}$ have *two* irreducible parts: besides the fully antisymmetric pieces appearing in (4.14) there are the hook diagram contributions. Furthermore, it was shown in ref. [41] that $X^{(4)}, Y^{(4)}$ and $X^{(7)}, Y^{(7)}$ are in the kernel of the supersymmetry variations (4.40) provided that

$$X^{(4)}_{[m|abc]} = 0 \,, \quad X^{(7)}_{[m|abcdef]} = 0 \,. \tag{4.43}$$

That is, the fully antisymmetric parts (the four-form and seven-form field strengths) are determined, but the hook diagram contributions can be chosen freely, as they drop out in the supersymmetry variations of the fermions in (4.39). Note that $\Xi_{m|npq}$ and $\Xi_{m|npqrst}$ that appear in the generalized affine connection in (4.21) and (4.22) are precisely of the hook-type, hence providing a *geometrical* explanation for the ambiguities found in [41].

As for the remaining SO(7) part $X^{(3)}_{m|ab}$, which was not considered in ref. [41], the first expression in equations (4.40) reduces to

$$X^{(3)}_{a|bc} \, \Gamma^{abc}_{AB} + \left(2 X^{(3)}_{a|ab} + \frac{4}{3} Y^{(3)}_{a|ab} \right) \Gamma^{b}_{AB} \, - \, Y^{(3)}_{a|bb} \Gamma^{a}_{AB} = 0 \,. \tag{4.44}$$

Whence we read off the condition

$$Y^{(3)}_{m|ab} = -\frac{3}{2} \, X^{(3)}_{m|ab}. \tag{4.45}$$

With this identification the second line in (4.40) becomes

$$X^{(3)}_{a|bc} \left(2 \Gamma^{[a}_{[AB} \Gamma^{b]c}_{C]D} - \Gamma^{c}_{[AB} \Gamma^{ab}_{C]D} \right) - X^{(3)}_{a|bb} \Gamma^{a}_{[AB} \delta_{C]D} = 0 \,. \tag{4.46}$$

We now see that all terms in (4.44) and (4.46) except the last ones involving $X^{(3)}_{a|bb}$ cancel, provided we demand that

$$X^{(3)}_{[a|b]c} = 0 \,. \tag{4.47}$$

To interpret the remaining term let us check the difference between the expressions for the connection coefficients given in ref. [3], equation (4.14), and in ref. [8], equations (4.17) and (4.18). These connections are fully covariant under internal diffeomorphisms. The difference is thus

$$X^{(3)}_{m|ab} = \frac{1}{2} \left(e^{n}{}_{b} \partial_{m} e_{n\,a} + \omega_{m\,ab} \right) = \frac{1}{2} \, e^{n}{}_{a} e_{p\,b} \Gamma^{p}_{mn} \,, \tag{4.48}$$

where we have used the usual vielbein postulate satisfied by the siebenbein and Γ^{p}_{mn} is the usual Christoffel symbol. Hence (4.47) is indeed satisfied for a torsion-free affine connection. The only extra term in the supersymmetry variations then comes from the 'leftover' term in (4.46) which is just a density term proportional to $\Gamma_{km}{}^{k}$, which is required here because the supersymmetry parameter is a density. This is the same term that was obtained

above with the connections (4.14) just from $Q'_{m\,ab}$ and $P'_{m\,ab}$ alone. We thus see that the switch from (4.14) to (4.17) and (4.18) reintroduces the density term proportional to $\Gamma_{km}{}^k$ that was absent in ref. [3]. In other words, even the density term which is there with the correct weight if the GVP is formulated with the usual affine connection as in ref. [8] can be absorbed into a redefinition of $Q_{m\,A}{}^B$ and $P_m{}^{ABCD}$, as they were originally given in ref. [3]. In fact we are free to also choose any *interpolating* solution where the coefficient of the density term changes, as part of it is absorbed into $Q_{m\,A}{}^B$, while the other into $P_m{}^{ABCD}$.

Let us also point out how the apparent discrepancy between (3.19), where $\Gamma_{km}{}^k \propto e^{-1}\partial_m e$ (with e the usual vierbein determinant), and the above result, where $\Gamma_{km}{}^k \propto \Delta^{-1}\partial_m\Delta$, is resolved: while in (3.29) the contribution proportional to $\Gamma_{KM}{}^K$ cancels with the weight assignments given there, the contribution proportional to $\Gamma_{km}{}^k$ here can be eliminated by shifting back to the non-covariant connections Q'_m and P'_m, and only then the two pictures can be made to agree. Otherwise the two sets of connections (both of which are consistent) simply reflect the unavoidable ambiguities identified in section 3.1.

Let us emphasize once again that the connections given in equations (4.17) and (4.18) satisfy *all required covariance properties* of generalized or exceptional geometry provided we break up $\mathcal{V}$ by choosing the specific 'frame' as derived from D=11 supergravity. First of all, the covariance under local SU(8) follows by the same arguments as in ref. [3]: as given, these expressions correspond to objects in a special SU(8) gauge (namely the one that accords with the D=11 theory), such that $Q_{m\,A}{}^B$ transforms as a proper SU(8) connection (for the SO(7) subgroup this is anyhow obvious). Secondly, $P_m{}^{ABCD}$ transforms covariantly when we apply an SU(8) rotation that moves us out of the given gauge. Furthermore, these objects are also covariant under generalized diffeomorphisms: for the 7-dimensional internal diffeomorphisms this is manifestly true, while the fact that they do not transform at all under the remaining generalized diffeomorphisms with parameters ξ_{mn}, ξ^{mn} and ξ_m is consistent with the formulae (17) and (18) of ref. [8] because $Q_M = P_M = 0$ for $M \neq m$. Of course, these statements apply only to the specific 'frame' as derived from $D = 11$ supergravity, that we have adopted here, where the connections have non-vanishing coefficients only along the seven internal dimensions. However, it is straightforward to see that the manipulations we are now going to perform on these specific connections to bring them in line with the constructions described in the two foregoing sections are themselves fully covariant and therefore *preserve these covariance properties*.

Let us point out once more that the existence of covariant connections is possible here because we have given the connections explicitly in terms of $D = 11$ fields. It is not possible to achieve if all quantities are to be expressed only in terms of the generalized vielbein $\mathcal{V}$ and its derivatives in an $E_{7(7)}$-covariant manner, as we already saw in the foregoing section (and will explain again for a simplified example in appendix D).

4.4 Non-metricity and redefinition of the generalized connection

In order to understand how the appearance of P_M on the right-hand side of the GVP (4.24) can be reconciled with the absence in the corresponding relation given previously in equation (3.12), it is useful to recall that similar ambiguities arise in standard differen-

tial geometry. While the vielbein postulate is usually quoted as

$$\partial_m e_n{}^a + \omega_m{}^a{}_b e_n{}^b - \Gamma_{mn}^p e_p{}^a = 0 \tag{4.49}$$

with Γ_{mn}^p the Christoffel symbols, there is a more general expression

$$\partial_m e_n{}^a + \omega_m{}^a{}_b e_n{}^b - \Gamma_{mn}^p e_p{}^a = T_{mn}{}^p e_p{}^a + P_m{}^a{}_b e_n{}^b, \tag{4.50}$$

where Γ_{mn}^p is no longer given by the Christoffel symbols, $T_{mn}{}^p = T_{[mn]}{}^p$ is referred to as the torsion and $P_{m\,ab} = P_{m\,(ab)}$ is referred to as the non-metricity, as it 'measures' the failure of the metric to be covariantly constant (see for example ref. [57]). Notice that there is quite a lot of freedom in the definition of the various objects in the equation above. For example, the antisymmetric part of the affine connection $\Gamma_{[mn]}^p$ can be absorbed into a redefinition of T_{mn}^p so that $\Gamma_{mn}^p = \Gamma_{(mn)}^p$. Similarly, the non-metricity can be absorbed into a redefinition of the affine connection and the torsion:

$$\begin{aligned}
\Gamma_{mn}^p &\longrightarrow \Gamma_{mn}^p - P_{(m}{}^c{}_{|d|}\, e_{n)}{}^d e^p{}_c\,, \\
T_{mn}{}^p &\longrightarrow T_{mn}{}^p - P_{[m}{}^c{}_{|d|}\, e_{n]}{}^d e^p{}_c\,.
\end{aligned} \tag{4.51}$$

Furthermore, the fully anti-symmetric part of the torsion can be absorbed into a redefinition of the spin connection

$$\omega_{m\,ab} \longrightarrow \omega_{m\,ab} - T_{mnp}\, e^n{}_a\, e^p{}_b\,. \tag{4.52}$$

Hence, in differential geometry there is a great deal of freedom in how one defines various structures such as non-metricity, torsion and the affine and spin connections.

In complete analogy with this discussion, connection coefficient P_M can be absorbed into a redefinition of $\mathbf{\Gamma}_M$ in the internal GVP, equation (4.24):

$$\mathbf{\Gamma}_{MN}{}^P \longrightarrow \tilde{\mathbf{\Gamma}}_{MN}{}^P = \mathbf{\Gamma}_{MN}{}^P + i\left(\mathcal{V}_N{}^{AB} P_{M\,ABCD} \mathcal{V}^{PCD} - \mathcal{V}_{N\,AB} P_M{}^{ABCD} \mathcal{V}^P{}_{CD}\right) \tag{4.53}$$

so that the internal GVP becomes

$$\partial_M \mathcal{V}_{N\,AB} - \tilde{\mathbf{\Gamma}}_{MN}{}^P \mathcal{V}_{P\,AB} + Q_M{}^C{}_{[A} \mathcal{V}_{N\,B]C} = 0\,. \tag{4.54}$$

We note that this shift only changes the affine connection, but *does not affect the* SU(8) *connection* $Q_{M\,A}{}^B$. The GVP is now of the form of (3.12) in section 3, but the connections are still different. In particular, the $Q_{M\,A}{}^B$ and $\tilde{\mathbf{\Gamma}}_{MN}{}^P$ are still non-zero only for the first seven components given by equations (4.17). However, by removing the non-metricity in the affine connection we have reintroduced torsion in $\tilde{\mathbf{\Gamma}}$ where there was none before, in analogy to ordinary differential geometry. Therefore, in order to recover a torsion-free affine connection we follow the same procedure as in section 3.1, and accordingly redefine the affine connection once more, as follows:

$$Q_{M\,A}{}^B \longrightarrow \widehat{\mathcal{Q}}_{M\,A}{}^B \equiv Q_{M\,A}{}^B + \mathbb{Q}_{M\,A}{}^B, \tag{4.55}$$

$$\tilde{\mathbf{\Gamma}}_{MN}{}^P \longrightarrow \widehat{\mathbf{\Gamma}}_{MN}{}^P \equiv \tilde{\mathbf{\Gamma}}_{MN}{}^P + i\left(\mathcal{V}^P{}_{AB}\mathbb{Q}_{M\,A}{}^C \mathcal{V}_N{}^{BC} - \mathcal{V}^{PAB}\mathbb{Q}_{M\,A}{}^C \mathcal{V}_{NBC}\right), \tag{4.56}$$

where, modulo the remaining ambiguity $U_{MA}{}^B$, the modification $\mathbb{Q}_M$ is now chosen to obtain precisely the connection $\mathcal{Q}$ in section 3, namely

$$\mathbb{Q}_{MA}{}^B = R_{MA}{}^B + U_{MA}{}^B + W_{MA}{}^B + \frac{2i}{3}\,\mathbf{\Gamma}_{MN}{}^P \mathcal{V}_{PAC} \mathcal{V}^{NCB}. \qquad (4.57)$$

With the redefinitions (4.56), we have now brought the GVP into the standard form

$$\partial_M \mathcal{V}_{NAB} - \widehat{\mathbf{\Gamma}}_{MN}{}^P \mathcal{V}_{PAB} + \mathcal{Q}_{M[A}^C \mathcal{V}_{NB]C} = 0, \qquad (4.58)$$

with the following properties:

- the affine connection $\widehat{\mathbf{\Gamma}}_{MN}{}^P$ is torsion-free, an SU(8) singlet and transforms properly under generalized diffeomorphisms.

- The SU(8) connection $\mathcal{Q}_{MA}{}^B$ transforms as a connection under SU(8), and as a generalized vector under generalized diffeomorphisms.

- The connections have non-vanishing components for *all* 56 components, and this is necessary for the supersymmetry variations of the fermions to be expressible in terms of the SU(8) connection $\mathcal{Q}_{MA}{}^B$ alone (see the previous section).

- The remaining differences between the above connections and the ones obtained in the previous section are all contained in the hook-type ambiguity.

Modulo the ambiguity, these connections are now equivalent to the connections defined in section 3, namely $\widehat{\mathbf{\Gamma}} \cong \Gamma$. We should point out that, with the formulae at hand, we could in principle proceed to work out explicit expressions for $\mathcal{Q}_{MA}{}^B$ and $\Gamma_{MN}{}^P$ in terms of the $D = 11$ fields. However, after the redefinitions these expressions will be very complicated, and by themselves not very illuminating.

The trace of the affine connection $\mathbf{\Gamma}$ is given by the determinant of the siebenbein [8],

$$\mathbf{\Gamma}_{KM}{}^K = \frac{3}{2} \partial_M \log \Delta \,. \qquad (4.59)$$

The connection used to construct the exceptional geometry in section 3 is required to be compatible with the vierbein density, (3.18), which implies equation (3.19). This condition can be satisfied by the torsion-free connection by choosing W in equation (4.57) appropriately. In particular the trace of $\mathbf{\Gamma}$ drops out of $\Gamma_{KM}{}^K$:

$$\begin{aligned}
\Gamma_{KM}{}^K &= \tilde{\mathbf{\Gamma}}_{KM}{}^K + i\Big(\mathcal{V}^K{}_{AB}\mathbb{Q}_K{}^A{}_C \mathcal{V}_M{}^{BC} - \mathcal{V}^{KAB}\mathbb{Q}_{KA}{}^C \mathcal{V}_{MBC}\Big) \\
&= i\left(\mathcal{V}^K{}_{AB}W_K{}^A{}_C \mathcal{V}_M{}^{BC} - \mathcal{V}^{KAB}W_{KA}{}^C \mathcal{V}_{MBC}\right).
\end{aligned}$$

The W given in equation (3.24) ensures that the affine connection Γ satisfies the condition (3.19). Note that the part of the fermion supersymmetry transformations given by the internal connection are independent of the vierbein determinant. This remains so despite the contribution from W, which is cancelled by the density contributions in the covariant derivative ∇_M of weighted tensors in the supersymmetry transformations.

4.5 Connections and fermion supersymmetry transformations

In section 3.2, we give the fermion supersymmetry transformations (3.34) in terms of the torsion-free connection constructed in section 3.1. Solving the section condition to obtain the $D = 11$ supergravity, the fermion supersymmetry transformation should yield those of the SU(8) invariant reformulation [3], (4.10). Using the definition of the covariant derivative (3.3) and equations (3.11) and (3.22), transformations (3.34) become

$$\delta_\epsilon \psi_\mu^A = 2\,\mathcal{D}_\mu \epsilon^A + \frac{1}{4}\mathcal{F}_{\rho\sigma}^{-\,AB}\gamma^{\rho\sigma}\gamma_\mu\epsilon_B + i\,e_{\nu\,\beta}\partial_M e_\rho{}^\beta \mathcal{V}^{M\,AB}\gamma^{\nu\rho}\gamma_\mu\epsilon_B - 4\,i\,\mathcal{V}^{M\,AB}\partial_M\left(\gamma_\mu\epsilon_B\right)$$
$$- 2\,i\,\mathcal{V}^{M\,AB}q_{M\,B}{}^C\gamma_\mu\epsilon_C - 2\,i\,\mathcal{V}^M{}_{CD}p_M{}^{ABCD}\gamma_\mu\epsilon_B\,,$$

$$\delta_\epsilon\chi^{ABC} = -2\sqrt{2}\,\mathcal{P}_\mu{}^{ABCD}\gamma^\mu\epsilon_D + \frac{3\sqrt{2}}{4}\mathcal{F}_{\mu\nu}^{-\,[AB}\gamma^{\mu\nu}\epsilon^{C]} + 3\sqrt{2}\,i\,e_{\mu\,\beta}\partial_M e_\nu{}^\beta\mathcal{V}^{M\,[AB}\gamma^{\mu\nu}\epsilon^{C]}$$
$$- 12\sqrt{2}i\,\mathcal{V}^{M[AB}\,\partial_M\epsilon^{C]} + 6\sqrt{2}\,i\,\mathcal{V}^{M[AB}q_{M\,D}{}^{C]}\epsilon^D - 8\sqrt{2}\,i\,\mathcal{V}^M{}_{DE}p_M{}^{ABCD}\epsilon^E$$
$$- 6\sqrt{2}\,i\,\mathcal{V}^M{}_{DE}p_M{}^{DE[AB}\epsilon^{C]}\,. \tag{4.60}$$

In this form, the supersymmetry transformations (3.34) reduce to the following expressions upon use of the canonical solution of the section condition

$$\delta\psi_\mu^A = 2\left(\partial_\mu - B_\mu{}^m\partial_m - \frac{1}{4}\partial_m B_\mu{}^m\right)\epsilon^A + \frac{1}{2}\omega_\mu{}^{\alpha\beta}\gamma_{\alpha\beta}\epsilon^A + \mathcal{Q}_\mu{}^A{}_B\epsilon^B$$
$$+ \frac{1}{4}\mathcal{F}_{\alpha\beta}^{-\,AB}\gamma^{\alpha\beta}\gamma_\mu\epsilon_B - \frac{1}{4}e^{m\,AB}e_{\nu\,\beta}\partial_m e_\rho{}^\beta\gamma^{\nu\rho}\gamma_\mu\epsilon_B$$
$$+ e^{m\,AB}\partial_m\left(\gamma_\mu\epsilon_B\right) + \frac{1}{2}e^{m\,AB}q_{m\,B}{}^C\gamma_\mu\epsilon_C - \frac{1}{2}e^m{}_{CD}\,p_m{}^{ABCD}\gamma_\mu\epsilon_B,$$

$$\delta\chi^{ABC} = -2\sqrt{2}\mathcal{P}_\mu{}^{ABCD}\gamma^\mu\epsilon_D + \frac{3\sqrt{2}}{4}\mathcal{F}_{\alpha\beta}^{-\,[AB|}\gamma^{\alpha\beta}\epsilon^{|C]} - \frac{3\sqrt{2}}{4}e_{\mu\,\beta}\partial_m e_\nu{}^\beta e^{m[AB}\gamma^{\mu\nu}\epsilon^{C]}$$
$$+ 3\sqrt{2}e^{m[AB}\partial_m\epsilon^{C]} - \frac{3\sqrt{2}}{2}e^{m[AB}q_{m\,D}{}^{C]}\epsilon^D - \frac{3\sqrt{2}}{2}e^m{}_{DE}\,p_m{}^{DE[AB}\epsilon^{C]}$$
$$- 2\sqrt{2}e^m{}_{DE}\,p_m{}^{ABCD}\epsilon^E. \tag{4.61}$$

Comparing the supersymmetry transformations above that come from the supersymmetric EFT with the canonical solution of the section condition with those of the $D = 11$ theory as written in ref. [3], transformation (4.10), we find that they are identical upon identifying $\frac{1}{8}\mathcal{F}_{\alpha\beta}{}^{AB}$ with $\mathcal{G}_{\alpha\beta}{}^{AB}$ and Q', P' with q, p, respectively.

First, let us consider the relation between $\mathcal{F}_{\alpha\beta}{}^{AB}$ and $\mathcal{G}_{\alpha\beta}{}^{AB}$. Note that $\mathcal{F}_{\alpha\beta}{}^{AB}$ satisfies a twisted self-duality condition, which means that on-shell

$$\mathcal{F}_{\alpha\beta}^{-\,AB} = \mathcal{F}_{\alpha\beta}{}^{AB}.$$

The $\mathcal{G}_{\alpha\beta}{}^{AB}$, however, does not satisfy a twisted self-duality condition and in order to modify it so that it does, we need to add to it the Hodge dual of the field strengths, *viz.*

$$\mathcal{G}_{\alpha\beta AB} \equiv -\frac{i}{16}\Delta^{1/2}e_{[\alpha}{}^\mu e_{\beta]}{}^\nu(\partial_\mu - B_\mu{}^m\partial_m)B_\nu{}^n\Gamma_{nAB} + \frac{\sqrt{2}}{64}i\Delta^{-1/2}F_{\alpha\beta mn}\Gamma_{AB}^{mn}$$
$$+ \frac{\sqrt{2}}{64\cdot 5!}\Delta^{-1/2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta m_1...m_5}\Gamma_{m_1...m_5\,AB} + i\Delta^{1/2}\epsilon_{\alpha\beta\gamma\delta}X^{\gamma\delta|n}\Gamma_{nAB}, \tag{4.62}$$

where $X_{\alpha\beta|m}$ would correspond to the field strength of the field dual to $B_\mu{}^m$. However, since the first term in the expression above is not exact, $B_\mu{}^m$ cannot be dualized in the usual way. This is why the new field $B_{\mu\nu M}$ is necessary in the definition of $\mathcal{F}_{\mu\nu}{}^M$, (2.13), schematically "eating up" the non-exact terms to allow dualization.

Regarding the relation between Q', P' and q, p: as explained in section 4.3, the Q' and P' are related to Q and P by the usual Christoffel symbol associated with the siebenbein. Moreover, the Q and P are related to q and p by the generalized affine connection $\mathbf{\Gamma}$,

$$Q_{m\,A}{}^B = q_{m\,A}{}^B - \frac{2i}{3}\,\mathbf{\Gamma}_{mN}{}^P \mathcal{V}_{P\,AC} \mathcal{V}^{N\,CB},$$
$$P_{mABCD} = p_{mABCD} + i\,\mathbf{\Gamma}_{mN}{}^P \mathcal{V}_{PAB} \mathcal{V}^N{}_{CD}. \tag{4.63}$$

In both cases, the redefinitions correspond to hook-type redefinitions to which the supersymmetry transformations are insensitive, as explained in section 4.3. Therefore, at the level of the supersymmetry transformations, the two sets of connection coefficients are equivalent.

The fermion supersymmetry transformations of a truncation of the $D = 11$ theory have been studied in ref. [19], where they are also given in terms of a generalized SU(8) connection constructed in ref. [18]. In this paper, we use a connection that allows us to express the fermion supersymmetry transformations covariantly in terms of the 56-bein, rather than its components. This is done by using some of the components in the **1280** representation, to which supersymmetry transformations are insensitive to [18] (see also section 3). Therefore, the connection $\mathcal{Q} - U$ still contains terms, not expressible in terms of the 56-bein and its derivatives, that are in the **1280** representation. These terms are precisely the difference between the $\mathcal{Q} - U$ and the unambiguous part of the connection of ref. [18]. In practice, an explicit expression of this difference is rather complicated.

Acknowledgments

A Notations and conventions

The index notation used in this paper is as follows:

- $\mu, \nu, \ldots$ and $\alpha, \beta, \ldots$ denote $D = 4$ spacetime and tangent space indices, respectively.

- $m, n, \ldots$ and $a, b, \ldots$ denote $D = 7$ spacetime and tangent space indices, respectively.

- $M, N, \ldots$ label the fundamental (**56**) of $E_{7(7)}$.

- $\boldsymbol{\alpha}$ labels the adjoint (**133**) of $E_{7(7)}$.

- $A, B, \ldots$ denote SU(8) indices.

Furthermore, the following notations are used for covariant derivatives:

- $D_\mu = \partial_\mu - \mathbb{L}_{\mathcal{A}_\mu}$ denotes the $E_{7(7)}$-covariant derivative.

- $\mathcal{D}_\mu = D_\mu + \omega_\mu{}^\alpha{}_\beta + \mathcal{Q}_\mu{}^A{}_B$ denotes the $E_{7(7)}$-covariant derivative that is also covariant with respect to the local SO(1,3) and SU(8) symmetries.

- $\nabla_\mu = \mathcal{D}_\mu + \Gamma^\rho_{\mu\nu}$ is the fully covariant derivative.

Analogously,

- $\mathcal{D}_M = \partial_M + \omega_M{}^\alpha{}_\beta + \mathcal{Q}_M{}^A{}_B$ denotes derivative that is also covariant with respect to the local SO(1,3) and SU(8) symmetries.

- $\nabla_M = \mathcal{D}_M + \Gamma^P_{MN}$ is the fully covariant derivative,

and $\widehat{\mathcal{D}}_M$ and $\widehat{\nabla}_M$ are defined with the modified spin connection $\widehat{\omega}_M$.

B Useful identities

In this appendix we collect a handful of useful relations and identities in order to deal with the $E_{7(7)}$ projectors (2.17) and the section constraint (2.1) upon contractions with the 56-bein. Let us first note the projector identity

$$
\mathbb{P}^M{}_N{}^P{}_Q \, \mathcal{V}_{PAB} \, \mathcal{V}^{QCD} = \frac{1}{3} \, \mathcal{V}_{NE[A} \, \mathcal{V}^{ME[C} \, \delta_{B]}{}^{D]} + \frac{1}{3} \, \mathcal{V}^M{}_{E[A} \, \mathcal{V}_N{}^{E[C} \, \delta_{B]}{}^{D]}
$$
$$
- \frac{1}{24} \left(\mathcal{V}_{NEF} \, \mathcal{V}^{MEF} + \mathcal{V}^M{}_{EF} \, \mathcal{V}_N{}^{EF} \right) \delta^{CD}_{AB} \, . \tag{B.1}
$$

As a consistency check, we may calculate the trace of this relation

$$
\mathbb{P}^M{}_N{}^P{}_Q \, \mathcal{V}_{PAB} \, \mathcal{V}^{QCB} = \frac{1}{2} \, \mathcal{V}_{NAB} \, \mathcal{V}^{MCB} + \frac{1}{2} \, \mathcal{V}^M{}_{AB} \, \mathcal{V}_N{}^{CB}
$$
$$
- \frac{1}{16} \left(\mathcal{V}_{NEF} \, \mathcal{V}^{MEF} + \mathcal{V}^M{}_{EF} \, \mathcal{V}_N{}^{EF} \right) \delta_A{}^C \, , \tag{B.2}
$$

confirming that $\mathbb{P}^M{}_N{}^P{}_Q$ acts as an identity on the right hand side. Similarly, one finds that

$$
\mathbb{P}^M{}_N{}^P{}_Q \, \mathcal{V}_{PAB} \, \mathcal{V}^Q{}_{CD} = \frac{1}{2} \, \mathcal{V}_{N[AB} \, \mathcal{V}^M{}_{CD]} - \frac{1}{48} \, \epsilon_{ABCDEFGH} \mathcal{V}_N{}^{EF} \, \mathcal{V}^{MGH} \, . \tag{B.3}
$$

The section constraint (2.1) states that

$$(\mathbb{P}_{\mathbf{1+133}})_{PQ}{}^{MN} \, \partial_M \otimes \partial_N = 0 \,, \tag{B.4}$$

where **133** and **1** are in the symmetric and antisymmetric tensor product, respectively. Contracting this equation with the 56-bein, we obtain explicitly

$$\mathcal{V}^{(M}{}_{AC}\mathcal{V}^{N)BC} \, \partial_M \otimes \partial_N = \frac{1}{8} \, \delta_A^B \, \mathcal{V}^{(M}{}_{CD}\mathcal{V}^{N)CD} \, \partial_M \otimes \partial_N \,,$$

$$\mathcal{V}^M{}_{[AB}\mathcal{V}^N{}_{CD]} \, \partial_M \otimes \partial_N = \frac{1}{24} \, \epsilon_{ABCDEFGH} \, \mathcal{V}^{MEF}\mathcal{V}^{NGH} \, \partial_M \otimes \partial_N \,. \tag{B.5}$$

C   The supersymmetry algebra

In this appendix, we show that the commutator of supersymmetry transformations (3.32)–(3.34) closes into the supersymmetry algebra (3.35). For the commutator on the external and internal vielbeine $e_\mu{}^\alpha$ and $\mathcal{V}_M{}^{AB}$ we have seen in section 3.2 above that closure of the algebra is a direct consequence of the vanishing torsion conditions (2.25) and (3.13), respectively. Here, we complete the algebra on the vectors $\mathcal{A}_\mu{}^M$ and two-forms $\mathcal{B}_{\mu\nu\,\alpha}$ and $\mathcal{B}_{\mu\nu\,M}$.

We start with the vector fields, for which the commutator of two supersymmetry transformations yields

$$\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \, \mathcal{A}_\mu{}^M &= -8i \, \mathcal{D}_\mu \left(\mathcal{V}^{MAB}\bar{\epsilon}_{2\,A}\epsilon_{1\,B} \right) + 16 \, \mathcal{V}^N{}_{AB} \, \mathcal{V}^{MAB} \, \bar{\epsilon}_2^C \, \gamma_\mu \, \widehat{\nabla}_N \epsilon_{1\,C} \\
&\quad + 32 \, \mathcal{V}^N{}_{CA} \, \mathcal{V}^{MAB} \, \bar{\epsilon}_2^C \, \gamma_\mu \, \nabla_N \epsilon_{1\,B} + 32 \, \mathcal{V}^{MAB}\mathcal{V}^K{}_{BC} \, \bar{\epsilon}_{2\,A} \, \widehat{\nabla}_K \left(\gamma_\mu \epsilon_1^C \right) + \text{c.c.} \\
&= \mathcal{D}_\mu \Lambda^M + 4 \, g_{\mu\nu}\mathcal{M}^{MN}\partial_N \left(\bar{\epsilon}_2^A \, \gamma^\nu \epsilon_{1\,A} \right) + 8 \, \mathcal{M}^{MN} \left(\bar{\epsilon}_2^A \, \gamma_\alpha \epsilon_{1\,A} \right) e_{\mu\beta} \, e^{\nu[\alpha}\widehat{\nabla}_N e_\nu{}^{\beta]} \\
&\quad + 8i \, \Omega^{MN} \left(\bar{\epsilon}_2^A \, \gamma_\mu \widehat{\nabla}_N \epsilon_{1\,A} - \widehat{\nabla}_N \bar{\epsilon}_2^A \, \gamma_\mu \, \epsilon_{1\,A} \right) \\
&\quad + 32 \left(\mathcal{V}^{MAB}\mathcal{V}^K{}_{BC} + \mathcal{V}^M{}_{BC}\mathcal{V}^{KAB} + \frac{1}{8} \, \delta_C^A \, \mathcal{M}^{MK} \right) \nabla_K \left(\bar{\epsilon}_2^C \gamma_\mu \epsilon_{1\,A} \right). \tag{C.1}
\end{aligned}$$

In the first line, we recognize the action of a gauge transformation together with the non-covariant contribution $g_{\mu\nu}\mathcal{M}^{MN}\partial_N\xi^\nu$ of the diffeomorphism action (2.34). The third term can be reduced using (3.10). Let us rewrite the last term of (C.1) as

$$\begin{aligned}
&32 \, \nabla_K \left\{ \left(\mathcal{V}^{MAB}\mathcal{V}^K{}_{BC} + \mathcal{V}^M{}_{BC}\mathcal{V}^{KAB} + \frac{1}{8} \, \delta_C^A \, \mathcal{M}^{MK} \right) \left(\bar{\epsilon}_2^C \gamma_\mu \epsilon_{1\,A} \right) \right\} \\
&= 32 \, \partial_K \left\{ \left(\mathcal{V}^{MAB}\mathcal{V}^K{}_{BC} + \mathcal{V}^M{}_{BC}\mathcal{V}^{KAB} + \frac{1}{8} \, \delta_C^A \, \mathcal{M}^{MK} \right) \left(\bar{\epsilon}_2^C \gamma_\mu \epsilon_{1\,A} \right) \right\} \\
&\quad - 32 \left(\mathcal{V}^K{}_{BC}\mathcal{D}_K\mathcal{V}^{MAB} + \mathcal{V}^{KAB}\mathcal{D}_K\mathcal{V}^M{}_{BC} - \frac{1}{8} \, \delta_C^A \, (\text{trace}) \right) \left(\bar{\epsilon}_2^C \gamma_\mu \epsilon_{1\,A} \right) \\
&\quad + 8 \left(e^{-1}\partial_K e \right) \left(\mathcal{V}^K{}_{BC}\mathcal{V}^{MAB} + \mathcal{V}^{KAB}\mathcal{V}^M{}_{BC} + \frac{1}{8} \, \delta_C^A \, \mathcal{M}^{MK} \right) \left(\bar{\epsilon}_2^C \gamma_\mu \epsilon_{1\,A} \right) \\
&= 12 \, (t^{\boldsymbol{\alpha}})^{MN} \, \partial_N \Xi_{\mu\,\boldsymbol{\alpha}} - \frac{8}{3} \, \Omega^{MN} \left(\mathcal{V}^K{}_{BC} \, \mathcal{D}_N\mathcal{V}_K{}^{AB} + \mathcal{V}^{K\,AB} \, \mathcal{D}_N\mathcal{V}_{KBC} \right) \left(\bar{\epsilon}_2^C \gamma_\mu \epsilon_{1\,A} \right),
\end{aligned}$$

reproducing the parameter $\Xi_{\mu\,\boldsymbol{\alpha}}$ from (3.37), and where we have used (3.19) in the first equality and the vanishing torsion condition (3.21) in the second. Together, we obtain

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\,\mathcal{A}_\mu{}^M = {}& \mathcal{D}_\mu \Lambda^M + g_{\mu\nu}\mathcal{M}^{MN}\partial_N \xi^\nu - \frac{1}{2}\xi^\nu \mathcal{F}_{\mu\nu}{}^M - 12\,(t^{\boldsymbol{\alpha}})^{MN}\,\partial_N \Xi_{\mu\,\boldsymbol{\alpha}} \\
& + 8i\,\Omega^{MN}\left(\bar{\epsilon}_2^A\,\gamma_\mu \widehat{\mathcal{D}}_N \epsilon_{1\,A} - \widehat{\mathcal{D}}_N \bar{\epsilon}_2^A\,\gamma_\mu\,\epsilon_{1\,A}\right) \\
& - \frac{8}{3}\,\Omega^{MN}\left(\mathcal{V}^K{}_{BC}\,\mathcal{D}_N \mathcal{V}_K{}^{AB} + \mathcal{V}^{KAB}\,\mathcal{D}_N \mathcal{V}_{K\,BC}\right)\left(\bar{\epsilon}_2^C \gamma_\mu \epsilon_{1\,A}\right). \qquad \text{(C.2)}
\end{aligned}
$$

We observe, that we can simultaneously drop the SU(8) connection part in the last two lines since they mutually cancel. The spin connection $\widehat{\omega}_M{}^{\alpha\beta}$ in the second line yields additional contributions which explicitly carry the field strength $\mathcal{F}_{\mu\nu}{}^M$ and can be simplified using the twisted self-duality equation (2.15):

$$
-\,i\,\Omega^{MN}\,\varepsilon_{\mu\nu\rho\sigma}\,\bar{\epsilon}_2^A\,\gamma^\nu\,\epsilon_{1A}\,\mathcal{M}_{NK}\,\mathcal{F}^{\rho\sigma K} = -\frac{1}{2}\,\xi^\nu\,\mathcal{F}_{\mu\nu}{}^M\,. \qquad \text{(C.3)}
$$

In total, the commutator (C.2) then takes the expected form

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\,\mathcal{A}_\mu{}^M = {}& \xi^\nu \mathcal{F}_{\nu\mu}{}^M + g_{\mu\nu}\mathcal{M}^{MN}\partial_N \xi^\nu + \mathcal{D}_\mu \Lambda^M + 12\,(t^{\boldsymbol{\alpha}})^{MN}\,\partial_N \Xi_{\mu\,\boldsymbol{\alpha}} \\
& + \frac{1}{2}\,\Omega^{MN}\,\Xi_{\mu\,N}\,, \qquad \text{(C.4)}
\end{aligned}
$$

with the last term corresponding to the action of a tensor gauge transformation (2.19) with parameter $\Xi_{\mu\,M}$ from (3.42).

Next, let us check the commutator of supersymmetry transformations on the two-forms $\mathcal{B}_{\mu\nu\,\boldsymbol{\alpha}}$. First, we note that to lowest order in the fermions the terms descending from variation of the $(t_{\boldsymbol{\alpha}})_{MN}\,\mathcal{A}_{[\mu}{}^M \delta_\epsilon \mathcal{A}_{\nu]}{}^N$ contribution in (3.32) simply reproduce the corresponding terms of type $(t_{\boldsymbol{\alpha}})_{MN}\,\mathcal{A}_{[\mu}{}^M\,[\delta_1, \delta_2]\mathcal{A}_{\nu]}{}^N$ in the action of gauge transformations (2.19) and diffeomorphisms (2.34), by virtue of the closure of the algebra (C.2) on the vector fields. We can thus in the following ignore all terms that carry explicit gauge fields $\mathcal{A}_\mu{}^M$. With some calculation the various remaining terms organize into

$$
\begin{aligned}
[\delta_1, \delta_2]\,\mathcal{B}_{\mu\nu\,\boldsymbol{\alpha}} = {}& -\frac{8}{3}\,(t_{\boldsymbol{\alpha}})^{PQ}\Big(-\mathcal{V}_{P\,AB}\mathcal{V}_{Q\,CD}\,\bar{\epsilon}_2^{[A}\,\gamma_{\mu\nu}\,\mathcal{P}_\rho{}^{BCD]E}\gamma^\rho\epsilon_{1\,E} \qquad\qquad\text{(C.5)} \\
& \qquad\qquad + 2\,\mathcal{V}_{P\,BC}\mathcal{V}_Q{}^{AC}\,\bar{\epsilon}_{2\,A}\,\gamma_{[\mu}\,\mathcal{D}_{\nu]}\epsilon_1^B \\
& \qquad\qquad - 6i\mathcal{V}_{P\,AB}\mathcal{V}_{Q\,CD}\,\bar{\epsilon}_2^{[A}\,\gamma_{\mu\nu}\,\mathcal{V}^{MBC}\,\widehat{\nabla}_M \epsilon_1^{D]} \\
& \qquad\qquad - 4i\,\mathcal{V}_{P\,BC}\mathcal{V}_Q{}^{AC}\,\bar{\epsilon}_{2\,A}\gamma_{[\mu}\mathcal{V}^{MBD}\,\widehat{\nabla}_M(\gamma_{\nu]}\epsilon_{1\,D}) + \text{c.c.}\Big) - (1 \leftrightarrow 2) \\
= {}& 2\mathcal{D}_{[\mu}\Xi_{\nu]\boldsymbol{\alpha}} + \frac{1}{3}\,(t_{\boldsymbol{\alpha}})^{PQ}\,\mathcal{V}_{P\,AB}\mathcal{V}_{Q\,CD}\,\mathcal{P}^{\sigma ABCD}\,e\varepsilon_{\mu\nu\rho\sigma}\,\xi^\rho + (t_{\boldsymbol{\alpha}})_{MN}\,\Lambda^M \mathcal{F}_{\mu\nu}{}^N \\
& - \frac{32}{3}(t_{\boldsymbol{\alpha}})^{PQ}\,\partial_M\left(i\mathcal{V}_{PAC}\mathcal{V}_Q{}^{BC}\mathcal{V}^M{}_{BD}\bar{\epsilon}_2^A\gamma_{\mu\nu}\epsilon_1^D + \text{c.c.}\right) \\
& - \frac{4}{3}(t_{\boldsymbol{\alpha}})^{PQ}\Big(-12\mathcal{V}_{PCD}\,\bar{\epsilon}_2^C\gamma_{\mu\nu}\partial_Q\epsilon_1^D + 4i\mathcal{V}_{PAC}\mathcal{V}^{NCD}\partial_Q\mathcal{V}_{NDB}\,\bar{\epsilon}_2^A\gamma_{\mu\nu}\epsilon_1^B \\
& \qquad\qquad + 3\mathcal{V}_{PCD}\Omega_{Q\rho\sigma}\,\bar{\epsilon}_2^C\gamma_{\mu\nu}\gamma^{\rho\sigma}\epsilon_1^D + \text{c.c.} - (1 \leftrightarrow 2)\Big) \\
= {}& 2\mathcal{D}_{[\mu}\Xi_{\nu]\boldsymbol{\alpha}} + \xi^\rho\,\mathcal{H}_{\rho\mu\nu\,\boldsymbol{\alpha}} + (t_{\boldsymbol{\alpha}})_{MN}\,\Lambda^M \mathcal{F}_{\mu\nu}{}^N + \partial_M \Omega_{\mu\nu}{}^M{}_{\boldsymbol{\alpha}} + (t_{\boldsymbol{\alpha}})_M{}^N \Omega_{\mu\nu\,N}{}^M\,,
\end{aligned}
$$

with the gauge parameters Λ^M and $\Xi_{\mu\,\alpha}$ defined in (3.37) above, and the shift parameters $\Omega_{\mu\nu}{}^M{}_\alpha$, $\Omega_{\mu\nu N}{}^M$ given in (3.42). Finally, we have used the first-order duality equations (2.39) for the last equality in (C.5) in order to reproduce on-shell the transformation (2.34) under external diffeomorphisms. Together, we confirm the supersymmetry algebra (3.35) on the two-forms $\mathcal{B}_{\mu\nu\,\alpha}$.

Closure of the supersymmetry algebra on the vector fields and two-forms $\mathcal{B}_{\mu\nu\,\alpha}$ thus has not only determined the supersymmetry transformation rules but also uniquely fixed all the gauge parameters appearing on the right hand side of (3.35). The remaining commutator for the constrained two-forms $\mathcal{B}_{\mu\nu\,M}$ thus becomes a consistency check of the entire construction with no more free or adjustable parameters to be determined. Indeed, closure of two supersymmetry transformations on $\mathcal{B}_{\mu\nu\,M}$ into (3.35) can be shown by a rather lengthy calculation of which we will give only a few essential ingredients here.

As for $\mathcal{B}_{\mu\nu\,\alpha}$, we can consistently ignore all terms that carry explicit gauge fields $\mathcal{A}_\mu{}^M$ which separately organize into the correct contributions due to closure (C.2) on the vector fields. After some calculation, we then find for the remaining commutator

$$
\begin{aligned}
[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\,\mathcal{B}_{\mu\nu\,M} = {}& 2\,\mathcal{D}_{[\mu}\Xi_{\nu]M} - 4i\,\xi^\rho\,e\varepsilon_{\mu\nu\rho\sigma}\,R_{M\tau}{}^{\sigma\tau} \\
& - 2i\,e\varepsilon_{\mu\nu\rho\sigma}\,\mathcal{D}^\rho\left(g_{\sigma\lambda}\partial_M\xi^\lambda\right) - \frac{2}{3}\,e\varepsilon_{\mu\nu\rho\sigma}\,\mathcal{P}^{\rho ABCD}\,\mathcal{V}^P{}_{AB}\mathcal{D}_M\mathcal{V}_{PCD}\,\xi^\sigma \\
& + \frac{128}{3}\,i\,\bar{\epsilon}_2^{(A}\gamma_{[\mu}\widehat{\nabla}_L(\gamma_{\nu]}\epsilon_1^{B)})\mathcal{D}_M\mathcal{V}_{KCA}\mathcal{V}^L{}_{BD}\mathcal{V}^{KCD} + \text{c.c.} \\
& - 64i\,\bar{\epsilon}_2^C\,\epsilon_1^D\,e_{[\mu}{}^\alpha\widehat{\nabla}_K e_{\nu]\alpha}\mathcal{V}^{KAB}\mathcal{V}^N{}_{AB}\mathcal{D}_M\mathcal{V}_{NCD} + \text{c.c.} \\
& + 64\,\mathcal{V}^K{}_{CD}\,\bar{\epsilon}_2^C\,\gamma_{[\mu}\widehat{\mathcal{D}}_M\widehat{\nabla}_K(\gamma_{\nu]}\epsilon_1^D)) + \text{c.c.} \\
& - 64\,\mathcal{D}_M\bar{\epsilon}_2^C\,\gamma_{[\mu}\,\mathcal{V}^K{}_{CD}\widehat{\nabla}_K(\gamma_{\nu]}\epsilon_1^D) + \text{c.c.} \\
& - 16i\,\Omega_{MN}\,\mathcal{F}_{\rho[\mu}{}^N\bar{\epsilon}_2^C\,\gamma^\rho\widehat{\nabla}_K(\gamma_{\nu]}\epsilon_1^D)\mathcal{V}^K{}_{CD} + \text{c.c.} \\
& - 16\,e\varepsilon_{\mu\nu\rho}{}^\sigma\,\nabla_M\left(\bar{\epsilon}_2^C\gamma^\rho\mathcal{V}^N{}_{CD}\widehat{\nabla}_N(\gamma_\sigma\epsilon_1^D)\right) + \text{c.c.} \,.
\end{aligned}
\tag{C.6}
$$

Here the curvature in the second term refers to the curvature of the corresponding spin connections

$$
\begin{aligned}
R_{M\tau}{}^{\sigma\tau} &\equiv e_\alpha{}^\sigma e_\beta{}^\rho\left(\partial_M\omega_\rho{}^{\alpha\beta} - \mathcal{D}[A,\omega]_\rho\,\omega_M{}^{\alpha\beta}\right) \\
&= e_\alpha{}^\sigma e_\beta{}^\rho\left(\partial_M\omega_\rho{}^{\alpha\beta} - \mathcal{D}_\rho\left(e^{\tau[\alpha}\partial_M e_\tau{}^{\beta]}\right)\right).
\end{aligned}
\tag{C.7}
$$

In the calculation of (C.6), we have made use of

$$
\mathcal{D}_M\mathcal{V}^L{}_{AB} = \frac{2i}{3}\,\mathcal{V}^{KCD}\mathcal{D}_M\mathcal{V}_{KC[B}\mathcal{V}^L{}_{A]D} - i\mathcal{V}^{LCD}\mathcal{V}^K{}_{CD}\mathcal{D}_M\mathcal{V}_{KAB}\,,
\tag{C.8}
$$

$$
\iff \quad \mathcal{V}^P{}_{AB}\,\Gamma_{MP}{}^L = \frac{2i}{3}\,\mathcal{V}^{KCD}\mathcal{V}_{PC[A}\mathcal{V}^L{}_{B]D}\,\Gamma_{MK}{}^P + i\mathcal{V}^{LCD}\mathcal{V}^K{}_{CD}\mathcal{V}_{PAB}\,\Gamma_{MK}{}^P\,,
$$

as well as

$$
\begin{aligned}
8i\,e\varepsilon_{\mu\nu\rho}{}^\sigma\,\partial_M\left(\bar{\epsilon}_2^A\gamma^\rho\mathcal{D}_\sigma\epsilon_{1\,A}\right) + \text{c.c.} &= 8i\,e\varepsilon_{\mu\nu\rho}{}^\sigma\,\partial_M\nabla_\sigma\left(\bar{\epsilon}_2^A\gamma^\rho\epsilon_{1\,A}\right) \tag{C.9} \\
&= 8i\,e\varepsilon_{\mu\nu\rho}{}^\sigma\,\nabla_\sigma\partial_M\left(\bar{\epsilon}_2^A\gamma^\rho\epsilon_{1\,A}\right) + 2i\,e\varepsilon_{\mu\nu\rho}{}^\sigma\,\partial_M\Gamma_{\sigma\tau}{}^\rho\xi^\tau \\
&= -2i\,e\varepsilon_{\mu\nu\rho}{}^\sigma\,\mathcal{D}^\rho\left(g_{\sigma\lambda}\partial_M\xi^\lambda\right) + 2i\,e\varepsilon_{\mu\nu\sigma\tau}\,R_{M\rho}{}^{\sigma\tau}\,\xi^\rho\,,
\end{aligned}
$$

and

$$32i\left(\bar{\epsilon}_2^A\,\gamma_{[\mu}[\mathcal{D}_{\nu]},\widehat{\mathcal{D}}_M]_{\text{spin}}\,\epsilon_{1\,A}-\text{c.c.}\right)=-2ie\varepsilon_{\alpha\beta[\mu|\rho}\xi^\rho\,\widehat{R}_{M|\nu]}{}^{\alpha\beta}\;+\;\text{c.c.}$$

$$=-4ie\varepsilon_{\alpha\beta[\mu|\rho}\xi^\rho\,R_{M|\nu]}{}^{\alpha\beta}-2\Omega_{MN}\xi^\rho\,\mathcal{D}_{[\mu}\mathcal{F}_{\nu]\rho}{}^N$$

$$=-4ie\varepsilon_{\mu\nu\tau\rho}\xi^\rho\,R_{M\sigma}{}^{\sigma\tau}-2ie\varepsilon_{\mu\nu\sigma\tau}\xi^\rho\,R_{M\rho}{}^{\sigma\tau}$$

$$-2\Omega_{MN}\xi^\rho\,\mathcal{D}_{[\mu}\mathcal{F}_{\nu]\rho}{}^N\;. \tag{C.10}$$

Let us start by considering the first five terms of (C.6). After some further calculation and upon using the first-order duality equation (2.39), they reduce to

$$\longrightarrow 2\,\mathcal{D}_{[\mu}\Xi_{\nu]M}-4i\,\xi^\rho\,e\varepsilon_{\rho\mu\nu\sigma}R_{M\tau}{}^{\sigma\tau}$$

$$-2i\,e\varepsilon_{\mu\nu}{}^{\rho\sigma}\,\mathcal{D}_\rho\left(g_{\sigma\lambda}\partial_M\xi^\lambda\right)-\frac{2}{3}\,e\varepsilon_{\mu\nu\rho\sigma}\,\mathcal{P}^{\rho ABCD}\,\mathcal{V}^P{}_{AB}\mathcal{D}_M\mathcal{V}_{PCD}\,\xi^\sigma$$

$$=\ 2\,\mathcal{D}_{[\mu}\Xi_{\nu]M}+2i\xi^\rho\,e\varepsilon_{\rho\mu\nu\sigma}\left(\widehat{J}_M{}^\sigma+\frac{1}{3}\,\mathcal{P}^{\sigma ABCD}\,p_{M\,ABCD}\right)$$

$$-2i\,e\varepsilon_{\mu\nu\rho}{}^\sigma\,\mathcal{D}^\rho\left(g_{\sigma\lambda}\partial_M\xi^\lambda\right)$$

$$=\ 2\,\mathcal{D}_{[\mu}\Xi_{\nu]M}+\xi^\rho\,\mathcal{H}_{\rho\mu\nu\,M}-2i\,e\varepsilon_{\mu\nu\rho\sigma}g^{\sigma\tau}\,\mathcal{D}^\rho\left(g_{\tau\lambda}\,\partial_M\xi^\lambda\right)\;. \tag{C.11}$$

This exactly reproduces the expected transformation of $\mathcal{B}_{\mu\nu\,M}$ under external diffeomorphisms (2.34). Next, we collect all $\mathcal{F}^M$ terms on the right hand side of (C.6). This yields

$$[\delta_1,\delta_2]\,\mathcal{B}_{\mu\nu\,M}\Big|_{\mathcal{F}}=\frac{8}{3}\,\mathcal{V}_{NCB}\mathcal{V}^{KCD}\,\mathcal{D}_M\mathcal{V}_{KDA}\,\bar{\epsilon}_2^A\gamma_{[\mu}\gamma^{\rho\sigma}\gamma_{\nu]}\epsilon_1^B\,\mathcal{F}_{\rho\sigma}{}^N$$

$$-4\,\mathcal{V}^P{}_{AB}\mathcal{D}_M\mathcal{V}_{PCD}\,\bar{\epsilon}_2^A\,\gamma_{\mu\nu}\,\gamma^{\rho\sigma}\epsilon_1^B\mathcal{V}_N{}^{CD}\mathcal{F}_{\rho\sigma}{}^N$$

$$-4i\,\bar{\epsilon}_2^A\,\gamma_{[\mu}\mathcal{D}_M\left(\gamma^{\rho\sigma}\gamma_{\nu]}\epsilon_1^B\,\mathcal{F}_{\rho\sigma}{}^N\mathcal{V}_{NAB}\right)$$

$$+4i\,\mathcal{D}_M\bar{\epsilon}_2^A\,\gamma_{[\mu}\,\gamma^{\rho\sigma}\gamma_{\nu]}\epsilon_1^B\,\mathcal{F}_{\rho\sigma}{}^N\mathcal{V}_{NAB}$$

$$+i\,e\varepsilon_{\mu\nu\rho}{}^\sigma\,\mathcal{D}_M\left(\bar{\epsilon}_2^A\gamma^\rho\gamma^{\lambda\tau}\gamma_\sigma\epsilon_1^B\,\mathcal{F}_{\lambda\tau}{}^N\mathcal{V}_{NAB}\right)+\text{c.c.}\;. \tag{C.12}$$

After some further calculation, these terms may be brought into the form

$$=-\frac{32}{3}\mathcal{V}^{KCD}\mathcal{D}_M\mathcal{V}_{KAD}\,\mathcal{F}_{\mu\nu CB}\,\bar{\epsilon}_2^A\epsilon_1^B+16\mathcal{V}^K{}_{CD}\mathcal{D}_M\mathcal{V}_{KAB}\mathcal{F}_{\mu\nu}{}^{CD}\,\bar{\epsilon}_2^A\epsilon_1^B$$

$$+8i\,\mathcal{F}_{\mu\nu AB}\mathcal{D}_M(\bar{\epsilon}_2^A\epsilon_1^B)-8i\,\mathcal{V}^{KAB}\mathcal{V}_{KCD}\mathcal{D}_M(\mathcal{F}_{\mu\nu AB})\bar{\epsilon}_2^C\epsilon_1^D$$

$$=-8\mathcal{V}^{KCD}\mathcal{F}_{\mu\nu CD}\mathcal{D}_M(\mathcal{V}_{KAB}\bar{\epsilon}_2^A\epsilon_1^B)+8\mathcal{V}^K{}_{CD}\mathcal{F}_{\mu\nu}{}^{CD}\mathcal{D}_M(\mathcal{V}_{KAB}\bar{\epsilon}_2^A\epsilon_1^B)$$

$$+8\mathcal{D}_M(\mathcal{V}^{KCD}\mathcal{F}_{\mu\nu CD})\mathcal{V}_{KAB}\bar{\epsilon}_2^A\epsilon_1^B-8\mathcal{D}_M(\mathcal{V}^K{}_{CD}\mathcal{F}_{\mu\nu}{}^{CD})\mathcal{V}_{KAB}\bar{\epsilon}_2^A\epsilon_1^B$$

$$=\mathcal{F}_{\mu\nu}{}^K\partial_M\Lambda_K-\Lambda_K\partial_M\mathcal{F}_{\mu\nu}{}^K\;, \tag{C.13}$$

and precisely reproduce the gauge transformation (2.19) of the two-form $\mathcal{B}_{\mu\nu\,M}$.

It remains to show that all the remaining terms in (C.6) combine into the Ω transformations of (2.21) with parameter $\Omega_{\mu\nu\,M}{}^N$ from (3.42). This can be verified by a lengthy but direct computation. In the course of this computation, it is useful to explicitly develop the curvature

$$R_{MN}{}^{\alpha\beta}\equiv2\partial_{[M}\omega_{N]}{}^{\alpha\beta}+2\omega_{[M}{}^{\alpha\gamma}\omega_{N]\gamma}{}^\beta$$

$$=e^\nu{}_\gamma e^{\rho[\alpha}\partial_{[M}e_\nu{}^{\beta]}\partial_{N]}e_\rho{}^\gamma-\frac{1}{2}\,g^{\mu\nu}\partial_{[M}e_\nu{}^\alpha\partial_{N]}e_\mu{}^\beta-\frac{1}{2}\,e^{\nu\alpha}e^{\mu\beta}\partial_{[M}e_\nu{}^\gamma\partial_{N]}e_{\mu\gamma}\;, \tag{C.14}$$

from which one obtains

$$
\begin{aligned}
R_{MN\mu\nu} &\equiv R_{MN}{}^{\alpha\beta}\, e_{\mu\alpha}e_{\nu\beta} \\
&= -\frac{1}{2}\, g^{\lambda\kappa}\partial_{[M}g_{\mu\lambda}\partial_{N]}g_{\nu\kappa} = -\frac{1}{2}\, g^{\lambda\kappa}\nabla_{[M}g_{\mu\lambda}\nabla_{N]}g_{\nu\kappa}\,.
\end{aligned}
\tag{C.15}
$$

We conclude that the supersymmetry algebra consistently closes also on the field $\mathcal{B}_{\mu\nu\, M}$.

D Non-exceptional gravity

In this appendix we will illustrate in terms of a simple example (taken from standard differential geometry) how the difficulties encountered in constructing a fully covariant connection can be understood and resolved in our framework. The main point will be that fully covariant expressions can be obtained in terms of the $D = 11$ connections, but that these cannot be written just in terms of the generalized vielbein and its ordinary derivatives — unlike in ordinary differential geometry.

In standard differential geometry and in the absence of torsion, the spin connection is defined as

$$
\omega_{mab} = -\frac{1}{2}e_m{}^c\big(\Omega_{ab\,c} - \Omega_{bc\,a} - \Omega_{ca\,b}\big)
$$

with coefficients of anholonomy

$$
\Omega_{ab\,c} \equiv e_a{}^p e_b{}^q \partial_p e_{qc} - e_b{}^p e_a{}^q \partial_p e_{qc}\,.
$$

Now define the Cartan form

$$
S_{m\,ab} \equiv e_a{}^n \partial_m e_{nb}\,,
$$

which is the analogue of $\mathcal{V}^{-1}\partial\mathcal{V}$ in (3.23), and decompose this into a symmetric and an antisymmetric part

$$
q_{m\,ab} \equiv S_{m\,[ab]}\,, \quad p_{m\,ab} \equiv S_{m\,(ab)}\,.
$$

These are the same as the $q_{m\,ab}$ and $p_{m\,ab}$ in (4.14). Now a quick calculation shows that

$$
\omega_{m\,ab} = q_{m\,ab} - \big(e_a{}^p e_m{}^c p_{p\,bc} - e_b{}^p e_m{}^c p_{p\,ac}\big) \equiv q_{m\,ab} - 2p_{[a\,b]m}\,.
$$

Under an arbitrary diffeomorphism, the non-covariant contributions are

$$
\Delta^{nc}q_{m\,ab} = e_{[a}{}^r e_{q|b]}\partial_m\partial_r\xi^q\,, \quad \Delta^{nc}p_{m\,ab} = e_{(a}{}^r e_{q|b)}\partial_m\partial_r\xi^q
$$

and these two contributions cancel in the variation of $\omega_{m\,ab}$, as expected. So the spin connection is indeed a covariant object under diffeomorphisms, and we also know that it is the *only* such object that can be built from the vielbein and its derivative. Under local SO(1,3) we have

$$
\delta q_{m\,ab} = \partial_m\Lambda_{ab} + \Lambda_a{}^c q_{m\,cb} + \Lambda_b{}^c q_{m\,ac}\,, \quad \delta p_{m\,ab} = \Lambda_a{}^c p_{m\,cb} + \Lambda_b{}^c p_{m\,ac}\,,
$$

so $q_{m\,ab}$ and hence $\omega_{m\,ab}$ transform non-covariantly as SO(1,3) gauge fields, while $p_{m\,ab}$ is covariant under local SO(1,3).

Next we repeat this calculation in the $E_{7(7)}$ formalism, replacing the siebenbein by the 56-bein $\mathcal{V}^M{}_{AB}$ of exceptional geometry. To simplify things we set $A^{(3)} = A^{(6)} = 0$, and this will suffice to make clear our main point. Then the $E_{7(7)}$ 56-bein (whose components are explicitly given in (4.4)–(4.7)) simplifies to

$$\mathcal{V}^{m8}{}_{AB} = \frac{1}{8}\Delta^{-1/2}\Gamma^m_{AB}, \quad \mathcal{V}_{mnAB} = \frac{1}{8}\Delta^{-1/2}\Gamma_{mnAB},$$

$$\mathcal{V}^{mn}{}_{AB} = \frac{i}{4}\Delta^{1/2}\Gamma^{mn}_{AB}, \quad \mathcal{V}_{m8AB} = -\frac{i}{4}\Delta^{1/2}\Gamma_{mAB}. \tag{D.1}$$

Note that $\mathcal{V}^{m8}{}_{AB}$ and $\mathcal{V}^{mn}{}_{AB}$ are imaginary, while $\mathcal{V}_{m8AB}$ and $\mathcal{V}_{mnAB}$ are real (this is true only in this particular SU(8) gauge). By direct computation we find

$$q_{mA}{}^B = \frac{2i}{3}\mathcal{V}^{NBC}\partial_m\mathcal{V}_{NCA} = \frac{1}{2}q_{mab}\Gamma^{ab}_{AB},$$

$$p_{mABCD} = -i\mathcal{V}^N{}_{AB}\partial_m\mathcal{V}_{NCD} = -\frac{3}{4}p_{mab}\Gamma^a_{[AB}\Gamma^b_{CD]}. \tag{D.2}$$

As a check on the coefficients we compute (this is the combination appearing in the variation of the gravitino)

$$e^{mAC}q_{mC}{}^B - e^m{}_{CD}p_m{}^{ABCD} = -\frac{1}{2}\omega_{mab}(\Gamma^m\Gamma^{ab})_{AB} - \frac{1}{2}P_{maa}\Gamma^m_{AB}, \tag{D.3}$$

which is indeed the correct result. The last term proportional to $-\frac{1}{2}\Delta^{-1}\partial_m\Delta$ is just the density contribution proportional to Γ^p_{mp} that is required because the supersymmetry parameter ε is a density, showing again how the density contribution was absorbed into the connections given in ref. [3].

With this information we can now compute

$$R_{MA}{}^B = \frac{4i}{3}\left(\mathcal{V}^{NBC}\mathcal{V}_M{}^{DE}p_{NACDE} + \mathcal{V}^N{}_{AC}\mathcal{V}_{MDE}p_N{}^{BCDE}\right)$$

$$+ \frac{20i}{27}\left(\mathcal{V}^{NDE}\mathcal{V}_M{}^{BC}p_{NACDE} + \mathcal{V}^N{}_{DE}\mathcal{V}_{MAC}p_N{}^{BCDE}\right)$$

$$- \frac{7i}{27}\delta^B_A\left(\mathcal{V}^{NCD}\mathcal{V}_M{}^{EF}p_{NCDEF} + \mathcal{V}^N{}_{CD}\mathcal{V}_{MEF}p_N{}^{CDEF}\right). \tag{D.4}$$

This gives

$$R_{mA}{}^B = -\frac{1}{6}p_{abm}\Gamma^{ab}_{AB} + \frac{5}{54}p_{aab}\Gamma_{bmAB} + \frac{1}{27}p_{abb}\Gamma_{maAB},$$

$$R_{pqA}{}^B = -\frac{4i}{27}\Delta^{-1}p_{aa[p}\Gamma_{q]AB} - \frac{i}{3}\Delta^{-1}p_{[pq]a}\Gamma^a_{AB} + \frac{7i}{27}p_{[paa}\Gamma_{q]AB},$$

$$R^{pq}{}_A{}^B = \frac{1}{3}p_{ab}{}^{[p}\Gamma^{q]ab}_{AB} + \frac{5}{54}p_{aab}\Gamma^{bpq}_{AB} - \frac{1}{27}p_{abb}\Gamma^{apq}_{AB},$$

$$R^m{}_A{}^B = 0. \tag{D.5}$$

The last component drops out because for this term the first two lines in (D.4) give something proportional to δ^B_A, and hence are cancelled by the third term in the definition of $R_{MA}{}^B$. This shows very explicitly, that no matter how we combine expressions depending only on $\mathcal{V}$ and its derivative, there is no way of getting rid of p_{mab} and replacing

$q_{m\,ab} \to \omega_{m\,ab}$ by such manipulations, without 'breaking up' the 56-bein $\mathcal{V}$. In other words, full covariance cannot be achieved in this way, but requires the explicit introduction 'by hand' of the spin connection.

In principle we could extend the above calculation to non-vanishing form fields; but this will be far more tedious than the calculation just presented (and the resulting expressions will not be any prettier). Perhaps the only interesting aspect here is that, again, there appears to be *no* combination of $\mathcal{V}$'s and $\partial\mathcal{V}$'s that would produce the fully anti-symmetrized (exterior) derivatives on the 3-form and the 6-form field, and this is the reason why the hook-like contributions in the affine connection are needed. It is therefore very remarkable that the supersymmetric theory avoids this problem by picking precisely the combinations (3.28) where these terms drop out.

E Covariant SU(8) connection

In this appendix we provide yet more evidence that an SU(8) connection satisfying all desired covariance properties *cannot* be constructed in terms of only $\mathcal{V}$ and its derivative $\partial\mathcal{V}$. Namely, we will show by explicit computation how the SU(8) connection of section 3 can be made to transform as a generalized vector under generalized diffeomorphisms, which implies a unique expression for $U_{M\,A}{}^{B}$ in terms of $\mathcal{V}$ and its derivatives. However, the modifications required to achieve this come at the price of destroying the covariance under SU(8).

Let the SU(8) connection be

$$\mathcal{Q}_{MA}{}^{B} = q_{MA}{}^{B} + R_{MA}{}^{B} + U_{MA}{}^{B} + W_{MA}{}^{B}\,, \tag{E.1}$$

with $q_{MA}{}^{B}$, $R_{MA}{}^{B}$, and $W_{MA}{}^{B}$ given by (3.23) and (3.24), and we make the following choice for the undetermined part $U_{MA}{}^{B}$

$$
\begin{aligned}
U_{MA}{}^{B} \equiv &-\frac{2}{3}q_{MA}{}^{B} + \frac{2i}{3}\left(\mathcal{V}_{MCD}\mathcal{V}^{NBC}q_{NA}{}^{D} - \mathcal{V}_{M}{}^{CD}\mathcal{V}^{N}{}_{AC}q_{ND}{}^{B}\right)\\
&-\frac{34i}{189}\left(\mathcal{V}_{MAC}\mathcal{V}^{NCD}q_{ND}{}^{B} - \mathcal{V}_{M}{}^{BC}\mathcal{V}^{N}{}_{CD}q_{NA}{}^{D}\right)\\
&-\frac{20i}{189}\left(\mathcal{V}_{MAD}\mathcal{V}^{NBC}q_{NC}{}^{D} - \mathcal{V}_{M}{}^{BD}\mathcal{V}^{N}{}_{AC}q_{ND}{}^{C}\right)\\
&-\frac{2i}{27}\delta_{A}^{B}\left(\mathcal{V}_{MCD}\mathcal{V}^{NEC}q_{NE}{}^{D} - \mathcal{V}_{M}{}^{CD}\mathcal{V}^{N}{}_{EC}q_{ND}{}^{E}\right).
\end{aligned}
\tag{E.2}
$$

These are indeed all the objects that one can construct in terms of $\mathcal{V}$ and its derivative $\partial\mathcal{V}$. However, while the first term $q_{M\,A}{}^{B}$, $R_{MA}{}^{B}$, and $W_{MA}{}^{B}$ have indeed the required covariance properties of an SU(8) connection, the expression (E.2) for $U_{M\,A}{}^{B}$ does not, and will therefore violate SU(8) covariance if general covariance requires such a contribution.

To see that the full connection can be made to transform covariantly under generalized diffeomorphisms, consider the non-covariant contributions in the transformation of $q_{M\,A}{}^{B}$ and p_{MABCD}

$$\Delta^{\mathrm{nc}}q_{M\,A}{}^{B} = 8i\,\mathcal{V}^{NBC}\,\mathbb{P}^{K}{}_{N}{}^{S}{}_{R}\,\partial_{M}\partial_{S}\Lambda^{R}\,\mathcal{V}_{KCA},\tag{E.3}$$

$$\Delta^{\mathrm{nc}}p_{M}{}^{ABCD} = 12i\,\mathcal{V}^{NAB}\,\mathbb{P}^{K}{}_{N}{}^{S}{}_{R}\,\partial_{M}\partial_{S}\Lambda^{R}\,\mathcal{V}_{K}{}^{CD},\tag{E.4}$$

where we have used

$$\mathbb{P}^M{}_N{}^P{}_Q = \frac{1}{24}\left(2\delta^M_Q\delta^P_N + \delta^M_N\delta^P_Q - \Omega_{NQ}\Omega^{MP}\right) + (t_{\boldsymbol{\alpha}})_{NQ}(t^{\boldsymbol{\alpha}})^{MP} \tag{E.5}$$

and the section condition. Note that the covariant part of the transformations of q_M and p_M contain a weight term. So in fact they transform as generalized tensor densities of weight $-1/2$.

Furthermore,

$$\Delta^{\mathrm{nc}}R_{M\,A}{}^B = \mathcal{V}_{M\,CD}\left(-8\,\mathcal{V}^N_{AE}\mathcal{V}_R{}^{[BE|}\mathcal{V}^{S\,|CD]} + 10\,\delta_A^{[B|}\mathcal{V}^{N|CD|}\mathcal{V}^{S\,|EF]}\mathcal{V}_{R\,EF} \tag{E.6}\right.$$
$$-\frac{40}{9}\delta_A^C\mathcal{V}^N{}_{EF}\mathcal{V}_R{}^{[BD|}\mathcal{V}^{S\,|EF]} + \frac{40}{9}\delta_A^C\mathcal{V}^{N\,[BD|}\mathcal{V}^{S|EF]}\mathcal{V}_{R\,EF}$$
$$\left.+\frac{14}{9}\delta_B^A\mathcal{V}^N{}_{EF}\mathcal{V}_R{}^{[EF|}\mathcal{V}^{S|CD]} - \frac{14}{9}\delta_B^A\mathcal{V}^{N[EF|}\mathcal{V}^{S|CD]}\mathcal{V}_{R\,EF}\right)\partial_N\partial_S\Lambda^R + \text{c.c.}\,,$$

where we have used equations (E.4), (A.3) and

$$\mathcal{V}^M{}_{[AB}\mathcal{V}^N{}_{CD]}\partial_M\partial_N\cdot = \frac{1}{24}\epsilon_{ABCDEFGH}\mathcal{V}^{M\,EF}\mathcal{V}^{N\,GH}\partial_M\partial_N.$$

which can be proved using identity (A.3) and the section condition. Now using,

$$\mathcal{V}^{MAC}\mathcal{V}^N_{BC}\partial_M\partial_N\cdot = \frac{1}{8}\delta_B^A\,\mathcal{V}^{MCD}\mathcal{V}^N_{CD}\partial_M\partial_N\cdot\,, \tag{E.7}$$

which holds by identity (A.2) and the section condition, equation (E.6) can be simplified to:

$$\Delta^{\mathrm{nc}}R_{M\,A}{}^B =$$
$$-\frac{1}{3}\mathcal{V}_{M\,CD}\left(4\mathcal{V}^{NCD}\left[\mathcal{V}_{R\,AE}\mathcal{V}^{S\,BE} + \mathcal{V}^S{}_{AE}\mathcal{V}_R{}^{BE} - \frac{1}{36}\delta_A^B\left(4\mathcal{V}_{R\,EF}\mathcal{V}^{S\,EF} + 7\mathcal{V}^S{}_{EF}\mathcal{V}_R{}^{EF}\right)\right]\right.$$
$$+ 8\mathcal{V}^{NBC}\left[\mathcal{V}_{R\,AE}\mathcal{V}^{S\,DE} + \mathcal{V}^S{}_{AE}\mathcal{V}_R{}^{DE}\right] + \frac{1}{9}\delta_A^C\mathcal{V}^{NEF}\mathcal{V}^S{}_{EF}\mathcal{V}_R{}^{BD}$$
$$-\frac{8}{9}\delta_A^C\mathcal{V}^{N\,DE}\mathcal{V}^{S\,BF}\mathcal{V}_{R\,EF} - \frac{4}{9}\delta_A^C\mathcal{V}^{N\,BD}\left[\mathcal{V}_{R\,EF}\mathcal{V}^{S\,EF} - 5\mathcal{V}^S{}_{EF}\mathcal{V}_R{}^{EF}\right]$$
$$\left.+\frac{1}{9}\delta_A^B\mathcal{V}^{N\,EF}\mathcal{V}^S{}_{EF}\mathcal{V}_R{}^{CD} + \frac{8}{9}\delta_A^B\mathcal{V}^{N\,EC}\mathcal{V}^{S\,FD}\mathcal{V}_{R\,EF}\right)\partial_N\partial_S\Lambda^R + \text{c.c.}. \tag{E.8}$$

Similarly, using identities (A.2) and (E.7)

$$\Delta^{\mathrm{nc}}U_{M\,A}{}^B =$$
$$-\frac{1}{3}\mathcal{V}_{M\,CD}\left(8\,\mathcal{V}^{NCD}\left[\mathcal{V}_{R\,AE}\mathcal{V}^{S\,BE} + \mathcal{V}^S{}_{AE}\mathcal{V}_R{}^{BE} - \frac{1}{9}\delta_A^B\left(\mathcal{V}_{R\,EF}\mathcal{V}^{S\,EF} + \mathcal{V}^S{}_{EF}\mathcal{V}_R{}^{EF}\right)\right]\right.$$
$$- 8\,\mathcal{V}^{N\,BC}\left[\mathcal{V}_{R\,AE}\mathcal{V}^{S\,DE} + \mathcal{V}^S{}_{AE}\mathcal{V}_R{}^{DE}\right] - \frac{1}{9}\delta_A^C\mathcal{V}^{N\,EF}\mathcal{V}^S{}_{EF}\mathcal{V}_R{}^{BD}$$
$$+ \frac{8}{9}\delta_A^C\mathcal{V}^{N\,DE}\mathcal{V}^{S\,BF}\mathcal{V}_{R\,EF} - \frac{8}{9}\delta_A^C\mathcal{V}^{N\,BD}\left[\mathcal{V}_{R\,EF}\mathcal{V}^{S\,EF} + \mathcal{V}^S{}_{EF}\mathcal{V}_R{}^{EF}\right]$$
$$\left.- \frac{1}{9}\delta_A^B\mathcal{V}^{N\,EF}\mathcal{V}^S{}_{EF}\mathcal{V}_R{}^{CD} - \frac{8}{9}\delta_A^B\mathcal{V}^{N\,EC}\mathcal{V}^{S\,FD}\mathcal{V}_{R\,EF}\right)\partial_N\partial_S\Lambda^R + \text{c.c.}. \tag{E.9}$$

It is straightforward to verify that

$$
\begin{aligned}
\Delta^{\mathrm{nc}} \left(R_{M\,A}{}^{B} + U_{M\,A}{}^{B} \right) = \\
-\,4\mathcal{V}_{MCD} & \left(\mathcal{V}^{NCD} \left[\mathcal{V}_{R\,AE}\mathcal{V}^{S\,BE} + \mathcal{V}^{S}{}_{AE}\mathcal{V}_{R}{}^{BE} - \frac{1}{36}\,\delta_{A}^{B}\left(4\mathcal{V}_{R\,EF}\mathcal{V}^{S\,EF} + 5\mathcal{V}^{S}{}_{EF}\mathcal{V}_{R}{}^{EF}\right) \right] \right. \\
& \left. -\,\frac{1}{9}\,\delta_{A}^{C}\mathcal{V}^{N\,BD}\left[\mathcal{V}_{R\,EF}\mathcal{V}^{S\,EF} - \mathcal{V}^{S}{}_{EF}\mathcal{V}_{R}{}^{EF} \right] \right) \partial_{N}\partial_{S}\Lambda^{R} + \text{c.c.}\,, \\
= 8\left(\mathcal{V}_{MCD}\mathcal{V}^{NCD} - \mathcal{V}_{M}{}^{CD}\mathcal{V}^{N}{}_{CD} \right) & \mathbb{P}^{P}{}_{Q}{}^{S}{}_{R}\,\mathcal{V}_{PAE}\mathcal{V}^{QBE}\,\partial_{N}\partial_{S}\Lambda^{R} \\
-\,\frac{4i}{9}\left[\mathcal{V}_{MAC}\mathcal{V}^{NBC} + \mathcal{V}_{M}{}^{AC}\mathcal{V}^{N}{}_{BC} \right. & \left. - \frac{1}{8}\delta_{A}^{B}\left(\mathcal{V}_{MCD}\mathcal{V}^{NCD} + \mathcal{V}_{M}{}^{CD}\mathcal{V}^{N}{}_{CD} \right) \right] \partial_{N}\partial_{S}\Lambda^{S}\,, \\
= -\Delta^{\mathrm{nc}} q_{m\,A}{}^{B} - \Delta^{\mathrm{nc}} W_{m\,A}{}^{B}\,. & \tag{E.10}
\end{aligned}
$$

Therefore, $\mathcal{Q}_{M\,A}{}^{B}$ defined in equation (E.1) is a generalized tensor density of weight $-1/2$. However, as the term $U_{M\,A}{}^{B}$ itself depends on $q_{M\,A}{}^{B}$ in a definite manner, the total SU(8) connection no longer transforms properly under SU(8). As we explained, this conclusion can only be evaded if one drops the assumption that all parts of $\mathcal{Q}_{M}$ should be expressible in terms of $\mathcal{V}$ and $\partial_{M}\mathcal{V}$.

References

[1] E. Cremmer and B. Julia, *The $N = 8$ Supergravity Theory. 1. The Lagrangian*, *Phys. Lett.* **B 80** (1978) 48 [INSPIRE].

[2] E. Cremmer, B. Julia and J. Scherk, *Supergravity Theory in Eleven-Dimensions*, *Phys. Lett.* **B 76** (1978) 409 [INSPIRE].

[3] B. de Wit and H. Nicolai, *$d = 11$ Supergravity With Local SU(8) Invariance*, *Nucl. Phys.* **B 274** (1986) 363 [INSPIRE].

[4] H. Nicolai, *$D = 11$ Supergravity With Local SO(16) Invariance*, *Phys. Lett.* **B 187** (1987) 316 [INSPIRE].

[5] K. Koepsell, H. Nicolai and H. Samtleben, *An exceptional geometry for $D = 11$ supergravity?*, *Class. Quant. Grav.* **17** (2000) 3689 [hep-th/0006034] [INSPIRE].

[6] B. de Wit and H. Nicolai, *Deformations of gauged SO(8) supergravity and supergravity in eleven dimensions*, *JHEP* **05** (2013) 077 [arXiv:1302.6219] [INSPIRE].

[7] H. Godazgar, M. Godazgar and H. Nicolai, *Generalised geometry from the ground up*, *JHEP* **02** (2014) 075 [arXiv:1307.8295] [INSPIRE].

[8] H. Godazgar, M. Godazgar and H. Nicolai, *Einstein-Cartan Calculus for Exceptional Geometry*, *JHEP* **06** (2014) 021 [arXiv:1401.5984] [INSPIRE].

[9] W. Siegel, *Superspace duality in low-energy superstrings*, *Phys. Rev.* **D 48** (1993) 2826 [hep-th/9305073] [INSPIRE].

[10] C. Hull and B. Zwiebach, *Double Field Theory*, *JHEP* **09** (2009) 099 [arXiv:0904.4664] [INSPIRE].

[11] C. Hull and B. Zwiebach, *The gauge algebra of double field theory and Courant brackets*, *JHEP* **09** (2009) 090 [arXiv:0908.1792] [INSPIRE].

[12] O. Hohm, C. Hull and B. Zwiebach, *Background independent action for double field theory*, *JHEP* **07** (2010) 016 [arXiv:1003.5027] [INSPIRE].

[13] O. Hohm, C. Hull and B. Zwiebach, *Generalized metric formulation of double field theory*, *JHEP* **08** (2010) 008 [arXiv:1006.4823] [INSPIRE].

[14] O. Hohm and H. Samtleben, *Exceptional Form of $D = 11$ Supergravity*, *Phys. Rev. Lett.* **111** (2013) 231601 [arXiv:1308.1673] [INSPIRE].

[15] O. Hohm and H. Samtleben, *Exceptional Field Theory II: $E_{7(7)}$*, *Phys. Rev.* **D 89** (2014) 066017 [arXiv:1312.4542] [INSPIRE].

[16] C.M. Hull, *Generalised Geometry for M-theory*, *JHEP* **07** (2007) 079 [hep-th/0701203] [INSPIRE].

[17] P.P. Pacheco and D. Waldram, *M-theory, exceptional generalised geometry and superpotentials*, *JHEP* **09** (2008) 123 [arXiv:0804.1362] [INSPIRE].

[18] A. Coimbra, C. Strickland-Constable and D. Waldram, *$E_{d(d)} \times \mathbb{R}^+$ generalised geometry, connections and M-theory*, *JHEP* **02** (2014) 054 [arXiv:1112.3989] [INSPIRE].

[19] A. Coimbra, C. Strickland-Constable and D. Waldram, *Supergravity as Generalised Geometry II: $E_{d(d)} \times \mathbb{R}^+$ and M-theory*, *JHEP* **03** (2014) 019 [arXiv:1212.1586] [INSPIRE].

[20] N. Hitchin, *Generalized Calabi-Yau manifolds*, *Quart. J. Math. Oxford Ser.* **54** (2003) 281 [math/0209099] [INSPIRE].

[21] M. Gualtieri, *Generalized complex geometry*, math/0401221 [INSPIRE].

[22] P.C. West, *E_{11}, SL(32) and central charges*, *Phys. Lett.* **B 575** (2003) 333 [hep-th/0307098] [INSPIRE].

[23] C. Hillmann, *Generalized E(7(7)) coset dynamics and $D = 11$ supergravity*, *JHEP* **03** (2009) 135 [arXiv:0901.1581] [INSPIRE].

[24] D.S. Berman, H. Godazgar, M.J. Perry and P. West, *Duality Invariant Actions and Generalised Geometry*, *JHEP* **02** (2012) 108 [arXiv:1111.0459] [INSPIRE].

[25] O. Hohm, S.K. Kwak and B. Zwiebach, *Unification of Type II Strings and T-duality*, *Phys. Rev. Lett.* **107** (2011) 171603 [arXiv:1106.5452] [INSPIRE].

[26] O. Hohm, S.K. Kwak and B. Zwiebach, *Double Field Theory of Type II Strings*, *JHEP* **09** (2011) 013 [arXiv:1107.0008] [INSPIRE].

[27] O. Hohm and H. Samtleben, *U-duality covariant gravity*, *JHEP* **09** (2013) 080 [arXiv:1307.0509] [INSPIRE].

[28] A. Coimbra, C. Strickland-Constable and D. Waldram, *Supergravity as Generalised Geometry I: Type II Theories*, *JHEP* **11** (2011) 091 [arXiv:1107.1733] [INSPIRE].

[29] O. Hohm and S.K. Kwak, *$N = 1$ Supersymmetric Double Field Theory*, *JHEP* **03** (2012) 080 [arXiv:1111.7293] [INSPIRE].

[30] I. Jeon, K. Lee and J.-H. Park, *Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity*, *Phys. Rev.* **D 85** (2012) 081501 [*Erratum ibid.* **D 86** (2012) 089903] [arXiv:1112.0069] [inSPIRE].

[31] O. Hohm and S.K. Kwak, *Frame-like Geometry of Double Field Theory*, *J. Phys.* **A 44** (2011) 085404 [arXiv:1011.4101] [inSPIRE].

[32] O. Hohm and B. Zwiebach, *On the Riemann Tensor in Double Field Theory*, *JHEP* **05** (2012) 126 [arXiv:1112.5296] [inSPIRE].

[33] O. Hohm and B. Zwiebach, *Towards an invariant geometry of double field theory*, *J. Math. Phys.* **54** (2013) 032303 [arXiv:1212.1736] [inSPIRE].

[34] I. Jeon, K. Lee and J.-H. Park, *Stringy differential geometry, beyond Riemann*, *Phys. Rev.* **D 84** (2011) 044022 [arXiv:1105.6294] [inSPIRE].

[35] G. Aldazabal, M. Graña, D. Marqués and J.A. Rosabal, *Extended geometry and gauged maximal supergravity*, *JHEP* **06** (2013) 046 [arXiv:1302.5419] [inSPIRE].

[36] M. Cederwall, J. Edlund and A. Karlsson, *Exceptional geometry and tensor fields*, *JHEP* **07** (2013) 028 [arXiv:1302.6736] [inSPIRE].

[37] V.I. Ogievetsky, *Infinite-dimensional algebra of general covariance group as the closure of finite-dimensional algebras of conformal and linear groups*, *Lett. Nuovo Cim.* **8** (1973) 988 [inSPIRE].

[38] A.B. Borisov and V.I. Ogievetsky, *Theory of Dynamical Affine and Conformal Symmetries as Gravity Theory*, *Theor. Math. Phys.* **21** (1975) 1179 [inSPIRE].

[39] P.C. West, E_{11} *and M-theory*, *Class. Quant. Grav.* **18** (2001) 4443 [hep-th/0104081] [inSPIRE].

[40] B. de Wit and H. Nicolai, *The Consistency of the S^7 Truncation in $D = 11$ Supergravity*, *Nucl. Phys.* **B 281** (1987) 211 [inSPIRE].

[41] H. Nicolai and K. Pilch, *Consistent Truncation of $D = 11$ Supergravity on $AdS_4 \times S^7$*, *JHEP* **03** (2012) 099 [arXiv:1112.6131] [inSPIRE].

[42] H. Godazgar, M. Godazgar and H. Nicolai, *Nonlinear Kaluza-Klein theory for dual fields*, *Phys. Rev.* **D 88** (2013) 125002 [arXiv:1309.0266] [inSPIRE].

[43] B. de Wit, H. Nicolai and N.P. Warner, *The Embedding of Gauged $N = 8$ Supergravity Into $d = 11$ Supergravity*, *Nucl. Phys.* **B 255** (1985) 29 [inSPIRE].

[44] H. Godazgar, M. Godazgar and H. Nicolai, *Testing the non-linear flux ansatz for maximal supergravity*, *Phys. Rev.* **D 87** (2013) 085038 [arXiv:1303.1013] [inSPIRE].

[45] H. Godazgar, M. Godazgar and H. Nicolai, *The embedding tensor of Scherk-Schwarz flux compactifications from eleven dimensions*, *Phys. Rev.* **D 89** (2014) 045009 [arXiv:1312.1061] [inSPIRE].

[46] K. Lee, C. Strickland-Constable and D. Waldram, *Spheres, generalised parallelisability and consistent truncations*, arXiv:1401.3360 [inSPIRE].

[47] O. Hohm and H. Samtleben, *Exceptional Field Theory I: $E_{6(6)}$ covariant Form of M-theory and Type IIB*, *Phys. Rev.* **D 89** (2014) 066016 [arXiv:1312.0614] [inSPIRE].

[48] B. de Wit, H. Samtleben and M. Trigiante, *The Maximal $D = 4$ supergravities*, *JHEP* **06** (2007) 049 [arXiv:0705.2101] [inSPIRE].

[49] G. Dall'Agata, G. Inverso and M. Trigiante, *Evidence for a family of* SO(8) *gauged supergravity theories*, *Phys. Rev. Lett.* **109** (2012) 201301 [`arXiv:1209.0760`] [INSPIRE].

[50] G. Dall'Agata, G. Inverso and A. Marrani, *Symplectic Deformations of Gauged Maximal Supergravity*, *JHEP* **07** (2014) 133 [`arXiv:1405.2437`] [INSPIRE].

[51] D.S. Berman, M. Cederwall, A. Kleinschmidt and D.C. Thompson, *The gauge structure of generalised diffeomorphisms*, *JHEP* **01** (2013) 064 [`arXiv:1208.5884`] [INSPIRE].

[52] E. Bergshoeff, R. Kallosh, T. Ortín, D. Roest and A. Van Proeyen, *New formulations of D = 10 supersymmetry and D8 - O8 domain walls*, *Class. Quant. Grav.* **18** (2001) 3359 [`hep-th/0103233`] [INSPIRE].

[53] B. de Wit, H. Samtleben and M. Trigiante, *On Lagrangians and gaugings of maximal supergravities*, *Nucl. Phys.* **B 655** (2003) 93 [`hep-th/0212239`] [INSPIRE].

[54] O. Hohm and H. Samtleben, *Gauge theory of Kaluza-Klein and winding modes*, *Phys. Rev.* **D 88** (2013) 085005 [`arXiv:1307.0039`] [INSPIRE].

[55] B. de Wit and H. Nicolai, *N=8 Supergravity*, *Nucl. Phys.* **B 208** (1982) 323 [INSPIRE].

[56] O. Hohm and H. Samtleben, *Exceptional Field Theory III: $E_{8(8)}$*, *Phys. Rev.* **D 90** (2014) 066002 [`arXiv:1406.3348`] [INSPIRE].

[57] F.W. Hehl, P. Von Der Heyde, G.D. Kerlick and J.M. Nester, *General Relativity with Spin and Torsion: Foundations and Prospects*, *Rev. Mod. Phys.* **48** (1976) 393 [INSPIRE].