# Recurrence relations for toric $N=1$ superconformal blocks

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ABSTRACT: General 1-point toric blocks in all sectors of N=1 superconformal field theories are analyzed. The recurrence relations for blocks coefficients are derived by calculating their residues and large $\Delta$ asymptotics.

## Contents

## 1 Introduction

Correlation functions in any 2-dimensional CFT can be expressed in terms of three-point coupling constants and some universal, model independent functions called conformal blocks [1]. Such decompositions are not unique and the equivalence of various representations yields strong restrictions for coupling constants. Simplest conditions of this kind — the crossing symmetry of the 4-point function on the sphere and the modular invariance of the 1-point function on the torus — turned out to be sufficient for the consistency of all multi-point amplitudes on closed, oriented surfaces of arbitrary genus [2]. From this point of view the 4-point spheric and the 1-point toric conformal blocks are of main interest in any CFT. Both objects are defined as power series of corresponding modular parameters with coefficients expressed in terms of the inverse of the Gram matrix of a Virasoro algebra Verma module. In this form a direct calculation of higher order terms is prohibitively complicated.

An efficient recursive technique of calculating 4-point conformal blocks on the sphere was developed long time ago by Al. Zamolodchikov [3–5]. This method was used for numerical tests of crossing symmetry in the Liouville field theory [6] and in the $c \to 1$ limit of minimal models [7]. It was also applied for the numerical analysis of the classical limit of the conformal blocks and for the verification of new relations in the classical geometry of hyperbolic surfaces [8, 9]. Recently in the context of the AGT relation Zamolodchikov's method has been extended to the 1-point toric blocks [10, 11]. It found its application in one of the first proofs of the AGT correspondence [12]. It was also used in [11] to prove the relations between 1-point toric and 4-point spheric conformal blocks conjectured by Poghossian [10].

A recursion representation of 4-point spheric blocks in the N=1 superconformal field theories was first derived in the Neveu-Schwarz sector [14–18]. The extension to the Ramond sector initiated in [19] was recently completed in [20]. These results clarified the structure of the N=1 superconformal blocks and paved the way for investigations of their analytical properties [21–23]. They were also used for numerical verifications of crossing symmetry in the N=1 superconformal Liouville field theory [16, 17, 20].

In the present paper we address the problem of recursion representation of the 1-point toric conformal blocks in the N=1 SCFT. Our main motivation is the problem of modular invariance of 1-point functions on the torus in the N=1 superconformal Liouville field theory with the structure constants derived in [24, 25]. The corresponding problem in the Liouville theory was solved by showing that the modular invariance of a generic 1-point function on the torus is equivalent to the crossing symmetry of a special 4-point function on the sphere [26]. An essential step of this reasoning is a relation between the modular and the fusion matrices which can be derived using Poghosian identities [10, 11]. One may expect a similar, although more complicated mechanism in the N=1 superconformal case. The recurrence representations which in the bosonic case were basic tools in analyzing relations between toric and spheric N=1 blocks are first steps along this line. They are also of some interest for the recently discovered extension of the AGT relation where on the CFT side the N=1 superconformal Liouville field theory shows up [27–32].

The paper is organized as follows. Section 2 contains a detailed discussion of toric blocks in all sectors of N=1 superconformal field theories. In section 3 we calculate the residues of blocks coefficients. The method employed is a simple extension of the techniques developed for the spheric case. The main technical point is discussed in section 4 where we calculate the large $\Delta$ asymptotics. The derivation is based on properties of the Gram matrix and the matrix elements of chiral vertex operators. Proofs of these properties are given in appendices A and B. In section 5 we derive the recursion relations for N=1 superconformal toric blocks which are the main results of the present paper. Explicit formulae for first few block's coefficients are listed in appendix C.

## 2    1-point toric blocks in N=1 superconformal field theories

In the N=1 superconformal field theory on a torus the basic independent 1-point functions are those of super-primary NS fields $\phi_{\lambda,\bar\lambda}(z,\bar z)$ and their even primary descendants

$$\widetilde\phi_{\lambda,\bar\lambda}(z,\bar z) = \left\{ S_{-1/2}, \left[ \bar S_{-1/2}, \phi_{\lambda,\bar\lambda}(z,\bar z) \right] \right\}.$$

In our notation $\lambda, \bar{\lambda}$ parameterize the left and the right conformal weights of $\phi_{\lambda,\bar{\lambda}}(z, \bar{z})$

$$\Delta_\lambda = \frac{Q^2}{8} - \frac{\lambda^2}{8}, \qquad \Delta_{\bar{\lambda}} = \frac{Q^2}{8} - \frac{\bar{\lambda}^2}{8},$$

and $Q = b + b^{-1}$ is related to the central charge $c$ of the NSR algebra by $c = \frac{3}{2} + 3Q^2$.

## 2.1 NS and $\widetilde{\mathrm{NS}}$ sectors

The 1-point function of $\phi_{\lambda,\bar{\lambda}}$ on a torus with the modular parameter $\tau$ can be written as

$$\langle \phi_{\lambda,\bar{\lambda}} \rangle_{\mathrm{NS}} = (q\bar{q})^{-\frac{c}{24}} \sum_{(\Delta, \bar{\Delta})} \sum_{f, \bar{f} \in \frac{1}{2}\mathbb{N} \cup \{0\}} q^{\Delta+f} \, \bar{q}^{\bar{\Delta}+\bar{f}} \tag{2.1}$$

$$\times \sum_{\substack{f = |M|+|K|=|N|+|L| \\ \bar{f} = |\bar{M}|+|\bar{K}|=|\bar{N}|+|\bar{L}|}} \left[ B_\Delta^f \right]^{MK,NL} \left[ B_{\bar{\Delta}}^{\bar{f}} \right]^{\bar{M}\bar{K},\bar{N}\bar{L}} \langle \nu_{\Delta,MK} \otimes \nu_{\bar{\Delta},\bar{M}\bar{K}} | \phi_{\lambda,\bar{\lambda}}(1,1) | \nu_{\Delta,NL} \otimes \nu_{\bar{\Delta},\bar{N}\bar{L}} \rangle$$

where $q = e^{2\pi i\tau}$ and the sum runs over the whole spectrum of the NS theory. The matrices $\left[ B_\Delta^f \right]^{MK,NL}$, $\left[ B_{\bar{\Delta}}^{\bar{f}} \right]^{\bar{M}\bar{K},\bar{N}\bar{L}}$ are inverse to the Gram matrices

$$\left[ B_\Delta^f \right]_{MK,NL} = \langle \nu_{\Delta,MK} | \nu_{\Delta,NL} \rangle, \qquad \left[ B_{\bar{\Delta}}^{\bar{f}} \right]_{\bar{M}\bar{K},\bar{N}\bar{L}} = \langle \nu_{\bar{\Delta},\bar{M}\bar{K}} | \nu_{\bar{\Delta},\bar{N}\bar{L}} \rangle.$$

calculated in the standard bases in the corresponding NS Verma modules:

$$\nu_{\Delta,MK} = L_{-M} S_{-K} \nu_\Delta = L_{-m_j} \dots L_{-m_1} S_{-k_i} \dots S_{-k_1} \nu_\Delta,$$
$$k_i > \dots > k_1, \quad k_s \in \mathbb{N} - \tfrac{1}{2}, \qquad m_j \geqslant \dots \geqslant m_1, \quad m_r \in \mathbb{N}$$
$$L_0 \nu_\Delta = \Delta \nu_\Delta, \quad (-1)^F \nu_\Delta = \nu_\Delta, \quad L_m \nu_\Delta = S_k \nu_\Delta = 0.$$

For $K, L$ and $\bar{K}, \bar{L}$ of the same parity one has

$$\langle \nu_{\Delta,MK} \otimes \nu_{\bar{\Delta},\bar{M}\bar{K}} | \phi_{\lambda,\bar{\lambda}}(1,1) | \nu_{\Delta,NL} \otimes \nu_{\bar{\Delta},\bar{N}\bar{L}} \rangle =$$
$$\rho_{\mathrm{NN}}(\nu_{\Delta,MK}, \nu_\lambda, \nu_{\Delta,NL}) \, \rho_{\mathrm{NN}}(\nu_{\bar{\Delta},\bar{M}\bar{K}}, \nu_{\bar{\lambda}}, \nu_{\bar{\Delta},\bar{N}\bar{L}}) \, C_{\Delta,\bar{\Delta}}^{\lambda,\bar{\lambda}},$$
$$\langle \nu_{\Delta,MK} \otimes \nu_{\bar{\Delta},\bar{M}\bar{K}} | \tilde{\phi}_{\lambda,\bar{\lambda}}(1,1) | \nu_{\Delta,NL} \otimes \nu_{\bar{\Delta},\bar{N}\bar{L}} \rangle = \tag{2.2}$$
$$\rho_{\mathrm{NN}}^*(\nu_{\Delta,MK}, *\nu_\lambda, \nu_{\Delta,NL}) \, \rho_{\mathrm{NN}}^*(\nu_{\bar{\Delta},\bar{M}\bar{K}}, *\nu_{\bar{\lambda}}, \nu_{\bar{\Delta},\bar{N}\bar{L}}) \, \tilde{C}_{\Delta,\bar{\Delta}}^{\lambda,\bar{\lambda}},$$

where $C_{\Delta,\bar{\Delta}}^{\lambda,\bar{\lambda}}$, $\tilde{C}_{\Delta,\bar{\Delta}}^{\lambda,\bar{\lambda}}$ are the three-point constants

$$C_{\Delta,\bar{\Delta}}^{\lambda,\bar{\lambda}} = \langle \nu_\Delta \otimes \nu_{\bar{\Delta}} | \phi_{\lambda,\bar{\lambda}}(1,1) | \nu_\Delta \otimes \nu_{\bar{\Delta}}, \rangle, \quad \tilde{C}_{\Delta,\bar{\Delta}}^{\lambda,\bar{\lambda}} = \langle \nu_\Delta \otimes \nu_{\bar{\Delta}} | \tilde{\phi}_{\lambda,\bar{\lambda}}(1,1) | \nu_\Delta \otimes \nu_{\bar{\Delta}}, \rangle,$$

and $\rho_{\mathrm{NN}}$, $\rho_{\mathrm{NN}}^*$ are 3-point conformal blocks in the NS sector. In the formulae above and in the rest of the paper we follow the notation conventions of [23].

The toric conformal blocks are defined by

$$\mathcal{F}_\Delta^{-\lambda}(q) = q^{\Delta - \frac{c}{24}} \sum_{f \in \frac{1}{2}\mathbb{N}} q^f \, F_\Delta^{-\lambda,f},$$
$$F_\Delta^{-\lambda,f} = \sum_{M,K,N,L} \rho_{\mathrm{NN}}^-(\nu_{\Delta,MK}, {}_-\nu_\lambda, \nu_{\Delta,NL}) \left[ B_\Delta^f \right]^{MK,NL}, \tag{2.3}$$

where the symbol $_*$ stands for the star or the lack of it. The 1-point functions can then be represented as

$$\langle \phi_{\lambda,\bar{\lambda}} \rangle_{\text{NS}} = \sum_{(\Delta,\bar{\Delta})} \mathcal{F}_{\Delta}^{\lambda}(q) \, \mathcal{F}_{\bar{\Delta}}^{\bar{\lambda}}(\bar{q}) \, C_{\Delta,\bar{\Delta}}^{\lambda,\bar{\lambda}},$$

$$\langle \tilde{\phi}_{\lambda,\bar{\lambda}} \rangle_{\text{NS}} = \sum_{(\Delta,\bar{\Delta})} \mathcal{F}_{\Delta}^{*\lambda}(q) \, \mathcal{F}_{\bar{\Delta}}^{*\bar{\lambda}}(\bar{q}) \, \tilde{C}_{\Delta,\bar{\Delta}}^{\lambda,\bar{\lambda}}.$$

In $\widetilde{\textbf{NS}}$ sector one introduces the modified conformal blocks [38]

$$\widetilde{\mathcal{F}}_{\Delta}^{\lambda}(q) = q^{\Delta - \frac{c}{24}} \sum_{f} (-1)^{2f} q^{f} \, F_{\Delta}^{\lambda,f} \tag{2.4}$$

and the 1-point functions take the form

$$\langle \phi_{\lambda,\bar{\lambda}} \rangle_{\widetilde{\text{NS}}} = \sum_{(\Delta,\bar{\Delta})} \widetilde{\mathcal{F}}_{\Delta}^{\lambda}(q) \, \widetilde{\mathcal{F}}_{\bar{\Delta}}^{\bar{\lambda}}(\bar{q}) \, C_{\Delta,\bar{\Delta}}^{\lambda,\bar{\lambda}},$$

$$\langle \tilde{\phi}_{\lambda,\bar{\lambda}} \rangle_{\widetilde{\text{NS}}} = \sum_{(\Delta,\bar{\Delta})} \widetilde{\mathcal{F}}_{\Delta}^{*\lambda}(q) \, \widetilde{\mathcal{F}}_{\bar{\Delta}}^{*\bar{\lambda}}(\bar{q}) \, \tilde{C}_{\Delta,\bar{\Delta}}^{\lambda,\bar{\lambda}}.$$

## 2.2 R and $\widetilde{\textbf{R}}$ sectors

The toric 1-point functions in the $R$ sector read

$$\langle \phi_{\lambda,\bar{\lambda}} \rangle_{\text{R}} = (q\bar{q})^{-\frac{c}{24}} \sum_{(\beta,\bar{\beta})} \sum_{f,\bar{f}=0}^{\infty} q^{\Delta_{\beta}+f} \, \bar{q}^{\Delta_{\bar{\beta}}+\bar{f}} \tag{2.5}$$

$$\times \sum_{\substack{f=|M|+|K|=|N|+|L| \\ \#K+\#L \in 2\mathbb{N}}} \left[ B_{c\beta}^{f} \right]^{MK,NL} \sum_{\substack{\bar{f}=|\bar{M}|+|\bar{K}|=|\bar{N}|+|\bar{L}| \\ \#\bar{K}+\#\bar{L} \in 2\mathbb{N}}} \left[ B_{c\bar{\beta}}^{\bar{f}} \right]^{\bar{M}\bar{K},\bar{N}\bar{L}}$$

$$\times \left\langle L_{-M} S_{-K} \bar{L}_{-\bar{M}} \bar{S}_{-\bar{K}} w_{\beta,\bar{\beta}}^{+} \middle| \phi_{\lambda,\bar{\lambda}}(1,1) \middle| L_{-N} S_{-L} \bar{L}_{-\bar{N}} \bar{S}_{-\bar{L}} w_{\beta,\bar{\beta}}^{+} \right\rangle.$$

The matrices $\left[ B_{c\beta}^{f} \right]^{MK,NL}, \left[ B_{c\bar{\beta}}^{\bar{f}} \right]^{\bar{M}\bar{K},\bar{N}\bar{L}}$ are inverse to the Gram matrices

$$\left[ B_{c\beta}^{f} \right]_{MK,NL} = \left\langle w_{\beta,MK}^{+} \middle| w_{\beta,NL}^{+} \right\rangle, \qquad \left[ B_{c\bar{\beta}}^{\bar{f}} \right]_{\bar{M}\bar{K},\bar{N}\bar{L}} = \left\langle w_{\bar{\beta},\bar{M}\bar{K}}^{+} \middle| w_{\bar{\beta},\bar{N}\bar{L}}^{+} \right\rangle,$$

calculated in the standard bases in the corresponding R Verma modules $\mathcal{W}_{\beta}, \mathcal{W}_{\bar{\beta}}$:

$$w_{\beta,MK}^{+} = L_{-M} S_{-K} w_{\beta}^{+} = L_{-m_j} \dots L_{-m_1} S_{-k_i} \dots S_{-k_1} w_{\beta}^{+},$$

$$k_i > \dots > k_1, \quad k_s \in \mathbb{N} \cup \{0\}, \qquad m_j \geqslant \dots \geqslant m_1, \quad m_r \in \mathbb{N},$$

$$L_0 w_{\beta}^{+} = \Delta_{\beta} w_{\beta}^{+}, \quad \Delta_{\beta} = \frac{c}{24} - \beta^2,$$

$$S_0 w_{\beta}^{\pm} = i e^{\mp i \frac{\pi}{4}} \beta w_{\beta}^{\mp} \neq 0, \quad (-1)^{F} w_{\beta}^{+} = w_{\beta}^{+},$$

$$L_m w_{\beta}^{+} = S_k w_{\beta}^{+} = 0 \quad \text{for} \quad m, k > 0,$$

and

$$w_{\beta,\bar{\beta}}^{+} = \frac{1}{\sqrt{2}} \left( w_{\beta}^{+} \otimes w_{\bar{\beta}}^{+} - i \, w_{\beta}^{-} \otimes w_{\bar{\beta}}^{-} \right).$$

– 4 –

The chiral decompositions of $\phi_{\lambda,\bar{\lambda}}$ and $\tilde{\phi}_{\lambda,\bar{\lambda}}$ in the Ramond sector take the form [20]

$$
\begin{aligned}
\phi_{\lambda,\bar{\lambda}} &= C_{\beta,\bar{\beta}}^{\lambda\bar{\lambda}(+)}\left(V_{\mathsf{e}}^+\big[{}^{\Delta}_{\beta}{}_{\bar\beta}^{\bar\lambda}\big]\otimes V_{\mathsf{e}}^+\big[{}^{\Delta}_{\bar\beta}{}_{\bar\lambda}\big] - iV_{\mathsf{o}}^+\big[{}^{\Delta}_{\beta}{}_{\lambda}\big]\otimes V_{\mathsf{o}}^+\big[{}^{\Delta}_{\bar\beta}{}_{\bar\lambda}\big]\right)\\
&\quad + C_{\beta,\bar{\beta}}^{\lambda\bar{\lambda}(-)}\left(V_{\mathsf{e}}^-\big[{}^{\Delta}_{\beta}{}_{\lambda}\big]\otimes V_{\mathsf{e}}^-\big[{}^{\Delta}_{\bar\beta}{}_{\bar\lambda}\big] - iV_{\mathsf{o}}^-\big[{}^{\Delta}_{\beta}{}_{\lambda}\big]\otimes V_{\mathsf{o}}^-\big[{}^{\Delta}_{\bar\beta}{}_{\bar\lambda}\big]\right),\\
\tilde{\phi}_{\lambda,\bar{\lambda}} &= C_{\beta,\bar{\beta}}^{\lambda\bar{\lambda}(+)}\left(iV_{\mathsf{e}}^+\big[{}^{*\Delta}_{\beta}{}_{\lambda}\big]\otimes V_{\mathsf{e}}^+\big[{}^{*\Delta}_{\bar\beta}{}_{\bar\lambda}\big] + V_{\mathsf{o}}^+\big[{}^{*\Delta}_{\beta}{}_{\lambda}\big]\otimes V_{\mathsf{o}}^+\big[{}^{*\Delta}_{\bar\beta}{}_{\bar\lambda}\big]\right)\\
&\quad + C_{\beta,\bar{\beta}}^{\lambda\bar{\lambda}(-)}\left(iV_{\mathsf{e}}^-\big[{}^{*\Delta}_{\beta}{}_{\lambda}\big]\otimes V_{\mathsf{e}}^-\big[{}^{*\Delta}_{\bar\beta}{}_{\bar\lambda}\big] + V_{\mathsf{o}}^-\big[{}^{*\Delta}_{\beta}{}_{\lambda}\big]\otimes V_{\mathsf{o}}^-\big[{}^{*\Delta}_{\bar\beta}{}_{\bar\lambda}\big]\right),
\end{aligned}
\tag{2.6}
$$

where the chiral vertex operators are defined in terms of 3-point blocks [23]

$$
\begin{aligned}
\left\langle w_{\beta,MK}^+\right|V_{\mathsf{f}}^{\pm}\big[{}^{\Delta}_{\beta}{}_{\lambda}\big](z)\left|w_{\beta,NL}^+\right\rangle &= \rho_{\mathrm{RR,f}}^{(\pm)}(w_{\beta,MK}^+,\nu_\lambda,w_{\beta,NL}^+|z),\\
\left\langle w_{\beta,MK}^+\right|V_{\mathsf{f}}^{\pm}\big[{}^{*\Delta}_{\beta}{}_{\lambda}\big](z)\left|w_{\beta,NL}^+\right\rangle &= \rho_{\mathrm{RR,\bar{f}}}^{(\pm)}(w_{\beta,MK}^+,*\nu_\lambda,w_{\beta,NL}^+|z).
\end{aligned}
\tag{2.7}
$$

For $\#K + \#L \in 2\mathbb{N}$ one has in particular:

$$
\begin{aligned}
&\left\langle L_{-M}S_{-K}\bar{L}_{-\bar{M}}\bar{S}_{-\bar{K}}w_{\beta,\bar{\beta}}^+\right|\phi_{\lambda,\bar{\lambda}}(1,1)\left|L_{-N}S_{-L}\bar{L}_{-\bar{N}}\bar{S}_{-\bar{L}}w_{\beta,\bar{\beta}}^+\right\rangle\\
&= C_{\beta,\bar{\beta}}^{\lambda\bar{\lambda}(+)}\rho_{\mathrm{RR,e}}^{(+)}(w_{\beta,MK}^+,\nu_\lambda,w_{\beta,NL}^+)\,\rho_{\mathrm{RR,e}}^{(+)}(w_{\bar{\beta},\bar{M}\bar{K}}^+,\nu_{\bar\lambda},w_{\bar{\beta},\bar{N}\bar{L}}^+)\\
&\quad + C_{\beta,\bar{\beta}}^{\lambda\bar{\lambda}(-)}\rho_{\mathrm{RR,e}}^{(-)}(w_{\beta,MK}^+,\nu_\lambda,w_{\beta,NL}^+)\,\rho_{\mathrm{RR,e}}^{(-)}(w_{\bar{\beta},\bar{M}\bar{K}}^+,\nu_{\bar\lambda},w_{\bar{\beta},\bar{N}\bar{L}}^+),
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
&\left\langle L_{-M}S_{-K}\bar{L}_{-\bar{M}}\bar{S}_{-\bar{K}}w_{\beta,\bar{\beta}}^+\right|\tilde{\phi}_{\lambda,\bar{\lambda}}(1,1)\left|L_{-N}S_{-L}\bar{L}_{-\bar{N}}\bar{S}_{-\bar{L}}w_{\beta,\bar{\beta}}^+\right\rangle\\
&= iC_{\beta,\bar{\beta}}^{\lambda\bar{\lambda}(+)}\rho_{\mathrm{RR,o}}^{(+)}(w_{\beta,MK}^+,*\nu_\lambda,w_{\beta,NL}^+)\,\rho_{\mathrm{RR,o}}^{(+)}(w_{\bar{\beta},\bar{M}\bar{K}}^+,*\nu_{\bar\lambda},w_{\bar{\beta},\bar{N}\bar{L}}^+)\\
&\quad + iC_{\beta,\bar{\beta}}^{\lambda\bar{\lambda}(-)}\rho_{\mathrm{RR,o}}^{(-)}(w_{\beta,MK}^+,*\nu_\lambda,w_{\beta,NL}^+)\,\rho_{\mathrm{RR,o}}^{(-)}(w_{\bar{\beta},\bar{M}\bar{K}}^+,*\nu_{\bar\lambda},w_{\bar{\beta},\bar{N}\bar{L}}^+).
\end{aligned}
\tag{2.9}
$$

1-point function (2.5) can then be written as

$$
\langle\phi_{\lambda,\bar{\lambda}}\rangle_{\mathrm{R}} = \sum_{(\beta,\bar{\beta})}C_{\beta,\bar{\beta}}^{\lambda\bar{\lambda}(+)}\mathcal{F}_{\beta}^{\lambda(+)}(q)\,\mathcal{F}_{\bar{\beta}}^{\bar{\lambda}(+)}(\bar{q}) + \sum_{(\beta,\bar{\beta})}C_{\beta,\bar{\beta}}^{\lambda\bar{\lambda}(-)}\mathcal{F}_{\beta}^{\lambda(-)}(q)\,\mathcal{F}_{\bar{\beta}}^{\bar{\lambda}(-)}(\bar{q})
$$

where

$$
\mathcal{F}_{\beta}^{\lambda(\pm)}(q) = \mathcal{F}_{\beta,\mathsf{e}}^{\lambda(\pm)}(q) + \mathcal{F}_{\beta,\mathsf{o}}^{\lambda(\pm)}(q),
$$

$$
\mathcal{F}_{\beta,\mathsf{e/o}}^{\lambda(\pm)}(q) = q^{\Delta_\beta - \frac{c}{24}}\sum_{f=0}^{\infty}q^f F_{\beta,\mathsf{e/o}}^{\lambda(\pm),f},
$$

$$
F_{\beta,\mathsf{e}}^{\lambda(\pm),f} = \sum_{\substack{f=|M|+|K|=|N|+|L|\\ \#K,\#L\in 2\mathbb{N}}}\rho_{\mathrm{RR,e}}^{(\pm)}(w_{\beta,MK}^+,\nu_\lambda,w_{\beta,NL}^+)\Big[B_{c\beta}^f\Big]^{MK,NL},
\tag{2.10}
$$

$$
F_{\beta,\mathsf{o}}^{\lambda(\pm),f} = \sum_{\substack{f=|M|+|K|=|N|+|L|\\ \#K,\#L\in 2\mathbb{N}+1}}\rho_{\mathrm{RR,e}}^{(\pm)}(w_{\beta,MK}^+,\nu_\lambda,w_{\beta,NL}^+)\Big[B_{c\beta}^f\Big]^{MK,NL}.
$$

The representation for the 1-point toric function of $\tilde{\phi}_{\lambda,\bar{\lambda}}$ reads

$$\langle\tilde{\phi}_{\lambda,\bar{\lambda}}\rangle_{\mathrm{R}} = \sum_{(\beta,\bar{\beta})} iC_{\beta,\bar{\beta}}^{\lambda\,\bar{\lambda}(+)}\mathcal{F}_{\beta}^{*\lambda(+)}(q)\,\mathcal{F}_{\bar{\beta}}^{*\bar{\lambda}(+)}(\bar{q}) + \sum_{(\beta,\bar{\beta})} iC_{\beta,\bar{\beta}}^{\lambda\,\bar{\lambda}(-)}\mathcal{F}_{\beta}^{*\lambda(-)}(q)\,\mathcal{F}_{\bar{\beta}}^{*\bar{\lambda}(-)}(\bar{q})$$

where

$$\mathcal{F}_{\beta}^{*\lambda(\pm)}(q) = \mathcal{F}_{\beta,\mathrm{e}}^{*\lambda(\pm)}(q) + \mathcal{F}_{\beta,\mathrm{o}}^{*\lambda(\pm)}(q),$$

$$\mathcal{F}_{\beta,\mathrm{e/o}}^{*\lambda(\pm)}(q) = q^{\Delta_\beta - \frac{c}{24}} \sum_{f=0}^{\infty} q^f F_{\beta,\mathrm{e/o}}^{*\lambda(\pm),f},$$

$$F_{\beta,\mathrm{e}}^{*\lambda(\pm),f} = \sum_{\substack{f=|K|+|M|=|L|+|N| \\ \#K,\#L\in 2\mathbb{N}}} \rho_{\mathrm{RR,o}}^{(\pm)}(w_{\beta,MK}^+, *\nu_\lambda, w_{\beta,NL}^+)\left[B_{c\beta}^f\right]^{MK,NL}, \qquad (2.11)$$

$$F_{\beta,\mathrm{o}}^{*\lambda(\pm),f} = \sum_{\substack{f=|K|+|M|=|L|+|N| \\ \#K,\#L\in 2\mathbb{N}+1}} \rho_{\mathrm{RR,o}}^{(\pm)}(w_{\beta,MK}^+, *\nu_\lambda, w_{\beta,NL}^+)\left[B_{c\beta}^f\right]^{MK,NL}.$$

As in the case of 4-point blocks on the sphere [20] one can show

$$\mathcal{F}_{\beta,\mathrm{e}}^{\lambda(\pm)}(q) = \pm\mathcal{F}_{\beta,\mathrm{o}}^{\lambda(\pm)}(q),$$
$$\mathcal{F}_{\beta,\mathrm{e}}^{*\lambda(\pm)}(q) = \mp\mathcal{F}_{\beta,\mathrm{o}}^{*\lambda(\pm)}(q).$$

Hence

$$\langle\phi_{\lambda,\bar{\lambda}}\rangle_{\mathrm{R}} = 4\sum_{(\beta,\bar{\beta})} C_{\beta,\bar{\beta}}^{\lambda\,\bar{\lambda}(+)}\mathcal{F}_{c\beta,\mathrm{e}}^{\lambda(+)}(q)\,\mathcal{F}_{c\bar{\beta},\mathrm{e}}^{\bar{\lambda}(+)}(\bar{q}),$$

$$\langle\phi_{\lambda,\bar{\lambda}}\rangle_{\widetilde{\mathrm{R}}} = 4\sum_{(\beta,\bar{\beta})} C_{\beta,\bar{\beta}}^{\lambda\,\bar{\lambda}(-)}\mathcal{F}_{c\beta,\mathrm{e}}^{\lambda(-)}(q)\,\mathcal{F}_{c\bar{\beta},\mathrm{e}}^{\bar{\lambda}(-)}(\bar{q}),$$

$$\langle\tilde{\phi}_{\lambda,\bar{\lambda}}\rangle_{\mathrm{R}} = 4i\sum_{(\beta,\bar{\beta})} C_{\beta,\bar{\beta}}^{\lambda\,\bar{\lambda}(-)}\mathcal{F}_{c\beta,\mathrm{e}}^{*\lambda(-)}(q)\,\mathcal{F}_{c\bar{\beta},\mathrm{e}}^{*\bar{\lambda}(-)}(\bar{q}),$$

$$\langle\tilde{\phi}_{\lambda,\bar{\lambda}}\rangle_{\widetilde{\mathrm{R}}} = 4i\sum_{(\beta,\bar{\beta})} C_{\beta,\bar{\beta}}^{\lambda\,\bar{\lambda}(+)}\mathcal{F}_{c\beta,\mathrm{e}}^{*\lambda(+)}(q)\,\mathcal{F}_{c\bar{\beta},\mathrm{e}}^{*\bar{\lambda}(+)}(\bar{q}),$$

and it is enough to consider the even blocks alone.

## 3 Residues

### 3.1 NS and $\widetilde{\mathrm{NS}}$ sectors

The method to derive the recursion relations is essentially the same as in the Virasoro algebra case [11]. The blocks' coefficients (2.3) are polynomials in the external weight $\Delta_\lambda$ and rational functions of the intermediate weight $\Delta$ and the central charge $c$. The poles in $\Delta$ are given by the Kac determinant formula for the NS Verma modules ($r+s \in 2\mathbb{N}$):

$$\Delta_{rs} = \frac{1-rs}{4} + \frac{1-r^2}{8}b^2 + \frac{1-s^2}{8}\frac{1}{b^2}, \qquad c = \frac{3}{2} + 3\left(b+\frac{1}{b}\right)^2. \qquad (3.1)$$

They are related to the null states

$$|\chi_{rs}\rangle = D_{rs}|\Delta_{rs}\rangle.$$

in the Verma modules $\mathcal{V}_{\Delta_{rs}}$. For a generic value of the central charge the modules $\mathcal{V}_{\Delta_{rs}+\frac{rs}{2}}$ are irreducible and the poles are simple [14], hence:

$$F_{\Delta}^{-\lambda,f} = \mathrm{h}_{\Delta}^{-\lambda,f} + \sum_{\substack{1 < rs \leq 2f \\ r+s \in 2\mathbb{N}}} \frac{R_{rs}^{-\lambda,f}}{\Delta - \Delta_{rs}}. \tag{3.2}$$

Following the method of [14] one gets

$$R_{rs}^{-\lambda,f} = \lim_{\Delta \to \Delta_{rs}} (\Delta - \Delta_{rs}) F_{\Delta}^{-\lambda,f} \tag{3.3}$$

$$= A_{rs} \times \sum_{|K|+|M|=|L|+|N|=f-\frac{rs}{2}} \rho_{\mathrm{NN}}^-(L_{-M}S_{-K}\chi_{rs}, {}_-\nu_\lambda, L_{-N}S_{-L}\chi_{rs}) \left[B_{\Delta_{rs}+\frac{rs}{2}}^{f-\frac{rs}{2}}\right]^{KM,LN},$$

where the coefficients $A_{rs}$ are given by:

$$\frac{1}{A_{rs}} = \lim_{\Delta \to \Delta_{rs}} \frac{\left\langle \Delta | D_{rs}^\dagger D_{rs} | \Delta \right\rangle}{\Delta - \Delta_{rs}}. \tag{3.4}$$

The exact formula for $A_{rs}(c)$ was proposed by A. Belavin and Al. Zamolodchikov in [35]. It reads

$$A_{rs}(c) = 2^{rs-2} \prod_{m=1-r}^{r} \prod_{n=1-s}^{s} \left(mb + nb^{-1}\right)^{-1} \tag{3.5}$$

where $m+n \in 2\mathbb{Z}$, $(m,n) \neq (0,0), (r,s)$. The corresponding expression in the bosonic case was first conjectured by Al. Zamolodchikov [4, 5] and recently proved by Yanagida [34].

In order to calculate the residues we shall use the factorization property of the 3-point blocks [14]. For $|K| + |L| \in \mathbb{N}$ one has in particular

$$\rho_{\mathrm{NN}}^-(L_{-M}S_{-K}\chi_{rs}, {}_-\nu_\lambda, L_{-N}S_{-L}\chi_{rs}) =$$

$$\rho_{\mathrm{NN}}^-(L_{-M}S_{-K}\nu_{\Delta_{rs}+rs}, {}_-\nu_\lambda, L_{-N}S_{-L}\nu_{\Delta_{rs}+rs}) \, \rho_{\mathrm{NN}}^-(\chi_{rs}, {}_-\nu_\lambda, \chi_{rs}) \,.$$

By the same token

$$\rho_{\mathrm{NN}}^-(\chi_{rs}, {}_-\nu_\lambda, \chi_{rs}) = \begin{cases} \rho_{\mathrm{NN}}^-(\chi_{rs}, {}_-\nu_\lambda, \nu_{\Delta_{rs}+\frac{rs}{2}}) \, \rho_{\mathrm{NN}}^-(\nu_{\Delta_{rs}}, {}_-\nu_\lambda, \chi_{rs}) \text{ for } \frac{rs}{2} \in \mathbb{N}, \\ \widetilde{\rho_{\mathrm{NN}}^-}(\chi_{rs}, {}_-\nu_\lambda, \nu_{\Delta_{rs}+\frac{rs}{2}}) \, \rho_{\mathrm{NN}}^-(\nu_{\Delta_{rs}}, \widetilde{{}_-\nu_\lambda}, \chi_{rs}) \text{ for } \frac{rs}{2} \in \mathbb{N} - \frac{1}{2}, \end{cases} \tag{3.6}$$

where $\widetilde{\rho_{\mathrm{NN}}} = \rho_{\mathrm{NN}}^*$, $\widetilde{\rho_{\mathrm{NN}}^*} = \rho_{\mathrm{NN}}$ etc. The 3-point blocks in the formula above can be expressed in terms of the fusion polynomials

$$P_c^{rs}\begin{bmatrix} \Delta_2 \\ \Delta_1 \end{bmatrix} = X_{\mathrm{e}}^{rs}(\lambda_1 + \lambda_2) X_{\mathrm{e}}^{rs}(\lambda_1 - \lambda_2),$$
$$P_c^{rs}\begin{bmatrix} *\Delta_2 \\ \Delta_1 \end{bmatrix} = X_{\mathrm{o}}^{rs}(\lambda_1 + \lambda_2) X_{\mathrm{o}}^{rs}(\lambda_1 - \lambda_2), \tag{3.7}$$

where

$$X_{\mathrm{e}}^{rs}(\lambda) = \prod_{p=1-r}^{r-1} \prod_{q=1-s}^{s-1} \left( \frac{\lambda - pb - qb^{-1}}{2\sqrt{2}} \right),$$

$$X_{\mathrm{o}}^{rs}(\lambda) = \prod_{p'=1-r}^{r-1} \prod_{q'=1-s}^{s-1} \left( \frac{\lambda - p'b - q'b^{-1}}{2\sqrt{2}} \right), \tag{3.8}$$

and the products run over:

$$\begin{aligned}
p &= 1 - r + 2k, & q &= 1 - s + 2l, & k + l &\in 2\mathbb{N} \cup \{0\}, \\
p' &= 1 - r + 2k, & q' &= 1 - s + 2l, & k + l &\in 2\mathbb{N} + 1.
\end{aligned} \tag{3.9}$$

Using the relations

$$\rho_{\mathrm{NN}}^-(\chi_{rs}, {}_-\nu_2, \nu_1) = \rho_{\mathrm{NN}}^-(\nu_1, {}_-\nu_2, \chi_{rs}) = P_c^{rs}\begin{bmatrix} {}_-\Delta_2 \\ \Delta_1 \end{bmatrix} \text{ for } \frac{rs}{2} \in \mathbb{N},$$

$$\rho_{\mathrm{NN}}^-(\chi_{rs}, \widetilde{{}_-\nu_2}, \nu_1) = (-1)^{|\widetilde{{}_-\nu_2}|}\rho_{\mathrm{NN}}^-(\nu_1, \widetilde{{}_-\nu_2}, \chi_{rs}) = P_c^{rs}\begin{bmatrix} \widetilde{{}_-\Delta_2} \\ \Delta_1 \end{bmatrix} \text{ for } \frac{rs}{2} \in \mathbb{N} - \tfrac{1}{2},$$

one gets

$$R_\Delta^{-\lambda, f} = A_{rs}\, F_{\Delta_{rs}+\frac{rs}{2}}^{-\lambda, f-\frac{rs}{2}} \times \begin{cases} P_c^{rs}\begin{bmatrix} {}_-\Delta_\lambda \\ \Delta_{rs}+\frac{rs}{2} \end{bmatrix} P_c^{rs}\begin{bmatrix} {}_-\Delta_\lambda \\ \Delta_{rs} \end{bmatrix} & \text{for } \frac{rs}{2} \in \mathbb{N}, \\[2mm] P_c^{rs}\begin{bmatrix} \widetilde{{}_-\Delta_\lambda} \\ \Delta_{rs}+\frac{rs}{2} \end{bmatrix} P_c^{rs}\begin{bmatrix} {}_-\Delta_\lambda \\ \Delta_{rs} \end{bmatrix} \widetilde{{}_-s_{rs}} & \text{for } \frac{rs}{2} \in \mathbb{N} - \tfrac{1}{2}, \end{cases} \tag{3.10}$$

where $s_{rs} = 1$, $*s_{rs} = (-1)^{rs}$.

## 3.2 R and $\widetilde{\mathrm{R}}$ sectors

The blocks' coefficients (2.10), (2.11) are polynomials in the external weight $\Delta_\lambda$ and rational functions of $\beta$ and the central charge $c$. The poles in $\beta$ are given by the Kac determinant formula for the positive parity subspace of an R Verma module. They are located at $\pm\beta_{rs}$ where

$$\beta_{rs} = \frac{1}{2\sqrt{2}}\left( rb + s\frac{1}{b} \right)$$

and $r + s \in 2\mathbb{N} + 1$. The poles are related to the positive parity null states

$$\left| \chi_{rs}^+ \right\rangle = D_{rs}\left| w_{rs}^+ \right\rangle.$$

For a generic value of the central charge the modules $\mathcal{W}_{\beta_{rs}+\frac{rs}{2}}$ are irreducible and the poles are simple [20], hence:

$$F_{\beta,\mathrm{e}}^{-\lambda(\pm),f} = h_\beta^{-\lambda(\pm),f} + \sum_{\substack{1 < rs \leqslant 2f \\ r+s \in 2\mathbb{N}+1}} \left( \frac{R_{rs+}^{-\lambda(\pm),f}}{\beta - \beta_{rs}} + \frac{R_{rs-}^{-\lambda(\pm),f}}{\beta + \beta_{rs}} \right). \tag{3.11}$$

Calculating the residues in the standard way [20] one gets

$$
\begin{aligned}
R_{rs\pm}^{-\lambda(\pm),f} &= \lim_{\beta \to \pm\beta_{rs}} (\beta \mp \beta_{rs}) F_{\beta,\mathrm{e}}^{-\lambda(\pm),f} \\
&= \mp \frac{1}{2\beta_{rs}} A_{rs} \\
&\quad \times \sum_{\substack{f=|K|+|M|=|L|+|N| \\ |K|,|L|\in 2\mathbb{N}}} \rho_{\mathrm{RR,f}}^{(\pm)}(L_{-M}S_{-K}D_{rs}w_{\beta_{rs}}^{+}, {}_{-\nu_\lambda}, L_{-N}S_{-L}D_{rs}w_{\beta_{rs}}^{+}) \Big[B_{c\,\beta'_{rs}}^{f}\Big]^{MK,NL},
\end{aligned}
$$

where

$$
\beta'_{rs} = \frac{(-1)^s}{2\sqrt{2}} \left(rb - s\frac{1}{b}\right)
$$

corresponds to the conformal weight $\Delta_{rs} + \frac{rs}{2}$ and the parity of the 3-point block is $\mathrm{f} = \mathrm{e}$ and $\mathrm{f} = \mathrm{o}$ in the case of $R_{rs\pm}^{\lambda(\pm),f}$ and $R_{rs\pm}^{*\lambda(\pm),f}$, respectively. If we assume the normalization

$$
D_{rs} = (L_{-1})^{\frac{rs}{2}} + \ldots
$$

the coefficient $A_{rs}$ is given by formula (3.5) with $m+n \in 2\mathbb{Z}+1$ [35]. For this normalization the odd null state $\chi_{rs}^{-} = \frac{\mathrm{e}^{i\frac{\pi}{4}}}{i\beta'_{rs}} S_0 \chi_{rs}^{+}$ can be expressed as $\chi_{rs}^{-} = D_{rs}w_{rs}^{-}$ [36]. Using this observation and the properties of the 3-point blocks one obtains the following factorization formulae [20]

$$
\begin{aligned}
&\rho_{\mathrm{RR,e}}^{(\pm)}(L_{-M}S_{-K}D_{rs}w_{\beta_{rs}}^{+}, \nu_\lambda, L_{-N}S_{-L}D_{rs}w_{\beta_{rs}}^{+}) \\
&\quad = \rho_{\mathrm{RR,e}}^{(\pm)}(L_{-M}S_{-K}w_{\beta'_{rs}}^{+}, \nu_\lambda, L_{-N}S_{-L}w_{\beta'_{rs}}^{+})\rho_{\mathrm{RR,e}}^{(\pm)}(D_{rs}w_{\beta_{rs}}^{+}, \nu_\lambda, D_{rs}w_{\beta_{rs}}^{+}), \\
&\rho_{\mathrm{RR,o}}^{(\pm)}(L_{-M}S_{-K}D_{rs}w_{\beta_{rs}}^{+}, *\nu_\lambda, L_{-N}S_{-L}D_{rs}w_{\beta_{rs}}^{+}) \\
&\quad = \rho_{\mathrm{RR,o}}^{(\pm)}(L_{-M}S_{-K}w_{\beta'_{rs}}^{+}, *\nu_\lambda, L_{-N}S_{-L}w_{\beta'_{rs}}^{+})\rho_{\mathrm{RR,e}}^{(\pm)}(D_{rs}w_{\beta_{rs}}^{+}, \nu_\lambda, D_{rs}w_{\beta_{rs}}^{+}), \\
&\rho_{\mathrm{RR,e}}^{(\pm)}(D_{rs}w_{\beta_{rs}}^{+}, \nu_\lambda, D_{rs}w_{\beta_{rs}}^{+}) \\
&\quad = \rho_{\mathrm{RR,e}}^{(\pm)}(D_{rs}w_{\beta_{rs}}^{+}, \nu_\lambda, w_{\beta'_{rs}}^{+})\rho_{\mathrm{RR,e}}^{(\pm)}(w_{\beta_{rs}}^{+}, \nu_\lambda, D_{rs}w_{\beta_{rs}}^{+}).
\end{aligned}
$$

In terms of the fusion polynomials

$$
P_c^{rs}\begin{bmatrix} \Delta_\lambda \\ \pm\beta \end{bmatrix} = X_{\mathrm{e}}^{rs}(2\sqrt{2}\beta \mp \lambda)X_{\mathrm{o}}^{rs}(2\sqrt{2}\beta \pm \lambda)
$$

one then has

$$
\rho_{\mathrm{RR,e}}^{(\pm)}(w_\beta^{+}, \nu_\lambda, D_{rs}w_{\beta_{rs}}^{+}) = \rho_{\mathrm{RR,e}}^{(\pm)}(D_{rs}w_{\beta_{rs}}^{+}, \nu_\lambda, w_\beta^{+}) = P_c^{rs}\begin{bmatrix} \Delta_\lambda \\ \pm\beta \end{bmatrix}
$$

and

$$
R_{rs\pm}^{-\lambda(\pm),f} = \mp \frac{1}{2\beta_{rs}} A_{rs} P_c^{rs}\begin{bmatrix} \Delta_\lambda \\ \pm\beta'_{rs} \end{bmatrix} P_c^{rs}\begin{bmatrix} \Delta_\lambda \\ \pm\beta_{rs} \end{bmatrix} F_{\beta'_{rs},\mathrm{e}}^{-\lambda(\pm),f-\frac{rs}{2}}.
$$

Since the residues at $\pm\beta_{rs}$ differ by sign one simply gets

$$
F_{\beta,\mathrm{e}}^{-\lambda(\pm),f} = h_\beta^{-\lambda(\pm),f} + \sum_{\substack{1 < rs \leqslant 2f \\ r+s\in 2\mathbb{N}+1}} \frac{A_{rs} P_c^{rs}\begin{bmatrix} \Delta_\lambda \\ \pm\beta'_{rs} \end{bmatrix} P_c^{rs}\begin{bmatrix} \Delta_\lambda \\ \pm\beta_{rs} \end{bmatrix}}{\Delta_\beta - \Delta_{rs}} F_{\beta'_{rs},\mathrm{e}}^{-\lambda(\pm),f-\frac{rs}{2}}. \tag{3.12}
$$

Let us note that recursions for the blocks $F_{\beta,\mathrm{e}}^{\lambda(\pm),f}$ and $F_{\beta,\mathrm{e}}^{*\lambda(\pm),f}$ are very similar. The only difference is the function $h_{\beta}^{-\lambda(\pm),f}$ which we determine in the next section.

## 4 Large $\Delta$ asymptotics

In order to complete the derivation of recurrence relations one needs the large $\Delta$ asymptotics of conformal blocks. Their rigorous calculation turned out however to be more difficult that in the Virasoro algebra case [11]. The method presented in this section is based on properties of the Gram matrix and the matrix elements of the chiral vertex operators, collected in Propositions 1–4. Their proofs are given in appendices A and B.

### 4.1 NS and $\widetilde{\mathrm{NS}}$ sectors

Let $\mathcal{B}_f$ denotes the standard basis of level $f$ subspace of the NS Verma module:

$$\mathcal{B}_f = \{L_{-M} S_{-K} \nu_\Delta \ : \ |M| + |K| = f\}. \tag{4.1}$$

It is convenient to use a simplified notation for elements of this basis

$$\mathcal{B}_f = \{u_i\}_{i=1}^{\dim \mathcal{B}_f}.$$

**Proposition 1.** *Let $Q$ be a polynomial in $\Delta$ and let $\deg_\Delta Q$ denotes its degree. Then:*

1. *for any $u_i = L_{-M} S_{-K} \nu_\Delta \in \mathcal{B}_f$:*

$$\deg_\Delta \langle u_i | u_i \rangle = \#M + \#K;$$

2. *for any $u_i, u_j \in \mathcal{B}_f, u_i \neq u_j$:*

$$\deg_\Delta \langle u_i | u_j \rangle < \max \left\{ \deg_\Delta \langle u_i | u_i \rangle, \deg_\Delta \langle u_j | u_j \rangle \right\};$$

3. *the product of the diagonal terms is the only highest degree term in the determinant of the Gram matrix with respect to the base $\mathcal{B}_f$ i.e.*

$$\deg_\Delta \left( \det \left[ \langle u_i, u_j \rangle \right] - \prod_{i=1}^{\dim \mathcal{B}_f} \langle u_i, u_i \rangle \right) < \deg_\Delta \det \left[ \langle u_i, u_j \rangle \right].$$

**Proposition 2.** *For any $u_i, u_j \in \mathcal{B}_f$, $u_i \neq u_j$:*

$$\deg_\Delta \rho_{\mathrm{NN}}(u_i, \nu_\lambda, u_j) < \max \left\{ \deg_\Delta \langle u_i | u_i \rangle, \deg_\Delta \langle u_j | u_j \rangle \right\},$$
$$\deg_\Delta \rho_{\mathrm{NN}}^*(u_i, *\nu_\lambda, u_j) < \max \left\{ \deg_\Delta \langle u_i | u_i \rangle, \deg_\Delta \langle u_j | u_j \rangle \right\}.$$

By Proposition 2, for off diagonal elements

$$\deg_\Delta \rho_{\mathrm{NN}}^-(u_i, {}_-\nu_\lambda, u_j) < \deg_\Delta \langle u_i | u_i \rangle \quad \text{or} \quad \deg_\Delta \rho_{\mathrm{NN}}^-(u_i, {}_-\nu_\lambda, u_j) < \deg_\Delta \langle u_j | u_j \rangle. \tag{4.2}$$

Suppose the first inequality holds. The minor $M_{ji}$ of the Gram matrix can be represented as

$$M_{ji} = \sum_\tau \text{sgn}(\tau) \langle u_1 | u_{\tau(1)} \rangle \ldots \langle u_{i-1} | u_{\tau(i-1)} \rangle \langle u_{i+1} | u_{\tau(i+1)} \rangle \ldots$$

where the sum runs over permutations $\tau$ such that $\tau(i) = j$. By Proposition 1:

$$\deg_\Delta \langle u_k | u_{\tau(k)} \rangle \leqslant \deg_\Delta \langle u_k | u_k \rangle$$

hence, for every permutation $\tau$ :

$$\deg_\Delta \big( \langle u_1 | u_{\tau(1)} \rangle \ldots \langle u_{i-1} | u_{\tau(i-1)} \rangle \langle u_{i+1} | u_{\tau(i+1)} \rangle \ldots \big) \leqslant \sum_{k \neq j} \deg_\Delta \langle u_k | u_k \rangle.$$

Taking into account the first inequality of (4.2) one thus gets

$$\deg_\Delta \big( \rho_{\text{NN}}^-(u_i, {}_-\nu_\lambda, u_j) M_{ji} \big) < \sum_k \deg_\Delta \langle u_k | u_k \rangle = \deg_\Delta \det \Big[ \langle u_i, u_j \rangle \Big].$$

and

$$\lim_{\Delta \to \infty} \rho_{\text{NN}}^-(u_i, {}_-\nu_\lambda, u_j) B_f^{ij} = (-1)^{i+j} \lim_{\Delta \to \infty} \frac{\rho_{\text{NN}}^-(u_i, {}_-\nu_\lambda, u_j) M_{ji}}{\det \Big[ \langle u_i, u_j \rangle \Big]} = 0. \tag{4.3}$$

If the second inequality of (4.2) holds one follows the same reasoning with a different minor representation:

$$M_{ji} = \sum_\tau \text{sgn}(\tau) \langle u_{\tau(1)} | u_1 \rangle \ldots \langle u_{\tau(j-1)} | u_{j-1} \rangle \langle u_{\tau(j+1)} | u_{j+1} \rangle \ldots$$

where the sum runs over permutations of $\tau$ such that $\tau(j) = i$. Thus for $i \neq j$ :

$$\lim_{\Delta \to \infty} \rho_{\text{NN}}^-(u_i, {}_-\nu_\lambda, u_j) B^{ij} = 0. \tag{4.4}$$

One easily shows (see the proof of Proposition 2 in appendix A) that the term of the highest $\Delta$ degree in $\rho_{\text{NN}}^-(u_i, {}_-\nu_\lambda, u_i)$ is equal to $\langle u_i | u_i \rangle$. Hence

$$\lim_{\Delta \to \infty} \rho_{\text{NN}}^-(u_i, {}_-\nu_\lambda, u_i) B^{ii} = 1. \tag{4.5}$$

(There is no summation over repeated indices in formulae (4.4), (4.5).) One finally gets

$$\text{h}_\Delta^{-\lambda,f} = \lim_{\Delta \to \infty} F_\Delta^{-\lambda,f} = \lim_{\Delta \to \infty} \left( \sum_{i,j=1}^{\dim \mathcal{B}_f} \rho_{\text{NN}}^-(u_i, {}_-\nu_\lambda, u_j) B_f^{ij} \right)$$

$$= \sum_{i=1}^{\dim \mathcal{B}_f} 1 = \dim \mathcal{B}_f = p_{\text{NS}}(f) \tag{4.6}$$

where $p_{\text{NS}}(f)$ is defined by the generating function

$$\sum_{f \in \frac{1}{2} \mathbb{N} \cup \{0\}} p_{\text{NS}}(f) q^f = \prod_{n=1}^\infty \frac{1 + q^{n - \frac{1}{2}}}{1 - q^n} = q^{\frac{c}{24} - \Delta} \chi_{\text{NS}}^\Delta(q)$$

and $\chi^\Delta_{\mathrm{NS}}(q)$ is the character of the NS Verma module [37–40]

$$\chi^\Delta_{\mathrm{NS}}(q) = \mathrm{Tr}_{\mathrm{NS}}\, q^{L_0 - \frac{c}{24}} = q^{\Delta - \frac{c}{24} + \frac{1}{16}}\, \eta(q)^{-\frac{3}{2}}\, \sqrt{\theta_3(q^{1/2})}\,.$$

For "twisted" blocks (2.4) asymptotic (4.6) implies

$$\widetilde{\mathrm{h}}^{-\lambda,f}_\Delta = (-1)^{2f} \lim_{\Delta \to \infty} F^{-\lambda,f}_\Delta = (-1)^{2f} p_{\mathrm{NS}}(f) = p_{\widetilde{\mathrm{NS}}}(f).$$

The generating function for $p_{\widetilde{\mathrm{NS}}}(f)$ takes the form

$$\sum_{f \in \frac{1}{2}\mathbb{N} \cup \{0\}} p_{\widetilde{\mathrm{NS}}}(f) q^f = \prod_{n=1}^{\infty} \frac{1 - q^{n - \frac{1}{2}}}{1 - q^n} = q^{\frac{c}{24} - \Delta} \chi^\Delta_{\widetilde{\mathrm{NS}}}(q)$$

where $\chi^\Delta_{\widetilde{\mathrm{NS}}}(q)$ is a modified character [38–40]

$$\chi^\Delta_{\widetilde{\mathrm{NS}}}(q) = \mathrm{Tr}_{\mathrm{NS}}\, (-)^F q^{L_0 - \frac{c}{24}} = q^{\Delta - \frac{c}{24} + \frac{1}{16}}\, \eta(q)^{-\frac{3}{2}}\, \sqrt{\theta_4(q^{1/2})}.$$

## 4.2 R and $\widetilde{\mathrm{R}}$ sectors

In order to calculate the large $\Delta_\beta$ behavior of the Ramond toric blocks we shall chose a special basis $\mathcal{B}_f$ of level $f$ even subspace of the Ramond Verma module. It is defined by:

$$\begin{aligned}
\mathcal{B}_f &= \mathcal{B}^+_f \cup \mathcal{B}^-_f, \\
\mathcal{B}^+_f &= \{L_{-M} S_{-K} w^+_\beta \;:\; |M| + |K| = f,\; \#K \in 2\mathbb{N} \cup \{0\}\}, \\
\mathcal{B}^-_f &= \{L_{-M} S_{-K} w^-_\beta \;:\; |M| + |K| = f,\; \#K \in 2\mathbb{N} + 1\},
\end{aligned} \qquad (4.7)$$

where the string of generators $S_{-K}$ does not include $S_0$. We shall also use a simplified notation for elements of the basis above:

$$\mathcal{B}_f = \{u_i\}_{i=1}^{\dim \mathcal{B}_f}, \qquad \mathcal{B}^\pm_f = \{u^\pm_k\}_{k=1}^{\dim \mathcal{B}^\pm_f}.$$

As it is shown in appendix B the subsets $\mathcal{B}^\pm_f$ are composed of the same number of elements,

$$\dim \mathcal{B}^+_f = \dim \mathcal{B}^-_f. \qquad (4.8)$$

**Proposition 3.** *Let $Q$ be a polynomial in $\beta$ and let $\deg_\beta Q$ denotes its degree. Then:*

1. *for any $u^+_k = L_{-M} S_{-K} w^+_\beta \in \mathcal{B}^+_f$ and for any $u^-_l = L_{-M} S_{-K} w^-_\beta \in \mathcal{B}^-_f$ :*

$$\deg_\beta \langle u^+_k | u^+_k \rangle = 2(\#M + \#K), \qquad \deg_\beta \langle u^-_l | u^-_l \rangle = 2(\#M + \#K);$$

2. *for any $u^+_k \in \mathcal{B}^+_n$, $u^-_l \in \mathcal{B}^-_n$ :*

$$\deg_\beta \langle u^+_k | u^-_l \rangle < \min \left\{ \deg_\beta \langle u^-_l | u^-_l \rangle, \deg_\beta \langle u^+_k | u^+_k \rangle \right\};$$

3. *for any $u^\pm_k, u^\pm_l \in \mathcal{B}^\pm_n$, $u^\pm_k \neq u^\pm_l$ :*

$$\deg_\beta \langle u^\pm_k | u^\pm_l \rangle < \max \left\{ \deg_\beta \langle u^\pm_l | u^\pm_l \rangle, \deg_\beta \langle u^\pm_k | u^\pm_k \rangle \right\};$$

4. *the product of the diagonal terms is the only highest degree term in the determinant of the Gram matrix with respect to the base $\mathcal{B}_n$ i.e.*

$$\deg_\beta \left( \det \left[ \langle u_i, u_j \rangle \right] - \prod_{i=1}^{\dim \mathcal{B}_n} \langle u_i, u_i \rangle \right) < \deg_\beta \det \left[ \langle u_i, u_j \rangle \right].$$

Let us recall that matrix elements of arbitrary chiral vertex operators $V(\nu_\lambda), V(*\nu_\lambda)$ between even states $u_i, u_j \in \mathcal{W}_\beta$ can be decomposed as [23]:

$$\langle u_i | V(\nu_\lambda) | u_j \rangle = \tag{4.9}$$
$$\rho_{\mathrm{RR,e}}^{++}(u_i, \nu_\lambda, u_j) \langle w_\beta^+ | V(\nu_\lambda) | w_\beta^+ \rangle + \rho_{\mathrm{RR,e}}^{--}(u_i, \nu_\lambda, u_j) \langle w_\beta^- | V(\nu_\lambda) | w_\beta^- \rangle,$$
$$\langle u_i | V(*\nu_\lambda) | u_j \rangle = \tag{4.10}$$
$$\rho_{\mathrm{RR,o}}^{+-}(u_i, *\nu_\lambda, u_j) \langle w_\beta^+ | V(\nu_\lambda) | w_\beta^- \rangle + \rho_{\mathrm{RR,o}}^{-+}(u_i, *\nu_\lambda, u_j) \langle w_\beta^- | V(\nu_\lambda) | w_\beta^+ \rangle.$$

The decompositions above can be seen as defining the forms $\rho_{\mathrm{RR,e}}^{\pm\pm}$, $\rho_{\mathrm{RR,o}}^{\pm\mp}$. They are related to 3-point blocks (2.7) by

$$\rho_{\mathrm{RR,e}}^{(\pm)}(u_i^\pm, \nu_\lambda, u_j^\pm) = \rho_{\mathrm{RR,e}}^{++}(u_i^\pm, \nu_\lambda, u_j^\pm) \pm \rho_{\mathrm{RR,e}}^{--}(u_i^\pm, \nu_\lambda, u_j^\pm),$$
$$\rho_{\mathrm{RR,o}}^{(\pm)}(u_i^\pm, *\nu_\lambda, u_j^\pm) = \rho_{\mathrm{RR,o}}^{+-}(u_i^\pm, *\nu_\lambda, u_j^\pm) \pm i \rho_{\mathrm{RR,o}}^{-+}(u_i^\pm, *\nu_\lambda, u_j^\pm). \tag{4.11}$$

There holds:

**Proposition 4.** *Let* $\deg_\beta \langle w_\beta^\pm | V(\nu_\lambda) | w_\beta^\pm \rangle = \deg_\beta \langle w_\beta^\pm | V(\nu_\lambda) | w_\beta^\mp \rangle = 0$. *Then:*

1. *for any* $u_k^+, u_l^+ \in \mathcal{B}_f$, $u_k^+ \neq u_l^+$ :

$$\deg_\beta \rho_{\mathrm{RR,e}}^{++}(u_k^+, \nu_\lambda, u_l^+) < \max \left\{ \deg_\beta \langle u_k^+ | u_k^+ \rangle, \deg_\beta \langle u_l^+ | u_l^+ \rangle \right\},$$
$$\deg_\beta \beta^{-1} \rho_{\mathrm{RR,o}}^{+-}(u_k^+, *\nu_\lambda, u_l^+) < \max \left\{ \deg_\beta \langle u_k^+ | u_k^+ \rangle, \deg_\beta \langle u_l^+ | u_l^+ \rangle \right\},$$
$$\deg_\beta \beta^{-1} \rho_{\mathrm{RR,o}}^{-+}(u_k^+, *\nu_\lambda, u_l^+) < \max \left\{ \deg_\beta \langle u_k^+ | u_k^+ \rangle, \deg_\beta \langle u_l^+ | u_l^+ \rangle \right\};$$

2. *for any* $u_k^-, u_l^- \in \mathcal{B}_f$, $u_k^- \neq u_l^-$ :

$$\deg_\beta \rho_{\mathrm{RR,e}}^{--}(u_k^-, \nu_\lambda, u_l^-) < \max \left\{ \deg_\beta \langle u_k^- | u_k^- \rangle, \deg_\beta \langle u_l^- | u_l^- \rangle \right\},$$
$$\deg_\beta \beta^{-1} \rho_{\mathrm{RR,o}}^{+-}(u_k^-, *\nu_\lambda, u_l^-) < \max \left\{ \deg_\beta \langle u_k^- | u_k^- \rangle, \deg_\beta \langle u_l^- | u_l^- \rangle \right\},$$
$$\deg_\beta \beta^{-1} \rho_{\mathrm{RR,o}}^{-+}(u_k^-, *\nu_\lambda, u_l^-) < \max \left\{ \deg_\beta \langle u_k^- | u_k^- \rangle, \deg_\beta \langle u_l^- | u_l^- \rangle \right\};$$

3. *for any* $u_k^\pm, u_l^\pm \in \mathcal{B}_f$ :

$$\deg_\beta \rho_{\mathrm{RR,e}}^{++}(u_k^-, \nu_\lambda, u_l^-) < \min \left\{ \deg_\beta \langle u_k^- | u_k^- \rangle, \deg_\beta \langle u_l^- | u_l^- \rangle \right\},$$
$$\deg_\beta \rho_{\mathrm{RR,e}}^{--}(u_k^+, \nu_\lambda, u_l^+) < \min \left\{ \deg_\beta \langle u_k^+ | u_k^+ \rangle, \deg_\beta \langle u_l^+ | u_l^+ \rangle \right\};$$

4. *for any* $u_k^\pm, u_l^\pm \in \mathcal{B}_f$ :

$$\deg_\beta \rho_{\mathrm{RR,e}}^{++}(u_k^\pm, \nu_\lambda, u_l^\mp) < \min \left\{ \deg_\beta \langle u_k^\pm | u_k^\pm \rangle, \deg_\beta \langle u_l^\mp | u_l^\mp \rangle \right\},$$
$$\deg_\beta \rho_{\mathrm{RR,e}}^{--}(u_k^\pm, \nu_\lambda, u_l^\mp) < \min \left\{ \deg_\beta \langle u_k^\pm | u_k^\pm \rangle, \deg_\beta \langle u_l^\mp | u_l^\mp \rangle \right\},$$
$$\deg_\beta \beta^{-1} \rho_{\mathrm{RR,o}}^{+-}(u_k^\pm, *\nu_\lambda, u_l^\mp) < \min \left\{ \deg_\beta \langle u_k^\pm | u_k^\pm \rangle, \deg_\beta \langle u_l^\mp | u_l^\mp \rangle \right\},$$
$$\deg_\beta \beta^{-1} \rho_{\mathrm{RR,o}}^{-+}(u_k^\pm, *\nu_\lambda, u_l^\mp) < \min \left\{ \deg_\beta \langle u_k^\pm | u_k^\pm \rangle, \deg_\beta \langle u_l^\mp | u_l^\mp \rangle \right\}.$$

It follows from Proposition 4 that for any $u_i, u_j \in \mathcal{B}_f$, $u_i \neq u_j$:

$$\deg_\beta \rho_{\mathrm{RR,e}}^{\pm\pm}(u_i, \nu_\lambda, u_j) < \deg_\beta \langle u_i | u_i \rangle \quad \text{or} \quad \deg_\beta \rho_{\mathrm{RR,e}}^{\pm\pm}(u_i, \nu_\lambda, u_j) < \deg_\beta \langle u_j | u_j \rangle$$

and

$$\deg_\beta \beta^{-1} \rho_{\mathrm{RR,o}}^{\pm\mp}(u_i, *\nu_\lambda, u_j) < \deg_\beta \langle u_i | u_i \rangle \quad \text{or} \quad \deg_\beta \beta^{-1} \rho_{\mathrm{RR,o}}^{\pm\mp}(u_i, *\nu_\lambda, u_j) < \deg_\beta \langle u_j | u_j \rangle.$$

Following the same steps as in the previous subsection we thus get for $i \neq j$:

$$\begin{aligned} \lim_{\beta \to \infty} \rho_{\mathrm{RR,e}}^{\pm\pm}(u_i, \nu_\lambda, u_j) B^{ij} &= 0, \\ \lim_{\beta \to \infty} \beta^{-1} \rho_{\mathrm{RR,o}}^{\pm\mp}(u_i, \nu_\lambda, u_j) B^{ij} &= 0. \end{aligned} \tag{4.12}$$

Let

$$\begin{bmatrix} B_{++}^{ij} & B_{+-}^{ij} \\ B_{-+}^{ij} & B_{--}^{ij} \end{bmatrix}$$

be the matrix inverse to the Gram matrix

$$\begin{bmatrix} \langle u_i^+ | u_j^+ \rangle & \langle u_i^+ | u_j^- \rangle \\ \langle u_i^- | u_j^+ \rangle & \langle u_i^- | u_j^- \rangle \end{bmatrix}.$$

By Proposition 4, for any $u_k^\pm \in \mathcal{B}_f^\pm$:

$$\deg_\beta \rho_{\mathrm{RR,e}}^{\mp\mp}(u_k^\pm, \nu_\lambda, u_k^\pm) < \deg \langle u_k^\pm, u_k^\pm \rangle$$

and we also get

$$\lim_{\beta \to \infty} \rho_{\mathrm{RR,e}}^{\mp\mp}(u_k^\pm, \nu_\lambda, u_k^\pm)) B_{\pm\pm}^{kk} = 0. \tag{4.13}$$

Since

$$\begin{aligned} \rho_{\mathrm{RR,o}}^{+-}(w_\beta^+, *\nu_\lambda, w_\beta^+) &= \rho_{\mathrm{RR,o}}^{+-}(w_\beta^-, *\nu_\lambda, w_\beta^-) = i\beta e^{-i\frac{\pi}{4}} = \beta e^{+i\frac{\pi}{4}}, \\ \rho_{\mathrm{RR,o}}^{-+}(w_\beta^+, *\nu_\lambda, w_\beta^+) &= \rho_{\mathrm{RR,o}}^{-+}(w_\beta^-, *\nu_\lambda, w_\beta^-) = i\beta e^{+i\frac{\pi}{4}} = -\beta e^{-i\frac{\pi}{4}}, \end{aligned} \tag{4.14}$$

the leading terms in the large $\beta$ limit take the form:

$$\begin{aligned} \rho_{\mathrm{RR,e}}^{\pm\pm}(u_k^\pm, \nu_\lambda, u_k^\pm) &= \langle u_k^\pm | u_k^\pm \rangle + \ldots, \\ \rho_{\mathrm{RR,o}}^{+-}(u_k^\pm, *\nu_\lambda, u_k^\pm) &= \beta e^{i\frac{\pi}{4}} \langle u_k^\pm | u_k^\pm \rangle + \ldots, \\ \rho_{\mathrm{RR,o}}^{-+}(u_k^\pm, *\nu_\lambda, u_k^\pm) &= -\beta e^{-i\frac{\pi}{4}} \langle u_k^\pm | u_k^\pm \rangle + \ldots. \end{aligned}$$

This yields

$$\begin{aligned} \lim_{\beta \to \infty} \rho_{\mathrm{RR,e}}^{\pm\pm}(u_k^\pm, \nu_\lambda, u_k^\pm) B_{\pm\pm}^{kk} &= 1, \\ \lim_{\beta \to \infty} \beta^{-1} \rho_{\mathrm{RR,o}}^{+-}(u_k^\pm, *\nu_\lambda, u_k^\pm) &= e^{i\frac{\pi}{4}}, \\ \lim_{\beta \to \infty} \beta^{-1} \rho_{\mathrm{RR,o}}^{-+}(u_k^\pm, *\nu_\lambda, u_k^\pm) &= -e^{-i\frac{\pi}{4}}. \end{aligned} \tag{4.15}$$

For forms (4.11), equations (4.13) and (4.15) give

$$\lim_{\beta\to\infty}\rho_{\mathrm{RR,e}}^{(\pm)}(u_k^+,\nu_\lambda,u_k^+)B_{++}^{kk}=1,$$

$$\lim_{\beta\to\infty}\rho_{\mathrm{RR,e}}^{(\pm)}(u_k^-,\nu_\lambda,u_k^-)B_{--}^{kk}=\pm1,$$

$$\lim_{\beta\to\infty}\beta^{-1}\rho_{\mathrm{RR,o}}^{(+)}(u_k^\pm,*\nu_\lambda,u_k^\pm)B_{\pm\pm}^{kk}=0,$$

$$\lim_{\beta\to\infty}\beta^{-1}\rho_{\mathrm{RR,o}}^{(-)}(u_k^\pm,*\nu_\lambda,u_k^\pm)B_{\pm\pm}^{kk}=2\mathrm{e}^{\frac{i\pi}{4}}.$$

(4.16)

Using (4.12) and (4.16) we finally get (the case $f=0$ is special as $\mathcal{B}_0^-=\varnothing$):

$$h_\beta^{\lambda(+),f}=\lim_{\beta\to\infty}F_{\beta,\mathrm{e}}^{\lambda(+),f}=\lim_{\beta\to\infty}\sum_{i=1}^{\dim\mathcal{B}_f}\rho_{\mathrm{RR,e}}^{(+)}(u_i,\nu_\lambda,u_i)B^{ii} \qquad (4.17)$$

$$=\lim_{\beta\to\infty}\sum_{k=1}^{\dim\mathcal{B}_f^\pm}\left(\rho_{\mathrm{RR,e}}^{(+)}(u_k^+,\nu_\lambda,u_k^+)B_{++}^{kk}+\rho_{\mathrm{RR,e}}^{(+)}(u_k^-,\nu_\lambda,u_k^-)B_{--}^{kk}\right)$$

$$=\dim\mathcal{B}_f=p_{\mathrm{R}}(f),$$

where $p_{\mathrm{R}}(f)$ can be computed by means of generating function

$$\sum_{q=0}^\infty p_{\mathrm{R}}(f)q^f=\prod_{n=1}^\infty\frac{1+q^n}{1-q^n}=\tfrac{1}{2}q^{\frac{c}{24}-\Delta}\chi_{\mathrm{R}}^\Delta(q)$$

and $\chi_{\mathrm{R}}^\Delta(q)$ is the character of the Ramond Verma module [37–40]

$$\chi_{\mathrm{R}}^\Delta(q)=\mathrm{Tr}_{\mathrm{R}}\,q^{L_0-\frac{c}{24}}=q^{\Delta-\frac{c}{24}}\,\eta(q)^{-\frac{3}{2}}\,\sqrt{2\,\theta_2(q^{1/2})}.$$

Similarly:

$$h_\beta^{\lambda(-),f}=\lim_{\beta\to\infty}F_{\beta,\mathrm{e}}^{\lambda(-),f}=\delta_{f,0},$$

$$\beta^{-1}h_\beta^{*\lambda(+),f}=\lim_{\beta\to\infty}\beta^{-1}F_{\beta,\mathrm{e}}^{*\lambda(+),f}=0, \qquad (4.18)$$

$$\beta^{-1}h_\beta^{*\lambda(-),f}=\lim_{\beta\to\infty}\beta^{-1}F_{\beta,\mathrm{e}}^{\lambda(-),f}=2\,\mathrm{e}^{\frac{i\pi}{4}}\,p_{\mathrm{R}}(f).$$

## 5 Elliptic blocks and recurrence relations

### 5.1 NS and $\widetilde{\mathrm{NS}}$ sectors

The large $\Delta$ asymptotic (4.6) suggests the following definition of the elliptic blocks in NS sector:

$$\mathcal{F}_\Delta^{-\lambda}(q)=q^{\Delta-\frac{c}{24}+\frac{1}{16}}\,\eta(q)^{-\frac{3}{2}}\,\sqrt{\theta_3(q^{1/2})}\,\mathcal{H}_\Delta^{-\lambda}(q)\,,$$

$$\mathcal{H}_\Delta^{-\lambda}(q)=\sum_{f\in\frac{1}{2}\mathbb{N}\cup\{0\}}q^f\,H_\Delta^{-\lambda,f}$$

(5.1)

and in the $\widetilde{\mathbf{NS}}$ sector:

$$\widetilde{\mathcal{F}}_{\Delta}^{-\lambda}(q) = q^{\Delta - \frac{c}{24} + \frac{1}{16}}\, \eta(q)^{-\frac{3}{2}}\, \sqrt{\theta_4(q^{1/2})}\, \widetilde{\mathcal{H}}_{\Delta}^{-\lambda}(q),$$

$$\widetilde{\mathcal{H}}_{\Delta}^{-\lambda}(q) = \sum_{f \in \frac{1}{2}\mathbb{N}\cup\{0\}} q^f\, \widetilde{H}_{\Delta}^{-\lambda,f} = \sum_{f \in \frac{1}{2}\mathbb{N}\cup\{0\}} (-1)^{2f}\, q^f\, H_{\Delta}^{-\lambda,f}.$$

Coefficients $H_{\Delta}^{-\lambda,f}$ have the same analytic properties as coefficients $F_{\Delta}^{-\lambda,f}$ and formula (3.10) yields the recursive relation:

$$H_{\Delta}^{-\lambda,f} = \delta_0^f + \sum_{\substack{rs \in 2\mathbb{N} \\ 1 < rs \leqslant 2f}} \frac{A_{rs} P_c^{rs}\left[\begin{smallmatrix} -\Delta_\lambda \\ \Delta_{rs} + \frac{rs}{2} \end{smallmatrix}\right] P_c^{rs}\left[\begin{smallmatrix} -\Delta_\lambda \\ \Delta_{rs} \end{smallmatrix}\right]}{\Delta - \Delta_{rs}} H_{\Delta_{rs}+\frac{rs}{2}}^{-\lambda,f-\frac{rs}{2}} \qquad (5.2)$$

$$+ \sum_{\substack{rs \in 2\mathbb{N}+1 \\ 1 < rs \leqslant 2f}} \frac{A_{rs} P_c^{rs}\left[\begin{smallmatrix} -\widetilde{\Delta}_\lambda \\ \Delta_{rs} + \frac{rs}{2} \end{smallmatrix}\right] P_c^{rs}\left[\begin{smallmatrix} -\Delta_\lambda \\ \Delta_{rs} \end{smallmatrix}\right] -\widetilde{s_{rs}}}{\Delta - \Delta_{rs}} H_{\Delta_{rs}+\frac{rs}{2}}^{-\lambda,f-\frac{rs}{2}}.$$

## 5.2 R and $\widetilde{\mathrm{R}}$ sectors

The large $\beta$ behavior of the blocks with R intermediate states, (4.17) and (4.18), lead to the following definition of the elliptic blocks:

$$\mathcal{F}_{\beta,\mathrm{e}}^{\lambda(+)}(q) = \frac{1}{\sqrt{2}}\, q^{\Delta - \frac{c}{24}}\, \eta(q)^{-\frac{3}{2}}\, \sqrt{\theta_2(q^{1/2})}\, \mathcal{H}_{\beta,\mathrm{e}}^{\lambda(+)}(q),$$

$$\mathcal{F}_{\beta,\mathrm{e}}^{\lambda(-)}(q) = q^{\Delta - \frac{c}{24}}\, \mathcal{H}_{\beta,\mathrm{e}}^{\lambda(-)}(q),$$

$$\mathcal{F}_{\beta,\mathrm{e}}^{*\lambda(-)}(q) = \mathrm{e}^{i\frac{\pi}{4}}\, \sqrt{2}\,\beta\, q^{\Delta - \frac{c}{24}}\, \eta(q)^{-\frac{3}{2}}\, \sqrt{\theta_2(q^{1/2})}\, \mathcal{H}_{\beta,\mathrm{e}}^{*\lambda(-)}(q).$$

Since $h_\beta^{*\lambda(+),f}$ vanishes, recursive relation (3.12) implies that $\mathcal{F}_{\beta,\mathrm{e}}^{*\lambda(+)}$ is identically zero

$$\mathcal{F}_{\beta,\mathrm{e}}^{*\lambda(+)}(q) = 0.$$

It follows that all 1-point functions of $\tilde{\phi}_{\lambda,\bar\lambda}$ vanish in the $\widetilde{R}$ sector:

$$\langle \tilde{\phi}_{\lambda,\bar\lambda} \rangle_{\widetilde{\mathrm{R}}} = 0.$$

Using recursive relation (3.12) and asymptotics (4.18) one can also show that

$$\mathcal{H}_{\beta,\mathrm{e}}^{*\lambda(-)}(q) = \mathcal{H}_{\beta,\mathrm{e}}^{\lambda(-)}(q).$$

There are thus only two independent elliptic blocks

$$\mathcal{H}_{\beta,\mathrm{e}}^{\lambda(\pm)}(q) = \sum_{f \in \mathbb{N}\cup\{0\}} q^f\, H_{\beta,\mathrm{e}}^{\lambda(\pm),f}$$

with coefficients satisfying the recursive relation

$$H_{\beta,\mathrm{e}}^{\lambda(\pm),f} = \delta_0^f + \sum_{\substack{1 < rs \leqslant 2f \\ r+s \in 2\mathbb{N}+1}} \frac{A_{rs} P_c^{rs}\left[\begin{smallmatrix} \Delta_\lambda \\ \pm\beta'_{rs} \end{smallmatrix}\right] P_c^{rs}\left[\begin{smallmatrix} \Delta_\lambda \\ \pm\beta_{rs} \end{smallmatrix}\right]}{\Delta - \Delta_{rs}} H_{\beta'_{rs},\mathrm{e}}^{\lambda(\pm),f-\frac{rs}{2}}. \qquad (5.3)$$

## Acknowledgments

## A  Neveu-Schwarz sector

In this appendix we shall prove the propositions of section 4.1.

**Proof of proposition 1.**  Part 1 is a simple consequence of the NS algebra. By the same token one has

$$\deg_\Delta \langle u_i | u_j \rangle \leqslant \min \left\{ \deg_\Delta \langle u_i | u_i \rangle, \deg_\Delta \langle u_j | u_j \rangle \right\}.$$

Let $u_i = L_{-M} S_{-K} \nu_\Delta$ and $u_j = L_{-N} S_{-L} \nu_\Delta$. If $\#M + \#K \neq \#N + \#L$, then part 2 follows from part 1 and the inequality above.

Suppose $\#M + \#K = \#N + \#L$. In this case the inequality of part 2 is also satisfied. Indeed, calculating the scalar product

$$\langle L_{-M} S_{-K} \nu_\Delta | L_{-N} S_{-L} \nu_\Delta \rangle = \langle \nu_\Delta | (S_{-K})^\dagger (L_{-M})^\dagger L_{-N} S_{-L} \nu_\Delta \rangle$$

by the NS algebra rules one can get the maximal degree $\#M + \#K$ if, and only if $L_{-M} S_{-K} \nu_\Delta = L_{-N} S_{-L} \nu_\Delta$.

In order to prove part 3 let us observe that by part 1 the product of diagonal terms $Q = \prod_{i=1}^{\dim \mathcal{B}_f} \langle u_i | u_i \rangle$ is of a maximal degree i.e.

$$\deg_\Delta Q = \deg_\Delta \det \left[ \langle u_i | u_j \rangle \right].$$

Any other term in the expression for the determinant of $\left[ \langle u_i | u_j \rangle \right]$ takes the form

$$P_\sigma = \prod_{i=1}^{\dim \mathcal{B}_f} \langle u_i | u_{\sigma(i)} \rangle,$$

where $\sigma$ is a nontrivial permutation. Let us assume that for all $i$

$$\deg_\Delta \langle u_i | u_{\sigma(i)} \rangle = \deg_\Delta \langle u_i | u_i \rangle,$$

hence $\deg_\Delta Q = \deg_\Delta P_\sigma$. On the other hand by Prop. 1.2 the equations above imply that

$$\deg_\Delta \langle u_i | u_{\sigma(i)} \rangle < \deg_\Delta \langle u_{\sigma(i)} | u_{\sigma(i)} \rangle$$

for all $i$ and therefore $\deg_\Delta Q > \deg_\Delta P$ in contradiction with our assumption. It follows that for an arbitrary nontrivial permutation $\sigma$ there exists at least one $i$ such that

$$\deg_\Delta \langle u_i | u_{\sigma(i)} \rangle < \deg_\Delta \langle u_i | u_i \rangle$$

hence $\deg_\Delta P_\sigma < \deg_\Delta Q$. □

**Proof of proposition 2.** Let $V(\_\nu_\lambda) : \mathcal{V}_\Delta \to \mathcal{V}_\Delta$ be an NS chiral vertex operator with a conformal weight $\Delta_\lambda$. For any $u_i, u_j \in \mathcal{B}_f$ of the same parity one has

$$\langle u_i | V(\nu_\lambda) | u_j \rangle = \rho_{\mathrm{NN}}(u_i, \nu_\lambda, u_j) \langle \nu_\Delta | V(\nu_\lambda) | \nu_\Delta \rangle,$$
$$\langle u_i | V(*\nu_\lambda) | u_j \rangle = \rho^*_{\mathrm{NN}}(u_i, *\nu_\lambda, u_j) \langle \nu_\Delta | V(*\nu_\lambda) | \nu_\Delta \rangle.$$

If we assume
$$\deg_\Delta \langle \nu_\Delta | V(\nu_\lambda) | \nu_\Delta \rangle = \deg_\Delta \langle \nu_\Delta | V(*\nu_\lambda) | \nu_\Delta \rangle = 0$$

then

$$\deg_\Delta \rho_{\mathrm{NN}}(u_i, \nu_\lambda, u_j) = \deg_\Delta \langle u_i | V(\nu_\lambda) | u_j \rangle,$$
$$\deg_\Delta \rho^*_{\mathrm{NN}}(u_i, *\nu_\lambda, u_j) = \deg_\Delta \langle u_i | V(*\nu_\lambda) | u_j \rangle$$

and it is enough to consider the matrix elements $\langle u_i | V(\_\nu_\lambda) | u_j \rangle$. By Proposition 1.2

$$\deg_\Delta \langle u_i | u_j \rangle < \max \{ \deg_\Delta \langle u_i | u_i \rangle, \deg_\Delta \langle u_j | u_j \rangle \}.$$

Suppose
$$\max \{ \deg_\Delta \langle u_i | u_i \rangle, \deg_\Delta \langle u_j | u_j \rangle \} = \deg_\Delta \langle u_j | u_j \rangle.$$

Calculating matrix elements $\langle u_i | V(\_\nu_\lambda) | u_j \rangle$ one can use the Ward identities to move all the NS algebra generators to the right.
Let $u_i = L_{-M} S_{-K} \nu_\Delta$, $u_j = L_{-N} S_{-L} \nu_\Delta$, then

$$\deg_\Delta \langle u_i | u_i \rangle = \#M + \#K \leqslant \deg_\Delta \langle u_j | u_j \rangle.$$

The matrix elements $\langle u_i | V(\_\nu_\lambda) | u_j \rangle$ can be represented as a linear combination of

$$\langle \nu_\Delta | V(\_\nu_\lambda) | (S_{-K})^\dagger (L_{-M})^\dagger u_j \rangle$$

and terms of the form

$$\langle \nu_\Delta | V(\nu_\lambda) | (S_{-K})^\dagger \widehat{(L_{-M})}^\dagger u_j \rangle, \qquad \langle \nu_\Delta | V(*\nu_\lambda) | (S_{-K})^\dagger \widehat{(L_{-M})}^\dagger u_j \rangle,$$

where $(S_{-K})^\dagger \widehat{(L_{-M})}^\dagger$ denotes product $(S_{-K})^\dagger (L_{-M})^\dagger$ with at least one generator removed. The coefficients of this combination are independent of $\Delta$.

Using Ward identities one easily checks that for arbitrary $L_{-P} S_{-Q} \nu_\Delta$

$$\deg_\Delta \langle \nu_\Delta | V(\_\nu_\lambda) | L_{-P} S_{-Q} \nu_\Delta \rangle = 0.$$

This in order imply

$$\deg_\Delta \langle \nu_\Delta | V(\_\nu_\lambda) | (S_{-K})^\dagger (L_{-M})^\dagger u_j \rangle \leqslant \#M + \#K,$$

and

$$\deg_\Delta \langle \nu_\Delta | V(\_\nu_\lambda) | (S_{-K})^\dagger \widehat{(L_{-M})}^\dagger u_j \rangle < \#M + \#K.$$

On the other hand

$$\langle \nu_\Delta | V(\_\nu_\lambda) | (S_{-K})^\dagger (L_{-M})^\dagger u_j \rangle = \langle \nu_\Delta | V(\_\nu_\lambda) | \nu_\Delta \rangle \langle u_i | u_j \rangle$$

and by assumption

$$\deg_\Delta \langle \nu_\Delta | V(\_\nu_\lambda) | (S_{-K})^\dagger (L_{-M})^\dagger u_j \rangle < \deg_\Delta \langle u_j | u_j \rangle.$$

Hence

$$\deg_\Delta \langle u_i | V(\_\nu_\lambda) | u_j \rangle < \deg_\Delta \langle u_j | u_j \rangle.$$

If $\max\{\deg_\Delta \langle u_i | u_i \rangle, \deg_\Delta \langle u_j | u_j \rangle\} = \deg_\Delta \langle u_i | u_i \rangle$ one can repeat the calculations moving all the NS generators to the left. One thus gets

$$\deg_\Delta \langle u_i | V(\_\nu_\lambda) | u_j \rangle \ < \ \max\{\deg_\Delta \langle u_i | u_i \rangle, \deg_\Delta \langle u_j | u_j \rangle\}. \qquad \square$$

## B Ramond sector

We shall first prove eq. (4.8).

Let $q(k,n)$ be the number of partitions of $n$ in $k$ distinct parts. The corresponding generating function reads

$$\sum_{k,n=0}^{\infty} q(n,k) y^k q^n = \prod_{i=1}^{\infty} (1 + y q^i).$$

For $y = -1$ it counts the difference between the number of partitions in an even number of unequal parts and the number of partitions in an odd number of unequal parts. Hence

$$\sum_{n=0}^{\infty} (\dim \mathcal{B}_n^+ - \dim \mathcal{B}_n^-) q^n = \prod_{i=1}^{\infty} \frac{1}{1 - q^i} \prod_{i=1}^{\infty} (1 - q^i) = 1. \qquad \square$$

**Proof of proposition 3.** Part 1 is a simple consequence of the Ramond algebra. Part 2 follows from part 1 and the observation that maximal possible degree of $\langle u_k^+ | u_l^- \rangle$ is odd while the diagonal elements of the Gram matrix are of even degrees. The proof of part 3 parallels the proof of Proposition 1, part 2 while part 4 is proved along the same lines as Proposition 1, part 3. $\qquad \square$

**Proof of proposition 4.** We shall prove part 1 using the same method as in the proof of Proposition 2.

In the case of interest eqs. (4.9) and (4.10) take the form

$$\langle u_k^+ | V(\nu_\lambda) | u_l^+ \rangle = \rho_{\mathrm{RR,e}}^{++}(u_k^+, \nu_\lambda, u_l^+) \langle w_\beta^+ | V(\nu_\lambda) | w_\beta^+ \rangle + \rho_{\mathrm{RR,e}}^{--}(u_k^+, \nu_\lambda, u_l^+) \langle w_\beta^- | V(\nu_\lambda) | w_\beta^- \rangle$$

and

$$\langle u_k^+ | V(*\nu_\lambda) | u_l^+ \rangle = \rho_{\mathrm{RR,o}}^{+-}(u_k^+, *\nu_\lambda, u_l^+) \langle w_\beta^+ | V(\nu_\lambda) | w_\beta^- \rangle + \rho_{\mathrm{RR,o}}^{-+}(u_k^+, *\nu_\lambda, u_l^+) \langle w_\beta^- | V(\nu_\lambda) | w_\beta^+ \rangle.$$

By Proposition 3:

$$\deg_\beta \langle u_k^+|u_l^+\rangle < \max\{\deg_\beta \langle u_l^+|u_l^+\rangle, \deg_\beta \langle u_k^+|u_k^+\rangle\}.$$

Suppose

$$\max\{\deg_\beta \langle u_l^+|u_l^+\rangle, \deg_\beta \langle u_k^+|u_k^+\rangle\} = \deg_\beta \langle u_k^+|u_k^+\rangle \qquad \text{(B.1)}$$

and let $u_k^+ = L_{-M}S_{-K}w_\beta^+, u_l^+ = L_{-N}S_{-L}w_\beta^+$ (with $\#K, \#L \in 2\mathbb{N}$). In order to calculate $\langle u_k^+|V_\lambda(\nu)|u_l^+\rangle$ one can use the Ward identities to move all the Ramond algebra generators to the right representing it as linear combination (with $\beta-$independent coefficients) of

$$\langle w^+|V(\nu_\lambda)|(S_{-K})^\dagger (L_{-M})^\dagger u_l^+\rangle$$

and terms of the form

$$\langle w^+|V(\nu_\lambda)|(S_{-K})^\dagger \widehat{(L_{-M})}^\dagger u_l^+\rangle, \quad \langle w^+|V(*\nu_\lambda)|(S_{-K})^\dagger \widehat{(L_{-M})}^\dagger u_l^+\rangle.$$

For arbitrary $L_{-P}S_{-Q}w_\beta^\pm$ the Ward identities give

$$\deg_\beta \langle w^+|V(_-\nu_\lambda)|L_{-P}S_{-Q}w_\beta^\pm\rangle \leqslant 1,$$

what in turn implies

$$\deg_\beta \langle w^+|V(_-\nu_\lambda)|(S_{-K})^\dagger \widehat{(L_{-M})}^\dagger u_l^+\rangle < 2(\#M + \#K) = \deg\langle u_k^+|u_k^+\rangle.$$

On the other hand

$$\langle w^+|V(\nu_\lambda)|(S_{-K})^\dagger (L_{-M})^\dagger u_l^+\rangle = \langle w^+|V(\nu_\lambda)|w^+\rangle\langle u_k|u_l^+\rangle$$

and by assumption

$$\deg\langle w^+|V(\nu_\lambda)|(S_{-K})^\dagger (L_{-M})^\dagger u_l^+\rangle < \deg\langle u_k^+|u_k^+\rangle.$$

If $\max\{\deg_\beta \langle u_l^+|u_l^+\rangle, \deg_\beta \langle u_k^+|u_k^+\rangle\} = \deg_\beta \langle u_l^+|u_l^+\rangle$ one follows similar calculations moving all the Ramond generators to the left.

Taking into account decomposition (4.9) one gets in particular

$$\deg \rho_{\text{RR,e}}^{++}(u_k^+, \nu_\lambda, u_l^+) < \max\left\{\deg\langle u_k^+|u_k^+\rangle, \deg\langle u_l^+|u_l^+\rangle\right\}.$$

Since the terms

$$\langle w^+|V(\nu_\lambda)|S_K^\dagger L_M^\dagger u_l^+\rangle, \quad \langle S_L^\dagger L_N^\dagger u_k^+|V(\nu_\lambda)|w^+\rangle,$$

do not contribute to $\rho_{\text{RR,e}}^{--}$ one also has the second inequality of part 3:

$$\deg \rho_{\text{RR,e}}^{--}(u_k^+, \nu_\lambda, u_l^+) < \min\left\{\deg\langle u_k^+|u_k^+\rangle, \deg\langle u_l^+|u_l^+\rangle\right\}.$$

The matrix element $\langle u_k^+|V(*\nu_\lambda)|u_l^+\rangle$ can be analyzed in a similar way. Suppose that equation (B.1) holds. As before one has

$$\deg\langle w^+|V(_-\nu_\lambda)|(S_{-K})^\dagger \widehat{(L_{-M})}^\dagger u_l^+\rangle < 2(\#M + \#K).$$

On the other hand

$$\langle w^+|V(*\nu_\lambda)|(S_{-K})^\dagger(L_{-M})^\dagger u_l^+\rangle = \langle w^+|V(*\nu_\lambda)|w^+\rangle\langle u_k^+|u_l^+\rangle$$
$$= \beta\, e^{i\frac{\pi}{4}}\left(\langle w^+|V(\nu_\lambda)|w^-\rangle + i\langle w^-|V(\nu_\lambda)|w^+\rangle\right)\langle u_k^+|u_l^+\rangle$$

and by assumption

$$\deg\beta^{-1}\langle w^+|V(*\nu_\lambda)|S_K^\dagger L_M^\dagger u_l^+\rangle < \deg\langle u_k^+|u_k^+\rangle,$$

hence

$$\deg\beta^{-1}\rho_{\mathrm{RR,o}}^{+-}(u_k^+, *\nu_\lambda, u_l^+) < \deg\langle u_k^+|u_k^+\rangle,$$
$$\deg\beta^{-1}\rho_{\mathrm{RR,o}}^{-+}(u_k^+, *\nu_\lambda, u_l^+) < \deg\langle u_k^+|u_k^+\rangle.$$

The rest of the proof parallels the considerations above.  □

## C  Some explicite formulae for blocks' coefficients

First elliptic block coefficients of the NS sector as defined in (5.1):

$$H_\Delta^{\lambda,0} = 1, \qquad H_\Delta^{\lambda,\frac{1}{2}} = \frac{\Delta_\lambda}{2\Delta}, \qquad H_\Delta^{\lambda,1} = \frac{\Delta_\lambda^2}{2\Delta},$$

$$H_\Delta^{\lambda,\frac{3}{2}} = \frac{\Delta_\lambda\Big(16b^2\Delta^2 + (2+5b^2+2b^4)\Delta_\lambda^2 + 2\Delta(2+2b^4+b^2(8+(-8+\Delta_\lambda)\Delta_\lambda))\Big)}{2\Delta(1+2b^2+2\Delta)(2+b^2(1+2\Delta))}$$

$$H_\Delta^{\lambda,2} = \Delta_\lambda\Big(6(1+b^8)\Delta_\lambda(1+4\Delta+\Delta_\lambda^2) + (b^2+b^6)\big(-64\Delta(1+2\Delta)$$
$$+(3+4\Delta)(17+54\Delta)\Delta_\lambda - 24(1+4\Delta)\Delta_\lambda^2 + 11(3+2\Delta)\Delta_\lambda^3\big)$$
$$+b^4\big(-32\Delta(5+4\Delta(1+\Delta)) + 2(51+2\Delta(101+92\Delta+48\Delta^2))\Delta_\lambda$$
$$-4(3+4\Delta)(5+6\Delta)\Delta_\lambda^2 + (57+2\Delta(19+8\Delta))\Delta_\lambda^3\big)\Big) \times$$
$$\times\Big(4\Delta(1+2b^2+2\Delta)(2+b^2(1+2\Delta))(3+3b^4+b^2(6+8\Delta))\Big)^{-1}$$

$$H_\Delta^{*\lambda,0} = 1, \qquad H_\Delta^{*\lambda,\frac{1}{2}} = \frac{1-2\Delta_\lambda}{4\Delta}, \qquad H_\Delta^{*\lambda,1} = \frac{(1-2\Delta_\lambda)^2}{8\Delta},$$

$$H_\Delta^{*\lambda,\frac{3}{2}} = -\Big((-1+2\Delta_\lambda)\big(2(8\Delta+(1-2\Delta_\lambda)^2) + 2b^4(8\Delta+(1-2\Delta_\lambda)^2)$$
$$+b^2(64\Delta^2+5(1-2\Delta_\lambda)^2+\Delta(34-72\Delta_\lambda+8\Delta_\lambda^2)))\big)\Big)\times$$
$$\times\Big(16\Delta(1+2b^2+2\Delta)(2+b^2(1+2\Delta))\Big)^{-1}$$

$$H_\Delta^{*\lambda,2} = (-1+2\Delta_\lambda)\Big(6(1+b^8)(-1+2\Delta_\lambda)(5+16\Delta-4\Delta_\lambda+4\Delta_\lambda^2)$$
$$+(b^2+b^6)\big(32\Delta^2(5+54\Delta_\lambda) + 2\Delta(-119+986\Delta_\lambda-516\Delta_\lambda^2+88\Delta_\lambda^3)$$
$$+3(-63+202\Delta_\lambda-196\Delta_\lambda^2+88\Delta_\lambda^3)\big)$$
$$+b^4\big(256\Delta^3(1+6\Delta_\lambda) + 16\Delta^2(-17+190\Delta_\lambda-60\Delta_\lambda^2+8\Delta_\lambda^3)$$
$$+2\Delta(-35+1730\Delta_\lambda-836\Delta_\lambda^2+3(-115+386\Delta_\lambda-388\Delta_\lambda^2+152\Delta_\lambda^3))\big)\Big)\times$$
$$\times\Big(64\Delta(1+2b^2+2\Delta)(2+b^2(1+2\Delta))(3+3b^4+b^2(6+8\Delta))\Big)^{-1}$$

First R blocks coefficients as defined in (2.10) and (2.11):

$$F_{\beta,e}^{\lambda(+),0} = 1$$

$$F_{\beta,e}^{\lambda(+),1} = \frac{6(2+5b^2+2b^4)(3+4(-1+\Delta_\lambda)\Delta_\lambda)+64(3+3b^4+b^2(3-6\Delta_\lambda+2\Delta_\lambda^2))\Delta_\beta+512b^2\Delta_\beta^2}{(3+6b^2+16\Delta_\beta)(6+b^2(3+16\Delta_\beta))}$$

$$F_{\beta,e}^{\lambda(+),2} = 4\Big(180(1+b^8)\Big(11+2(-1+\Delta_\lambda)\Delta_\lambda+16\Delta_\beta\Big)\Big(3+4(-1+\Delta_\lambda)\Delta_\lambda+16\Delta_\beta\Big)$$

$$+b^4\Big(15(3855+2\Delta_\lambda(-3077+5\Delta_\lambda(1021+4\Delta_\lambda(-124+37\Delta_\lambda))))$$

$$+32(6078+\Delta_\lambda(-8961+\Delta_\lambda(9083+224(-12+\Delta_\lambda)\Delta_\lambda)))\Delta_\beta$$

$$+512(779+\Delta_\lambda(-337+\Delta_\lambda(353+4(-12+\Delta_\lambda)\Delta_\lambda)))\Delta_\beta^2$$

$$+8192(14+\Delta_\lambda(-11+5\Delta_\lambda))\Delta_\beta^3+65536\Delta_\beta^4\Big)$$

$$+12(b^2+b^6)\Big(16\Delta_\lambda^4(59+24\Delta_\beta)-8\Delta_\lambda^3(361+496\Delta_\beta)$$

$$+8\Delta_\lambda^2(823+128\Delta_\beta(13+7\Delta_\beta))+(13+48\Delta_\beta)(213+32\Delta_\beta(7+8\Delta_\beta))$$

$$-32\Delta_\lambda(136+21\Delta_\beta(17+16\Delta_\beta)))\Big)\Big)$$

$$\times\Big((3+6b^2+16\Delta_\beta)(11+30b^2+16\Delta_\beta)(6+b^2(3+16\Delta_\beta))(30+b^2(11+16\Delta_\beta))\Big)^{-1}$$

$$F_{\beta,e}^{\lambda(-),0} = 1$$

$$F_{\beta,e}^{\lambda(-),1} = \frac{32(4(\Delta_\lambda-2)\Delta_\lambda+3)\Delta b^2+6\left(2b^4+5b^2+2\right)\left(4\Delta_\lambda^2-1\right)}{(6b^2+16\Delta+3)\left((16\Delta+3)b^2+6\right)}$$

$$F_{\beta,e}^{\lambda(-),2} = 4(-1+2\Delta_\lambda)\Big(180(1+b^8)(1+2\Delta_\lambda)(1+2\Delta_\lambda^2+8\Delta)$$

$$+12(b^2+b^6)\Big(3(39+88\Delta)+4\big(\Delta_\lambda(89+2\Delta_\lambda(-33+59\Delta_\lambda))$$

$$+4\Delta_\lambda(149+2\Delta_\lambda(-47+6\Delta_\lambda))\Delta+64(-3+8\Delta_\lambda)\Delta^2\big)\Big)$$

$$+b^4\Big(4\Delta_\lambda^3(2775+128\Delta(7+2\Delta))-2\Delta_\lambda^2(4725+2432\Delta(7+2\Delta))$$

$$-9(-305+8\Delta(31+32\Delta(1+8\Delta)))$$

$$+6\Delta_\lambda(1525+8\Delta(1047+32\Delta(39+8\Delta))))\Big)\Big)$$

$$\Big((3+6b^2+16\Delta)(11+30b^2+16\Delta)(6+b^2(3+16\Delta))(30+b^2(11+16\Delta))\Big)^{-1}$$

$$F_{\beta,e}^{*\lambda(-),0} = 2\beta e^{i\frac{\pi}{4}}$$

$$F_{\beta,e}^{*\lambda(-),1} = 8\beta e^{i\frac{\pi}{4}}\Big(6+12\Delta_\lambda^2+48\Delta+6b^4(1+2\Delta_\lambda^2+8\Delta)$$

$$+b^2\big(15+6(4-3\Delta_\lambda)\Delta_\lambda+8\Delta+32\Delta_\lambda(2+\Delta_\lambda)\Delta+128\Delta^2\big)\Big)$$

$$\times\Big((3+6b^2+16\Delta)(6+b^2(3+16\Delta))\Big)^{-1}$$

$$F_{\beta,\mathrm{e}}^{*\lambda(-),2} = 8\beta e^{i\frac{\pi}{4}} \bigg( 180(1+b^8)(1+2\Delta_\lambda^2+8\Delta_\beta)(21+4\Delta_\lambda^2+32\Delta_\beta)$$

$$+ b^4 \Big( 15\big(2387+2\Delta_\lambda(-122+5\Delta_\lambda(699+4\Delta_\lambda(-50+37\Delta_\lambda))))\big)$$

$$+ 8\big(29337+4\Delta_\lambda(-5794+\Delta_\lambda(7627+224(-10+\Delta_\lambda)\Delta_\lambda)))\big)\Delta_\beta$$

$$+ 512\big(809+\Delta_\lambda(-214+\Delta_\lambda(327+4(-10+\Delta_\lambda)\Delta_\lambda)))\big)\Delta_\beta^2$$

$$+ 2048(71+20(-2+\Delta_\lambda)\Delta_\lambda)\Delta_\beta^3+65536\Delta_\beta^4 \Big)$$

$$+ 12(b^2+b^6)\Big( 1729-200\Delta_\lambda^3(5+16\Delta_\beta)+16\Delta_\lambda^4(59+24\Delta_\beta)$$

$$-2\Delta_\lambda(61+128\Delta_\beta(21+34\Delta_\beta))+8\Delta_\beta(1677+32\Delta_\beta(65+48\Delta_\beta))$$

$$+4\Delta_\lambda^2(1167+32\Delta_\beta(87+56\Delta_\beta)) \Big) \bigg)$$

$$\times \Big( \big(3+6b^2+16\Delta_\beta\big)\big(11+30b^2+16\Delta_\beta\big)\big(6+b^2(3+16\Delta_\beta)\big)\big(30+b^2(11+16\Delta_\beta)\big) \Big)^{-1}$$

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