The differential of all two-loop MHV amplitudes in $\mathcal{N} = 4$ Yang-Mills theory

John Golden^a and Marcus Spradlin^{a,b}

^a*Department of Physics, Brown University, Box 1843,
Providence, RI 02912-1843, U.S.A.*

^b*Theory Division, Physics Department, CERN,
1211 Geneva 23, Switzerland*

E-mail: john.golden@brown.edu, marcus_spradlin@brown.edu

ABSTRACT: We present an explicit analytic calculation of the differential of the planar n -particle, two-loop MHV scattering amplitude in $\mathcal{N} = 4$ super Yang-Mills theory. The result is expressed only in terms of the polylogarithm functions $\text{Li}_k(-x)$, for $k = 1, 2, 3$, with arguments x belonging to the special class of dual conformal cross-ratios known as cluster $\mathcal{X}$ -coordinates. The surprising fact that these amplitudes may be expressed in this way provides a striking example of the manner in which the cluster structure on the kinematic configuration space underlies the structure of amplitudes in SYM theory.

KEYWORDS: Scattering Amplitudes, Differential and Algebraic Geometry

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1 Introduction

Insights gleaned from difficult calculations have driven much of the recent progress in unlocking the structure of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory. The synergy between technical and conceptual advances frequently enables the undertaking of previously impossible calculations; the often surprising results of which in turn fuel a deeper understanding of the theory. Indeed, in SYM theory the search for such insights, rather than the need for busywork, is precisely the motivation for carrying out arduous new calculations.

In this note we complete the analytic calculation of the differential $dR_n^{(2)}$ of the planar two-loop n -particle MHV amplitudes (or, more properly, the ‘remainder functions’ — with infrared divergences subtracted off in a standard way [1, 2]). Reasonably efficient numerical techniques for evaluating $R_n^{(2)}$ have been available for some time [3], and analytic formulas are known in two-dimensional kinematics [4] but a breakthrough towards unlocking the general analytic structure of these amplitudes was made by Caron-Huot in [5]. By considering the dual superconformal symmetry of a certain generalization of scattering amplitudes depending on twice the usual number of Grassmann variables, he was able to express $dR_n^{(2)}$ in terms of a certain combination of one-fold integrals. Here we complete the evaluation of these integrals and present analytic expressions for $dR_n^{(2)}$. We find that they can be expressed completely in terms of the functions $\text{Li}_k(-x)$, for $k = 1, 2, 3$ with arguments x always belonging to the set of cluster $\mathcal{X}$ -coordinates on the kinematic configuration space $\text{Conf}_n(\mathbb{P}^3)$. This provides strong support for the suggestion of [7] that the cluster structure of this space underlies the structure of amplitudes in SYM theory.

While this computation may seem modest in scope, we feel that it is a useful example in which to showcase the power of the two most recent additions to the amplitudeologist’s toolkit: positivity and cluster $\mathcal{X}$ -coordinates. In particular, the positive domain, a subset of the kinematic domain $\text{Conf}_n(\mathbb{P}^3)$, is evidently the natural habitat for amplitudes in SYM theory, and cluster $\mathcal{X}$ -coordinates provide a natural set of arguments for these amplitudes to ‘depend on’. It is our hope that other analytic formulae in SYM theory can be unlocked using these or similar approaches. In particular, it would be very interesting to see if the

monodromies of $R_n^{(2)}$, which were computed in integral form in [8] (see also [9]), can be similarly expressed only in terms of $\text{Li}_k(-x)$.

Following ample motivation (but minimal review), we present our analytic expression for $dR_n^{(2)}$. For additional introductory material, we refer the reader to [7] and the references therein.

2 Motivation and review

In [5], Caron-Huot expressed the differential of the planar two-loop n -particle MHV amplitude in terms of momentum twistors [6] as

$$dR_n^{(2)} = \sum_{i,j} C_{i,j} d\log\langle i-1 \ i \ i+1 \ j \rangle \quad (2.1)$$

and presented a means of calculating the coefficient functions $C_{i,j}$. Already the fact that the differential may be expressed in the form of eq. (2.1) is a nontrivial all-loop-order prediction of [5].

Thanks to the full dihedral symmetry of the amplitude in the particle labels, it is sufficient to focus on $C_{2,i}$. This was given as a sum of four components $C_{2,i}^{(a)}$, for $a = 1, 2, 3, 4$, two of which were represented analytically as

$$C_{2,i}^{(1)} = \log u \times \sum_{j=2}^{i-1} \sum_{k=i}^{n+1} \left[\text{Li}_2(1 - u_{j,k,k-1,j+1}) + \log \frac{x_{j,k}^2}{x_{j+1,k}^2} \log \frac{x_{j,k}^2}{x_{j,k-1}^2} \right] \quad (2.2)$$

where for our purposes it is sufficient to take $x_{i,j}^2 = \langle i \ i+1 \ j \ j+1 \rangle$, and

$$C_{2,i}^{(4)} = -2 \text{Li}_3(1 - \frac{1}{u}) - \text{Li}_2(1 - \frac{1}{u}) \log u - \frac{1}{6} \log^3 u + \frac{\pi^2}{6} \log u, \quad (2.3)$$

where

$$u_{i,j,k,l} = \frac{\langle i \ i+1 \ j \ j+1 \rangle \langle k \ k+1 \ l \ l+1 \rangle}{\langle i \ i+1 \ k \ k+1 \rangle \langle j \ j+1 \ l \ l+1 \rangle} \quad (2.4)$$

and $u \equiv u_{2,i-1,i,1}$. $C_{2,i}^{(2)}$ has the integral representation

$$\begin{aligned} C_{2,i}^{(2)} = & \sum_{j=4}^{i-2} \int_1^2 d_X \log \frac{\langle X i-1 i \rangle}{\langle X i j \rangle} \left(\text{Li}_2(1 - u_{X,i-1,i,j}) - \text{Li}_2(1 - u_{X,i-1,i,j-1}) \right. \\ & + \text{Li}_2(1 - u_{i,j-1,j,i-1}) + \text{Li}_2(1 - u_{X,j,j-1,i-1}) - \text{Li}_2(1 - u_{X,j,j-1,i}) \\ & \left. + \log u_{X,i-1,i,j} \log u_{X,j,j-1,i-1} \right) \\ & + \sum_{j=4}^{i-2} \int_1^2 d_X \log \frac{\langle X i-1 i \rangle}{\langle X i i+1 \rangle} \left(\text{Li}_2(1 - u_{i-1,1,X,j}) + \text{Li}_2(1 - u_{i,j,j-1,1}) \right. \\ & + \text{Li}_2(1 - u_{1,j,j-1,X}) - \text{Li}_2(1 - u_{i-1,1,X,j-1}) - \text{Li}_2(1 - u_{i,j,j-1,X}) \\ & \left. + \log u_{i-1,1,X,j} \log u_{i,j,j-1,1} \right) \\ & + \int_1^2 d_X \log \frac{\langle X i-1 i \rangle}{\langle X 34 \rangle} (\text{Li}_2(1 - u_{2,i,i-1,3}) - \text{Li}_2(1 - u_{X,i,i-1,3})) \end{aligned} \quad (2.5)$$

and $C_{2,i}^{(3)}$ may be determined from $C_{2,i}^{(2)}$ by imposing dihedral symmetry on eq. (2.1), which fixes

$$C_{2,i}^{(3)}(1, 2, \dots, n) = C_{2,4+n-i}^{(2)}(3, 2, \dots, 4). \quad (2.6)$$

While eqs. (2.2) and (2.3) along with the integral representation in eq. (2.5) were sufficient to compute the symbol of $dR_n^{(2)}$ in [5], any attempt at an analytic expression for $dR_n^{(2)}$ would have encountered two major obstacles:

Choosing a ‘good’ kinematic domain. First of all, n -particle scattering amplitudes in SYM theory are multi-valued functions on the $3(n-5)$ -dimensional kinematic configuration space $\text{Conf}_n(\mathbb{P}^3)$ (we refer the reader to [7] for a review of this notation), and in computing scattering amplitudes it is often very difficult to get all of the branch cuts in the right place. In particular, Mathematica’s `Integrate[]` function has no *a priori* understanding of $\text{Conf}_n(\mathbb{P}^3)$ and cannot be expected to easily return an analytic formula valid on the whole of this domain — at best it can produce numerical results at specifically inputted kinematic points, or an analytic formula valid on some subdomain of its own choosing. On physical grounds, amplitudes should be real-valued and singularity-free throughout the Euclidean domain, a subset of $\text{Conf}_n(\mathbb{P}^3)$. Even for what is in some sense the simplest non-trivial multi-loop amplitude, $R_6^{(2)}$, finding a formula with these properties was by far the hardest part of [10]. For $n > 6$, where the kinematic configuration space is far more complicated (due to Gram determinant constraints amongst dual conformal cross-ratios), finding explicit formulas valid throughout even just the Euclidean domain seems like a daunting challenge. Instead of giving up all hope, one would be content to find expressions valid even in some subset of the Euclidean domain. But, which subset should one look at? What principle, either mathematical or physical, could make any one subset more worthy of attention than another?

Choosing ‘good’ Li_k arguments. The second complication has to do with the class of iterated integrals of the type which define generalized polylogarithm functions and which appear in the result of [5]. Such functions can be partially characterized by their symbols, but integrating a function of this type can generate a function whose symbol contains entries that are, in general, arbitrary algebraic functions of the entries in the symbol of the original function. Already Caron-Huot’s result for the symbol of $R_n^{(2)}$ contains rather non-trivial algebraic functions on $\text{Conf}_n(\mathbb{P}^3)$, and one might have worried that even more complicated functions could appear after integration. Yet, true believers in SYM theory know well that it only ever produces very special polylogarithmic functions, not ‘general’ ones. One would therefore have every reason to expect only relatively straightforward expressions to actually appear in amplitudes. Unfortunately, unless one knows in advance what to look out for, the plethora of identities amongst polylogarithm functions makes it difficult to identify any particular set of arguments as preferred over any other set.

3 Tools and techniques

Fortunately, related recent advances have shown how to overcome both obstacles at the same time.

Cluster $\mathcal{X}$ -coordinates. We begin by addressing the question of Li_k arguments by noting that in [7] it was shown via examples at $n = 6, 7$, and suggested more generally, that (in a sense made precise in that paper) two-loop MHV amplitudes only ‘depend on’ certain very special variables. These variables are called cluster $\mathcal{X}$ -coordinates on the kinematic configuration space $\text{Conf}_n(\mathbb{P}^3)$, which may be realized as the Grassmannian quotient $\text{Gr}(4, n)/(\mathbb{C}^*)^{n-1}$. For the purpose of this paper, the notion of ‘depending on’ only certain variables can be stated very clearly: in support of [7], we find that the differential $dR_n^{(2)}$ of all two-loop MHV amplitudes can be expressed only in terms of the functions $\text{Li}_k(-x)$, for $k = 1, 2, 3$ with arguments x always belonging to the set of cluster $\mathcal{X}$ -coordinates on $\text{Conf}_n(\mathbb{P}^3)$.

In fact, we find that (in line with the expectation expressed in the previous section) only relatively tame $\mathcal{X}$ -coordinates appear in $dR_n^{(2)}$, for any n . The most complicated of these is the cross-ratio

$$\frac{\langle 12i-1i \rangle \langle 123j \rangle \langle 2j-1jj+1 \rangle \langle j-1ji-1i \rangle}{\langle 12j-1j \rangle \langle 2ji-1i \rangle \langle i-1i(123) \cap (j-1jj+1) \rangle} \quad (3.1)$$

which depends on the eight points $\{1, 2, 3, j-1, j, j+1, i-1, i\}$. Here we use the notation

$$\langle ab(cde) \cap (fgh) \rangle \equiv \langle acde \rangle \langle b f g h \rangle - \langle bcde \rangle \langle a f g h \rangle. \quad (3.2)$$

The fact that the complexity of two-loop MHV amplitudes stabilizes (in this sense) already at $n = 8$ is manifest in the result of [5]. This implies that a full understanding of the cluster structure on $\text{Conf}_8(\mathbb{P}^3)$ should be sufficient to understand the structure of all such amplitudes.

The positive domain. We now turn to the question of the kinematical domain, where the sole dependence on cluster $\mathcal{X}$ -coordinates plays a fortuitous role. A salient feature of cluster $\mathcal{X}$ -coordinates is that they are positive-valued everywhere in the positive domain, which is the subset of the Euclidean domain defined by imposing that $\langle abcd \rangle > 0$ whenever $a < b < c < d$. With the differential $dR_n^{(2)}$ expressed completely in terms of the functions $\text{Li}_k(-x)$, it therefore becomes manifestly real-valued and singularity free throughout the positive domain. (Moreover, we believe that $R_n^{(2)}$ itself is also positive-valued throughout the positive domain, but that is another matter.) The importance of positivity for scattering amplitudes was first emphasized in [11], though in a somewhat different context. Their Grassmannian for the integrand involves both ‘external’ kinematic data as well as ‘internal’ loop integration variables, but it is now clear that positivity in the ‘external’ data alone plays a similarly important role for the fully integrated amplitudes which we study here.

Before proceeding let us add one crucial disclaimer. As reviewed in [7], for $n > 7$ the cluster algebra associated to $\text{Conf}_n(\mathbb{P}^3)$ has an infinite number of cluster $\mathcal{X}$ -coordinates (although of course only a finite number actually appear in $dR_n^{(2)}$). Moreover there is no known general classification of these coordinates, and given any cross-ratio there does not exist a general algorithm for determining whether or not it actually is a cluster $\mathcal{X}$ -coordinate. Therefore, for $n > 7$ we use the empirical criterion discussed in section 6.6 of [7]: we say that a cross-ratio x is a cluster $\mathcal{X}$ -coordinate if $1+x$ factors into a product of four-brackets as a consequence of Plücker relations and if $x > 0$ everywhere inside the positive domain.

Given the two advances we have reviewed — the understanding that the natural domain on which to study multi-loop amplitudes in SYM theory is the positive domain, and that the natural set of arguments appearing inside the Li_k functions for two-loop MHV amplitudes are cluster $\mathcal{X}$ -coordinates — analytically integrating eq. (2.5) transforms from being merely possible to being inevitable.

4 Results

In this section we present the Li_3 and $\text{Li}_2 \times \text{Li}_1$ contributions to $dR_n^{(2)}$. There are, in addition, terms of the form Li_1^3 and $\pi^2 \text{Li}_1$ which are too numerous to efficiently display here. For this reason we attach to this submission a Mathematica notebook which contains the necessary expressions and which can construct the full analytic formula for $dR_n^{(2)}$ for any given n .

While the original integral representation of $C_{i,j}$ required the general object $u_{i,j,k,l}$, the integrated result can be written in terms of the slightly smaller class of cross-ratios defined by $u_{i,j,k} = u_{i,j,k,i-1}$. The cluster $\mathcal{X}$ -coordinates related to the three-index u 's (in the same sense that the v_i and the u_i used in [7] for $n = 6$ are related to each other) are

$$v_{i,j,k} = \frac{1}{u_{i,j,k}} - 1 = -\frac{\langle i(i-1 \ i+1)(j \ j+1)(k \ k+1) \rangle}{\langle i-1 \ i \ k \ k+1 \rangle \langle i \ i+1 \ j \ j+1 \rangle}. \quad (4.1)$$

Here we use the notation

$$\langle a(bc)(de)(fg) \rangle \equiv \langle abde \rangle \langle acfg \rangle - \langle abfg \rangle \langle acde \rangle. \quad (4.2)$$

For any n , $v_{i,j,k}$ is a cluster $\mathcal{X}$ -coordinate as long as $i < j < k \pmod{n}$. We also find it useful to define another type of ratio,

$$w_{i,j,k} = \frac{v_{i,k-1,k} u_{i,k-1,k}}{v_{j,k-1,k}} \quad (4.3)$$

which is a cluster $\mathcal{X}$ -coordinate when $i = j \pm 1$. Because we are interested only in objects of the form $\text{Li}(-x)$, where x is a cluster $\mathcal{X}$ -coordinate, we will make one more definition

$$\text{li}_k(x) = \text{Li}_k(-x) \quad (4.4)$$

purely in the interest of cleaning up the notation.

We begin with the essentially semantic task of converting $C^{(1)}$ and $C^{(4)}$ into $\mathcal{X}$ -coordinate form. Expressing $C_{2,i}^{(4)}$ in the desired form is trivial:

$$C_{2,i}^{(4)} = -2 \text{li}_3(v_{2,i-1,i}) - \text{li}_2(v_{2,i-1,i}) \text{li}_1(v_{2,i-1,i}) + \mathcal{O}(\text{li}_1^3). \quad (4.5)$$

$C^{(1)}$ is slightly more difficult, since it involves non-conformal objects such as $\log \frac{x_{j,k}^2}{x_{j,k-1}^2}$. Of course, the expression as a whole is conformally invariant, but it takes some combinatoric effort to recast things as an explicit sum over conformal cross-ratios. This is only a problem for the $\mathcal{O}(\text{Li}_1^3)$ ratios, our results for this are in the attached Mathematica notebook. For

$\text{Li}_2 \times \text{Li}_1$, we find that the ratios appearing are in fact already in cluster $\mathcal{X}$ -coordinate form, giving

$$C_{2,i}^{(1)} = -\text{li}_1(v_{2,i-1,i}) \left(\sum_{j=2}^{i-2} \sum_{k=i}^n \text{li}_2(v_{j+1,k,k+1}) - \sum_{k=i+2}^n \text{li}_2(v_{k,2,i-1}) - \sum_{j=4}^{i-2} \text{li}_2(v_{j,i,1}) \right) + \mathcal{O}(\text{li}_1^3). \quad (4.6)$$

This sum looks slightly different than eq. (2.2) because we find it useful to make explicit the behavior of $1 - u_{j,k,k-1,j+1}$ at various upper and lower limits of the j, k summation indices. The double-sum of eq. (4.6) contains some boundary terms which diverge — specifically, when j, k equal $2, n$ or $i - 2, i$. However, these divergences cancel when the $\mathcal{O}(\text{Li}_1^3)$ terms are added. The attached Mathematica file identifies and discards these canceling divergent terms.

We now turn to the main focus of this note: the evaluation of the remaining integrals in $C_{2,i}^{(2)}$. For fixed i, j , the terms on the first four lines of eq. (2.5) altogether depend on at most nine distinct momentum twistors. We can therefore focus our efforts entirely on evaluating the integrals for the case $n = 9, i = 8, j = 5$, and then replace $\{4, 5, 6, 7, 8, 9\} \rightarrow \{j-1, j, j+1, i-1, i, i+1\}$ to recover the summand for arbitrary i, j . One may immediately worry about divergences at boundary terms in the sum, such as $j = i - 2$; while these cases do create individual terms that are divergent, these divergences cancel out in the full function.

In order to present our result for $C_{2,i}^{(2)}$ in a succinct fashion, let us define the following permutation operators acting on the particle labels:

$$\sigma_a = Z_{a-1} \leftrightarrow Z_{a+1} \quad (4.7)$$

$$\sigma_{a,b} = 1 - \sigma_a - \sigma_b + \sigma_a \sigma_b \quad (4.8)$$

$$\sigma_{a,b,c} = 1 - \sigma_a - \sigma_b - \sigma_c + \sigma_a \sigma_b + \sigma_a \sigma_c + \sigma_b \sigma_c - \sigma_a \sigma_b \sigma_c \quad (4.9)$$

as well as the four cross-ratios

$$\begin{aligned} R_1 &= \frac{\langle 12j-1j \rangle \langle 2ji-1i \rangle \langle j-1jj+1i \rangle}{\langle 2j-1jj+1 \rangle \langle 12ji \rangle \langle j-1ji-1i \rangle}, \\ R_2 &= \frac{\langle 2i-1ii+1 \rangle \langle 12ij \rangle \langle i-1ij-1j \rangle}{\langle 12i-1i \rangle \langle 2ij-1j \rangle \langle i-1ii+1j \rangle}, \\ R_3 &= \frac{\langle 123j \rangle \langle 12i-1i \rangle \langle 2j-1ji \rangle}{\langle 123i \rangle \langle 12j-1j \rangle \langle 2i-1ij \rangle}, \\ R_4 &= \frac{\langle 12i-1i \rangle \langle 123j \rangle \langle 2j-1jj+1 \rangle \langle j-1ji-1i \rangle}{\langle 12j-1j \rangle \langle 2ji-1i \rangle \langle i-1i(123) \cap (j-1jj+1) \rangle}. \end{aligned} \quad (4.10)$$

These ratios have the following $1 + R_i$ factorizations:

$$\begin{aligned} 1 + R_1 &= \frac{\langle 2j-1ji \rangle \langle j(j-1j+1)(12)(i-1i) \rangle}{\langle 2j-1jj+1 \rangle \langle 12ji \rangle \langle j-1ji-1i \rangle}, \\ 1 + R_2 &= \frac{\langle 2ji-1i \rangle \langle i(i-1i+1)(12)(j-1j) \rangle}{\langle 12i-1i \rangle \langle 2ij-1j \rangle \langle i-1ii+1j \rangle}, \end{aligned}$$

$$\begin{aligned}
 1 + R_3 &= -\frac{\langle 12ji \rangle \langle 2(13)(j-1j)(i-1i) \rangle}{\langle 123i \rangle \langle 12j-1j \rangle \langle 2ji-1i \rangle}, \\
 1 + R_4 &= -\frac{\langle 2(13)(j-1j)(i-1i) \rangle \langle j(j-1j+1)(12)(i-1i) \rangle}{\langle 12j-1j \rangle \langle 2ji-1i \rangle \langle i-1i(123) \cap (j-1jj+1) \rangle},
 \end{aligned} \tag{4.11}$$

None of the R_i depend on more than eight points, and we have checked that they are all $\mathcal{X}$ -coordinates of the $\text{Gr}(4, 8)$ cluster algebra (for more information on generating cluster $\mathcal{X}$ -coordinates via mutation, see [7]).

Note that there is additional symmetry present in the arguments of R_1 and R_2 , which are related via $R \rightarrow 1/R$ combined with $i \leftrightarrow j$.

Given these definitions, we find that $C_{2,i}^{(2)}$ has the following relatively simple Li_3 contribution:

$$\begin{aligned}
 &\text{li}_3(v_{2,i-1,i}) - \text{li}_3(w_{3,2,i}) + \text{li}_3(w_{2,3,i}) \\
 &\quad - \sum_{j=4}^{i-2} \sigma_{2,i,j} \left(\text{li}_3(R_1) + \text{li}_3(R_2) + \text{li}_3(R_3) - \text{li}_3(R_4) \right).
 \end{aligned} \tag{4.12}$$

The three symmetries lead to 32 distinct Li_3 's in each term in the j sum. This presentation has discarded all terms which cancel telescopically in the sum. One can alternatively incorporate the non-summand terms into the summand by including some telescopic cancellations, thus increasing the total number of Li_3 terms (still all with cluster $\mathcal{X}$ -coordinates as arguments) in the summand to 48.

While there are no remaining telescopic cancellations in eq. (4.12), the full 32 Li_3 terms do not appear for all values of i and j .¹ For example, six of the Li_3 terms go to zero at the upper limit of the sum, $j = i - 2$.

It is interesting to note that the evaluation of the integrals in eq. (2.5), for the general case $n = 9, i = 8, j = 5$, necessarily produces Li_3 's with *non*-cluster arguments. However, these terms cancel telescopically in the j sum and are zero at the $j = 4$ and $j = i - 2$ boundaries.

Next we turn to the $\text{Li}_2 \times \text{Li}_1$ contribution to $C_{2,i}^{(2)}$, which we represent here as

$$\begin{aligned}
 &\text{li}_1(v_{2,i-1,i}) (\text{li}_2(v_{i,1,3}) - \text{li}_2(v_{3,i-1,i})) - (\text{li}_1(v_{2,i-1,i}) + \text{li}_1(v_{3,i-1,i})) \text{li}_2(w_{2,3,i}) \\
 &\quad + \sum_{j=4}^{i-2} \sigma_{2,j} \left(\text{li}_1(v_{i,2,j-1}) (\text{li}_2(R_2) - \text{li}_2(R_3)) + \text{li}_1(v_{i,1,2}) \sigma_i \text{li}_2(R_2) \right. \\
 &\quad + \frac{1}{2} \left(\text{li}_1(v_{j,i-1,1}) + \left(\frac{1}{2} \text{li}_1(v_{j,1,2}) - \text{li}_1(v_{2,j-1,i-1}) \right) (1 - \sigma_i) - \text{li}_1(v_{j,i,1}) \sigma_i \right) \times \\
 &\quad \quad \quad \left. \times (\text{li}_2(R_1) - \text{li}_2(R_3) + \text{li}_2(R_4)) \right) \\
 &\quad + \text{li}_1(v_{i,1,2}) (1 - \sigma_j) ((1 - \sigma_i) \sigma_2 \text{li}_2(R_2) - (\sigma_2 + \sigma_i) \text{li}_2(R_3) + \sigma_i \text{li}_2(R_4)).
 \end{aligned} \tag{4.13}$$

¹The exact counting is: for $n > 6$ the total number of Li_3 terms in $C_{2,i}^{(2)}$ is given by $32i - 169$ for $5 < i < n$ and $26n - 141$ for $i = n$. For $n = 6$, $C_{2,6}^{(2)}$ has 11 Li_3 terms. $C_{2,i}^{(2)} = 0$ for $i \leq 5$.

Note that $\sigma_a v_{a,b,c}$ is not a cluster $\mathcal{X}$ -coordinate, so some of the terms in eq. (4.13) aren't obviously in the form $\text{li}_k(x)$. However, this obstruction is easily removed by the relationship $\sigma_a \text{li}_1(v_{a,b,c}) = -\text{li}_1(v_{a,b,c})$.

We have checked our full result for $C_{2,i}^{(2)}$ against numerical integrations of (2.5) for several hundred random kinematic points in the positive Grassmannian for $n \leq 12$.

It is important to emphasize that our particular expression for eqs. (4.13) is far from being unique. Some additional symmetry can be introduced by adding terms that telescopically cancel, as was the case with the Li_3 sum. Furthermore, there exist numerous non-trivial identities amongst the Li_2 terms in eq. (4.13), introducing additional redundancy. This particular representation was chosen simply because it was the most typographically concise presentation we could deduce.

We conclude this note with a brief discussion of parity (see appendix A of [7] for a thorough introduction). The parity invariance of $dR_n^{(2)}$ follows from the fact that $*C_{i,j} = C_{j,i}$ and $\sum_i C_{i,j} = 0$. These properties can be made manifest at the level of the symbol, as was done in [5], but our integrated results sacrifice explicit parity invariance for brevity. Of course, parity invariance in our functional representation can be confirmed through the application of (numerous) polylogarithm identities.

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References

- [1] C. Anastasiou, Z. Bern, L.J. Dixon and D. Kosower, *Planar amplitudes in maximally supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **91** (2003) 251602 [[hep-th/0309040](#)] [[INSPIRE](#)].
- [2] Z. Bern, L.J. Dixon and V.A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev. D* **72** (2005) 085001 [[hep-th/0505205](#)] [[INSPIRE](#)].
- [3] C. Anastasiou et al., *Two-loop polygon Wilson loops in $N = 4$ SYM*, *JHEP* **05** (2009) 115 [[arXiv:0902.2245](#)] [[INSPIRE](#)].
- [4] P. Heslop and V.V. Khoze, *Analytic results for MHV Wilson loops*, *JHEP* **11** (2010) 035 [[arXiv:1007.1805](#)] [[INSPIRE](#)].
- [5] S. Caron-Huot, *Superconformal symmetry and two-loop amplitudes in planar $N = 4$ super Yang-Mills*, *JHEP* **12** (2011) 066 [[arXiv:1105.5606](#)] [[INSPIRE](#)].
- [6] A. Hodges, *Eliminating spurious poles from gauge-theoretic amplitudes*, *JHEP* **05** (2013) 135 [[arXiv:0905.1473](#)] [[INSPIRE](#)].

- [7] J. Golden, A.B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Motivic amplitudes and cluster coordinates*, [arXiv:1305.1617](#) [[INSPIRE](#)].
- [8] D. Gaiotto, J. Maldacena, A. Sever and P. Vieira, *Pulling the straps of polygons*, *JHEP* **12** (2011) 011 [[arXiv:1102.0062](#)] [[INSPIRE](#)].
- [9] A. Sever and P. Vieira, *Multichannel conformal blocks for polygon Wilson loops*, *JHEP* **01** (2012) 070 [[arXiv:1105.5748](#)] [[INSPIRE](#)].
- [10] A.B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, *Classical polylogarithms for amplitudes and Wilson loops*, *Phys. Rev. Lett.* **105** (2010) 151605 [[arXiv:1006.5703](#)] [[INSPIRE](#)].
- [11] N. Arkani-Hamed et al., *Scattering amplitudes and the positive grassmannian*, [arXiv:1212.5605](#) [[INSPIRE](#)].