Multidimensional Dirac strings and the Witten index of SYMCS theories with groups of higher rank

A.V. Smilga

*SUBATECH, Université de Nantes,
4 rue Alfred Kastler, BP 20722, Nantes 44307, France*

E-mail: smilga@subatech.in2p3.fr

ABSTRACT: We discuss generalized Dirac strings associated with a given Lie group. They live on $\mathcal{C}^r$ (r being the rank of the group). Such strings show up in the effective Born-Oppenheimer Hamiltonian for 3d supersymmetric Yang-Mills-Chern-Simons theories, brought up by the gluon loops. We calculate accurately the number of the vacuum states in the effective Hamiltonian associated with these strings. We also show that these states are *irrelevant* for the final SYMCS vacuum counting. The Witten index of SYMCS theories depends thus only on the strings generated by fermion loops and carrying fractional generalized fluxes.

KEYWORDS: Supersymmetric gauge theory, Field Theories in Lower Dimensions, Chern-Simons Theories, Supersymmetric Effective Theories

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1 Introduction

The Lagrangian of pure $3d$ $\mathcal{N} = 1$ supersymmetric Yang-Mills-Chern-Simons theory reads

$$\mathcal{L} = \frac{1}{g^2} \left\langle -\frac{1}{2} F_{\mu\nu}^2 + i\bar{\psi}\not{D}\psi \right\rangle + \kappa \left\langle \epsilon^{\mu\nu\rho} \left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) - \bar{\psi}\psi \right\rangle. \quad (1.1)$$

The conventions are: $\epsilon^{012} = 1$, $D_\mu \mathcal{O} = \partial_\mu \mathcal{O} - i[A_\mu, \mathcal{O}]$ (such that A_μ is Hermitian), ψ_α is a 2-component Majorana $3d$ spinor belonging to the adjoint representation of the gauge group, and $\langle \dots \rangle$ stands for the color trace. γ -matrices can be chosen as $\gamma^0 = \sigma^2$, $\gamma^1 = i\sigma^1$, $\gamma^2 = i\sigma^3$.

This is a $3d$ theory and the gauge coupling constant g^2 carries the dimension of mass. The requirement for the functional integral to be invariant under certain large (non-contractible) gauge transformations (see e.g. ref. [1] for a nice review) leads to the quantization condition

$$\kappa = \frac{k}{4\pi}. \quad (1.2)$$

For the unitary $SU(N)$ gauge groups, the *level* k must be integer if N is even and half-integer if N is odd. For an arbitrary group, this depends on whether the adjoint Casimir eigenvalue c_V is even or odd.

The index of this theory

$$I = \text{Tr}\{(-1)^F e^{-\beta H}\} \quad (1.3)$$

was evaluated in [2] with the result

$$I(k, N) = [\text{sgn}(k)]^{N-1} \binom{|k| + N/2 - 1}{N-1}, \quad (1.4)$$

for $SU(N)$ gauge group. This is valid for $|k| \geq N/2$. For $|k| < N/2$, the index vanishes and supersymmetry is broken. In the simplest $SU(2)$ case, the index is just

$$I(k, 2) = k. \quad (1.5)$$

For $SU(3)$, it is

$$I(k, 3) = \frac{k^2 - 1/4}{2}. \quad (1.6)$$

The result (1.4) was derived in [2] by considering the theory in a large spatial volume, $g^2 L \gg 1$, where the vacuum dynamics is determined by the Chern-Simons term with the coupling renormalized due *only* to fermion loops¹

$$k \rightarrow k - \frac{N}{2}. \quad (1.7)$$

The number (1.4) is nothing but the full number of states in the topological pure CS theory with the renormalized level (1.7).

For an arbitrary group, the general recipe is

$$I_{\text{SYMCS}}^{\text{full}}(k) = I_{\text{SYMCS}}^{\text{tree}}\left(k - \frac{c_V}{2}\right), \quad (1.8)$$

where c_V is the adjoint Casimir eigenvalue.

In refs. [3, 4] (see also the review [5]), the result (1.4) was reproduced with another method [6] by considering the theory in a *small* spatial box, $g^2 L \ll 1$, and studying the dynamics of the corresponding Born-Oppenheimer Hamiltonian. (We also evaluated the index for $Sp(2r)$ and G_2 , see eqs. (4.51), (4.49).)

If imposing the periodic boundary conditions for all fields, the slow variables in the effective BO Hamiltonian are just the zero Fourier modes of the spatial components of the *Abelian* vector potential (belonging to the Cartan subalgebra) $C_{j=1,2}^{a=1,\dots,r}$ and its superpartners $\psi^a = \psi_{1-2}^a$. The motion in the field space $\{C_j^a\} \equiv \{\mathbf{C}_j\}$ is actually finite because the shifts

$$\mathbf{C}_j \rightarrow \mathbf{C}_j + 4\pi n_j \mathbf{a}/L, \quad (1.9)$$

¹Note that the renormalized level is always integer.

where $\mathbf{a}$ are the coroots of the group and n_j are integer numbers, amount to contractible gauge transformations. The wave functions are invariant with respect to these transformations up to certain phase factors [3, 7],

$$\begin{aligned}\Psi(\mathbf{x} + \mathbf{a}, \mathbf{y}) &= e^{-2\pi i k \mathbf{a} \mathbf{y}} \Psi(\mathbf{x}, \mathbf{y}), \\ \Psi(\mathbf{x}, \mathbf{y} + \mathbf{a}) &= e^{2\pi i k \mathbf{a} \mathbf{x}} \Psi(\mathbf{x}, \mathbf{y}),\end{aligned}\tag{1.10}$$

where $\mathbf{x} = \mathbf{C}_1 L / (4\pi)$, $\mathbf{y} = \mathbf{C}_2 L / (4\pi)$. It is enough then to consider the motion over the (multidimensional) *dual torus* $T_G \times T_G$ with the maximal torus T_G representing the elementary cell of the coroot lattice. For $SU(2)$ when $r = 1$, the dual torus corresponds to $x, y \in [0, 1]$.

The presence of the phase factors (1.10) represents a certain complication compared to the $4d$ case where such factors are absent. The effective BO Hamiltonian is also somewhat more complicated than in the nonchiral $4d$ theories (where it is just a free Laplacian). Here it represents a multidimensional generalization of the Landau-Dubrovnik-Krichever-Novikov Hamiltonian describing the motion in a planar magnetic field [8, 9]. For the group of rank r , the effective Hamiltonian reads

$$H = \frac{g^2}{2L^2} \left[(P_j^a + \mathcal{A}_j^a)^2 + \mathcal{B}^{ab} (\psi^a \bar{\psi}^b - \bar{\psi}^b \psi^a) \right] \tag{1.11}$$

with the matrix-valued $B^{ab} = \epsilon_{ij} \partial_i^a \mathcal{A}_j^b$ [and the effective vector potentials $\mathcal{A}_j^a(\mathbf{C}_j)$ having nothing to do, of course, with the gauge fields of the original theory (1.1)]

But the most serious $3d$ complication is that it *is* not enough here to analyze the effective Hamiltonian to the leading BO order, but one-loop corrections should also be taken into account. At the tree level, the magnetic field is homogeneous,

$$\begin{aligned}\mathcal{A}_j^a &= -\frac{\kappa L^2}{2} \epsilon_{jk} C_k^a, \\ \mathcal{B}^{ab} &= \kappa L^2 \delta^{ab}.\end{aligned}\tag{1.12}$$

The loops bring about corrections that are singular at the “corners” of the dual torus with $\mathbf{C}_1$ and $\mathbf{C}_2$ coinciding with the nodes of the lattice generated by the fundamental coweights. The number of such nodes in the dual torus is equal to the square of the order of the center of the group. We will illustrate and explain these assertions later.

For $r = 1$, the matrix-valued magnetic field becomes an ordinary one and the extra singular loop-induced fields represent thin vortices² placed in each of the $2 \cdot 2 = 4$ dual torus corners, $x, y = 0, 1/2$. For gluon loops, these lines carry the flux $+1$. In the limit $g^2 L \rightarrow 0$, they become infinitely narrow Dirac strings. For fermion loops, the flux of each line is $-1/2$. Half-integer magnetic fluxes are not admissible, they are not compatible with supersymmetry [10]. This refers, however, only to *net* fluxes and in our case the net magnetic flux of four lines with fluxes $-1/2$ each is quite integer, $\Phi_{\text{net}}^{\text{ferm}} = -2$.

An accurate analysis of refs. [4, 5] (see also section 4.4) displays that the integer fluxes are irrelevant for the vacuum counting (Dirac strings are not observable) and the result (1.5)

²Their width is of order of mass $\sim g^2$, which is much less than the size of the dual torus $4\pi/L$.

for the index is obtained from the tree-level result $I^{\text{tree}}(k, 2) = k + 1$ by the substitution $k \rightarrow k - 1$ rather than $k \rightarrow k + 1$ as one should have written if gluon loops were taken into account.

Gluon loops should be irrelevant for any group — this is the only way to reproduce the result (1.4) and the general recipe (1.8). However, an explicit and rigorous demonstration of this fact was lacking up to now. This paper is written to fill out this gap.

Section 2 represents a mathematical introduction. We describe the properties of the ordinary θ functions and also of certain special θ functions of several variables that are quasiperiodic with respect to shifts along the coroot lattice of a Lie group. We will use these functions in a subsequent analysis.

In section 3, we introduce and discuss the multidimensional Dirac strings — a generalization of usual flux lines that is relevant for non-Abelian theories.

Section 4 is devoted to index calculations. First, we calculate the generalized magnetic fluxes — index integrals associated with individual multidimensional strings living on $\mathcal{C}^r$. As the spectrum in this case is continuous, these fluxes do not have a direct index interpretation, but they might represent an independent interest. Then we calculate the index of the strings placed on the maximal complex torus and finally discuss how loop corrections and the associated multidimensional strings affect the index of SYMCS theories.

There is also a technical appendix where we prove the relation (2.22) between the adjoint Casimir eigenvalue and the simple coroots. Though this relation seems to be known to mathematicians, we were not able to find its explicit proof in the literature.

2 θ functions

We review here certain mathematical facts concerning the properties of θ functions living on the coroot lattice of a Lie group that we will use in the following. We have no doubt that they are known to mathematicians, though we were not able to find a manual with their clear exposition.

To begin with, let us remind some properties of the ordinary θ functions [11]. They are analytic functions on a torus playing the same role there as the ordinary polynomials do for the Riemann sphere. The polynomials have a pole at infinity and θ functions satisfy certain nontrivial quasiperiodic boundary conditions with respect to shifts along the cycles of the torus. A generic torus is characterized by a complex modular parameter τ , but we will stick to the simplest choice $\tau = i$ so that the torus represents a square $x, y \in [0, 1]$ ($z = x + iy$) glued around.

The simplest basic θ -function satisfies the boundary conditions

$$\begin{aligned}\theta(z + 1) &= \theta(z), \\ \theta(z + i) &= e^{\pi(1-2iz)}\theta(z).\end{aligned}\tag{2.1}$$

This defines a *unique* (up to a constant complex factor) analytic function. Its explicit form is

$$\theta(z) = \sum_{n=-\infty}^{\infty} \exp\{-\pi n^2 + 2\pi i n z\}.\tag{2.2}$$

This function (call it theta function of level 1 and introduce an alternative notation $\theta(z) \equiv Q^1(z)$) has only one zero in the square $x, y \in [0, 1]$ — right in its middle, $\theta(\frac{1+i}{2}) = 0$.

It will also be convenient for us in the following to use the function

$$\pi(z) = \frac{\theta\left(z - \frac{1+i}{2}\right)}{\theta'\left(\frac{1+i}{2}\right)}. \quad (2.3)$$

It has zeroes on the square lattice including the origin, where it behaves as $\pi(z) = z + O(z^2)$. It satisfies the boundary conditions

$$\begin{aligned} \pi(z+1) &= \pi(z), \\ \pi(z+i) &= -e^{-2i\pi z} \pi(z). \end{aligned} \quad (2.4)$$

The function $\pi(z)$ is expressed into the function $\sigma(z)$ defined in eq. (8.171) of ref. [12] with the choice $\omega_1 = 1/2, \omega_2 = i/2$ for the half-periods as

$$\pi(z) = \exp\left\{-\zeta\left(\frac{1}{2}\right)z^2 + i\pi z\right\} \sigma(z), \quad (2.5)$$

[the function $\zeta(z)$ to be defined later in (4.10)].

For any integer $q > 0$, one can define theta functions of level q satisfying

$$\begin{aligned} Q^q(z+1) &= Q^q(z), \\ Q^q(z+i) &= e^{q\pi(1-2iz)} Q^q(z). \end{aligned} \quad (2.6)$$

A product of two such functions of levels q and q' gives a function of level $q+q'$.

The functions satisfying (2.6) lie in vector space of dimension q . The basis in this vector space can be chosen as

$$\begin{aligned} Q_m^q(z) &\equiv \theta_{m/q,0}(qz, iq) \text{ (Mumford's notation)} \\ &= \sum_{n=-\infty}^{\infty} \exp\left\{-\pi q \left(n + \frac{m}{q}\right)^2 + 2\pi i q z \left(n + \frac{m}{q}\right)\right\}, \\ m &= 0, \dots, q-1. \end{aligned} \quad (2.7)$$

Generically, a function $Q^q(z)$ has q simple zeros. A particular function of level 4,

$$\Pi(z) = Q_3^4(z) - Q_1^4(z) = C e^{-2\pi i z} \pi(2z) \quad (2.8)$$

will play a special role in our discussion.³

$\Pi(z)$ is odd in z and has four zeros at the corners $z = 0, \frac{1}{2}, \frac{i}{2}, \frac{1+i}{2}$. Notice that on top of (2.6) it also satisfies certain quasiperiodicity conditions with respect to half-integer shifts,

$$\begin{aligned} \Pi(z+1/2) &= -\Pi(z) \\ \Pi(z+i/2) &= -e^{\pi-4\pi i z} \Pi(z). \end{aligned} \quad (2.9)$$

³The function (2.8) and its analogs for higher groups to be discussed later were introduced in [13–15] where the spectrum of pure CS theory was studied.

Figure 1. Coroot lattice for $SU(3)$. The points marked by $\square$ and $\triangle$ are fundamental coweights.

The vacuum wave functions of the theory (1.1) with $SU(2)$ gauge group are expressed into θ functions (2.7). For a group of rank $r > 1$, we need θ functions of r complex variables satisfying certain quasiperiodic boundary conditions on the coroot lattice of the corresponding group. To treat them, we will choose an inductive pragmatic approach listing only some necessary (and sufficient for us) facts and discussing first the simplest $SU(3)$ case and generalizing to other groups afterwards.

2.1 $SU(3)$

Let $h = z^3\lambda^3 + z^8\lambda^8$ be an element of the Cartan subalgebra of the complexified $su(3)$ algebra. The coroot lattice is depicted in figure 1. We will be interested in the functions $\theta(\mathbf{z})$ satisfying the following quasiperiodicity conditions

$$\begin{aligned}\theta(\mathbf{z} + \mathbf{a}) &= \theta(\mathbf{z} + \mathbf{b}) = \theta(\mathbf{z}), \\ \theta(\mathbf{z} + i\mathbf{a}) &= \exp\{k[2\pi - 4\pi i\mathbf{z}\mathbf{a}]\}\theta(\mathbf{z}), \\ \theta(\mathbf{z} + i\mathbf{b}) &= \exp\{k[2\pi - 4\pi i\mathbf{z}\mathbf{b}]\}\theta(\mathbf{z}),\end{aligned}\tag{2.10}$$

where

$$\mathbf{a} = (1, 0), \quad \mathbf{b} = (-1/2, \sqrt{3}/2)\tag{2.11}$$

are the simple coroots. It follows that

$$\theta(\mathbf{z} + i\mathbf{a} + i\mathbf{b}) = \exp\{k[2\pi - 4\pi i\mathbf{z}(\mathbf{a} + \mathbf{b})]\}\theta(\mathbf{z}).$$

The property $\exp\{2\pi i\mathbf{a}\boldsymbol{\lambda}\} = \exp\{2\pi i\mathbf{b}\boldsymbol{\lambda}\} = 1$ holds. With the metric choice

$$\langle h, g \rangle = \frac{1}{2}\text{Tr}\{hg\} = \mathbf{h}\mathbf{g},\tag{2.12}$$

the simple coroots have the length 1.

Speaking of the *roots* (defined according to $[h, E_\alpha] = \alpha(h)E_\alpha$ for positive root vectors E_α), they represent the linear forms

$$\begin{aligned}\alpha_a(\mathbf{z}) &= 2\mathbf{a} \cdot \mathbf{z} \equiv 2z^{(a)}, & \alpha_b(\mathbf{z}) &= 2\mathbf{b} \cdot \mathbf{z} \equiv 2z^{(b)}, \\ \alpha_{a+b}(\mathbf{z}) &= 2(\mathbf{a} + \mathbf{b}) \cdot \mathbf{z} \equiv 2z^{(a+b)}\end{aligned}\tag{2.13}$$

such that $\alpha_a(\mathbf{a}) = \alpha_b(\mathbf{b}) = \alpha_{a+b}(\mathbf{a} + \mathbf{b}) = 2$.

For a given integer level k , the functions satisfying (2.10) form the vector space of dimension $3k^2$. The product of two functions of levels k, k' gives a function of level $k + k'$.

Three basis functions of level $k = 1$ can be chosen in the form

$$\begin{aligned}\Psi_0 &= \sum_{\mathbf{n}} \exp \{-2\pi \mathbf{n}^2 + 4\pi i \mathbf{z} \mathbf{n}\} , \\ \Psi_{\Delta} &= \sum_{\mathbf{n}} \exp \{-2\pi (\mathbf{n} + \Delta)^2 + 4\pi i \mathbf{z} (\mathbf{n} + \Delta)\} , \\ \Psi_{\square} &= \sum_{\mathbf{n}} \exp \{-2\pi (\mathbf{n} + \square)^2 + 4\pi i \mathbf{z} (\mathbf{n} + \square)\} ,\end{aligned}\tag{2.14}$$

where the sum runs over the nodes of the lattice and

$$\Delta = (2\mathbf{a} + \mathbf{b})/3, \quad \square = (\mathbf{a} + 2\mathbf{b})/3\tag{2.15}$$

are the fundamental coweights.⁴

There are two functions of this kind representing a particular interest. One of them is expressed as

$$\begin{aligned}\Theta^{\text{SU}(3)}(\mathbf{z}) &= \Pi\left(\frac{z^8}{\sqrt{3}}\right) \Pi\left(\frac{z^3}{2} + \frac{z^8}{2\sqrt{3}}\right) \Pi\left(\frac{z^3}{2} - \frac{z^8}{2\sqrt{3}}\right) \\ &= C \exp\left\{-2\pi i \left(z^3 + \frac{z^8}{\sqrt{3}}\right)\right\} \pi\left(\frac{2z^8}{\sqrt{3}}\right) \pi\left(z^3 + \frac{z^8}{\sqrt{3}}\right) \pi\left(z^3 - \frac{z^8}{\sqrt{3}}\right)\end{aligned}\tag{2.16}$$

with $\Pi(z)$ defined in (2.8) [the arguments of the three functions π in the r.h.s. of eq.(2.16) being the fundamental weights]. It satisfies the boundary conditions (2.10) with $k = 1$ and represents thus a certain linear combination of the functions (2.14)

In contrast to the functions (2.14), the function (2.16) has a simple structure of zeroes. It has the zeroes of the 3-d order in the nodes of the lattice in figure 1 and simple zeroes on its edges, $\frac{2z^8}{\sqrt{3}} = p + iq$ or $z^3 + \frac{z^8}{\sqrt{3}} = p + iq$ or $z^3 - \frac{z^8}{\sqrt{3}} = p + iq$.

Consider now the function

$$\Pi^{\text{SU}(3)}(\mathbf{z}) = \Pi(z^3) \Pi\left(\frac{-z^3 + z^8\sqrt{3}}{2}\right) \Pi\left(\frac{z^3 + z^8\sqrt{3}}{2}\right)\tag{2.17}$$

In this case, the arguments of the three functions Π are the positive roots (2.13) with the factor 2 removed. One can observe that the function $\Pi^{\text{SU}(3)}(\mathbf{z})$ satisfies the boundary conditions (2.10) with $k = 3$.

The structure of the zeroes of the function (2.17) is similar for that of the function (2.16), but is more dense. It has zeroes of the third order at the nodes of the coroot lattice and also at the points

$$\begin{cases} \text{Re}(\mathbf{z}) = \mathbf{0}, \Delta, \square \\ \text{Im}(\mathbf{z}) = \mathbf{0}, \Delta, \square \end{cases},\tag{2.18}$$

⁴A fundamental coweight is an element of the Cartan subalgebra with zero projections on all simple coroots but one, the nonzero projection being equal to 1/2 in our normalization, $\Delta\mathbf{a} = \square\mathbf{b} = 1/2$. Note that $\exp\{2\pi i \Delta^a \lambda^a\} = \text{diag}(e^{-2\pi i/3}, e^{-2\pi i/3}, e^{-2\pi i/3})$ and $\exp\{2\pi i \square^a \lambda^a\} = \text{diag}(e^{2\pi i/3}, e^{2\pi i/3}, e^{2\pi i/3})$ — the elements of the center of $\text{SU}(3)$ (as was mentioned, for the coroots, these exponentials give the unit matrix).

the fundamental coweights $\Delta, \square$ being defined in (2.15). There are altogether 9 such points in $T_{\text{SU}(3)} \times T_{\text{SU}(3)}$.

2.2 $\text{SU}(N)$, $\text{Sp}(4)$, G_2 , etc.

These definitions and observations can be generalized for any group.

- (i) $\text{SU}(N)$. Consider the coroot lattice of $\text{SU}(N)$ generated by its $N - 1$ simple coroots $\mathbf{a}^s$. Consider the functions $\theta(\mathbf{z})$ that are periodic under the shifts $\mathbf{z} \rightarrow \mathbf{z} + \mathbf{a}^s$ and satisfy the conditions

$$\theta(\mathbf{z} + i\mathbf{a}^s) = \exp\{k[2\pi - 4\pi i \mathbf{z} \mathbf{a}^s]\} \theta(\mathbf{z}). \quad (2.19)$$

The functions satisfying (2.19) form the vector space of dimension Nk^{N-1} .

Consider the function

$$\Pi^{\text{SU}(N)}(\mathbf{z}) = \prod_p \Pi \left[\frac{\alpha_p(\mathbf{z})}{2} \right] \quad (2.20)$$

where the product runs over all $N(N - 1)/2$ positive roots.

One can be convinced that this function satisfies the boundary conditions (2.19) with $k = c_V = N$. Indeed, consider the shift $\mathbf{z} \rightarrow \mathbf{z} + i\mathbf{a}$ for a some particular coroot $\mathbf{a}$. The argument in one of the factors in (2.20) [with $\alpha_a(\mathbf{z})$] is shifted by i . There are also $2(N - 2)$ positive roots $\alpha_{\bar{p}}$ with $\alpha_{\bar{p}}(\mathbf{a}) = -1$. Thus, the argument of $2(N - 2)$ factors in (2.20) is shifted by $-i/2$. We can use now the boundary conditions (2.6) and (2.9). Let us concentrate on the $\mathbf{z}$ -independent term in the exponent. It is equal to

$$4\pi + 2(N - 2)\pi = 2N\pi, \quad (2.21)$$

which matches (2.19) with $k = N$. The linear in $\mathbf{z}$ terms also match.

The identity (2.21) is actually a manifestation of the general identity⁵

$$\sum_{p=\text{positive roots}} [\alpha_p(\mathbf{h})]^2 = 2c_V \mathbf{h}^2 \quad (2.22)$$

valid not only for coroots $\mathbf{a}$ in $\text{SU}(N)$, but for any element $\mathbf{h}$ in the Cartan subalgebra of any Lie algebra.

There are N^2 special points where the function (2.20) has zeroes of order $N(N - 1)/2$: $\text{Re}(\mathbf{z})$ and $\text{Im}(\mathbf{z})$ can be zero or coincide with one of $N - 1$ fundamental coweights of $\text{SU}(N)$.

- (ii) $\text{Sp}(4)$. It is convenient to choose the following orthogonal basis in the Cartan subalgebra, $e_1 = \text{diag}(0, 1, -1, 0)$, $e_2 = \text{diag}(1, 0, 0, -1)$. In this basis, the simple coroots $\alpha^\vee = e_1, \beta^\vee = e_2 - e_1$ are represented by the vectors $\mathbf{a} = \{1, 0\}$ and $\mathbf{b} = \{-1, 1\}$. These simple coroots generate the coroot lattice depicted in figure 2. The maximal

Figure 2. Coroot lattice for $\text{Sp}(4)$.

torus represents just a square [not a rhombus as for $\text{SU}(3)$]. There are two short $(\mathbf{a}, \mathbf{b} + \mathbf{a})$ and two long $(\mathbf{b}, \mathbf{b} + 2\mathbf{a})$ positive coroots.

The relevant θ functions satisfy the boundary conditions

$$\begin{aligned}\theta(\mathbf{z} + i\mathbf{a}) &= \exp\{k[2\pi - 4\pi i\mathbf{z}\mathbf{a}]\}\theta(\mathbf{z}), \\ \theta(\mathbf{z} + i\mathbf{b}) &= \exp\{k[4\pi - 4\pi i\mathbf{z}\mathbf{b}]\}\theta(\mathbf{z}).\end{aligned}\tag{2.23}$$

with an integer k . They form the vector space of dimension $4k^2$. The appearance of $e^{4\pi k}$ rather than $e^{2\pi k}$ in the exponential factor in the second line is due to the fact that $\mathbf{b}$ is a long coroot, $\mathbf{b}^2 = 2$, while $\mathbf{a}^2 = 1$.

The conditions (2.23) imply that

$$\theta(\mathbf{z} + i\mathbf{c}) = \exp\{k[2\pi\mathbf{c}^2 - 4\pi i\mathbf{z}\mathbf{c}]\}\theta(\mathbf{z})\tag{2.24}$$

for all other vectors $\mathbf{c}$ in the coroot lattice.

Consider the function

$$\Pi^{\text{Sp}(4)}(\mathbf{z}) = \Pi(z_1)\Pi(z_2)\Pi\left(\frac{z_1 + z_2}{2}\right)\Pi\left(\frac{z_1 - z_2}{2}\right).\tag{2.25}$$

As its analogs written before, the function (2.25) is represented in the form (2.20). It satisfies the boundary conditions (2.23) with $k = c_V[\text{Sp}(4)] = 3$, which follows again from (2.22).

The function (2.25) has zeroes of the 4-th order at four special points on $T_{\text{Sp}(4)} \times T_{\text{Sp}(4)}$: $\text{Re}(z_1) = \text{Re}(z_2) = 0, 1/2$ and $\text{Im}(z_1) = \text{Im}(z_2) = 0, 1/2$, the point $z_1 = z_2 = 1/2$ corresponding to the nontrivial element $-\mathbb{1}$ of the center Z_2 of $\text{Sp}(4)$.

⁵Though it is known to experts [16], we were not able to find it in the standard textbooks. Its elementary proof is outlined in appendix.

Besides, it has zeroes of the second order when only $\Pi(z_1)$ and $\Pi(z_2)$ vanish. There are 12 such zeroes at

$$\begin{aligned} \begin{cases} z_1 = 0 \\ z_2 = \frac{1}{2}, \frac{i}{2}, \frac{1+i}{2} \end{cases}, & \begin{cases} z_1 = \frac{1}{2} \\ z_2 = 0, \frac{i}{2}, \frac{1+i}{2} \end{cases}, \\ \begin{cases} z_1 = \frac{i}{2} \\ z_2 = 0, \frac{1}{2}, \frac{1+i}{2} \end{cases}, & \begin{cases} z_1 = \frac{1+i}{2} \\ z_2 = 0, \frac{1}{2}, \frac{i}{2} \end{cases}. \end{aligned} \quad (2.26)$$

- (iii) G_2 . The coroot lattice has the same hexagonal form as for $SU(3)$, but it is generated now by the simple coroots $\mathbf{a}$ and $\mathbf{b} = (-3/2, \sqrt{3}/2)$. There are 6 positive coroots: 3 short coroots $\mathbf{a}, \mathbf{b} + \mathbf{a}, \mathbf{b} + 2\mathbf{a}$ and 3 long coroots $\mathbf{b}, \mathbf{b} + 3\mathbf{a}, 2\mathbf{b} + 3\mathbf{a}$. The θ functions satisfying (2.10) satisfy also the property

$$\theta(\mathbf{z} + i\mathbf{b}) = \exp\{k[6\pi - 4\pi i\mathbf{z}\mathbf{b}]\}\theta(\mathbf{z}), \quad (2.27)$$

the norm $\mathbf{b}^2 = 3$ giving the term $2\pi k\mathbf{b}^2 = 6\pi k$ in the exponent.

In virtue of (2.22), the function

$$\begin{aligned} \Pi^{G_2}(\mathbf{z}) = \Pi(z^3) \Pi\left(\frac{-z^3 + z^8\sqrt{3}}{2}\right) \Pi\left(\frac{z^3 + z^8\sqrt{3}}{2}\right) \times \\ \Pi\left(\frac{z^8}{\sqrt{3}}\right) \Pi\left(\frac{z^3\sqrt{3} + z^8}{2\sqrt{3}}\right) \Pi\left(\frac{z^3\sqrt{3} - z^8}{2\sqrt{3}}\right) \end{aligned} \quad (2.28)$$

(we kept the $SU(3)$ notation for the components of $\mathbf{z}$) satisfies the boundary conditions (2.10) (and its colloraries (2.24) for all other vectors in the coroot lattice) with $k = c_V[G_2] = 4$.

There is only one special point on $T_{G_2} \times T_{G_2}$ ($\mathbf{z} = 0$) where the function (2.28) has a zero of 6-th order. This conforms to the fact that G_2 has no centre.

Then there are 8 zeroes of the 3-d order at the points

$$\begin{aligned} \begin{cases} \text{Re}(\vec{z}) = 0, \Delta, \square, \\ \text{Im}(\vec{z}) = \Delta, \square \end{cases}, & \begin{cases} \text{Re}(\vec{z}) = \Delta, \square, \\ \text{Im}(\vec{z}) = 0 \end{cases}, \end{aligned} \quad (2.29)$$

where only the factors $\Pi(z^3)$ and $\Pi\left(\frac{\pm z^3 + z^8\sqrt{3}}{2}\right)$ corresponding to the long roots of G_2 vanish.

Finally, there are the zeroes of the 2-nd order when only two factors in (2.28) corresponding to a pair of orthogonal long and short roots, like $\Pi(z^3)$ and $\Pi\left(\frac{z^8}{\sqrt{3}}\right)$, vanish. There are 9 such points,

$$\begin{aligned} \begin{cases} z^3 = 0 \\ z^8 = \frac{\sqrt{3}}{2}, \frac{i\sqrt{3}}{2}, \frac{(1+i)\sqrt{3}}{2} \end{cases}, & \begin{cases} z^3 = \pm\frac{1}{4} \\ z^8 = \frac{\sqrt{3}}{4} \end{cases}, & \begin{cases} z^3 = \pm\frac{i}{4} \\ z^8 = \frac{i\sqrt{3}}{4} \end{cases}, & \begin{cases} z^3 = \pm\frac{(1+i)}{4} \\ z^8 = \frac{(1+i)\sqrt{3}}{4} \end{cases}. \end{aligned} \quad (2.30)$$

Generalization of these constructions to all other groups is straightforward.

3 Dirac strings and multidimensional Dirac strings

3.1 Effective theory

The effective wave functions depend on r slow complex bosonic variables $z^a = x^a + iy^a$, their conjugates, and their holomorphic fermionic superpartners ψ^a . The effective theory belongs to the class of complex supersymmetric sigma models introduced in [17] and studied in details in [18]. In our case, the metric is flat and the superfield action reads

$$S = \int dt d^2\theta \left[-\frac{1}{4} D Z^a \bar{D} \bar{Z}^a + W(Z^a, \bar{Z}^a) \right], \quad (3.1)$$

where

$$D = \frac{\partial}{\partial\theta} - i\bar{\theta} \frac{\partial}{\partial t}, \quad \bar{D} = -\frac{\partial}{\partial\bar{\theta}} + i\theta \frac{\partial}{\partial t}$$

are the supersymmetric covariant derivatives and $Z^a = z^a + \sqrt{2}\theta\psi^a - i\theta\bar{\theta}\dot{z}^a$ depending on $t_L = t - i\theta\bar{\theta}$ and $\bar{Z}^a = \bar{z}^a - \sqrt{2}\bar{\theta}\bar{\psi}^a + i\theta\bar{\theta}\dot{\bar{z}}^a$ depending on $t_R = t + i\theta\bar{\theta}$ are chiral superfields, $\bar{D}Z^a = D\bar{Z}^a = 0$. The particular form of the real prepotential $W(Z^a, \bar{Z}^a)$ will be shortly revealed.

The nilpotent Nöther supercharges derived from (3.1) are

$$Q = \sqrt{2}(P_a + i\partial_a W)\psi^a, \quad \bar{Q} = \sqrt{2}(\bar{P}_a - i\bar{\partial}_a W)\bar{\psi}^a, \quad (3.2)$$

where $\partial_a = (\partial_1^a - i\partial_2^a)/2$ and $\bar{\partial}_a = (\partial_1^a + i\partial_2^a)/2$ are holomorphic and antiholomorphic derivatives. The Hamiltonian is

$$H = (\bar{P}_a - i\bar{\partial}_a W)(P_a + i\partial_a W) - 2(\partial_a \bar{\partial}_b W)\psi^a \bar{\psi}^b. \quad (3.3)$$

Being multiplied by a proper constant, it can be expressed in the form (1.11). Note, however, that the effective vector potentials $\mathcal{A}_j^a$ entering (1.11) are not arbitrary, but are derived from a single real prepotential, $\mathcal{A}_j^a \propto \epsilon_{jk} \partial_k^a W$. At the tree level, $\mathcal{A}_j^a$ have the form (1.12). This corresponds to $W^{\text{tree}} = -\pi k \bar{z}^a z^a$. Loop corrections bring about extra effective gauge fields. Let us discuss their structure first in the simple $SU(2)$ ($r = 1$) case and then for the groups of higher rank.

3.2 $SU(2)$

As was mentioned before, gluon and fermion loops bring about thin vortices of fluxes $+1$ and $-1/2$, respectively. In our problem, there are only two spatial dimensions, but one can imagine the existence of the third orthogonal direction where the vortices (representing now fluxes lines) extend. For the flux $+1$, the physics of such a line is the same as for a *Dirac string*, and that is how we will call it, also in two dimensions.

The Dirac string piercing the origin $z = 0$ corresponds to

$$W^{\text{string}} = -\frac{1}{2} \ln(z\bar{z}) \quad (3.4)$$

such that the holomorphic potentials are

$$\mathcal{A} = \frac{\mathcal{A}_1 - i\mathcal{A}_2}{2} = i\partial W = -\frac{i}{2z}, \quad \bar{\mathcal{A}} = -i\bar{\partial} W = \frac{i}{2\bar{z}}, \quad (3.5)$$

and

$$\mathcal{A}_j = -\frac{\epsilon_{jk}x_k}{\mathbf{x}^2}. \quad (3.6)$$

The supercharges are

$$Q = -i\sqrt{2}\left(\frac{\partial}{\partial z} + \frac{1}{2z}\right)\psi, \quad \bar{Q} = -i\sqrt{2}\left(\frac{\partial}{\partial \bar{z}} - \frac{1}{2\bar{z}}\right)\bar{\psi}. \quad (3.7)$$

As was mentioned, the motion over the dual torus is finite. But let us forget it for a while and consider the operators (3.7) acting on the wave functions that live on the infinite complex plane. The spectrum of the corresponding Hamiltonian

$$H = -\left(\bar{\partial} - \frac{1}{2\bar{z}}\right)\left(\partial + \frac{1}{2z}\right) \quad (3.8)$$

(the Hamiltonians in the sectors $F = 0$ and $F = 1$ are the same) is then continuous. It is easy to see that the spectrum of (3.8) coincides with the spectrum of the free Laplacian $-\bar{\partial}\partial$: all the eigenstates of H are obtained from the eigenstates of $-\bar{\partial}\partial$ by multiplying the latter by the factor $\sqrt{\bar{z}/z} = e^{-i\phi}$.⁶

Also for a finite motion, adding a Dirac string at some point does not affect the spectrum of the Hamiltonian and the wave functions are multiplied by the factor $e^{-i\phi}$. This means that an infinitely thin vortex of unit flux is actually not observable. Non-observability of Dirac strings in the problem of motion of a scalar or a spinor particle in the field of a conventional 3-dimensional Dirac monopole with properly quantized charge is, of course, a well-known fact [20, 21].

3.3 SU(3) and higher groups

For SU(3), the index a in (3.1) takes two values, $a = 3, 8$. Let us choose

$$W(z^a, \bar{z}^a) = -\frac{1}{2}\{\ln(\mathbf{z}\mathbf{a}) + \ln(\mathbf{z}\mathbf{b}) + \ln[\mathbf{z}(\mathbf{a} + \mathbf{b})]\} + \text{c.c.} \quad (3.10)$$

with $\mathbf{a}, \mathbf{b}$ defined in (2.11). The supercharges acquire the form

$$Q = -i\sqrt{2}(\partial_a + i\mathcal{A}_a)\psi^a, \quad \bar{Q} = -i\sqrt{2}(\bar{\partial}_a + i\bar{\mathcal{A}}_a)\bar{\psi}^a \quad (3.11)$$

with

$$\begin{aligned} \mathcal{A}_3 &= \partial_3 W = -\frac{i}{2}\left(\frac{1}{z^3} + \frac{1}{z^3 + z^8\sqrt{3}} + \frac{1}{z^3 - z^8\sqrt{3}}\right) \\ \mathcal{A}_8 &= \partial_8 W = \frac{i\sqrt{3}}{2}\left(\frac{1}{z^3 - z^8\sqrt{3}} - \frac{1}{z^3 + z^8\sqrt{3}}\right). \end{aligned} \quad (3.12)$$

The vector potentials (3.12) represent the SU(3) counterpart of the standard Dirac string (3.5). They live on $\mathcal{C}^2$ and, in contrast to the usual Dirac string, are singular

Figure 3. Singularities of the potential for the SU(3) string. Complex z^a are represented by their real parts.

not just at one point, but on 3 planes $z^3 = 0$ and $z^3 = \pm\sqrt{3}z^8$ (see figure 3). As is clear, this object enjoys $O(2)$, but not $O(4)$ symmetry.

The supercharge Q gives zero when acting on the function

$$f_{\text{SU}(3)}(z^a, \bar{z}^a) = \sqrt{\frac{\bar{z}^3[(\bar{z}^3)^2 - 3(\bar{z}^8)^2]}{z^3[(z^3)^2 - 3(z^8)^2]}}. \quad (3.13)$$

The function (3.13) is uniquely defined on $\mathcal{C}^2$. It is the SU(3) counterpart of the factor $e^{-i\phi}$ for SU(2). Note now that the spectrum of the Hamiltonian with the SU(3) Dirac string added coincides with the spectrum of the Hamiltonian without such string. The wave functions of the former are obtained from the wave functions of the latter by multiplication over the factor (3.13). The Dirac string (3.12) is unobservable!

A generalization for an arbitrary group is straightforward. We should consider instead of (3.10) the function

$$W(z^a, \bar{z}^a) = -\frac{1}{2} \sum_p \ln[\alpha_p(\mathbf{z})] + \text{c.c.}, \quad (3.14)$$

where the sum runs over all positive roots. Note that there are *three different* generalized Dirac strings living on $\mathcal{C}^2$ corresponding to three different simple groups of rank 2, there are 3 different strings for $\mathcal{C}^3$, etc. The Hamiltonian involving an extra multidimensional Dirac string has the same spectrum as the Hamiltonian without such string, with the wave functions being multiplied by the (uniquely defined on $\mathcal{C}^r$) factor

$$f_G(\mathbf{z}, \bar{\mathbf{z}}) = \sqrt{\prod_p \frac{\alpha_p(\bar{\mathbf{z}})}{\alpha_p(\mathbf{z})}}. \quad (3.15)$$

⁶We exclude from the spectrum the singular quasinormalizable zero energy state with the wave function [19]

$$\Psi_0 = \frac{1}{\sqrt{\bar{z}z}}. \quad (3.9)$$

The conventional 3-dimensional Dirac strings associated with the monopoles also have a multidimensional generalization. Multidimensional analogs of monopoles were constructed in [23]. They appear when constructing the effective Hamiltonian in the chiral (3+1)-supersymmetric gauge theories. This Hamiltonian depends on $3r$ slow variables. The effective multidimensional vector potentials that live in $\mathcal{R}^{3r}$ are singular on hyperplanes of codimension 2 whose structure is similar to that displayed in figure 3. The kinship of these two different problems is natural: in three dimensions, mass fermion term breaks parity and hence the theory (1.1) is chiral.

4 Index of the strings

4.1 SU(2)

As was mentioned, the string (3.5) carries the unit flux. To see this, one has to regularize it replacing (3.5) by

$$\mathcal{A} = -\frac{i\bar{z}}{2(\bar{z}z + m^2)}, \quad \bar{\mathcal{A}} = \frac{iz}{2(\bar{z}z + m^2)}. \quad (4.1)$$

The corresponding magnetic field is

$$B(\bar{z}, z) = 2i(\bar{\partial}\mathcal{A} - \partial\bar{\mathcal{A}}) = \frac{2m^2}{(\bar{z}z + m^2)^2}. \quad (4.2)$$

Hence

$$I = \frac{\Phi}{2\pi} = \frac{1}{2\pi} \int dz d\bar{z} B(\bar{z}, z) = 1. \quad (4.3)$$

The integral is saturated by the region of small $|z| \sim m$.

The famous Atiyah-Singer theorem seems to dictate for the spectrum of the Dirac operator (our supersymmetric problem is equivalent to the Dirac problem) in the field of unit flux to involve a zero mode. However, the best what we can get by solving the zero mode equation $Q\Psi = 0$ with the gauge field (4.1) is the function

$$\Psi_0 = \frac{F(\bar{z})}{\sqrt{\bar{z}z + m^2}} \quad (4.4)$$

with an arbitrary antiholomorphic $F(\bar{z})$. This is not a “catholic” zero mode because the normalization integral diverges logarithmically or worse. If choosing $F(\bar{z}) = 1$ and lifting the regularization, it goes to (3.9).

In fact, the AS theorem applies only to *compact* manifolds where the spectrum is discrete. Thus, to get a nice normalizable zero mode, we need to compactify the complex plane. It is usually done by replacing $\mathcal{C} \rightarrow S^2$ [20, 21], but compactification on the torus is also possible [22].

The latter implies nontrivial boundary conditions for the wave functions. In the presence of the magnetic flux, $\Phi = 2\pi q$ with integer nonzero q , the wave functions are not just periodic, but involve certain phase factors,

$$\begin{aligned} \Psi(x+1, y) &= e^{i\alpha(x,y)} \Psi(x, y), \\ \Psi(x, y+1) &= e^{i\beta(x,y)} \Psi(x, y) \end{aligned} \quad (4.5)$$

with the functions $\alpha(x, y)$ and $\beta(x, y)$ satisfying the condition

$$\alpha(x, y) + \beta(x + 1, y) - \alpha(x, y + 1) - \beta(x, y) = 2\pi q. \quad (4.6)$$

In the case under consideration, $q = 1$. Different choices for the phases α, β are possible. One of the choices was presented in (1.10) where one should replace $\mathbf{x} \rightarrow x, \mathbf{y} \rightarrow y, \mathbf{a} \rightarrow 1$ and set $2k = q = 1$. It is more convenient for us now to use an asymmetric choice $\alpha(x, y) = 0, \beta(x, y) = 2\pi x$. The toric zero mode should satisfy the boundary conditions (4.5) and behave as (4.4) near the origin. It is difficult to write an analytic expression for such a function in a generic massive case, but in the massless limit $m \rightarrow 0$, it can be easily done,

$$\Psi_0^{\text{torus}} = \sqrt{\frac{\bar{\pi}(z)}{\pi(z)}}. \quad (4.7)$$

For $|z| \ll 1$, the wave function (4.7) behaves as $\sqrt{\bar{z}/z} = e^{-i\phi}$.

The function (4.7) satisfies the equation

$$(\partial + i\mathcal{A})\Psi_0^{\text{torus}} = 0 \quad (4.8)$$

with

$$\mathcal{A} = -\frac{i\pi'(z)}{2\pi(z)} \quad (4.9)$$

If thinking in terms of the whole complex plane $\mathcal{C}$, the vector potential in (4.9) corresponds to a regular lattice of strings placed at $z = p + iq$ with integer p, q . Bearing in mind (2.5), it can be expressed as

$$\mathcal{A} = -\frac{i}{2} [\zeta(z) + i\pi - 2z\zeta(1/2)],$$

where $\zeta(z)$ is the Weierstrass zeta function,

$$\begin{aligned} \zeta(z) &= \frac{1}{z} + \sum_{\{pq\} \neq \{00\}} \left(\frac{1}{z + p + iq} - \frac{1}{p + iq} + \frac{z}{(p + iq)^2} \right) \\ &= \frac{1}{z} - \int_0^z \left(\mathcal{P}(u) - \frac{1}{u^2} \right) du. \end{aligned} \quad (4.10)$$

4.2 Strings on $\mathcal{C}^r$ and the associated index integrals.

a) $\text{SU}(N)$. For the effective Hamiltonian (1.11), the analog of (4.3) reads [3]

$$I = \frac{1}{(2\pi)^r} \int \prod_{a=1}^r \prod_{j=1}^2 dC_j^a \det \|\mathcal{B}^{ab}\|, \quad (4.11)$$

where the integral is done over the relevant range of C_j^a . The result (4.11) is obtained by replacing the functional integral for the index (1.3) by the ordinary one which is admissible in the semiclassical limit $\beta \rightarrow 0$ [24]. The determinant appears after integration over fermion variables.

For $SU(3)$, the integral (4.11) was calculated in [3]. For a (regularized) individual Dirac string (3.12) living on $\mathcal{C}^2$, the result is $I = 3$. Bearing in mind further generalizations for more complicated groups, let us describe this calculation in some more details choosing the simple coroot basis (cf. [23]),

$$\psi^s = \boldsymbol{\psi} \mathbf{a}^s, \quad z^s = \mathbf{z} \mathbf{a}^s. \quad (4.12)$$

We regularize the superpotential (3.14) and write (bearing in mind that $z^p = \frac{1}{2}\alpha_p(\mathbf{z})$ for the simply laced groups)

$$W = -\frac{1}{2} \sum_p \ln [|z^p|^2 + m^2]. \quad (4.13)$$

For $SU(3)$, the sum involves 3 terms. The vector potentials in the root basis are

$$\begin{aligned} \mathcal{A}^1 \equiv \mathcal{A}^{(a)} &= \mathcal{A}_3 + \frac{1}{\sqrt{3}} \mathcal{A}_8 = -\frac{i}{2} \left(\frac{\bar{z}^{(a)}}{\bar{z}^{(a)} z^{(a)} + m^2} + \frac{\bar{z}^{(a+b)}}{\bar{z}^{(a+b)} z^{(a+b)} + m^2} \right), \\ \mathcal{A}^2 \equiv \mathcal{A}^{(b)} &= \frac{2}{\sqrt{3}} \mathcal{A}_8 = -\frac{i}{2} \left(\frac{\bar{z}^{(b)}}{\bar{z}^{(b)} z^{(b)} + m^2} + \frac{\bar{z}^{(a+b)}}{\bar{z}^{(a+b)} z^{(a+b)} + m^2} \right), \end{aligned} \quad (4.14)$$

$$z^{(a+b)} = z^{(a)} + z^{(b)}.$$

They enter the effective supercharges expressed as

$$Q = -i\sqrt{2}\psi^{(a)} \left[\frac{\partial}{\partial z^{(a)}} + i\mathcal{A}^{(a)} \right] - i\sqrt{2}\psi^{(b)} \left[\frac{\partial}{\partial z^{(b)}} + i\mathcal{A}^{(b)} \right]. \quad (4.15)$$

The magnetic fields $\mathcal{B}^{ss'} = 2i \left(\bar{\partial}^s \mathcal{A}^{s'} - \partial^{s'} \bar{\mathcal{A}}^s \right)$ are

$$\begin{aligned} \mathcal{B}^{11} &= \mathcal{B}(\bar{z}^{(a)}, z^{(a)}) + \mathcal{B}(\bar{z}^{(a+b)}, z^{(a+b)}), \\ \mathcal{B}^{12} &= \mathcal{B}^{21} = \mathcal{B}(\bar{z}^{(a+b)}, z^{(a+b)}), \\ \mathcal{B}^{22} &= \mathcal{B}(\bar{z}^{(b)}, z^{(b)}) + \mathcal{B}(\bar{z}^{(a+b)}, z^{(a+b)}), \end{aligned} \quad (4.16)$$

where $\mathcal{B}(\bar{z}, z)$ is the universal function written in (4.2).

We obtain

$$I = \frac{1}{4\pi^2} \int \prod_{s=1,2} d\bar{z}^s dz^s (\mathcal{B}_a \mathcal{B}_b + \mathcal{B}_a \mathcal{B}_{a+b} + \mathcal{B}_b \mathcal{B}_{a+b}). \quad (4.17)$$

Each term in (4.17) gives a unit contribution and we obtain the result $I = 3$.

A similar calculation for $SU(4)$ gives the magnetic field matrix

$$\begin{pmatrix} \mathcal{B}_a + \mathcal{B}_{a+b} + \mathcal{B}_{a+b+c} & \mathcal{B}_{a+b} + \mathcal{B}_{a+b+c} & \mathcal{B}_{a+b+c} \\ \mathcal{B}_{a+b} + \mathcal{B}_{a+b+c} & \mathcal{B}_b + \mathcal{B}_{a+b} + \mathcal{B}_{b+c} + \mathcal{B}_{a+b+c} & \mathcal{B}_{b+c} + \mathcal{B}_{a+b+c} \\ \mathcal{B}_{a+b+c} & \mathcal{B}_{b+c} + \mathcal{B}_{a+b+c} & \mathcal{B}_c + \mathcal{B}_{b+c} + \mathcal{B}_{a+b+c} \end{pmatrix} \quad (4.18)$$

in obvious notations (a, b, c being the simple roots). In the determinant, only the products of magnetic fields associated with all *different* positive roots survive, each such product giving a contribution $+1$ to the index integral. The number of such products is

$$I = \left\| \begin{array}{ccc} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{array} \right\| = 16. \quad (4.19)$$

For $SU(5)$,⁷ we obtain

$$I = \left\| \begin{array}{cccc} 4 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 2 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 \end{array} \right\| = 125. \quad (4.20)$$

The numbers on the diagonal of this matrix are the numbers of the positive roots involving a given simple root a, b, c or d in their simple root expansion. The adjacent numbers are the numbers of positive roots involving in the expansion *two* roots (a, b) , (b, c) and (c, d) . For example, the positive roots involving both b and c are $b + c$, $a + b + c$, $b + c + d$, $a + b + c + d$, and the corresponding matrix element is 4. Next, 2 is the number of positive roots involving *three* roots (a, b, c) and (b, c, d) in the expansion. Finally, there is only one root, $a + b + c + d$, involving all four simple roots in the expansion.

For $SU(6)$, the result is

$$I = \left\| \begin{array}{ccccc} 5 & 4 & 3 & 2 & 1 \\ 4 & 8 & 6 & 4 & 2 \\ 3 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 8 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{array} \right\| = 1296. \quad (4.21)$$

For an arbitrary N , the index integral is (conjectured to be)

$$I = N^{N-2}. \quad (4.22)$$

b) $Sp(4)$. We keep the notations (4.12) with the simple coroots $\mathbf{b}, \mathbf{a}$ as in figure 2. There are altogether two long coroots $\mathbf{b}$, $\mathbf{b} + 2\mathbf{a}$ and two short ones $\mathbf{a}$, $\mathbf{b} + \mathbf{a}$. Note that, while, for $SU(N)$, z^p are all related to the roots as $z^p = \frac{1}{2}\alpha_p(\mathbf{z})$, for $Sp(4)$, this is true only for the short coroots (and long roots) whereas for the long coroots (and short roots), z^p and $\alpha_p(\mathbf{z})$ just coincide.

⁷To justify quite rigorously the estimates (4.20) and (4.21), one has to demonstrate that only the products of different positive roots survive in the determinant. To justify (4.22), one has to prove in addition that the observed pattern for the determinants (4.19)–(4.21) generalizes for any N . This is an interesting question to clarify.

Choose the superpotential as in (4.13). The supercharge has the same form as in (4.15) where now

$$\begin{aligned}\mathcal{A}^{(a)} &= -\frac{i}{2} \left(\frac{\bar{z}^{(a)}}{\bar{z}^{(a)}z^{(a)} + m^2} + \frac{\bar{z}^{(b+a)}}{\bar{z}^{(b+a)}z^{(b+a)} + m^2} + \frac{2\bar{z}^{(b+2a)}}{\bar{z}^{(b+2a)}z^{(b+2a)} + m^2} \right), \\ \mathcal{A}^{(b)} &= -\frac{i}{2} \left(\frac{\bar{z}^{(b)}}{\bar{z}^{(b)}z^{(b)} + m^2} + \frac{\bar{z}^{(b+a)}}{\bar{z}^{(b+a)}z^{(b+a)} + m^2} + \frac{\bar{z}^{(b+2a)}}{\bar{z}^{(b+2a)}z^{(b+2a)} + m^2} \right).\end{aligned}\quad (4.23)$$

Note the appearance of the coefficient 2 in the last term in $\mathcal{A}^{(a)}$. This corresponds to the coefficient 2 with which the simple coroot a enters in the expansion of the coroot $b + 2a$. The magnetic field determinant is

$$\begin{aligned}\left\| \begin{array}{cc} \mathcal{B}_a + \mathcal{B}_{b+a} + 4\mathcal{B}_{b+2a} & \mathcal{B}_{b+a} + 2\mathcal{B}_{b+2a} \\ \mathcal{B}_{b+a} + 2\mathcal{B}_{b+2a} & \mathcal{B}_b + \mathcal{B}_{b+a} + \mathcal{B}_{b+2a} \end{array} \right\| = \\ \mathcal{B}_a(\mathcal{B}_b + \mathcal{B}_{b+a} + \mathcal{B}_{b+2a}) + \mathcal{B}_{b+a}(\mathcal{B}_b + \mathcal{B}_{b+2a}) + 4\mathcal{B}_b\mathcal{B}_{b+2a}.\end{aligned}\quad (4.24)$$

The integral $(\int \mathcal{B}_a\mathcal{B}_b)/(4\pi^2)$ is equal to 1. Four other integrals of the products $\mathcal{B}_a\mathcal{B}_{b+2a}$, $\mathcal{B}_b\mathcal{B}_{b+a}$, $\mathcal{B}_a\mathcal{B}_{b+2a}$, and $\mathcal{B}_a\mathcal{B}_{b+a}$ are reduced to $\int \mathcal{B}_a\mathcal{B}_b$ by the variable change with a unit Jacobian and also give 1. On the other hand, the integral $(\int \mathcal{B}_b\mathcal{B}_{b+2a})/(4\pi^2)$ is equal to 1/4. We thus obtain the result

$$I = 6 \quad (4.25)$$

for the index integral.

c) G_2 . The positive coroots of G_2 are depicted in figure 4. The (regularized) supercharge has, again, the form (4.15) with

$$\begin{aligned}\mathcal{A}_a &= -\frac{i}{2} \left(\frac{\bar{z}^{(a)}}{\bar{z}^{(a)}z^{(a)} + m^2} + \frac{\bar{z}^{(b+a)}}{\bar{z}^{(b+a)}z^{(b+a)} + m^2} + \frac{2\bar{z}^{(b+2a)}}{\bar{z}^{(b+2a)}z^{(b+2a)} + m^2} \right. \\ &\quad \left. + \frac{3\bar{z}^{(b+3a)}}{\bar{z}^{(b+3a)}z^{(b+3a)} + m^2} + \frac{3\bar{z}^{(2b+3a)}}{\bar{z}^{(2b+3a)}z^{(2b+3a)} + m^2} \right), \\ \mathcal{A}_b &= -\frac{i}{2} \left(\frac{\bar{z}^{(b)}}{\bar{z}^{(b)}z^{(b)} + m^2} + \frac{\bar{z}^{(b+a)}}{\bar{z}^{(b+a)}z^{(b+a)} + m^2} + \frac{\bar{z}^{(b+2a)}}{\bar{z}^{(b+2a)}z^{(b+2a)} + m^2} \right. \\ &\quad \left. + \frac{\bar{z}^{(b+3a)}}{\bar{z}^{(b+3a)}z^{(b+3a)} + m^2} + \frac{2\bar{z}^{(2b+3a)}}{\bar{z}^{(2b+3a)}z^{(2b+3a)} + m^2} \right).\end{aligned}\quad (4.26)$$

The matrix of magnetic fields is

$$\begin{aligned}\mathcal{B}_{aa} &= \mathcal{B}_a + \mathcal{B}_{b+a} + 4\mathcal{B}_{b+2a} + 9\mathcal{B}_{b+3a} + 9\mathcal{B}_{2b+3a}, \\ \mathcal{B}_{ab} &= \mathcal{B}_{ba} = \mathcal{B}_{b+a} + 2\mathcal{B}_{b+2a} + 3\mathcal{B}_{b+3a} + 6\mathcal{B}_{2b+3a}, \\ \mathcal{B}_{bb} &= \mathcal{B}_b + \mathcal{B}_{b+a} + \mathcal{B}_{b+2a} + \mathcal{B}_{b+3a} + 4\mathcal{B}_{2b+3a}.\end{aligned}\quad (4.27)$$

As in the all cases considered above, only the products of magnetic fields corresponding to different coroots survive in the determinant.

Figure 4. Positive coroots for G_2 .

1. There are 27 products of the type $\mathcal{B}_b \mathcal{B}_{b+3a}$ where both coroots are long, each such product entering with the Jacobian factor $1/9$.
2. There are 3 products of the type $\mathcal{B}_a \mathcal{B}_{b+a}$ when both coroots are short. They enter with a unit factor.
3. There are altogether 18 products when one of the coroots is long and another short. 12 such products of the type $\mathcal{B}_a \mathcal{B}_{2b+3a}$ correspond to orthogonal coroots, they enter with the Jacobian factor $1/4$. 6 other long-short products are of the type $\mathcal{B}_a \mathcal{B}_{2b+3a}$. They enter with the unit Jacobian.

Adding all together, we obtain

$$I = \frac{27}{9} + 3 + \frac{12}{4} + 6 = 15. \quad (4.28)$$

4.3 Strings on the maximal tori

To make the spectrum discrete and the index well-defined, we have to compactify $\mathcal{C}^r \rightarrow T_G \times T_G$ where T_G is the maximal torus of the corresponding group. This amounts to solving the problem for a regular lattice of strings.

a) $SU(N)$. Consider $SU(3)$ first. Consider the lattice of $SU(3)$ Dirac strings placed at the nodes of the lattice in figure 1. As a first try, we replace (4.14) by the sums

$$\begin{aligned} \mathcal{A}^{(a)} &= -\frac{i}{2} \sum_{pq} \left(\frac{\bar{z}_{pq}^{(a)}}{\bar{z}_{pq}^{(a)} z_{pq}^{(a)} + m^2} + \frac{\bar{z}_{pq}^{(a+b)}}{\bar{z}_{pq}^{(a+b)} z_{pq}^{(a+b)} + m^2} \right), \\ \mathcal{A}^{(b)} &= -\frac{i}{2} \sum_{pq} \left(\frac{\bar{z}_{pq}^{(b)}}{\bar{z}_{pq}^{(b)} z_{pq}^{(b)} + m^2} + \frac{\bar{z}_{pq}^{(a+b)}}{\bar{z}_{pq}^{(a+b)} z_{pq}^{(a+b)} + m^2} \right), \end{aligned} \quad (4.29)$$

where $z_{pq}^{(a)} = (\mathbf{z} + p\mathbf{a} + q\mathbf{b}) \cdot \mathbf{a}$, etc with complex integer p, q .

Several remarks are in order here, however.

First of all, one should not sum over *all* p, q . Such sum would be infinite by two different reasons:

1. The projections $z_{pq}^{(a)}$ etc depend not on p, q separately, but on their certain combinations: $2p - q$ for $z_{pq}^{(a)}$, $2q - p$ for $z_{pq}^{(b)}$, and $p + q$ for $z_{pq}^{(a+b)}$. To avoid a ∞^2 -fold counting, one should only sum over *these* parameters:

$$\sum_{pq} \frac{\bar{z}_{pq}^{(a)}}{\bar{z}_{pq}^{(a)} z_{pq}^{(a)} + m^2} \rightarrow \sum'_{2p-q} \frac{\bar{z}_{pq}^{(a)}}{\bar{z}_{pq}^{(a)} z_{pq}^{(a)} + m^2}, \quad (4.30)$$

etc.

2. A *naive* sum in the R.H.S. of (4.30) still diverges. The prime put there means subtraction of a certain linear function $\alpha \mathbf{z} + \beta$ with infinite coefficients, like in (4.10).

Second, one can be easily convinced that the potentials (4.29) are singular not only at the nodes of the lattice, but also at the fundamental coweights points (2.18) translated with $p\mathbf{a} + q\mathbf{b}$. The torus $T_{\text{SU}(3)} \times T_{\text{SU}(3)}$ involves 9 such points. In other words, one could consider right from the beginning the sums (4.29) with $z_{pq}^{(a)} = (\mathbf{z} + p\Delta + q\Box) \cdot \mathbf{a}$, etc.

These potentials enter the supercharge (4.15) and the corresponding Hamiltonian. The wave functions satisfy the quasiperiodic boundary conditions with the same topology as the boundary conditions (1.10) with $k = 3$. As was the case for $\text{SU}(2)$, an asymmetric choice $\alpha_a(\mathbf{x}, \mathbf{y}) = 0$, $\beta_a(\mathbf{x}, \mathbf{y}) = 12\pi \mathbf{a}\mathbf{x}$ is more convenient for us. The spectrum is now discrete. There are $3 \cdot 9 = 27$ vacuum wave functions satisfying $Q\Psi = 0$ and these boundary conditions. In the massless limit, they acquire the form

$$|0\rangle_{\text{SU}(3) \text{ lattice}} = \sqrt{\frac{\Psi(\bar{\mathbf{z}})}{\Pi^{\text{SU}(3)}(\mathbf{z})}} \quad (4.31)$$

with $\Psi(\mathbf{z})$ being one of 27 theta functions satisfying the boundary conditions (2.10) with $k = 3$.

The index of such system is thus equal to 27. Note that all eigenfunctions (4.31) are singular (for nonzero mass, the singularity is smeared out).

It is the potentials (4.29) that are generated by the gluon loops in SYMCS with $\text{SU}(3)$ gauge group. However, one can also construct a less dense string lattice where the $\text{SU}(3)$ Dirac strings are placed *only* at the black blobs in figure 1 and not at the points (2.18). To this end, one should start not with the superpotential (4.13), but replace z^p there by the fundamental weights (with the factor $\frac{\sqrt{3}}{2}$), z^8 , $\frac{z^8}{2} \pm \frac{\sqrt{3}z^3}{2}$. The corresponding potentials have the same form as in (3.12), but with z^3 and z^8 interchanged.⁸

In this case, there is only one string in $T_{\text{SU}(3)} \times T_{\text{SU}(3)}$ and there are only three vacuum states. In the massless limit, their wave functions have the form

$$|0\rangle_{\text{spacy SU}(3) \text{ lattice}} = \sqrt{\frac{\Psi_{0,\Delta,\Box}(\bar{\mathbf{z}})}{\Theta^{\text{SU}(3)}(\mathbf{z})}} \quad (4.32)$$

(see (2.14), (2.16)).

⁸An alternative would be to stay with (3.12), but translate it over the lattice rotated by $\pi/2$ compared to figure 1.

A generalization to $SU(N)$ is straightforward. One can either place the properly defined $SU(N)$ strings at the nodes of the coroot lattice, in which case the index is N^{N-2} , or place them also at the fundamental coweights points giving $I = N^N$.

b) $Sp(4)$. As was mentioned after (2.25), the maximal torus of $Sp(4)$ involves a special point, $\mathbf{w}_1 = \mathbf{a} + \mathbf{b}/2$. This is a fundamental coweight corresponding to the element $-\mathbb{1}$ of the center Z_2 of $Sp(4)$. Bearing in mind further applications to SYMCS, consider the lattice of strings generated by the vectors $\mathbf{w}_1$ and $\mathbf{w}_2 = \mathbf{b}/2$. Each node of this lattice [there are four such nodes in $T_{Sp(4)} \times T_{Sp(4)}$] contributes 6 to the index. However, the fields

$$\begin{aligned}\mathcal{A}^{(a)} &= -\frac{i}{2} \sum'_{pq} \left(\frac{\bar{z}_{pq}^{(a)}}{\bar{z}_{pq}^{(a)} z_{pq}^{(a)} + m^2} + \frac{\bar{z}_{pq}^{(b+a)}}{\bar{z}_{pq}^{(b+a)} z_{pq}^{(b+a)} + m^2} + \frac{2\bar{z}_{pq}^{(b+2a)}}{\bar{z}_{pq}^{(b+2a)} z_{pq}^{(b+2a)} + m^2} \right), \\ \mathcal{A}^{(b)} &= -\frac{i}{2} \sum'_{pq} \left(\frac{\bar{z}_{pq}^{(b)}}{\bar{z}_{pq}^{(b)} z_{pq}^{(b)} + m^2} + \frac{\bar{z}_{pq}^{(b+a)}}{\bar{z}_{pq}^{(b+a)} z_{pq}^{(b+a)} + m^2} + \frac{\bar{z}_{pq}^{(b+2a)}}{\bar{z}_{pq}^{(b+2a)} z_{pq}^{(b+2a)} + m^2} \right)\end{aligned}\quad (4.33)$$

where $z_{pq}^{(a)} = (\mathbf{z} + p\mathbf{w}_1 + q\mathbf{w}_2) \cdot \mathbf{a}$, etc, and the symbol $\sum'_{pq}$ has the same meaning as in (4.30) [that is, for example, the first sum in the first line in (4.33) is actually the sum over the complex integer $p - q$ regularized as in (4.10)] are both singular in the massless limit not only at the nodes of this lattice, but also at the points (2.26) (with $p\mathbf{w}_1 + q\mathbf{w}_2$ added).

Consider one of these points, $\mathbf{z}_* = (\frac{1}{2}, 0)$. In this case, the projections $\mathbf{z}_{pq}^{(b)}$ and $\mathbf{z}_{pq}^{(b+2a)}$ associated with the long coroots are all nonzero and the corresponding contributions to the vector potentials (4.33) are not singular. One can disregard them. On the other hand, the short coroot projections $\mathbf{z}_*^{(b+a)}_{pq}$ vanish when $p + q = 0$ and $\mathbf{z}_*^{(a)}_{pq}$ vanish when $p - q = -1$.

It is convenient to pose now $\mathbf{z} = \mathbf{z}_* + \delta$ and express the supercharge in terms of $\delta^{(a)} = \delta_1$ and $\delta^{(a+b)} = \delta_2$, and similarly for ψ . Then at the vicinity of $\mathbf{z}_*$, one can neglect the nonsingular contributions due to the long coroots $\mathbf{b}$, $\mathbf{b} + 2\mathbf{a}$, and the supercharge acquires the form

$$Q = -i\sqrt{2} \left[\psi_1 \left(\frac{\partial}{\partial \delta_1} + \frac{1}{2\delta_1} \right) + \psi_2 \left(\frac{\partial}{\partial \delta_2} + \frac{1}{2\delta_2} \right) \right]. \quad (4.34)$$

In other words, the supercharge represents a sum of two $SU(2)$ supercharges and the Hamiltonian — the sum of two $SU(2)$ Hamiltonians. There is only one zero mode representing the product of the zero modes (4.7). The same is true for 11 other special points in (2.26).

The net index is equal to

$$I_{Sp(4)}^{\text{torus}} = 4 \cdot 6 + 12 = 36. \quad (4.35)$$

The result (4.35) refers to the lattice of strings generated by the vectors $\mathbf{w}_1$, $\mathbf{w}_2$. For the coroot lattice generated by $\mathbf{a}$ and $\mathbf{b}$, $T_{Sp(4)} \times T_{Sp(4)}$ involves just one genuine $Sp(4)$ string while the points $\mathbf{z} = \mathbf{w}_1, i\mathbf{w}_1, (1+i)\mathbf{w}_1$ involve, in contrast to what was the case for (4.29), only the ‘tensor products’ of two independent $SU(2)$ strings and contribute 1 to the index. We obtain

$$I_{Sp(4)}^{\text{coroot lattice}} = 6 + 3 = 9. \quad (4.36)$$

c) G_2 . G_2 has no center and the strings (4.26) are located only at the nodes of the coroot lattice in figure 4, $\mathbf{z} \rightarrow \mathbf{z} + p\mathbf{a} + q\mathbf{b}$. There is only one such node in $T_{G_2} \times T_{G_2}$ and this contributes 15 to the index.

Besides, there are 8 special points (2.29) where the function (2.28) has the zeroes of the 3-d order and simultaneously the vector potentials (4.26) translated over the lattice as in (4.29) and (4.33) are both singular. Consider for example the pole at $\mathbf{z}_* = \Delta = \mathbf{a} + \mathbf{b}/3$ ($\mathbf{b}$ being the coroot of G_2). It is not difficult to see that the only projections that vanish there are $z_{pq}^{(a)}$ with $2p - 3q = -1$, $z_{pq}^{(b+a)}$ with $-p + 3q = 0$, and $z_{pq}^{(b+2a)}$ with $p = -1$. They are all associated with the short coroots.

It follows then that, at the vicinity of $\mathbf{z}_*$ [and also at the vicinity of 7 other poles (2.29)], the gauge fields and the supercharge have the same form as for $SU(3)$. In other words, the contribution of each such pole to the index is 3.

Finally, the lattice sums $\sum_{pq} \mathcal{A}_{pq}^{(a,b)}$ are singular at nine points (2.30). The associated singularities are “simple”, like in (4.34). Each such singularity contributes 1 to the index. We finally obtain

$$I_{G_2}^{\text{torus}} = 15 + 3 \cdot 8 + 9 = 48. \quad (4.37)$$

4.4 Index of SYMCS

a) $SU(2)$. The tree-level effective Hamiltonian describes the motion in a homogeneous magnetic field of flux $2k$. As was mentioned in the Introduction, gluon loops bring about the flux lines placed at four corners

$$z = 0, \frac{1}{2}, \frac{i}{2}, \frac{1+i}{2}, \quad (4.38)$$

each line carrying the flux +1. The extra net flux +4 brings about 4 extra states in the Hamiltonian. However, as was mentioned above, these extra states become singular in the massless limit and should be disregarded. If we keep mass finite, these states are not singular [cf. (4.4)], but they have an essential support in the region $|z - z_{\text{pole}}| \sim m$ where the Born-Oppenheimer approximation (on the basis of which the whole method is based) does not apply, and these extra states sitting in the string cores should be disregarded by that reason [4, 5].

There are also fermion loops bringing extra flux lines at the corners (4.38), but the flux of each fermion-induced line is $-1/2$. The Schrödinger problem in the field of an *individual* fractional flux line is ill-defined, but when there are four such lines, the net flux is integer and the explicit expressions for the vacuum BO functions can be written. They have the form

$$\chi_m^{\text{eff}}(z, \bar{z}) \sim e^{-\pi k \bar{z} z} e^{\pi k \bar{z}^2} Q_m^{2k-2}(\bar{z}) \Pi^{3/4}(\bar{z}) \Pi^{-1/4}(z) \quad (4.39)$$

with the functions Q_m^q , Π defined in (2.7), (2.8). The fractional powers of Π are due to fractional fluxes, but the function (4.39) is uniquely defined and vanishes at the corners.

The parameter m (nothing to do with the mass!) changes from 1 to $2k - 2$ and we see thus $2k - 2$ vacuum states.⁹

Let us remind how (4.39) is derived [5]. Each corner (4.38) carries the flux $1 - 1/2 = 1/2$; in each such corner, for example near the origin in the region $m \ll |z| \ll 1$, the effective wave function satisfies the equation

$$\left(\frac{\partial}{\partial z} + \frac{1}{4z}\right)\chi^{\text{eff}} = 0 \quad (4.41)$$

and hence involves the factor $\sim z^{-1/4}$. The effective wave function that behaves in such a way in the vicinity of each corner (4.38) and satisfies the proper twisted boundary conditions corresponding to the net flux $2k + 2$ has the form

$$\chi \propto \frac{Q_m^{2k+2}(\bar{z})}{[\Pi(z)\Pi(\bar{z})]^{1/4}}. \quad (4.42)$$

The requirement for the wave function to be regular at the corners implies that $Q_m^{2k+2}(\bar{z})$ has zeroes there and this means that it can be represented as $\Pi(\bar{z})Q_m^{2k-2}(\bar{z})$. Which brings us to (4.39), where the exponential factors are the same as in the tree-level wave functions. The first factor takes its origin in the constant part of the magnetic field. The second factor (together with the first one and with other factors) makes the probability density $\sim |\chi^{\text{eff}}|^2$ periodic.

Note that the actual number of vacuum SYMCS states is, however, less than $2k - 2$ because we have to impose an additional constraint for the states to be gauge-invariant, which entails Weyl-invariance of the effective wave functions. For $\text{SU}(2)$, this means invariance under $z \rightarrow -z, \bar{z} \rightarrow -\bar{z}$. This leaves only $(k - 1) + 1 = k$ states, in accordance with (1.5).

b) $\text{SU}(N)$, $\text{Sp}(4)$, G_2 . Consider $\text{SU}(3)$ first. The gluon loop corrections generate Dirac strings (3.12) at 9 points (2.18). The fermion loops generate there *fractional* strings with the vector potentials involving an additional factor $1/2$. Similarly to what was mentioned for the $\text{SU}(2)$ case, the Schrödinger problem in the field of an individual fractional string is ill-defined. However, it is *well* defined when there are 9 such strings and, on top of that, also the constant magnetic field corresponding to a half-integer level k . In the full analogy with (4.39), one can derive

$$\chi_{\text{SU}(3)}^{\text{eff}}(\mathbf{z}, \bar{\mathbf{z}}) \propto \left[\Pi^{\text{SU}(3)}(\bar{\mathbf{z}})\right]^{3/4} \left[\Pi^{\text{SU}(3)}(\mathbf{z})\right]^{-1/4} \theta^{\text{SU}(3)}(\bar{\mathbf{z}}), \quad (4.43)$$

⁹This counting coincides with the heuristic

$$\text{relevant flux} = (2k)_{\text{tree}} - 4 \cdot \left(\frac{1}{2}\right)_{\text{ferm. loops}} = 2k - 2, \quad (4.40)$$

with the contribution of the gluon loops disregarded.

where $\theta^{\text{SU}(3)}(\bar{\mathbf{z}})$ is a theta function satisfying the boundary conditions (2.10) with $k_{\text{eff}} = k - \frac{3}{2}$ [cf. (1.7) !] . This gives

$$I = 3 \left(k - \frac{3}{2} \right)^2 \quad (4.44)$$

states and, after imposing the Weyl invariance condition, results in $\frac{k^2-1/4}{2}$ vacuum states in SYMCS theory, in accordance with (1.6).

Note that the counting (4.44) has a relationship to the value of the index, $I = 27$, for the lattice of $\text{SU}(3)$ Dirac strings including fundamental coweights, which was evaluated earlier. Indeed, disregard in (4.44) the constant field (set $k = 0$) and replace $3/2$ by 3 (go over from the fermion-induced fractional strings to gluon-induced integer strings). We obtain $I = 27$.

In a similar way, we obtain

$$I = N \left(k - \frac{N}{2} \right)^{N-1} \quad (4.45)$$

“pre-Weyl” states for $\text{SU}(N)$, which leads to (1.4).

Let us discuss $\text{Sp}(4)$. The effective wave function has exactly the same form as in (4.43) where one should replace the functions (2.17) by the functions (2.25), while the function $\theta^{\text{Sp}(4)}(\bar{\mathbf{z}})$ satisfies the boundary conditions (2.23) with $k_{\text{eff}} = k - \frac{3}{2}$ (the Casimir eigenvalues for $\text{SU}(3)$ and $\text{Sp}(4)$ are the same). This gives

$$I = 4 \left(k - \frac{3}{2} \right)^2 \quad (4.46)$$

pre-Weyl states. If setting $k = 0$ and replacing $3/2 \rightarrow 3$, we obtain 36 states, which conforms with the counting (4.35). When Weyl invariance requirement is imposed, only $\frac{k^2-1/4}{2}$ states is left,¹⁰ the same number as for $\text{SU}(3)$.

The effective wave functions for G_2 have a similar form. The difference with $\text{SU}(3)$ is that the tree value of k is shifted down not by $3/2$, but by $\frac{1}{2}c_V[G_2] = 2$. This gives

$$I = 3 (k - 2)^2 \quad (4.48)$$

pre-Weyl states. If setting $k = 0$ and replacing $2 \rightarrow 4$, we obtain 48 states, which conforms with the counting (4.37). After Weyl invariance requirement is imposed, the final result for the index is [5]

$$I_{G_2}^{\text{tree}}(k) = \left\{ \begin{array}{ll} \frac{(|k|+2)^2}{4} & \text{for even } k \\ \frac{(|k|+1)(|k|+3)}{4} & \text{for odd } k \end{array} \right\} . \quad (4.49)$$

The estimates (4.44), (4.45), (4.46), (4.48) confirm the recipe (1.8).

¹⁰This number is obtained setting $r = 2$ in the general tree-level result [3] for $\text{Sp}(2r)$,

$$I = \binom{k+r}{r} \quad (4.47)$$

and replacing $k \rightarrow k - 3/2$.

c) Other groups. In addition to the groups discussed above, the SYMCS index can be easily evaluated also for higher symplectic groups. At the tree level, the pre-Weyl counting for $\text{Sp}(2r)$ is $(2k)^r$. When Weyl invariance requirement is imposed, the result (4.47) is obtained. When loop effects are taken into account, the reasoning above for $\text{SU}(2)$ and $\text{SU}(3)$ can be repeated without change. We are led to the relation

$$\chi_{\text{Sp}(2r)}^{\text{eff}}(\mathbf{z}, \bar{\mathbf{z}}) \propto \left[\Pi^{\text{Sp}(2r)}(\bar{\mathbf{z}}) \right]^{3/4} \left[\Pi^{\text{Sp}(2r)}(\mathbf{z}) \right]^{-1/4} \theta^{\text{Sp}(2r)}(\bar{\mathbf{z}}), \quad (4.50)$$

where $\theta^{\text{Sp}(2r)}(\bar{\mathbf{z}})$ is a theta function depending on r complex arguments and satisfying an obvious generalization of the boundary conditions (2.23) with k replaced by $k - \frac{r+1}{2}$. This finally gives for positive k

$$I^{\text{Sp}(2r)} = \binom{k + \frac{r-1}{2}}{r}, \quad (4.51)$$

and $I^{\text{Sp}(2r)}(-k) = (-1)^r I^{\text{Sp}(2r)}(k)$.

For more complicated groups, the tree-level calculation has not been performed yet, but the $\text{SU}(2)$ reasoning regarding a proper account of loop corrections *can* be generalized for an arbitrary group. The recipe (1.8) is thus confirmed even though the L.H.S. and the R.H.S. of this relation are not yet known in a general case.

For $\text{SU}(N)$, $\text{Sp}(4)$, and G_2 , we also observed that the pre-Weyl SYMCS index counting matches well the evaluations of the index of the string lattices performed in section 4.3. One can *conjecture* that this matching works also for more complicated groups, but an *explicit* proof of this statement is not so easy. We have seen that, for non-unitary groups, the counting expected on the basis of the SYMCS analysis is reproduced after adding *different* nontrivial contributions to the index [see eqs.(4.35) and (4.37)].

It would be nice to explore this interesting conspiracy for higher orthogonal and exceptional groups.

5 Discussion

Our motivation to perform this study was the wish to prove more or less rigourously the assertion (1.8) for the groups of higher rank. We believe that this goal has now been achieved.

However, a byproduct of such a study — an analysis of generalized multidimensional Dirac strings might prove to be also interesting, even more interesting than this anticipated result. The index integrals associated with these generalized strings are nontrivial. Even for $\text{SU}(N)$, we were able to explicitly calculate it only for $N \leq 6$. The result (4.22) is a conjecture. In other cases, we performed this calculation only for the groups of rank 2.

It would be interesting to calculate these integrals also for other groups. Physics applications disregarding, each such integral represents an integer number associated with a given group. What is a mathematical nature of this number ? How can it be explained ? This question is akin to the question of the so called *principal contribution* to the index in maximally supersymmetric gauge matrix models. For a given simple group, the principal

contribution to the index represents a bizarre fractional number (see [25], [26] and references therein) whose mathematical nature is now obscure.

The problem of calculating the index for a lattice of such generalized Dirac strings proved to be also rather nontrivial. Here the final result was anticipated on the base of the SYMCS analysis. However, for non-unitary groups, it was obtained as a sum of different nontrivial contributions. This reminds the story for the Witten index of pure SYM theory in four dimensions. The physical arguments provide a universal answer: $I = c_V \equiv h^\vee$ for any gauge group. However, to reproduce this result for higher orthogonal and exceptional groups in the framework of BO analysis turned out to be a highly nontrivial task, and it took almost 20 years to finally resolve it [27–29]. The value c_V is obtained in these cases as a sum of different nontrivial contributions.

One can also recall the problem of index evaluation for SYMCS theories with matter. In [30], we verified in some special cases the generic conjecture of [31] for the index of a SYMCS theory with $SU(N)$ gauge group involving also matter multiplets in different representations. Also in this case one can observe a kind of conspiracy: a simple result is obtained as a sum of complicated individual terms.

It would be rather desirable to achieve a better understanding of all these conspiracies.

A Proof ¹¹ of the relation (2.22)

The Casimir operator is defined as

$$\hat{\mathcal{C}}x = [T^a, [T^a, x]], \quad (\text{A.1})$$

where T^a represent a particular orthogonal basis in the Lie algebra normalized with respect to the Killing form $\langle x, y \rangle$ normalized such that $\langle \alpha^\vee, \alpha^\vee \rangle = 4$ for *short* coroots. This corresponds to the normalization $\langle \alpha, \alpha \rangle$ for the long roots. For example, for $su(N)$ with x being the Hermitian $N \times N$ matrices, this Killing form coincides with $2\text{Tr}\{xy\}$. Note the difference by the factor 4 with the convention (2.12) used throughout the text ! It is known that the operator (A.1) is proportional to the unit matrix.

Choose now the Chevalley basis

$$T^a = \left\{ h^a, \frac{E_p + E_{-p}}{2\sqrt{d_p}}, \frac{i(E_{-p} - E_p)}{2\sqrt{d_p}} \right\}, \quad (\text{A.2})$$

where h^a belong to the Cartan subalgebra, $E_{\pm p}$ are positive and negative root vectors normalized such that $[E_p, E_{-p}] = \alpha_p^\vee$, $d_p = 1$ for short coroots and $d_p = 2$ or $d_p = 3$ for long coroots. Choose $x = h$ in the Cartan subalgebra. The Casimir operator (A.1) is then rewritten as

$$\hat{\mathcal{C}}h = \sum_p \frac{1}{d_p} [E_p, [E_{-p}, h]] = \sum_p \frac{1}{d_p} \alpha_p^\vee \alpha_p(h). \quad (\text{A.3})$$

¹¹We constructed it together with J.-L. Milhorat whose aid I appreciate.

Projecting on h with the generalization of the convention (2.12) (such that the length of the short coroots is normalized to unity) and using

$$\langle \alpha_p^\vee, h \rangle = \frac{d_p}{2} \alpha_p(h), \quad (\text{A.4})$$

we arrive at (2.22).

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